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CONTROL OF A LINEARIZED VISCOUS LIQUID–TANK SYSTEM WITH SURFACE TENSION*

IASSON KARAFYLLIS† AND MIROSLAV KRSTIC‡

Abstract. This paper studies the linearization of the viscous tank–liquid system. The linearization of the tank–liquid system gives a high-order partial differential equation, which is a combination of a wave equation with Kelvin–Voigt damping and a Euler–Bernoulli beam equation. The single input appears in two of the boundary conditions (boundary input). The paper provides results both for the open-loop system (existence/uniqueness of solutions and stability properties of the open-loop system) as well as results for the construction of feedback stabilizers. More specifically, the feedback design methodology is based on control Lyapunov functionals (CLFs). The proposed CLFs are modifications and augmentations of the total energy functionals for the tank–liquid system so that the dissipative effects of viscosity, friction, and surface tension are captured. By focusing on the linearized water–tank system, we are able to provide results that are not provided in the nonlinear case: (1) existence and uniqueness of solutions, (2) simultaneous presence of friction and surface tension, and (3) stabilization in a stronger norm, using a different CLF.

Key words. Saint-Venant model, shallow water equations, feedback stabilization, control Lyapunov functional, higher-order partial differential equations

MSC codes. 93C20, 93B52, 93D30, 35G46, 76D55, 76D45, 76D27

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1. Introduction.

In this paper, we consider the system described by the following partial differential equation (PDE) for $t>0$, $x\in(0,L)$,

$$\varphi_{tt}=c^2\varphi_{xx}-\sigma h^*\varphi_{xxxx}+\mu\varphi_{txx}-\bar{\kappa}\varphi_t; \tag{1.1}$$

the following ordinary differential equations (ODEs) for $t\geq 0$,

$$\dot{\xi}=w,\quad \dot{w}=-f; \tag{1.2}$$

and the following additional (boundary and nonlocal) conditions for $t\geq 0$:

$$\varphi_x(t,0)=\varphi_x(t,L)=0, \tag{1.3}$$

$$\varphi_{xxx}(t,0)=\varphi_{xxx}(t,L)=-\sigma^{-1}f(t), \tag{1.4}$$

$$\int_0^L\varphi(t,x)dx=\int_0^L\varphi_t(t,x)dx=0, \tag{1.5}$$

where $c,\sigma,h^*,\mu,L>0$, $\bar{\kappa}\geq 0$ are constants and $f(t)\in\mathbb{R}$ is an external (control) input. Model (1.1), (1.2), (1.3), (1.4), and (1.5) represents the linearization of the classical Saint-Venant model of liquid–tank motion in which the liquid is viscous with viscosity

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†Department of Mathematics, National Technical University of Athens, 15780 Athens, Greece (iasonkaraf@gmail.com).
‡Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411 USA (krstic@ucsd.edu).

μ, has surface tension σ, and exhibits friction $\bar{\kappa}$ with the tank walls. This particular linearized PDE–ODE model is reminiscent of both wave and Euler–Bernoulli (EB) beam equations, with Kelvin–Voigt damping and ordinary damping added.

Since such a wave-beam PDE combination has, to our knowledge, not been a subject of study as either an abstract system or as a liquid–tank model, it is important to develop results for it on well-posedness and stabilization, in particular norms. However, it is equally important to understand the model's linkage with various related PDE systems, of which some are simpler—the wave and EB PDEs—and some are related but more complex (linearized models of aeroelasticity/flutter). We discuss that next.

On the one hand, for $\sigma = 0$, one notes that (1.1) is a wave equation, with wave speed $c = \sqrt{gh^*}$, where $h^* > 0$ is the liquid height at equilibrium while g is the acceleration of gravity. Such a wave PDE governs the wave dynamics of a liquid that has no surface tension. On the other extreme, for $c = 0$ (namely, for $g = 0$), one recognizes that (1.1) is the EB beam equation, governing the dynamics of the surface of a thin layer/film of liquid with nonnegligible surface tension. The wave and beam effects appear combined in this model because of the interaction between surface tension (a beamlike effect) and the liquid motion (a wavelike effect).

The classical interaction between elastic structures and fluids appears in aeroelasticity [1, 48], also known under the alternative (but not precisely synonymous) names of flutter, flow-induced vibration, and fluid–structure interaction. Flutter arises in many systems, of which aircraft wings and gas turbine compressor blades are the most well-known examples. Flutter models in general are far more complex, even when linearized, than the wave–beam equation (1.1). The complexity of flutter models is due to the fact that, while in the liquid–tank system, there is no net equilibrium motion of the liquid in the tank relative to the fluid's surface, in the wing flutter, there is a considerable velocity differential between the wing/beam and the air flowing over the wing. In the extreme case of high Mach number flight, the elastic effect of the wing is entirely dominated by the fluid motion, and the flow-induced vibrations are approximated by a wave equation [48], with elasticity absent but antidamping present due to the instability induced by the flow.

Hence, the work we undertake here on (1.1), (1.2), (1.3), (1.4), and (1.5) should be seen in the broader context of control of fluid–structure interaction. Furthermore, it should be observed that the liquid–tank system with surface tension is a fluid–structure system of a very particular (limited) kind. In a tank, the dynamics of the water, constrained from the sides and bottom by the rigid tank and from above by the skinlike elastic effect of surface tension, have a "waterbed"-like quality. This is the quality of conservation of the mean height and velocity of the water (recall (1.5)), along with smoothing of the water surface by the surface tension.

Since (1.1), (1.2), (1.3), (1.4), and (1.5) is evidently a system of a very particular kind, with little history of prior study of the system, it is appropriate, and even helpful, to terminologically differentiate this system from its wave, beam, and other PDE "cousins." A name like, for instance, "waterbed PDE," seems appropriate and physically descriptive of this mathematical system. We shall occasionally refer to (1.1) as the (linear) "waterbed PDE."

Liquid–tank system: Application, modeling, and control. The liquid–tank system has an intensely practical motivation. For instance, in [49, 50], the challenges that arise in spacecraft operation because of the liquid fuel sloshing during spacecraft transfer and maneuvering, due to the dynamic interaction between the liquid and the rigid body, are described. One way of mitigating this interaction is through a design of devices that can achieve sloshing suppression (see, for instance, Chapter 3 in [29]). However, suppression of the unsteady motion of the liquid by active control is superior in achieving suppression because it requires no additional devices. The actuators that perform the transfer, and thus induce the fluid–body unsteady interaction, are available also for suppressing this interaction.

From a mathematical point of view, the description of sloshing is a highly nontrivial modeling problem. Two main approaches have been used for the modeling of free surface flows of incompressible liquids in the literature: (i) the use of the fluid momentum equations under the assumption of the irrotational flow of the liquid (see, for instance, [29, 30, 39]) and (ii) the use of the fluid momentum equations for the derivation of Saint-Venant models (see [3], the first paper by Adhémar Jean Claude Barré de Saint-Venant in 1871) by neglecting the fluid motion in the direction of the liquid height. The Saint-Venant model or shallow water model is a well-known mathematical model that has been used extensively, and many modifications of this model take into account various types of forces such as gravity, viscous stresses, surface tension, and friction forces (see [7, 8, 9, 12, 23, 37, 38, 39, 41, 42, 52]). In this review, we focus on sloshing induced by the movement of the container and on 1-D Saint-Venant models for the description of the liquid motion.

Control studies of the Saint-Venant model have focused on the inviscid Saint-Venant model (i.e., the model that ignores viscous stresses and surface tension) and its linearization around an equilibrium point (see [4, 5, 6, 13, 14, 15, 16, 18, 19, 20, 21, 22, 26, 27, 40, 44, 45]). The feedback design has been performed by employing either the backstepping methodology or the control Lyapunov functional (CLF) methodology. Although there is no inviscid liquid, the use of the inviscid Saint-Venant model is justified when studying the flow in rivers: In this case, the inertial, gravity, and friction forces are orders of magnitude larger than the viscous stresses, and the effect of viscosity is negligible.

However, when one studies the flow in a tank, there is no guarantee that the effect of viscosity is negligible because the velocity of the fluid is (expected to be) relatively small. To this purpose, viscous Saint-Venant models have been proposed and studied in [7, 8, 23, 32, 33, 34, 35, 37, 42, 52]. From a mathematical point of view, the effect of the viscosity is huge: In the case where surface tension is absent, the system is described by two ODEs, one first-order hyperbolic PDE, and one parabolic PDE, whereas in the inviscid case, we have two ODEs and two first-order hyperbolic PDEs.

From the point of view of applications, when one studies the flow in a tank, the avoidance of the phenomenon of liquid spilling out of the tank becomes as important as the sloshing problem. Thus, the solution of the spill-free and slosh-free movement problem by means of a robust feedback law becomes a significant mathematical problem with possibly important applications. The recent works [32, 33, 34, 35] study this particular feedback stabilization problem for the viscous Saint-Venant model without linearization around an equilibrium point. More specifically, in [32, 33], a feedback control law is constructed by employing the CLF methodology for the viscous Saint-Venant model without wall friction and surface tension (state feedback in [32] and output feedback in [33]). In [34, 35], it is shown that the same feedback control law proposed in [32] works even if friction forces or surface tension are present. It should be noticed that [32, 33, 34, 35] are the only works that guarantee a spill-free movement of the fluid.

Linearized viscous liquid–tank system (waterbed PDE). In the present work, we study the linearized version of the viscous liquid–tank system. The linearization gives the high-order PDE (1.1), which is a combination of a wave equation with Kelvin–

Voigt damping and an EB beam equation. The single input appears in two of the boundary conditions (boundary input). This particular ODE–PDE system has not been studied before in the literature and can also describe a waterbed–tank system, with the tank being actuated. Our work provides results both for the open-loop system (see Theorem 1 for existence/uniqueness of solutions and Theorem 2 for the stability properties of the open-loop system), as well as results for the construction of feedback stabilizers (see Theorem 4 below). There are important additional results that can be provided for the linearization compared to the nonlinear liquid–tank system:

(1) In the linearized case, we provide existence/uniqueness results for the closed-loop system. In the nonlinear case, we do not provide existence/uniqueness results for the corresponding closed-loop system.
(2) In the linearized case, we can study the situation where both friction and surface tension are present. In the nonlinear case, we cannot study the situation where both friction and surface tension are present.
(3) The state norm for which stabilization is achieved in the linearized case is *stronger* than the state norm for which stabilization is achieved in the nonlinear case. This difference is explained by the difference of the CLFs in the nonlinear and the linearized case. The Lyapunov functional in the linearized case is a linear combination of four functionals: (i) the Lyapunov function for the tank, (ii) the mechanical energy of the liquid, (iii) the modified mechanical energy of the liquid, and (iv) the energy of the liquid that is obtained if one considers the liquid as a beam. The first three functionals correspond to functionals that are also used for the construction of a Lyapunov functional in the nonlinear case in [32, 34, 35]. However, the last functional—the beam energy—has no nonlinear counterpart. The constructed CLF is a weighted H^2-quadratic functional with weights depending on the controller gains.

The paper is structured as follows. In section 2, we describe the mathematical ODE–PDE model of the liquid–tank system and its linearization. In section 3, we provide the main results of the paper and a detailed comparison with the results for the nonlinear case. Section 4 is devoted to the proofs of all results in the paper. Finally, in section 5, we give the concluding remarks of the present work as well as some problems that remain open and can be topics for future research.

Notation. Throughout the article, we adopt the following notation.

* $\mathbb{R}_+ = [0, +\infty)$ denotes the set of nonnegative real numbers.
* Let $S \subseteq \mathbb{R}^n$ be an open set, and let $A \subseteq \mathbb{R}^n$ be a set that satisfies $S \subseteq A \subseteq cl(S)$. By $C^0(A\,;\,\Omega)$, we denote the class of continuous functions on A that take values in $\Omega \subseteq \mathbb{R}^m$. By $C^k(A\,;\,\Omega)$, where $k \geq 1$ is an integer, we denote the class of functions on $A \subseteq \mathbb{R}^n$ that take values in $\Omega \subseteq \mathbb{R}^m$ and have continuous derivatives of order k. In other words, the functions of class $C^k(A;\Omega)$ are the functions that have continuous derivatives of order k in $S = int(A)$ that can be continued continuously to points in $\partial S \cap A$. When $\Omega = \mathbb{R}$, then we write $C^0(A)$ or $C^k(A)$.
* Let $I \subseteq \mathbb{R}$ be an interval, and let Y be a normed linear space. By $C^0(I\,;\,Y)$, we denote the class of continuous functions on I that take values in Y. By $C^1(I\,;\,Y)$, we denote the class of continuously differentiable functions on I that take values in Y.
* Let $I \subseteq \mathbb{R}$ be an interval, let $a < b$ be given constants, and let $u : I \times [a,b] \to \mathbb{R}$ be a given function. We use the notation $u[t]$ to denote the profile at certain $t \in I$; i.e., $(u[t])(x) = u(t,x)$ for all $x \in [a,b]$. When $u(t,x)$ is

(twice) differentiable with respect to $x \in [a,b]$, we use the notation $u_x(t,x)$ ($u_{xx}(t,x)$) for the (second) derivative of u with respect to $x \in [a,b]$. When $u(t,x)$ is differentiable with respect to t, we use the notation $u_t(t,x)$ for the derivative of u with respect to t; i.e., $u_t(t,x) = \frac{\partial u}{\partial t}(t,x)$. When $u[t] \in X$ for all $t \in I$, where X is a normed linear space with norm $\|\|_X$ and the mapping $I \ni t \to u[t] \in X$ is C^1—i.e., there exists a continuous mapping $v : I \to X$ with $\lim_{h\to 0}(\|h^{-1}(u[t+h] - u[t]) - v[t]\|_X) = 0$ for all $t \in I$—we use the notation u_t for v. Furthermore, when $u \in C^1(I;X)$ and the mapping $I \ni t \to u_t[t] \in X$ is C^1—i.e., there exists a continuous mapping $w : I \to X$ with $\lim_{h\to 0}(\|h^{-1}(u_t[t+h] - u_t[t]) - w[t]\|_X) = 0$ for all $t \in I$—we use the notation u_{tt} for w. Mixed derivatives are to be understood in this way. For example, when $u_x \in C^1(I;X)$ (i.e., when there exists a continuous mapping $\varphi : I \to X$ with $\lim_{h\to 0}(\|h^{-1}(u_x[t+h] - u_x[t]) - \varphi[t]\|_X) = 0$ for all $t \in I$), we use the notation u_{xt} for φ.

* Given a set $U \subseteq \mathbb{R}^n$, χ_U denotes the characteristic function of U, i.e., the function defined by $\chi_U(x) := 1$ for all $x \in U$ and $\chi_U(x) := 0$ for all $x \notin U$.
* Let $a < b$ be given constants. For $p \in [1,+\infty)$, $L^p(a,b)$ is the set of equivalence classes of Lebesgue measurable functions $u : (a,b) \to \mathbb{R}$ with $\|u\|_p := (\int_a^b |u(x)|^p dx)^{1/p} < +\infty$. The scalar product in $L^2(a,b)$ is denoted by $\langle \bullet, \bullet \rangle$; i.e., $\langle f,g \rangle = \int_a^b f(x)g(x)dx$ for all $f,g \in L^2(a,b)$. $L^\infty(a,b)$ is the set of equivalence classes of Lebesgue measurable functions $u : (a,b) \to \mathbb{R}$ with $\|u\|_\infty := ess\sup_{x\in(a,b)}(|u(x)|) < +\infty$. For an integer $k \geq 1$, $H^k(a,b)$ denotes the Sobolev space of functions in $L^2(a,b)$ with all its weak derivatives up to order $k \geq 1$ in $L^2(a,b)$.

2. The mathematical model.

We consider a 1-D model for the motion of a tank. The tank contains a viscous, Newtonian, incompressible liquid. The tank is subject to a force that can be manipulated. We assume that the liquid pressure is hydrostatic, and consequently, the liquid is modeled by the 1-D viscous Saint-Venant equations, whereas the tank obeys Newton's second law, and consequently, we consider the tank acceleration to be the control input.

2.1. The general 1-D model.

We next give a general 1-D model for the liquid–tank system that takes into account all possible forces exerted on the fluid: gravity, viscous stresses, surface tension, and friction. Let the position of the left side of the tank at time $t \geq 0$ be $a(t)$, and let the length of the tank be $L > 0$ (a constant). The equations describing the motion of the liquid within the tank are

$$H_t + (H\bar{v})_z = 0, \text{ for } t > 0,\ z \in [a(t), a(t)+L], \tag{2.1}$$

$$\begin{aligned} &(H\bar{v})_t + \left(H\bar{v}^2 + \frac{1}{2}gH^2\right)_z - \sigma H\left(\frac{H_{zz}}{(1+H_z^2)^{3/2}}\right)_z \\ &= \mu\left(H\bar{v}_z\right)_z - \kappa\left(H(t,z), \bar{v}(t,z) - \dot{a}(t)\right)\left(\bar{v}(t,z) - \dot{a}(t)\right) \\ &\text{for } t > 0,\ z \in (a(t), a(t)+L), \end{aligned} \tag{2.2}$$

where $H(t,z) > 0$, $\bar{v}(t,z) \in \mathbb{R}$ are the liquid level and the liquid velocity, respectively, at time $t \geq 0$ and position $z \in [a(t), a(t)+L]$, $\kappa \in C^0((0,+\infty)\times\mathbb{R};\mathbb{R}_+)$ is the friction coefficient that depends on the liquid level and the relative velocity of the fluid with respect to the tank, while $g,\mu > 0$, $\sigma \geq 0$ (constants) are the acceleration of gravity, the kinematic viscosity of the liquid, and the ratio of the surface tension and liquid density, respectively. In certain works, the term $(\frac{H_{zz}}{(1+H_z^2)^{3/2}})_z$ is replaced by H_{zzz} (see [7, 8, 9, 41]), but here, we use a more accurate description of the surface tension. Equations (2.1) and (2.2) can be derived by performing mass and momentum balances (from first principles, assuming that the liquid pressure is the combination of hydrostatic pressure and capillary pressure given by the Young–Laplace equation; see [17]). Various empirical relations have been used for the friction coefficient in the literature; see [6, 8, 21, 23, 26].

The liquid velocities at the walls of the tank must coincide with the tank velocity; i.e., we have

$$\bar{v}(t,a(t)) = \bar{v}(t,a(t)+L) = w(t) \text{ for } t \geq 0, \tag{2.3}$$

where $w(t) = \dot{a}(t)$ is the velocity of the tank at time $t \geq 0$. Moreover, since the tank acceleration is the control input, we get

$$\ddot{a}(t) = -f(t) \text{ for } t > 0, \tag{2.4}$$

where $-f(t)$, the control input to the problem, is equal to the force exerted on the tank at time $t \geq 0$ divided by the total mass of the tank. Using (2.1) and (2.3), it becomes clear that every classical solution of (2.1) and (2.3) satisfies $\frac{d}{dt}(\int_{a(t)}^{a(t)+L} H(t,z)dz) = 0$ for all $t > 0$. Therefore, the total mass of the liquid is constant. Thus, without loss of generality, we assume that the following equation holds:

$$\int_{a(t)}^{a(t)+L} H(t,z)dz \equiv m, \tag{2.5}$$

where $m > 0$ is the total mass of the liquid divided by the product of liquid density times the width of the tank. It should be emphasized that, for obvious physical reasons, the liquid level $H(t,z)$ must be positive for all times; i.e., we must have

$$\min_{x\in[0,L]} (H(t,a(t)+x)) > 0 \text{ for } t \geq 0. \tag{2.6}$$

For a complete mathematical model of the system in the case $\sigma > 0$ (the case where surface tension is present), we need two additional boundary conditions that describe the interaction between the liquid and the solid walls of the tank. There are many ways to describe the evolution of the angle of contact of a liquid with a solid boundary (see the detailed presentation in [36]). In [46, 47], the use of a constant contact angle was suggested based on energy arguments and the fact that there may be a discrepancy between the actual microscopic and the apparent macroscopic contact angle. Moreover, the assumption of a constant contact angle allows the well-posedness of the overall problem (at least for small data; see [46, 47, 54]). The constant contact angle approach has been used extensively in the literature (see, for instance, [28, 54, 55]). In this work, we adopt the constant contact angle approach by imposing a contact angle equal to $\pi/2$. Therefore, the model is accompanied by the following boundary conditions (written in a way that holds even in the case $\sigma = 0$, i.e., the case where surface tension is absent and the additional boundary conditions are not needed):

$$\sigma H_z(t,a(t)) = \sigma H_z(t,a(t)+L) = 0 \text{ for } t \geq 0. \tag{2.7}$$

Applying the transformation

$$\begin{aligned} v(t,x) &= \bar{v}(t,a(t)+x) - w(t), \\ h(t,x) &= H(t,a(t)+x), \\ \xi(t) &= a(t) - a^*, \end{aligned} \tag{2.8}$$

where $a^* \in \mathbb{R}$ is the specified position (a constant) to which we want to bring (and maintain) the left side of the tank, we obtain the following model:

$$\dot{\xi} = w, \quad \dot{w} = -f \text{ for } t \geq 0, \tag{2.9}$$

$$h_t + (hv)_x = 0 \text{ for } t > 0, \ x \in [0, L], \tag{2.10}$$

$$\begin{aligned}(hv)_t + \left(hv^2 + \frac{1}{2}gh^2\right)_x - \sigma h \left(\frac{h_{xx}}{(1 + h_x^2)^{3/2}}\right)_x \\ = \mu (hv_x)_x - \kappa(h, v)v + hf \\ \text{for } t > 0, \ x \in (0, L),\end{aligned} \tag{2.11}$$

$$v(t, 0) = v(t, L) = 0 \text{ for } t \geq 0, \tag{2.12}$$

$$\int_0^L h(t, x)dx = m \text{ for } t \geq 0, \tag{2.13}$$

$$\sigma h_x(t, 0) = \sigma h_x(t, L) = 0 \text{ for } t \geq 0, \tag{2.14}$$

$$\min_{x \in [0,L]} (h(t, x)) > 0 \text{ for } t \geq 0. \tag{2.15}$$

The open-loop system (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), and (2.15)—i.e., system (2.9), (2.10), (2.11), (2.12), (2.13), (2.14) and (2.15) with $f(t) \equiv 0$—allows a continuum of equilibria, namely, the points

$$h(x) \equiv h^*, \ v(x) \equiv 0 \text{ for } x \in [0, L], \tag{2.16}$$

$$\xi \in \mathbb{R}, w = 0, \tag{2.17}$$

where $h^* = m/L$. The existence of a continuum family of equilibrium points for the open-loop system given by (2.16) and (2.17), with the family parameterized by an arbitrary position of the tank while the liquid is at a unique and spatially constant height, implies that the desired equilibrium point—i.e., the equilibrium point with $\xi = 0$—is not asymptotically stable for the open-loop system.

2.2. The linearization of the liquid–tank system. Linearizing model (2.9), (2.10), (2.11), (2.12), (2.13), and (2.14) with $\sigma > 0$ around the equilibrium point $h(x) \equiv h^* = m/L$, $v(x) \equiv 0$ and setting $\varphi = h - h^* \chi_{[0,L]}$, we obtain the linear PDE–ODE model (1.1), (1.2), (1.3), (1.4), and (1.5), where $c = \sqrt{gh^*}$ and $\bar{\kappa} \geq 0$ are constants. The control system (1.1), (1.2), (1.3), (1.4), and (1.5) is a system that has not been studied so far in the literature. The control input appears in the ODEs (1.2) and in the boundary condition (1.4) (boundary control). It should be noticed that the control input is a boundary input for the linearized model (1.1), (1.2), (1.3), (1.4), and (1.5), while it is not a boundary input for the nonlinear system. This is not a result of the linearization but a result of the conversion of the linearized model to a single second-order in time PDE.

The distributed subsystem (1.1), (1.3), and (1.4) is a combination of an EB beam equation (with a Young modulus σh^*) and a wave equation (with a wave speed $c = \sqrt{gh^*}$) with additional internal Kelvin–Voigt damping and possible friction. Equation (1.1) appears in the study of incompressible fluids flowing underground in a fractured or fissured medium, where φ is the pressure of the fluid in the porous part of the medium; see [24, pp. 217–218]. If (1.1) is to be interpreted as a beam equation, then the boundary conditions (1.3) and (1.4) mean that the beam ends are subject to "a paired force actuation" and sliding orthogonally to the beam (see also [51]). Condition

(1.5) comes from the fact that the "mean displacement" and mean velocity in a vessel like a tank can neither be lost nor added. Therefore, in this setting, (1.1) and (1.5) model neither a fluid alone nor a beam but a fluid–beam system akin to a "waterbed," in which the fabric that constrains the motion of the water underneath it is analogous to the surface tension, as captured by the term $-\sigma h^* \varphi_{xxxx}$ in (1.1). Consequently, the overall system (1.1), (1.2), (1.3), (1.4), and (1.5) can be interpreted as a water–tank system where the water surface is effectively covered with a skin/fabric, whose dynamics are beamlike in interaction with the water. So, (1.1), (1.2), (1.3), (1.4), and (1.5) is a linearized waterbed–tank system, with the tank being actuated.

The open-loop eigenvalues of the distributed subsystem (1.1), (1.3), (1.4), and (1.5) are the roots of the equation

$$s^2 + \left(\mu \frac{n^2\pi^2}{L^2} + \bar{\kappa}\right) s + \frac{n^2\pi^2}{L^2}\left(c^2 + \sigma h^* \frac{n^2\pi^2}{L^2}\right) = 0 \text{ for } n = 1, 2, \ldots. \tag{2.18}$$

We next provide the eigenvalues in the case $\bar{\kappa} = 0$. If $\mu^2 \leq 4\sigma h^*$, then all eigenvalues are complex and are given by the following formula for $n = 1, 2, \ldots$:

$$s_n = -\frac{\mu}{2L^2} n^2\pi^2 \pm i \frac{n\pi}{L} \sqrt{\frac{4\sigma h^* - \mu^2}{4L^2} n^2\pi^2 + c^2}. \tag{2.19}$$

If $\mu^2 > 4\sigma h^*$, then the eigenvalues are real for $n \geq \frac{2cL}{\pi\sqrt{\mu^2 - 4\sigma h^*}}$ and are given by the following formula for all $n = 1, 2, \ldots$ with $n \geq \frac{2cL}{\pi\sqrt{\mu^2 - 4\sigma h^*}}$:

$$s_n = -\mu \frac{n^2\pi^2}{2L^2} \pm \frac{n\pi}{L} \sqrt{\frac{\mu^2 - 4\sigma h^*}{4L^2} n^2\pi^2 - c^2}. \tag{2.20}$$

In every case ($\mu^2 > 4\sigma h^*$ or $\mu^2 \leq 4\sigma h^*$), we have $\lim_{n\to+\infty}(Re(s_n)) = -\infty$, indicating the strong (internal) damping that is caused by the viscosity of the fluid.

When $\bar{\kappa} > 0$, then, in general, the eigenvalues are moved to the left in the complex plane due to the additional damping caused by the friction term $-\bar{\kappa}\varphi_t$ in the left-hand side of (1.1).

3. Main results.

This section provides the main results of the paper.

3.1. Results for the open-loop system.

Define

$$\bar{S} = \left\{ \varphi \in H^2(0, L) : \varphi'(0) = \varphi'(L) = 0 \right\}. \tag{3.1}$$

For the distributed subsystem (1.1), (1.3), (1.4), and (1.5), we are in a position to show the following results, which guarantee well-posedness and exponential stability.

Theorem 1 (existence/uniqueness of solutions for the open-loop system). *For every $\varphi_0 \in \bar{S} \cap H^4(0, L)$, $p_0 \in \bar{S}$, and $f \in C^3(\mathbb{R}_+)$ with $\varphi_0'''(0) = \varphi_0'''(L) = -\sigma^{-1} f(0)$, there exists a unique function $\varphi \in C^0(\mathbb{R}_+; \bar{S} \cap H^4(0, L)) \cap C^1(\mathbb{R}_+; \bar{S}) \cap C^2(\mathbb{R}_+; L^2(0, L))$ with $\varphi[0] = \varphi_0$, $\varphi_t[0] = p_0$ that satisfies* (1.1), (1.3), *and* (1.4) *for all $t \geq 0$. Moreover, if $\int_0^L \varphi_0(x)dx = \int_0^L p_0(x)dx = 0$, then* (1.5) *holds.*

Theorem 2 (stability properties of the open-loop system). *There exist constants $\bar{M}, \bar{\lambda}, \Gamma > 0$ such that, for every $f \in C^0(\mathbb{R}_+)$ and for every function $\varphi \in C^0(\mathbb{R}_+; \bar{S} \cap H^4(0, L)) \cap C^1(\mathbb{R}_+; \bar{S}) \cap C^2(\mathbb{R}_+; L^2(0, L))$ that satisfies* (1.1), (1.3), (1.4), *and* (1.5) *for all $t \geq 0$, the following estimate holds:*

$$P(t) \leq \bar{M} \exp\left(-\bar{\lambda} t\right) P(0) + \Gamma \max_{0 \leq s \leq t} \left(\exp\left(-\bar{\lambda}(t - s)\right) |f(s)|\right) \text{ for } t \geq 0, \tag{3.2}$$

where

$$P(t) := \left(\|\varphi[t]\|_2^2 + \|\varphi_x[t]\|_2^2 + \|\varphi_{xx}[t]\|_2^2 + \|\varphi_t[t]\|_2^2 \right)^{1/2} \text{ for } t \geq 0. \tag{3.3}$$

Remarks. (a) Theorem 1 shows that condition (1.5) defines a positively invariant subspace of the state space $(\bar{S} \cap H^4(0,L)) \times \bar{S}$. When the initial condition satisfies (1.5), then the solution satisfies (1.5) for all $t \geq 0$. (b) Estimate (3.2) shows that the distributed subsystem (1.1), (1.3), (1.4), and (1.5) satisfies the input-to-state stability property (see [31]) with respect to the boundary input f in the $H^2(0,L) \times L^2(0,L)$ norm of the state (φ, φ_t).

3.2. Feedback stabilization. Theorem 2 shows that the linearized model (1.1), (1.2), (1.3), (1.4), and (1.5) is the interconnection of a double integrator ODE subsystem (recall (1.2)) with the exponentially stable distributed subsystem (1.1), (1.3), (1.4), and (1.5). The only connection of the ODE subsystem with the PDE subsystem is the control input that appears in both subsystems (otherwise, the subsystems are completely independent). This structural feature allows us to consider two different ways of stabilizing the equilibrium point $(\xi, w) = 0 \in \mathbb{R}^2$, $(\varphi, \varphi_t) = 0 \in \bar{S} \times L^2(0,L)$ of system (1.1), (1.2), (1.3), (1.4), and (1.5).

First way of stabilization. We can stabilize exponentially the equilibrium point $(\xi, w) = 0 \in \mathbb{R}^2$, $(\varphi, \varphi_t) = 0 \in \bar{S} \times L^2(0,L)$ of system (1.1), (1.2), (1.3), (1.4), and (1.5) by completely ignoring the liquid dynamics and using the feedback law

$$f(t) = k_1 \xi(t) + k_2 w(t), \tag{3.4}$$

where $k_1, k_2 > 0$ are constants. Then, using Theorem 1 and Theorem 2, we conclude that, for every $(\xi_0, w_0) \in \mathbb{R}^2$, $\varphi_0 \in \bar{S} \cap H^4(0,L)$, $p_0 \in \bar{S}$ with $\int_0^L \varphi_0(x)dx = \int_0^L p_0(x)dx = 0$ and $\varphi_0'''(0) = \varphi_0'''(L) = -\sigma^{-1}(k_1\xi_0 + k_2 w_0)$, there exist unique functions $\varphi \in C^0(\mathbb{R}_+; \bar{S} \cap H^4(0,L)) \cap C^1(\mathbb{R}_+; \bar{S}) \cap C^2(\mathbb{R}_+; L^2(0,L))$, $(\xi, w) \in C^\infty(\mathbb{R}_+; \mathbb{R}^2)$ with $(\xi(0), w(0)) = (\xi_0, w_0)$, $\varphi[0] = \varphi_0$, $\varphi_t[0] = p_0$ that satisfy (1.1), (1.2), (1.3), (1.4), and (1.5) and (3.4) for all $t \geq 0$. Moreover, there exist constants $\tilde{M}, \tilde{\lambda} > 0$ such that the following estimate holds for ≥ 0:

$$\begin{aligned} &\sqrt{\xi^2(t) + w^2(t) + P^2(t)} \\ &\quad \leq \tilde{M} \exp\left(-\tilde{\lambda}\, t\right) \sqrt{\xi^2(0) + w^2(0) + P^2(0)}, \end{aligned} \tag{3.5}$$

where $P(t)$ is defined by (3.3).

However, there is a problem for the feedback law (3.4). Since we have ignored completely the liquid dynamics, it is possible that the overshoot for the state component φ (i.e., the deviation of the liquid level from the equilibrium level) is large even if the liquid starts from an almost slosh-free initial condition. In other words, it is possible that the feedback law (3.4) agitates strongly the liquid causing sloshing during a transient period.

Second way of stabilization. The following result plays a fundamental role in what follows.

THEOREM 3 (well-posedness of the closed-loop system under an arbitrary feedback law). *Let $B, C \in \mathbb{R}$ be constants, and let $\tilde{r}, \tilde{p} \in C^0([0,L])$ be given functions. Suppose that*

$$\langle \tilde{g}, \tilde{r} \rangle \geq \frac{12L^4}{\pi^4}\sqrt{\frac{2}{L}}\sum_{n\ odd}\frac{|\langle \phi_n, \tilde{r}\rangle|}{n^4},$$
(3.6)
$$\langle \tilde{g}', \tilde{p} \rangle \geq \frac{12L^4}{\pi^4}\sqrt{\frac{2}{L}}\sum_{n\ odd}\frac{|\langle \phi_n', \tilde{p}\rangle|}{n^4},$$

where $\phi_n(x) = \sqrt{\frac{2}{L}}\cos(n\pi\frac{x}{L})$ *for* $n = 1, 2, \ldots$ *and*

$$\tilde{g}(x) := x^3 - \frac{3L}{2}x^2 + \frac{L^3}{4} \text{ for } x \in [0, L]. \tag{3.7}$$

Then, for every $(\xi_0, w_0) \in \mathbb{R}^2$, $\varphi_0 \in \bar{S} \cap H^4(0, L)$, $u_0 \in \bar{S}$ *with* $\int_0^L \varphi_0(x)dx = \int_0^L u_0(x)dx = 0$, $\varphi_0'''(0) = \varphi_0'''(L) = -\sigma^{-1}(B\xi_0 + Cw_0 + \langle u_0, \tilde{r}\rangle + \langle \varphi_0', \tilde{p}\rangle)$, *there exist unique functions* $\varphi \in C^0(\mathbb{R}_+; \bar{S} \cap H^4(0, L)) \cap C^1(\mathbb{R}_+; \bar{S}) \cap C^2(\mathbb{R}_+; L^2(0, L))$, $(\xi, w) \in C^1(\mathbb{R}_+; \mathbb{R}^2)$ *with* $(\xi(0), w(0)) = (\xi_0, w_0)$, $\varphi[0] = \varphi_0$, $\varphi_t[0] = u_0$ *that satisfy, for all* $t \geq 0$, (1.1), (1.2), (1.3), (1.4), *and* (1.5) *under the control law*

$$f(t) = B\xi(t) + Cw(t) + \langle \varphi_t[t], \tilde{r}\rangle + \langle \varphi_x[t], \tilde{p}\rangle\,. \tag{3.8}$$

Remark. Theorem 3 states that the closed-loop system (1.1), (1.2), (1.3), (1.4), and (1.5) under a feedback law of the form (3.8) is well-posed when inequalities (3.6) are valid.

We next consider the family of feedback laws given by the following formula:

$$\begin{aligned} f(t) &= K\left(k_5^2 w(t) + k_5\xi(t)\right) \\ &\quad - K\left(h^*(k_3 + k_4)\int_0^L x\varphi_t(t, x)dx - k_3\mu h^*\left(\varphi(t, L) - \varphi(t, 0)\right)\right), \end{aligned} \tag{3.9}$$

where $K, k_3, k_4, k_5 > 0$ are the control parameters with

$$k_5^{-3} < \min\left(\frac{c^2}{4k_3\mu\,(h^*)^2\,L}, \frac{\mu\pi^2\left(\mu\pi^2 + 2K\,(h^*)^2\,L^3k_4\right)}{8K\,(h^*)^4\,L^6(k_4 + k_3)^2}, \frac{K}{4}\right). \tag{3.10}$$

Notice that the family of feedback laws (3.9) corresponds to the linearization of the nonlinear feedback laws that were used for the nonlinear system (2.9), (2.10), (2.11), (2.12), (2.13), and (2.14) in [32, 34, 35]. It should be noticed that the family of feedback laws (3.9) and (3.10) is independent of the surface tension coefficient $\sigma > 0$ and the friction coefficient $\bar{\kappa} \geq 0$. Moreover, contrary to (3.4), the feedback law (3.9) is strongly affected by the liquid momentum and the liquid level. Consequently, it is expected that the feedback law (3.9) does not cause agitation of the liquid and tries to compensate between the two control objectives of bringing the tank to a specified position and having the liquid at rest. For the family of feedback laws (3.9) and (3.10), we are in a position to prove the following result.

THEOREM 4 (exponential stabilization by means of liquid-dependent feedback). *Let* $K, k_3, k_4, k_5 > 0$ *be given constants for which* (3.10) *holds. Then, for every* $(\xi_0, w_0) \in \mathbb{R}^2$, $\varphi_0 \in \bar{S} \cap H^4(0, L)$, $u_0 \in \bar{S}$ *with* $\int_0^L \varphi_0(x)dx = \int_0^L u_0(x)dx = 0$ *and*

$$\begin{aligned} \sigma\varphi_0'''(0) &= \sigma\varphi_0'''(L) \\ &= -K\left(k_5^2 w_0 + k_5\xi_0 - h^*(k_3 + k_4)\int_0^L xu_0(x)dx - k_3\mu h^*\left(\varphi_0(L) - \varphi_0(0)\right)\right), \end{aligned}$$

there exist unique functions $\varphi \in C^0(\mathbb{R}_+;\bar{S}\cap H^4(0,L))\cap C^1(\mathbb{R}_+;\bar{S})\cap C^2(\mathbb{R}_+;L^2(0,L))$, $(\xi,w)\in C^1(\mathbb{R}_+;\mathbb{R}^2)$ *with* $(\xi(0),w(0))=(\xi_0,w_0)$, $\varphi[0]=\varphi_0$, $\varphi_t[0]=u_0$ *that satisfy* (1.1), (1.2), (1.3), (1.4), *and* (1.5) *and* (3.9) *for all* $t\geq 0$. *Moreover, there exist constants* $\hat{M},\hat{\lambda}>0$ *such that the following estimate holds for* $t\geq 0$*:*

$$\sqrt{\xi^2(t)+w^2(t)+P^2(t)}\leq \hat{M}\exp\left(-\hat{\lambda}\,t\right)\sqrt{\xi^2(0)+w^2(0)+P^2(0)}, \tag{3.11}$$

where $P(t)$ *is defined by* (3.3).

The proof of Theorem 4 is based on the following Lyapunov functional:

$$\begin{aligned}\tilde{W}\left(\xi,w,\varphi,\varphi_t\right)=&\frac{1}{2}\xi^2+\frac{k_5^2}{2}\left(w+k_5^{-1}\xi\right)^2\\&+\frac{\mu}{K\left(h^*\right)^2L}\left(\frac{1}{2}\left\|\varphi_t\right\|_2^2+\frac{c^2}{2}\left\|\varphi'\right\|_2^2+\frac{\sigma h^*}{2}\left\|\varphi''\right\|_2^2\right)\\&+k_4\left(\frac{1}{2}\left\|\theta\right\|_2^2+\frac{c^2}{2}\left\|\varphi\right\|_2^2+\frac{\sigma h^*}{2}\left\|\varphi'\right\|_2^2\right)\\&+k_3\left(\frac{1}{2}\left\|\theta-\mu\varphi'\right\|_2^2+\frac{c^2+\bar{\kappa}\mu}{2}\left\|\varphi\right\|_2^2+\frac{\sigma h^*}{2}\left\|\varphi'\right\|_2^2\right),\end{aligned} \tag{3.12}$$

where

$$\theta(x)=\int_0^x\varphi_t(s)ds\ \text{ for } x\in[0,L]. \tag{3.13}$$

More specifically, we show that there exists a constant $\omega>0$ such that the solutions of the closed-loop system (1.1), (1.2), (1.3), (1.4), and (1.5) and (3.9) satisfy the differential inequality $\frac{d}{dt}(\tilde{W}(\xi(t),w(t),\varphi[t],\varphi_t[t]))\leq -2\omega\,\tilde{W}(\xi(t),w(t),\varphi[t],\varphi_t[t])$ for all $t\geq 0$.

There are major differences between the results for the linearized system and the results for the nonlinear system in [32, 34, 35].

(1) In the linearized case, we provide existence/uniqueness results for the closed-loop system. In the nonlinear case, existence/uniqueness results for the corresponding closed-loop system are not provided in [32, 34, 35].

(2) In the linearized case, we can study the situation where both friction and surface tension are present. In the nonlinear case, the situation where both friction and surface tension are present is not studied in [32, 34, 35].

(3) There is a big difference in the state norm for which we achieve stabilization. In the nonlinear case, stabilization is achieved in the state norm

$$P_1=\left(\xi^2+w^2+\left\|h-h^*\chi_{[0,L]}\right\|_2^2+\left\|h_x\right\|_2^2+\left\|v\right\|_2^2\right)^{1/2}.$$

On the other hand, in the linearized case, we achieve stabilization in the state norm

$$P_2=\left(\xi^2+w^2+\left\|\varphi\right\|_2^2+\left\|\varphi_x\right\|_2^2+\left\|\varphi_{xx}\right\|_2^2+\left\|\varphi_t\right\|_2^2\right)^{1/2}.$$

In order to compare the two norms, we notice that the linearization of (2.9), (2.9), (2.10), (2.11), (2.12), (2.13), and (2.14) gives $\varphi=h-h^*\chi_{[0,L]}$ as well as the equation $\varphi_t+h^*v_x=0$. Consequently, the state norm P_2 corresponds to the norm

$$P_3=\left(\xi^2+w^2+\left\|h-h^*\chi_{[0,L]}\right\|_2^2+\left\|h_x\right\|_2^2+\left\|h_{xx}\right\|_2^2+\left\|v_x\right\|_2^2\right)^{1/2}.$$

Therefore, the state norm for which stabilization is achieved in the linearized case is *stronger* than the state norm for which stabilization is achieved in the nonlinear case.

(4) The Lyapunov functional used for the linearized case does not correspond exactly to the functionals used for the nonlinear case in [32, 34, 35]. The Lyapunov functional in the linearized case is a linear combination of four functionals: (i) the function $\frac{1}{2}\xi^2+\frac{1}{2a^2}(w+a\xi)^2$, which is the Lyapunov function for the tank; (ii) the functional $\frac{1}{2}\|\theta\|_2^2 + \frac{c^2}{2}\|\varphi\|_2^2 + \frac{\sigma h^*}{2}\|\varphi_x\|_2^2$, which corresponds to the mechanical energy of the liquid; (iii) the functional $\frac{1}{2}\|\theta - \mu\varphi_x\|_2^2 + \frac{c^2+\bar{\kappa}\mu}{2}\|\varphi\|_2^2 + \frac{\sigma h^*}{2}\|\varphi_x\|_2^2$, which corresponds to the modified mechanical energy of the liquid; and (iv) the functional $\frac{1}{2}\|\varphi_t\|_2^2 + \frac{c^2}{2}\|\varphi_x\|_2^2 + \frac{\sigma h^*}{2}\|\varphi_{xx}\|_2^2$, which is the energy of the liquid if one considers the liquid as a beam described by the beamlike equation (1.1). The first three functionals correspond to functionals that are also used for the construction of a Lyapunov functional in the nonlinear case in [32, 34, 35]. However, the last functional, the beam energy $\frac{1}{2}\|\varphi_t\|_2^2 + \frac{c^2}{2}\|\varphi_x\|_2^2 + \frac{\sigma h^*}{2}\|\varphi_{xx}\|_2^2$, has no nonlinear counterpart. This also explains the difference in the state norm for which stabilization is achieved.

4. Proofs. This section provides the proofs of all main results.

Proof of Theorem 1. Let arbitrary $\varphi_0 \in \bar{S} \cap H^4(0,L)$, $p_0 \in \bar{S}$, and $f \in C^3(\mathbb{R}_+)$ with $\varphi_0'''(0) = \varphi_0'''(L) = -\sigma^{-1} f(0)$ be given. We perform the following transformation:

$$\varphi(t,x) = u(t,x) + \bar{r}(x) f(t) \text{ for } t \geq 0,\ x \in [0,L]\,, \tag{4.1}$$

where $\bar{r} : [0,L] \to \mathbb{R}$ is a smooth function that satisfies

$$\bar{r}'(0) = \bar{r}'(L) = \int_0^L \bar{r}(x)dx = 0 \ \text{ and } \ \bar{r}'''(0) = \bar{r}'''(L) = -\sigma^{-1}. \tag{4.2}$$

Then, using (4.1) and (4.2) and (1.1), (1.3), and (1.4), we get the problem

$$u_{tt} = c^2 u_{xx} - \sigma h^* u_{xxxx} + \mu u_{txx} - \bar{\kappa} u_t + \bar{g} \text{ for } t > 0,\ x \in (0,L)\,, \tag{4.3}$$

$$u_x(t,0) = u_x(t,L) = u_{xxx}(t,0) = u_{xxx}(t,L) = 0 \text{ for } t \geq 0, \tag{4.4}$$

where

$$\bar{g}(t,x) = \left(c^2 \bar{r}''(x) - \sigma h^* \bar{r}^{(4)}(x)\right) f(t) + \left(\mu \bar{r}''(x) - \bar{\kappa}\bar{r}(x)\right) \dot{f}(t) - \bar{r}(x)\ddot{f}(t). \tag{4.5}$$

Defining

$$p = u_t, U = \begin{pmatrix} u \\ p \end{pmatrix}, F = \begin{pmatrix} 0 \\ \bar{g} \end{pmatrix}, \tag{4.6}$$

we get from (4.3) and (4.4) the initial-value problem

$$\dot{U} + AU = F \tag{4.7}$$

with

$$U[0] = U_0 = \begin{pmatrix} u_0 \\ p_0 \end{pmatrix},\ u_0(x) = \varphi_0(x) - \bar{r}(x) f(0) \text{ for } x \in [0,L]\,, \tag{4.8}$$

where $A : D(A) \to X_1$ is the linear unbounded operator

$$A = \begin{bmatrix} 0 & -1 \\ \sigma h^* \frac{d^4}{dx^4} - c^2 \frac{d^2}{dx^2} & -\mu \frac{d^2}{dx^2} + \bar{\kappa} \end{bmatrix} \tag{4.9}$$

with X_1 being the real Hilbert space $X_1 = \bar{S} \times L^2(0,L)$ with scalar product defined for all $U = \binom{u}{p} \in X_1, \bar{U} = \binom{\bar{u}}{\bar{p}} \in X_1$

$$\left(U, \bar{U}\right) = \langle u, \bar{u}\rangle + c^2 \langle u', \bar{u}'\rangle + \sigma h^* \langle u'', \bar{u}''\rangle + \langle p, \bar{p}\rangle \tag{4.10}$$

and $D(A) \subset X_1$ being the linear space

$$D(A) = \left\{ U = \begin{pmatrix} u \\ p \end{pmatrix} \in \bar{S}^2 \; : \; u'' \in \bar{S} \right\}. \tag{4.11}$$

Notice that definitions (4.8) and (4.11); the fact that $\varphi_0 \in \bar{S} \cap H^4(0,L)$, $p_0 \in \bar{S}$, and $\varphi_0'''(0) = \varphi_0'''(L) = -\sigma^{-1} f(0)$; and properties (4.2) guarantee that $U_0 \in D(A)$.

We next show that there exists $\bar{q} \geq 0$ such that the operator $A + \bar{q}I$, where I is the identity operator, is a maximal monotone operator. Using (4.9) and (4.10), we get, for all $U = \binom{u}{p} \in D(A)$ and $\bar{q} \geq 0$,

$$\begin{aligned} ((A+\bar{q}I)U,U) = &- \langle u,p\rangle - c^2 \langle u',p'\rangle - \sigma h^* \langle u'',p''\rangle + \sigma h^* \langle p, u^{(4)}\rangle - c^2 \langle p, u''\rangle \\ &- \mu \langle p,p''\rangle + (\bar{q}+\bar{\kappa}) \langle p,p\rangle + \bar{q}\langle u,u\rangle + \bar{q}c^2 \langle u',u'\rangle + \bar{q}\sigma h^* \langle u'',u''\rangle . \end{aligned} \tag{4.12}$$

Since $U = \binom{u}{p} \in D(A)$, it follows from (3.1) and (4.11) that $u'(0) = u'(L) = p'(0) = p'(L) = u'''(0) = u'''(L) = 0$. Consequently, integration by parts implies that $-\langle p, u''\rangle = \langle u', p'\rangle$, $-\langle p, p''\rangle = \langle p', p'\rangle$, and $\langle p, u^{(4)}\rangle = \langle u'', p''\rangle$, and therefore, we get from (4.12) for all $U = \binom{u}{p} \in D(A)$ and $\bar{q} \geq 0$ that

$$((A+\bar{q}I)U,U) = - \langle u,p\rangle + \mu \|p'\|_2^2 + (\bar{q}+\bar{\kappa}) \|p\|_2^2 + \bar{q}\|u\|_2^2 + \bar{q}c^2 \|u'\|_2^2 + \bar{q}\sigma h^* \|u''\|_2^2 . \tag{4.13}$$

The Cauchy–Schwarz inequality implies that $-\langle u,p\rangle \geq -\|u\|_2 \|p\|_2 \geq -\frac{1}{2}\|p\|_2^2 - \frac{1}{2}\|u\|_2^2$. Consequently, we get from (4.13) for all $U = \binom{u}{p} \in D(A)$ and $\bar{q} \geq 1/2$ that

$$((A+\bar{q}I)U,U) \geq 0. \tag{4.14}$$

Let arbitrary $\binom{f_1}{f_2} \in X_1$ be given. By virtue of (4.9), the equation $(A + (\bar{q}+1)I)U = \binom{f_1}{f_2}$ gives

$$\begin{aligned} &(\bar{q}+1)u - p = f_1, \\ &\sigma h^* u^{(4)} - c^2 u'' - \mu p'' + (\bar{q}+1+\bar{\kappa})p = f_2. \end{aligned} \tag{4.15}$$

The system (4.15) gives the equation

$$\sigma h^* u^{(4)} - \left(c^2 + \mu(\bar{q}+1)\right) u'' + (\bar{q}+1)(\bar{q}+1+\bar{\kappa})u = f_3, \tag{4.16}$$

where $f_3 = f_2 + (\bar{q}+1+\bar{\kappa})f_1 - \mu f_1''$. Using Fourier series, we find that, for every $f_3 \in L^2(0,L)$ and every $\bar{q} \geq 0$, (4.16) has a solution $u \in \bar{S}$ with $u'' \in \bar{S}$, which is given by the following equation for $x \in [0,L]$:

$$
(4.17)\qquad \begin{aligned} u(x) &= \frac{1}{(\bar{q}+1)(\bar{q}+1+\bar{\kappa})L}\int_0^L f_3(z)dz \\ &+\sum_{n=1}^{\infty}\frac{2L^3\cos\left(n\pi\frac{x}{L}\right)\int_0^L f_3(z)\cos\left(n\pi\frac{z}{L}\right)dz}{\sigma h^* n^4\pi^4+L^2\left(c^2+\mu(\bar{q}+1)\right)n^2\pi^2+L^4(\bar{q}+1)(\bar{q}+1+\bar{\kappa})}. \end{aligned}
$$

Using (4.15) and (4.16), we conclude that, for every $\binom{f_1}{f_2}\in X_1$ and every $\bar{q}\geq 0$, there exists $U=\binom{u}{p}\in D(A)$ such that $(A+(\bar{q}+1)I)U=\binom{f_1}{f_2}$. Therefore, using (4.14), we conclude that the operator $A+\bar{q}I$ is a maximal monotone operator for $\bar{q}\geq 1/2$. The proof is finished by applying Theorem 7.10 on p. 198 in [10]. The proof is complete. ∎

Proof of Theorem 2. Let $f\in C^0(\mathbb{R}_+)$ and an arbitrary function

$$
\varphi\in C^0\left(\mathbb{R}_+;\bar{S}\cap H^4(0,L)\right)\cap C^1\left(\mathbb{R}_+;\bar{S}\right)\cap C^2\left(\mathbb{R}_+;L^2(0,L)\right)
$$

that satisfies (1.1), (1.3), (1.4), and (1.5) for all $t\geq 0$ be given. Define

$$
(4.18)\qquad \theta(t,x)=\int_0^x\varphi_t(t,s)ds \text{ for } t\geq 0,\ x\in[0,L].
$$

Using (1.1), (1.3), (1.4), and (1.5) and definition (4.18), we conclude that the following equations hold:

$$
(4.19)\qquad \theta_t=c^2\varphi_x-\sigma h^*\varphi_{xxx}-h^*f+\mu\varphi_{tx}-\bar{\kappa}\theta \quad \text{for } t\geq 0,
$$

$$
(4.20)\qquad \theta(t,0)=\theta(t,L)=0 \text{ for } t\geq 0.
$$

Using (1.1), (1.3), (1.4), (4.19), (4.20), and integration by parts, we conclude that the following equations hold for $t\geq 0$:

$$
(4.21)\qquad \begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\varphi_t[t]\|_2^2+\frac{c^2}{2}\|\varphi_x[t]\|_2^2+\frac{\sigma h^*}{2}\|\varphi_{xx}[t]\|_2^2\right) \\ &=-\mu\|\varphi_{tx}[t]\|_2^2-\bar{\kappa}\|\varphi_t[t]\|_2^2+h^*\left\langle\varphi_{tx}[t],\chi_{[0,L]}\right\rangle f(t), \end{aligned}
$$

$$
(4.22)\qquad \begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\theta[t]\|_2^2+\frac{c^2}{2}\|\varphi[t]\|_2^2+\frac{\sigma h^*}{2}\|\varphi_x[t]\|_2^2\right) \\ &=-\bar{\kappa}\|\theta[t]\|_2^2-\mu\|\varphi_t[t]\|_2^2-h^*\left\langle\theta[t],\chi_{[0,L]}\right\rangle f(t), \end{aligned}
$$

$$
(4.23)\qquad \begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|\theta[t]-\mu\varphi_x[t]\|_2^2+\frac{c^2+\bar{\kappa}\mu}{2}\|\varphi[t]\|_2^2+\frac{\sigma h^*}{2}\|\varphi_x[t]\|_2^2\right) \\ &=-\mu c^2\|\varphi_x[t]\|_2^2-\mu\sigma h^*\|\varphi_{xx}[t]\|_2^2-\bar{\kappa}\|\theta[t]\|_2^2 \\ &\quad -h^*\left\langle\theta[t]-\mu\varphi_x[t],\chi_{[0,L]}\right\rangle f(t). \end{aligned}
$$

Define the mapping

$$
(4.24)\qquad \begin{aligned} V_1(t)&=\frac{1}{2}\|\varphi_t[t]\|_2^2+\frac{c^2+2\sigma h^*}{2}\|\varphi_x[t]\|_2^2+\frac{\sigma h^*}{2}\|\varphi_{xx}[t]\|_2^2+\frac{1}{2}\|\theta[t]\|_2^2 \\ &+\frac{2c^2+\bar{\kappa}\mu}{2}\|\varphi[t]\|_2^2+\frac{1}{2}\|\theta[t]-\mu\varphi_x[t]\|_2^2. \end{aligned}
$$

We notice that $V_1(t)$, as defined by (4.24), is nothing else but the sum of the quantities whose time derivatives appear in the left-hand sides of (4.21), (4.22), and (4.23). Therefore, we get from (4.21), (4.22), (4.23), and (4.24) for all $t \geq 0$ that

$$\begin{aligned}\dot{V}_1(t) = &-\mu \left\|\varphi_{tx}[t]\right\|_2^2 - (\mu+\bar{\kappa}) \left\|\varphi_t[t]\right\|_2^2 - 2\bar{\kappa} \left\|\theta[t]\right\|_2^2 \\ &- \mu c^2 \left\|\varphi_x[t]\right\|_2^2 - \mu\sigma h^* \left\|\varphi_{xx}[t]\right\|_2^2 - h^* \left\langle 2\theta[t] - \mu\varphi_x[t] - \varphi_{tx}[t], \chi_{[0,L]}\right\rangle f(t).\end{aligned} \tag{4.25}$$

Using Wirtinger's inequality, (1.3) and (1.5) and (4.18) and (4.20), we get, for all $t \geq 0$,

$$\left\|\varphi_x[t]\right\|_2^2 \leq \frac{L^2}{\pi^2} \left\|\varphi_{xx}[t]\right\|_2^2, \tag{4.26}$$

$$\left\|\theta[t]\right\|_2^2 \leq \frac{L^2}{\pi^2} \left\|\varphi_t[t]\right\|_2^2, \tag{4.27}$$

$$\left\|\varphi[t]\right\|_2^2 \leq \frac{L^2}{\pi^2} \left\|\varphi_x[t]\right\|_2^2, \tag{4.28}$$

$$\left\|\varphi_t[t]\right\|_2^2 \leq \frac{L^2}{\pi^2} \left\|\varphi_{tx}[t]\right\|_2^2. \tag{4.29}$$

Using definitions (3.3), (4.24), (4.18), and (4.27), we conclude that there exist constants $K_2 > K_1 > 0$ (independent of $t \geq 0$ and the solution φ) such that the following inequalities hold for all $t \geq 0$:

$$K_1 P^2(t) \leq V_1(t) \leq K_2 P^2(t). \tag{4.30}$$

The Cauchy–Schwarz inequality and (4.27) give the inequalities $|\langle \varphi_{tx}[t], \chi_{[0,L]}\rangle| \leq \sqrt{L}\|\varphi_{tx}[t]\|_2$, $|\langle \varphi_x[t], \chi_{[0,L]}\rangle| \leq \sqrt{L}\|\varphi_x[t]\|_2$, and $|\langle \theta[t], \chi_{[0,L]}\rangle| \leq \frac{L\sqrt{L}}{\pi}\|\varphi_t[t]\|_2$ for all $t \geq 0$. Consequently, using the previous inequalities and the elementary inequalities

$$2h^* \frac{L\sqrt{L}}{\pi} \left\|\varphi_t[t]\right\|_2 |f(t)| \leq \frac{\mu+\bar{\kappa}}{2} \left\|\varphi_t[t]\right\|_2^2 + \frac{2\left(h^*\right)^2 L^3}{\pi^2(\mu+\bar{\kappa})} |f(t)|^2,$$

$$h^* \mu\sqrt{L} \left\|\varphi_x[t]\right\|_2 |f(t)| \leq \frac{\mu c^2}{2} \left\|\varphi_x[t]\right\|_2^2 + \frac{\left(h^*\right)^2 \mu L}{2c^2} |f(t)|^2,$$

$$h^* \sqrt{L} \left\|\varphi_{tx}[t]\right\|_2 |f(t)| \leq \frac{\mu}{2} \left\|\varphi_{tx}[t]\right\|_2^2 + \frac{\left(h^*\right)^2 L}{2\mu} |f(t)|^2,$$

we obtain from (4.25) for all $t \geq 0$ that

$$\begin{aligned}\dot{V}_1(t) \leq &-\mu \left\|\varphi_{tx}[t]\right\|_2^2 - (\mu+\bar{\kappa}) \left\|\varphi_t[t]\right\|_2^2 - 2\bar{\kappa} \left\|\theta[t]\right\|_2^2 \\ &- \mu c^2 \left\|\varphi_x[t]\right\|_2^2 - \mu\sigma h^* \left\|\varphi_{xx}[t]\right\|_2^2 + 2h^* \frac{L\sqrt{L}}{\pi} \left\|\varphi_t[t]\right\|_2 |f(t)| \\ &+ h^* \mu\sqrt{L} \left\|\varphi_x[t]\right\|_2 |f(t)| + h^* \sqrt{L} \left\|\varphi_{tx}[t]\right\|_2 |f(t)| \\ \leq &-\frac{\mu}{2} \left\|\varphi_{tx}[t]\right\|_2^2 - \frac{\mu+\bar{\kappa}}{2} \left\|\varphi_t[t]\right\|_2^2 - \frac{\mu c^2}{2} \left\|\varphi_x[t]\right\|_2^2 \\ &- \mu\sigma h^* \left\|\varphi_{xx}[t]\right\|_2^2 + \frac{\left(h^*\right)^2 L}{2\mu} \left(\frac{4\mu L^2}{\pi^2(\mu+\bar{\kappa})} + \frac{\mu^2}{c^2} + 1\right) |f(t)|^2.\end{aligned} \tag{4.31}$$

Using (4.28), (4.29), and (4.31), we get for all $t \geq 0$ that

$$\begin{aligned} \dot{V}_1(t) \leq & -\frac{\mu\pi^2 + (\mu+\bar{\kappa})L^2}{2L^2} \|\varphi_t[t]\|_2^2 - \frac{\mu c^2}{4} \|\varphi_x[t]\|_2^2 - \frac{\mu c^2}{4} \|\varphi_x[t]\|_2^2 \\ & - \mu\sigma h^* \|\varphi_{xx}[t]\|_2^2 + \frac{(h^*)^2 L}{2\mu} \left(\frac{4\mu L^2}{\pi^2(\mu+\bar{\kappa})} + \frac{\mu^2}{c^2} + 1 \right) |f(t)|^2 \\ \leq & -\frac{\mu\pi^2 + (\mu+\bar{\kappa})L^2}{2L^2} \|\varphi_t[t]\|_2^2 - \frac{\mu c^2 \pi^2}{4L^2} \|\varphi[t]\|_2^2 - \frac{\mu c^2}{4} \|\varphi_x[t]\|_2^2 \\ & - \mu\sigma h^* \|\varphi_{xx}[t]\|_2^2 + \frac{(h^*)^2 L}{2\mu} \left(\frac{4\mu L^2}{\pi^2(\mu+\bar{\kappa})} + \frac{\mu^2}{c^2} + 1 \right) |f(t)|^2 . \end{aligned} \tag{4.32}$$

Definition (3.3) and (4.32) imply that there exists a constant $K_3 > 0$ (independent of $t \geq 0$ and the solution φ) such that the following inequality holds for all $t \geq 0$:

$$\dot{V}_1(t) \leq -K_3 P^2(t) + \frac{(h^*)^2 L}{2\mu} \left(\frac{4\mu L^2}{\pi^2(\mu+\bar{\kappa})} + \frac{\mu^2}{c^2} + 1 \right) |f(t)|^2 . \tag{4.33}$$

Using (4.30) and (4.33), we conclude that there exists a constant $K_4 > 0$ (independent of $t \geq 0$ and the solution φ) such that the following inequality holds for all $t \geq 0$:

$$\dot{V}_1(t) \leq -K_4 V_1(t) + \frac{(h^*)^2 L}{2\mu} \left(\frac{4\mu L^2}{\pi^2(\mu+\bar{\kappa})} + \frac{\mu^2}{c^2} + 1 \right) |f(t)|^2 . \tag{4.34}$$

Estimate (3.2) for appropriate constants $\bar{M}, \bar{\lambda}, \Gamma > 0$ is a direct consequence of differential inequality (4.34) and inequalities (4.30). The proof is complete. ◻

Proof of Theorem 3. The proof follows a similar notation with the proof of Theorem 1 (for example, we have states u, U, scalar product $(\bullet, \bullet)$, identity operator I, etc.). However, the reader should not be tempted by an overlapping notation for different quantities to compare the different quantities in the proofs. The proofs of Theorem 1 and Theorem 3 are completely independent. Define the real Hilbert space

$$X_2 = \left\{ (\xi, w, \varphi, u) \,:\, (\xi, w) \in \mathbb{R}^2 \,,\, \varphi \in \bar{S} \,,\, u \in L^2(0,L) \,,\, \int_0^L \varphi(x)dx = \int_0^L u(x)dx = 0 \right\} \tag{4.35}$$

with scalar product defined for all $U = (\xi, w, \varphi, u) \in X_2$, $\bar{U} = (\bar{\xi}, \bar{w}, \bar{\varphi}, \bar{u}) \in X_2$

$$(U, \bar{U}) = \xi\bar{\xi} + w\bar{w} + \langle \varphi, \bar{\varphi} \rangle + c^2 \langle \varphi', \bar{\varphi}' \rangle + \sigma h^* \langle \varphi'', \bar{\varphi}'' \rangle + \langle u, \bar{u} \rangle . \tag{4.36}$$

Define the linear unbounded operator $\tilde{A} : D(\tilde{A}) \to X_2$ for all $U = (\xi, w, \varphi, u) \in D(\tilde{A})$ by means of the following equation:

$$\tilde{A}U = \left(-w, \quad B\xi + Cw + \langle u, \tilde{r} \rangle + \langle \varphi', \tilde{p} \rangle, \quad -u, \quad -c^2\varphi'' + \sigma h^* \varphi^{(4)} - \mu u'' + \bar{\kappa} u \right), \tag{4.37}$$

where $D(\tilde{A}) \subset X_2$ is the linear subspace

$$D(\tilde{A}) = \left\{ (\xi, w, \varphi, u) \in X_2 \,:\, \begin{array}{c} \varphi \in H^4(0,L) \,,\, u \in \bar{S}, \\ \sigma\varphi'''(0) = \sigma\varphi'''(L) = -B\xi - Cw - \langle u, \tilde{r} \rangle - \langle \varphi', \tilde{p} \rangle \end{array} \right\} . \tag{4.38}$$

It is clear from definitions (4.37) and (4.38) that we are seeking for a solution of the initial-value problem

$$\dot{U} + \tilde{A}U = 0 \tag{4.39}$$

with

$$U[0] = (\xi_0, w_0, \varphi_0, u_0)\,. \tag{4.40}$$

Notice that definitions (4.35), (4.38), and (4.40) and the fact that $(\xi_0, w_0) \in \mathbb{R}^2$, $\varphi_0 \in \bar{S} \cap H^4(0,L)$, $u_0 \in \bar{S}$ with $\int_0^L \varphi_0(x)dx = \int_0^L u_0(x)dx = 0$, $\varphi_0'''(0) = \varphi_0'''(L) = -\sigma^{-1}(B\xi_0 + Cw_0 + \langle u_0, \tilde{r}\rangle + \langle \varphi_0', \tilde{p}\rangle)$ imply that $U[0] = (\xi_0, w_0, \varphi_0, u_0) \in D(\tilde{A})$. The theorem is proved by applying the Hille–Yosida theorem (Theorem 7.4 on p. 185 in [10] and Remark 6 on p. 190 in [10]) to the initial-value problems (4.39) and (4.40). To this purpose, it suffices to show that the linear unbounded operator $\tilde{A} + \bar{q}\,I$, where $\tilde{A} : D(\tilde{A}) \to X_2$ is defined by (4.37) and (4.38), is a maximal monotone operator for some $\bar{q} \geq 0$. Indeed, by virtue of (3.1), (4.35), (4.36), (4.37), and (4.38) and by using integration by parts, the Cauchy–Schwarz inequality and the fact that $|u(L) - u(0)| \leq \sqrt{L}\|u'\|_2$ (a consequence of the Cauchy–Schwarz inequality and the fact that $u(L) - u(0) = \int_0^L u'(x)dx$), we have, for all $U = (\xi, w, \varphi, u) \in D(\tilde{A})$,

(4.41)
$$\begin{aligned}
\left(\tilde{A}U, U\right) &= -\xi w + B\xi w + Cw^2 + w\langle u, \tilde{r}\rangle + w\langle \varphi', \tilde{p}\rangle \\
&\quad - \langle \varphi, u\rangle - c^2\langle \varphi', u'\rangle - \sigma h^*\langle \varphi'', u''\rangle - c^2\langle u, \varphi''\rangle \\
&\quad + \sigma h^*\left\langle u, \varphi^{(4)}\right\rangle - \mu\langle u, u''\rangle + \bar{\kappa}\|u\|_2^2 \\
&= (B-1)\xi w + Cw^2 + w\langle u, \tilde{r}\rangle + w\langle \varphi', \tilde{p}\rangle + \mu\|u'\|_2^2 + \bar{\kappa}\|u\|_2^2 \\
&\quad - \langle \varphi, u\rangle - h^*(u(L) - u(0))(B\xi + Cw + \langle u, \tilde{r}\rangle + \langle \varphi', \tilde{p}\rangle) \\
&\geq -|B-1|\,|\xi|\,|w| - |C|\,w^2 - \|\tilde{r}\|_2\,|w|\,\|u\|_2 - \|\tilde{p}\|_2\,|w|\,\|\varphi'\|_2 + \mu\|u'\|_2^2 + \bar{\kappa}\|u\|_2^2 \\
&\quad - \|\varphi\|_2\,\|u\|_2 - |B|\,h^*\sqrt{L}\,\|u'\|_2\,|\xi| \\
&\quad - h^*|C|\sqrt{L}\,\|u'\|_2\,|w| - h^*\sqrt{L}\,\|u'\|_2\,\|\tilde{r}\|_2\,\|u\|_2 - h^*\sqrt{L}\,\|u'\|_2\,\|\tilde{p}\|_2\,\|\varphi'\|_2\,.
\end{aligned}$$

Using the inequalities

$$\begin{aligned}
|B|\,h^*\sqrt{L}\,\|u'\|_2\,|\xi| &\leq \frac{\mu}{4}\|u'\|_2^2 + \frac{1}{\mu}B^2(h^*)^2 L\xi^2, \\
h^*|C|\sqrt{L}\,\|u'\|_2\,|w| &\leq \frac{\mu}{4}\|u'\|_2^2 + \frac{1}{\mu}C^2(h^*)^2 Lw^2, \\
h^*\sqrt{L}\,\|u'\|_2\,\|\tilde{r}\|_2\,\|u\|_2 &\leq \frac{\mu}{4}\|u'\|_2^2 + \frac{1}{\mu}(h^*)^2 L\,\|\tilde{r}\|_2^2\,\|u\|_2^2, \\
h^*\sqrt{L}\,\|u'\|_2\,\|\tilde{p}\|_2\,\|\varphi'\|_2 &\leq \frac{\mu}{4}\|u'\|_2^2 + \frac{1}{\mu}(h^*)^2 L\,\|\tilde{p}\|_2^2\,\|\varphi'\|_2^2,
\end{aligned}$$

$$\begin{aligned}
|B-1|\,|\xi|\,|w| &\leq \frac{|B-1|}{2}\xi^2 + \frac{|B-1|}{2}w^2, \\
\|\tilde{r}\|_2\,|w|\,\|u\|_2 &\leq \frac{\|\tilde{r}\|_2}{2}\|u\|_2^2 + \frac{\|\tilde{r}\|_2}{2}w^2, \\
\|\tilde{p}\|_2\,|w|\,\|\varphi'\|_2 &\leq \frac{\|\tilde{p}\|_2}{2}\|\varphi'\|_2^2 + \frac{\|\tilde{p}\|_2}{2}w^2, \\
\|\varphi\|_2\,\|u\|_2 &\leq \frac{1}{2}\|u\|_2^2 + \frac{1}{2}\|\varphi\|_2^2,
\end{aligned}$$

we get from (4.36) and (4.41) that

$$
\begin{aligned}
&\left((\tilde{A}+\bar{q}I)U,U\right)\\
&\quad\geq \left(\bar{q}-\frac{|B-1|}{2}-\frac{L}{\mu}B^2(h^*)^2\right)\xi^2+\bar{q}\sigma h^*\|\varphi''\|_2^2\\
&\qquad+\left(\bar{q}-\frac{|B-1|+\|\tilde{r}\|_2+\|\tilde{p}\|_2}{2}-|C|-\frac{L}{\mu}C^2(h^*)^2\right)w^2+\left(\bar{q}-\frac{1}{2}\right)\|\varphi\|_2^2\\
&\qquad+\left(\bar{q}c^2-\frac{\|\tilde{p}\|_2}{2}-\frac{L}{\mu}(h^*)^2\|\tilde{p}\|_2^2\right)\|\varphi'\|_2^2+\left(\bar{q}-\frac{\|\tilde{r}\|_2+1}{2}-\frac{L}{\mu}(h^*)^2\|\tilde{r}\|_2^2\right)\|u\|_2^2.
\end{aligned}
\tag{4.42}
$$

It follows from (4.42) that, for $\bar{q}\geq 0$ sufficiently large, it holds that

$$
\left((\tilde{A}+\bar{q}I)U,U\right)\geq 0. \tag{4.43}
$$

Let arbitrary $f_1,f_2\in\mathbb{R}$, $f_3\in H^2(0,L), f_4\in L^2(0,L)$ with $f_3'(0)=f_3'(L)=0$ and $\int_0^L f_3(x)dx=\int_0^L f_4(x)dx=0$ be given. We investigate the existence of $U=(\xi,w,\varphi,u)\in D(\tilde{A})$ with $\tilde{A}U+(\bar{q}+1)U=(f_1,f_2,f_3,f_4)$. Using (4.37), we get the equations

$$
\begin{aligned}
&(\bar{q}+1)\xi-w=f_1,\\
&(\bar{q}+1)w+B\xi+Cw+\langle u,\tilde{r}\rangle+\langle\varphi',\tilde{p}\rangle=f_2,\\
&(\bar{q}+1)\varphi-u=f_3,\\
&(\bar{q}+1)u-c^2\varphi''+\sigma h^*\varphi^{(4)}-\mu u''+\bar{\kappa}u=f_4.
\end{aligned}
\tag{4.44}
$$

For $\bar{q}\geq 0$ sufficiently large (so that $(\bar{q}+1+C)(\bar{q}+1)+B>0$), we get from (4.44) that

$$
\begin{aligned}
\xi&=\frac{\bar{q}+1+C}{s(\bar{q})}f_1+\frac{1}{s(\bar{q})}f_2+\frac{1}{s(\bar{q})}\langle f_3,\tilde{r}\rangle-\frac{\bar{q}+1}{s(\bar{q})}\langle\varphi,\tilde{r}\rangle-\frac{1}{s(\bar{q})}\langle\varphi',\tilde{p}\rangle,\\
w&=-\frac{B}{s(\bar{q})}f_1+\frac{\bar{q}+1}{s(\bar{q})}f_2+\frac{\bar{q}+1}{s(\bar{q})}\langle f_3,\tilde{r}\rangle-\frac{(\bar{q}+1)^2}{s(\bar{q})}\langle\varphi,\tilde{r}\rangle-\frac{\bar{q}+1}{s(\bar{q})}\langle\varphi',\tilde{p}\rangle,\\
u&=(\bar{q}+1)\varphi-f_3,
\end{aligned}
\tag{4.45}
$$

where $s(\bar{q})=(\bar{q}+1+C)(\bar{q}+1)+B$ and $\varphi\in H^4(0,L)$ is a function that satisfies

$$
\sigma h^*\varphi^{(4)}-\left(c^2+\mu(\bar{q}+1)\right)\varphi''+(\bar{q}+1)(\bar{q}+1+\bar{\kappa})\varphi=f_5, \tag{4.46}
$$

$$
\begin{aligned}
&\varphi'(0)=\varphi'(L)=0,\\
&\varphi'''(0)=\varphi'''(L)=Z-\frac{(\bar{q}+1)^3}{\sigma s(\bar{q})}\langle\varphi,\tilde{r}\rangle-\frac{(\bar{q}+1)^2}{\sigma s(\bar{q})}\langle\varphi',\tilde{p}\rangle
\end{aligned}
\tag{4.47}
$$

with $Z=-\frac{(\bar{q}+1)B}{\sigma s(\bar{q})}f_1-\frac{C(\bar{q}+1)+B}{\sigma s(\bar{q})}f_2+\frac{(\bar{q}+1)^2}{\sigma s(\bar{q})}\langle f_3,\tilde{r}\rangle$ and $f_5=f_4-\mu f_3''+(\bar{q}+1+\bar{\kappa})f_3$. Notice that, by virtue of (4.46) and (4.47) and the fact that $f_3'(0)=f_3'(L)=0$ and $\int_0^L f_3(x)dx=\int_0^L f_4(x)dx=0$, it follows that $\int_0^L\varphi(x)dx=0$. Therefore, a solution of the boundary-value problem (4.46) and (4.47) gives (by means of (4.45)) a solution $U=(\xi,w,\varphi,u)\in D(\tilde{A})$ of the equation $\tilde{A}U+(\bar{q}+1)U=(f_1,f_2,f_3,f_4)$. Therefore, we next finish the proof by showing that, for sufficiently large $\bar{q}\geq 0$ (so that $s(\bar{q})=(\bar{q}+1+C)(\bar{q}+1)+B>0$), the boundary-value problem (4.46) and (4.47) has a solution $\varphi\in H^4(0,L)$ for every $Z\in\mathbb{R}$ and every $f_5\in L^2(0,L)$ with $\int_0^L f_5(x)dx=0$.

We look for a solution of the form $\varphi(x)=a_0(x^3-\frac{3L}{2}x^2+\frac{L^3}{4})+\sum_{n=1}^{\infty}a_n\phi_n(x)$, where $a_n\in\mathbb{R}$ for $n=1,2,\dots$ and $\phi_n(x)=\sqrt{\frac{2}{L}}\cos(n\pi\frac{x}{L})$ for $n=1,2,\dots$. Substituting this expression in (4.46) and (4.47) and recalling (3.7), we obtain the following:

$$a_n=\frac{L^4\langle f_5,\phi_n\rangle}{\tilde{\theta}(n,\bar{q})}+6a_0\sqrt{\frac{2}{L}}\frac{L^6\left((-1)^n-1\right)}{\tilde{\theta}(n,\bar{q})}b(n,\bar{q})\text{ for } n=1,2,\dots, \tag{4.48}$$

$$\begin{aligned}&\left(6+\frac{(\bar{q}+1)^3}{\sigma s(\bar{q})}\tilde{\Gamma}+\frac{(\bar{q}+1)^2}{\sigma s(\bar{q})}\tilde{\Delta}\right)a_0\\&\quad=Z-\frac{(\bar{q}+1)^2L^4}{\sigma s(\bar{q})}\sum_{n=1}^{\infty}\frac{\langle f_5,\phi_n\rangle}{\tilde{\theta}(n,\bar{q})}\left((\bar{q}+1)\langle\phi_n,\tilde{r}\rangle+\langle\phi_n',\tilde{p}\rangle\right),\end{aligned} \tag{4.49}$$

where

$$\tilde{\theta}(n,\bar{q})=\sigma h^*n^4\pi^4+L^2\left(c^2+\mu(\bar{q}+1)\right)n^2\pi^2+(\bar{q}+1)(\bar{q}+1+\bar{\kappa})L^4, \tag{4.50}$$

$$b(n,\bar{q})=\frac{\left(c^2+\mu(\bar{q}+1)\right)n^2\pi^2+L^2(\bar{q}+1)(\bar{q}+1+\bar{\kappa})}{n^4\pi^4}, \tag{4.51}$$

$$\begin{aligned}\tilde{\Gamma}&=\langle\tilde{g},\tilde{r}\rangle+6\sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}\frac{L^6\left((-1)^n-1\right)}{\tilde{\theta}(n,\bar{q})}b(n,\bar{q})\langle\phi_n,\tilde{r}\rangle,\\ \tilde{\Delta}&=\langle\tilde{g}',\tilde{p}\rangle+6\sqrt{\frac{2}{L}}\sum_{n=1}^{\infty}\frac{L^6\left((-1)^n-1\right)}{\tilde{\theta}(n,\bar{q})}b(n,\bar{q})\langle\phi_n',\tilde{p}\rangle.\end{aligned} \tag{4.52}$$

Notice that (4.48), (4.50), and (4.51) and the fact that $f_5\in L^2(0,L)$ guarantee that $\varphi(x)=a_0(x^3-\frac{3L}{2}x^2+\frac{L^3}{4})+\sum_{n=1}^{\infty}a_n\phi_n(x)$ is indeed a function of class $H^4(0,L)$. Therefore, the solvability of the boundary-value problem (4.46) and (4.47) depends on the solvability of (4.49). Since $\tilde{\theta}(n,\bar{q})=n^4\pi^4(\sigma h^*+L^2b(n,\bar{q}))$ (a consequence of (4.50) and (4.51)) and since $b(n,\bar{q})\geq 0$, we get from (4.52) that

$$\begin{aligned}\tilde{\Gamma}&\geq\langle\tilde{g},\tilde{r}\rangle-\frac{12L^4}{\pi^4}\sqrt{\frac{2}{L}}\sum_{n\ odd}\frac{|\langle\phi_n,\tilde{r}\rangle|}{n^4},\\ \tilde{\Delta}&\geq\langle\tilde{g}',\tilde{p}\rangle-\frac{12L^4}{\pi^4}\sqrt{\frac{2}{L}}\sum_{n\ odd}\frac{|\langle\phi_n',\tilde{p}\rangle|}{n^4}.\end{aligned} \tag{4.53}$$

Inequalities (3.6) and (4.53) guarantee that $\tilde{\Gamma}\geq 0$ and $\tilde{\Delta}\geq 0$. Consequently, (4.49) is solvable. Thus, we conclude that, for sufficiently large $\bar{q}\geq 0$ (so that $s(\bar{q})=(\bar{q}+1+C)(\bar{q}+1)+B>0$) and for every $f_1,f_2\in\mathbb{R}$, $f_3\in H^2(0,L)$, $f_4\in L^2(0,L)$ with $f_3'(0)=f_3'(L)=0$ and $\int_0^L f_3(x)dx=\int_0^L f_4(x)dx=0$, there exists $U=(\xi,w,\varphi,u)\in D(\tilde{A})$ with $\tilde{A}U+(\bar{q}+1)U=(f_1,f_2,f_3,f_4)$.

The proof is complete. ∎

Proof of Theorem 4. Let $(\xi_0,w_0)\in\mathbb{R}^2$, $\varphi_0\in\bar{S}\cap H^4(0,L)$, $u_0\in\bar{S}$ with $\int_0^L\varphi_0(x)dx=\int_0^L u_0(x)dx=0$, and

$$\begin{aligned}\sigma\varphi_0'''(0)&=\sigma\varphi_0'''(L)\\&=-K\left(k_5^2w_0+k_5\xi_0-h^*(k_3+k_4)\int_0^L xu_0(x)dx-k_3\mu h^*\left(\varphi_0(L)-\varphi_0(0)\right)\right)\end{aligned}$$

be given. We notice that the feedback law (3.9) takes the form (3.8) with

$$\begin{aligned} B &= Kk_5\,,\ C = Kk_5^2, \\ \tilde{r}(x) &= -Kh^*(k_3+k_4)x, \\ \tilde{p}(x) &= -Kk_3\mu h^*. \end{aligned} \tag{4.54}$$

For $\phi_n(x) = \sqrt{\frac{2}{L}}\cos(n\pi\frac{x}{L})$, $n = 1,2,\ldots$, we get from (4.54) for $n = 1,2,\ldots$ that

$$\begin{aligned} \langle \phi_n, \tilde{r}\rangle &= -Kh^*(k_3+k_4)\frac{L^2\left((-1)^n-1\right)}{n^2\pi^2}\sqrt{\frac{2}{L}}, \\ \langle \phi_n', \tilde{p}\rangle &= -Kk_3\mu h^*\sqrt{\frac{2}{L}}\left((-1)^n-1\right). \end{aligned} \tag{4.55}$$

Using (3.7), (4.54), and (4.55), we get

$$\begin{aligned} &\langle \tilde{g}, \tilde{r}\rangle = Kh^*(k_3+k_4)\frac{L^5}{20}, \\ &\frac{12L^4}{\pi^4}\sqrt{\frac{2}{L}}\sum_{n\ odd}\frac{|\langle \phi_n, \tilde{r}\rangle|}{n^4} = \frac{48L^5}{\pi^6}Kh^*(k_3+k_4)\sum_{n\ odd}\frac{1}{n^6}, \\ &\langle \tilde{g}', \tilde{p}\rangle = Kk_3\mu h^*\frac{L^3}{2}, \\ &\frac{12L^4}{\pi^4}\sqrt{\frac{2}{L}}\sum_{n\ odd}\frac{|\langle \phi_n', \tilde{p}\rangle|}{n^4} = \frac{48L^3}{\pi^4}Kk_3\mu h^*\sum_{n\ odd}\frac{1}{n^4}. \end{aligned} \tag{4.56}$$

Since $\sum_{n\ odd}\frac{1}{n^6} = \frac{\pi^6}{960}$ and $\sum_{n\ odd}\frac{1}{n^4} = \frac{\pi^4}{96}$, it follows from (4.56) that inequalities (3.6) hold. Therefore, Theorem 3 implies that there exist unique functions $\varphi \in C^0(\mathbb{R}_+;\bar{S}\cap H^4(0,L))\cap C^1(\mathbb{R}_+;\bar{S})\cap C^2(\mathbb{R}_+;L^2(0,L))$, $(\xi,w)\in C^1(\mathbb{R}_+;\mathbb{R}^2)$ with $(\xi(0),w(0)) = (\xi_0,w_0)$, $\varphi[0]=\varphi_0$, $\varphi_t[0]=u_0$ that satisfy (1.1), (1.2), (1.3), (1.4), (1.5), and (3.9) for all $t\geq 0$. The rest of the proof exploits the Lyapunov functional

$$\begin{aligned} \tilde{W}(t) = {}& \frac{1}{2}\xi^2(t) + \frac{k_5^2}{2}\left(w(t)+k_5^{-1}\xi(t)\right)^2 \\ &+ \frac{\mu}{K\left(h^*\right)^2L}\left(\frac{1}{2}\left\|\varphi_t[t]\right\|_2^2 + \frac{c^2}{2}\left\|\varphi_x[t]\right\|_2^2 + \frac{\sigma h^*}{2}\left\|\varphi_{xx}[t]\right\|_2^2\right) \\ &+ k_4\left(\frac{1}{2}\left\|\theta[t]\right\|_2^2 + \frac{c^2}{2}\left\|\varphi[t]\right\|_2^2 + \frac{\sigma h^*}{2}\left\|\varphi_x[t]\right\|_2^2\right) \\ &+ k_3\left(\frac{1}{2}\left\|\theta[t]-\mu\varphi_x[t]\right\|_2^2 + \frac{c^2+\bar{\kappa}\mu}{2}\left\|\varphi[t]\right\|_2^2 + \frac{\sigma h^*}{2}\left\|\varphi_x[t]\right\|_2^2\right), \end{aligned} \tag{4.57}$$

where θ is defined by (4.18). Using (4.21), (4.22), and (4.23) and definition (4.57), we get for all $t\geq 0$ that

$$\begin{aligned} \frac{d}{dt}\left(\tilde{W}(t)\right) = {}& -k_5^{-1}\xi^2(t) + k_5\left(w(t)+k_5^{-1}\xi(t)\right)^2 - \frac{\mu^2}{K\left(h^*\right)^2L}\left\|\varphi_{tx}[t]\right\|_2^2 \\ &- \left(\frac{\bar{\kappa}\mu}{K\left(h^*\right)^2L} + k_4\mu\right)\left\|\varphi_t[t]\right\|_2^2 + \frac{\mu}{Kh^*L}f(t)\int_0^L\varphi_{tx}(t,x)dx \\ &- \bar{\kappa}\left(k_3+k_4\right)\left\|\theta[t]\right\|_2^2 - k_3\mu c^2\left\|\varphi_x[t]\right\|_2^2 - k_3\mu\sigma h^*\left\|\varphi_{xx}[t]\right\|_2^2 \\ &- \Bigg(k_5^2w(t) + k_5\xi(t) + h^*(k_3+k_4)\int_0^L\theta(t,x)dx \\ &\quad -k_3\mu h^*\int_0^L\varphi_x(t,x)dx\Bigg)f(t). \end{aligned} \tag{4.58}$$

Using integration by parts and (4.18) and (4.20), we get from (3.9) for all $t \geq 0$ that

$$f(t) = K\left(k_5^2 w(t) + k_5\xi(t) + h^*(k_3+k_4)\int_0^L \theta(t,x)dx - k_3\mu h^*\int_0^L \varphi_x(t,x)dx\right). \tag{4.59}$$

Combining (4.58) and (4.59) and using the fact that $|\int_0^L \varphi_{tx}(t,x)dx| \leq \sqrt{L}\|\varphi_{tx}[t]\|_2$, we get for all $t \geq 0$ and $\bar{\gamma} \geq 0$ that

$$\begin{aligned}\frac{d}{dt}\left(\tilde{W}(t)\right) \leq & -k_5^{-1}\xi^2(t) - \bar{\gamma}\left(w(t)+k_5^{-1}\xi(t)\right)^2 - K^{-1}f^2(t)\\ & + (k_5+\bar{\gamma})\left(w(t)+k_5^{-1}\xi(t)\right)^2 - \frac{\mu^2}{K(h^*)^2 L}\|\varphi_{tx}[t]\|_2^2\\ & - \left(\frac{\bar{\kappa}\mu}{K(h^*)^2 L} + k_4\mu\right)\|\varphi_t[t]\|_2^2 + \frac{\mu}{Kh^*\sqrt{L}}|f(t)|\,\|\varphi_{tx}[t]\|_2\\ & - \bar{\kappa}(k_3+k_4)\|\theta[t]\|_2^2 - k_3\mu c^2\|\varphi_x[t]\|_2^2 - k_3\mu\sigma h^*\|\varphi_{xx}[t]\|_2^2.\end{aligned} \tag{4.60}$$

Combining (4.59) and (4.60) and using the fact that

$$\frac{\mu}{Kh^*\sqrt{L}}|f(t)|\,\|\varphi_{tx}[t]\|_2 \leq \frac{\mu^2}{2K(h^*)^2 L}\|\varphi_{tx}[t]\|_2^2 + \frac{1}{2K}f^2(t),$$

we get for all $t \geq 0$ and $\bar{\gamma} \geq 0$ that

$$\begin{aligned}\frac{d}{dt}\left(\tilde{W}(t)\right) \leq & -k_5^{-1}\xi^2(t) - \bar{\gamma}\left(w(t)+k_5^{-1}\xi(t)\right)^2\\ & + k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right)\left(K^{-1}f(t) - h^*(k_3+k_4)\int_0^L \theta(t,x)dx\right.\\ & \qquad \left. + k_3\mu h^*\int_0^L \varphi_x(t,x)dx\right)^2\\ & - \frac{\mu^2}{2K(h^*)^2 L}\|\varphi_{tx}[t]\|_2^2 - \left(\frac{\bar{\kappa}\mu}{K(h^*)^2 L} + k_4\mu\right)\|\varphi_t[t]\|_2^2 - \frac{1}{2K}f^2(t)\\ & - \bar{\kappa}(k_3+k_4)\|\theta[t]\|_2^2 - k_3\mu c^2\|\varphi_x[t]\|_2^2 - k_3\mu\sigma h^*\|\varphi_{xx}[t]\|_2^2.\end{aligned} \tag{4.61}$$

Using the fact that

$$\begin{aligned}&\left(K^{-1}f(t) - h^*(k_3+k_4)\int_0^L \theta(t,x)dx + k_3\mu h^*\int_0^L \varphi_x(t,x)dx\right)^2\\ &\quad \leq 2K^{-2}f^2(t) + 4k_3^2\mu^2(h^*)^2\left(\int_0^L \varphi_x(t,x)dx\right)^2 + 4(h^*)^2(k_3+k_4)^2\left(\int_0^L \theta(t,x)dx\right)^2,\end{aligned}$$

we obtain from (4.61) for all $t \geq 0$ and $\bar{\gamma} \geq 0$ that

$$\begin{aligned}\frac{d}{dt}\left(\tilde{W}(t)\right) \leq & -k_5^{-1}\xi^2(t) - \bar{\gamma}\left(w(t)+k_5^{-1}\xi(t)\right)^2 \\ & + 4k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right)k_3^2\mu^2(h^*)^2\left(\int_0^L \varphi_x(t,x)dx\right)^2 \\ & + 4k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right)(h^*)^2(k_3+k_4)^2\left(\int_0^L \theta(t,x)dx\right)^2 \\ & - \frac{\mu^2}{2K(h^*)^2 L}\|\varphi_{tx}[t]\|_2^2 - k_4\mu\|\varphi_t[t]\|_2^2 \\ & - 2K^{-2}\left(\frac{K}{4} - k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right)\right)f^2(t) \\ & - \bar{\kappa}(k_3+k_4)\|\theta[t]\|_2^2 - k_3\mu c^2\|\varphi_x[t]\|_2^2 - k_3\mu\sigma h^*\|\varphi_{xx}[t]\|_2^2. \end{aligned} \tag{4.62}$$

Using the fact that $(\int_0^L \varphi_x(t,x)dx)^2 \leq L\|\varphi_x[t]\|_2^2$ and $(\int_0^L \theta(t,x)dx)^2 \leq L\|\theta[t]\|_2^2$, we obtain from (4.62) for all $t \geq 0$ and $\bar{\gamma} \geq 0$ that

$$\begin{aligned}\frac{d}{dt}\left(\tilde{W}(t)\right) \leq & -k_5^{-1}\xi^2(t) - \bar{\gamma}\left(w(t)+k_5^{-1}\xi(t)\right)^2 \\ & + 4k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right)(h^*)^2(k_3+k_4)^2 L\|\theta[t]\|_2^2 - k_3\mu\sigma h^*\|\varphi_{xx}[t]\|_2^2 \\ & - \frac{\mu^2}{2K(h^*)^2 L}\|\varphi_{tx}[t]\|_2^2 - k_4\mu\|\varphi_t[t]\|_2^2 \\ & - 2K^{-2}\left(\frac{K}{4} - k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right)\right)f^2(t) \\ & - 4k_3^2\mu^2(h^*)^2 L\left(\frac{c^2}{4k_3\mu(h^*)^2 L} - k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right)\right)\|\varphi_x[t]\|_2^2. \end{aligned} \tag{4.63}$$

Using (4.27) and (4.29), we obtain from (4.63) for all $t \geq 0$ and $\bar{\gamma} \geq 0$ that

$$\begin{aligned}\frac{d}{dt}\left(\tilde{W}(t)\right) \leq & -k_5^{-1}\xi^2(t) - \bar{\gamma}\left(w(t)+k_5^{-1}\xi(t)\right)^2 \\ & - k_3\mu\sigma h^*\|\varphi_{xx}[t]\|_2^2 - 2K^{-2}\left(\frac{K}{4} - k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right)\right)f^2(t) \\ & - 4(h^*)^2(k_3+k_4)^2\frac{L^3}{\pi^2}\left(\frac{\mu\pi^2\left(2Kk_4(h^*)^2L^3+\mu\pi^2\right)}{8K(h^*)^4(k_3+k_4)^2L^6}\right. \\ & \left. - k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right)\right)\|\varphi_t[t]\|_2^2 \\ & - 4k_3^2\mu^2(h^*)^2 L\left(\frac{c^2}{4k_3\mu(h^*)^2 L} - k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right)\right)\|\varphi_x[t]\|_2^2. \end{aligned} \tag{4.64}$$

Inequality (3.10) implies that there exists $\bar{\gamma} > 0$ sufficiently small such that

$$K > 4k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right),$$

$$\frac{\mu\pi^2\left(2Kk_4(h^*)^2L^3+\mu\pi^2\right)}{8K(h^*)^4(k_3+k_4)^2L^6} > k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right),$$

and

$$\frac{c^2}{4k_3\mu(h^*)^2 L} > k_5^{-3}\left(1+k_5^{-1}\bar{\gamma}\right).$$

Consequently, there exists a constant $c_1 > 0$ (independent of the solution and independent of $t \geq 0$) such that the following inequality holds for all $t \geq 0$:

$$\frac{d}{dt}\left(\tilde{W}(t)\right) \leq -c_1\left(\xi^2(t) + \left(w(t) + k_5^{-1}\xi(t)\right)^2 + \|\varphi_x[t]\|_2^2 + \|\varphi_{xx}[t]\|_2^2 + \|\varphi_t[t]\|_2^2\right). \tag{4.65}$$

Using definition (4.57) and inequalities (4.27) and (4.28), we conclude that there exists a constant $c_2 > 0$ (independent of $t \geq 0$ and the solution) such that the following inequality holds for all $t \geq 0$:

$$\tilde{W}(t) \leq c_2\left(\xi^2(t) + \left(w(t) + k_5^{-1}\xi(t)\right)^2 + \|\varphi_x[t]\|_2^2 + \|\varphi_{xx}[t]\|_2^2 + \|\varphi_t[t]\|_2^2\right). \tag{4.66}$$

Combining (4.65) and (4.66), we obtain the differential inequality

$$\frac{d}{dt}\left(\tilde{W}(t)\right) \leq -c_2^{-1}c_1\tilde{W}(t)$$

for all $t \geq 0$, which directly gives the following estimate for all $t \geq 0$:

$$\tilde{W}(t) \leq \exp\left(-c_2^{-1}c_1 t\right)\tilde{W}(0). \tag{4.67}$$

The rest of the proof follows from estimate (4.67) and the fact that there exist constants $c_4 > c_3 > 0$ (independent of $t \geq 0$ and the solution φ, ξ, w) such that the following inequalities hold for all $t \geq 0$:

$$c_3\left(P^2(t) + \xi^2(t) + w^2(t)\right) \leq \tilde{W}(t) \leq c_4\left(P^2(t) + \xi^2(t) + w^2(t)\right). \tag{4.68}$$

The proof is complete. □

5. Conclusions. The present paper provided novel results for the linearization of the liquid–tank system, and we have shown that the same family of feedback laws that works for the nonlinear case can also achieve exponential stabilization for the linearization of the liquid–tank system, no matter what the value of the surface tension coefficient is. However, many new results are still needed for a complete study of the spill-free, slosh-free problem of liquid–tank transfer. We next present four problems that remain unresolved in the linearized case.

(1) *Extension to two dimensions.* Real tanks are not 1-D; the actual equations for the study of a real liquid–tank system involve an additional spatial dimension. The extension of the theoretical results to two spatial dimensions poses new challenges that will demand different approaches that have no analogues in the 1-D case (e.g., the appearance of vorticity). The extension may also involve different boundary conditions when surface tension is present (see [36, 43]).

(2) *Rapid stabilization.* It would be interesting to construct feedback laws that can achieve rapid stabilization of the linearized liquid–tank system, i.e., stabilization with an (arbitrarily) assignable, exponential convergence rate (see [11, 51] for the rapid stabilization of beam equations and see [53] for stabilization of linearized viscous flows). Notice that Theorem 4 does not guarantee an (arbitrarily) assignable, exponential convergence rate. If the rapid stabilization problem is not solvable, it would also be interesting to have a counterexample that shows that rapid stabilization is not possible.

(3) *Optimal control.* It would also be of interest to study optimal control problems that can be posed for the linearized liquid–tank system. The cost function may contain terms that penalize the control action (e.g., terms of the form $\int_0^T |f(t)|^2\,dt$) as well as terms that penalize sloshing (e.g., terms of the form $\int_0^T \|\varphi[t]\|_2^2\,dt$). The solution of optimal control problems can allow the computation of the optimal feedback gains for a given initial state.

(4) *Control Lyapunov functionals.* The feedback design for the linearized liquid–tank system is based on appropriate CLFs whose construction is based on energy arguments. It is possible that this methodology can be applied to similar PDE systems and that some general principles for the construction of CLFs can be stated.

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