Prescribed-Time Control of Nonlinear Systems With Linearly Vanishing Multiplicative Measurement Noise

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Abstract—We present new prescribed-time designs for strict-feedback nonlinear systems with multiplicative measurement noise. When the noise is small and linearly vanishing, we first propose a new postulated feedback to solve the prescribed-time mean-square stabilization problem, then redesign the control gains, which is not only optimal with respect to a meaningful cost functional but also globally stabilizes the closed-loop system in the prescribed-time. When the noise is arbitrary large but vanishing faster than linearly, we develop a new control scheme to make the system achieve prescribed-time mean-square stabilization. In contrast to the existing stochastic prescribed-time designs, the merit of our design is that it can effectively deal with multiplicative measurement noise. The existence of measurement noise makes the design rather challenging since the resulting process noise intensity, in a closed loop, depends on the feedback gains and even goes to infinity. Finally, two simulation examples are given to illustrate the designs.

Index Terms—Multiplicative measurement noise, nonlinear systems, prescribed-time control.

I. INTRODUCTION

Research on system control with sensor uncertainties has attracted much attention in the past two decades due to their wide engineering applications in areas such as circuits and electrical devices [1], biomedical equipment [2], and mechanical systems [3], [4], [5], [6], [7] focus on the output-feedback stabilization design for nonlinear systems with unknown measurement sensitivity, which are deterministic constants or bounded time-varying functions. As shown in [8], it is more reasonable to study the eye and arm movements based on the assumption that the neural control signals are corrupted by noise, which motivates the study of the stochastic sensor sensitivity. For linear systems whose white noise sources have intensities affinely related to the variance of the signal they corrupt, Oliveira and Skelton [9] provide necessary and sufficient conditions to guarantee the mean-square state-feedback stabilization. Recently, Li and Krstic [10] propose two designs to solve the mean-square stabilization problems for lower-triangular/upper-triangular nonlinear systems with multiplicative stochastic sensor uncertainty. It should be emphasized that [9] and [10] only achieve mean-square stabilization in an asymptotic sense. However, in many real applications, the control tasks require that the mean-square stabilization be achieved in prescribed-time, rather than as time goes to infinity.

Prescribed-time control has been receiving increasing attention due to its wide applications in tactical missile guidance [11] and other applications in which there exists a short, finite amount of time remaining to achieve control objectives. The advantage of such control is that it allows the user to prescribe the convergence times a priori and irrespective of initial conditions. There are fruitful results for the prescribed-time control of deterministic systems [12], [13], [14], [15], [16], [17], [18], [19], [20], [21]. When it turns to the stochastic prescribed-time control, Li and Krstic [22] propose a nonscaling backstepping state-feedback design, which is the first result on the prescribed-time mean-square stabilization and inverse optimality control for stochastic strict-feedback nonlinear systems; Li and Krstic [23] adopt scaled quartic Lyapunov functions to reduce the control effort in [22]; Li and Krstic [24] solve the prescribed-time output-feedback control problems for stochastic nonlinear systems without/with sensor uncertainty; Li and Krstic [25] propose a prescribed-time mean-nonovershooting stabilizing feedback law for stochastic nonlinear systems with noise that vanishes in finite time. Although [22], [23], [24], and [25] concentrate on the prescribed-time control of stochastic nonlinear systems, they do not consider systems with multiplicative measurement noise.

Noting that multiplicative measurement noise is ubiquitous in engineering, for example, multiplicative noise often appears in communication process modeling, especially for the cases of fading communication channels [26], [27], [28], [29], it is
imperative to study the prescribed-time control for nonlinear systems with multiplicative measurement noise.

As shown in [30], the inverse optimal control is motivated by the requirement that the stabilizing feedback control be optimal with respect to some meaningful cost function, without recourse to solving the steady-state Hamilton–Jacobi–Isaacs (HJI) partial differential equation. The work of Deng and Krstic [31] is the stochastic counterpart of the inverse optimality result of [30] for systems with deterministic uncertainties; [32] generalized the results in [31] from state-feedback to output-feedback; [33] addressed optimality and solved a differential game problem with the control and the noise covariance as opposing players; [34] presented a design of optimal controllers with respect to a meaningful cost function to globally asymptotically stabilize nonholonomic systems affine in stochastic disturbances. The inverse optimal controllers in [31], [32], [33], and [34] only ensure global asymptotic stability in probability. Recently, Li and Krstic [22], [23] solved the prescribed-time inverse optimal mean-square stabilization problems for stochastic strict-feedback nonlinear systems. However, all the stochastic inverse optimal designs in [22], [23], [31], [32], [33], and [34], have not considered multiplicative measurement noise.

Motivated by the above observations, we study the prescribed-time stabilization and prescribed-time inverse optimal assignment problems for strict-feedback nonlinear systems with multiplicative measurement noise. The contributions of this article are fourfold.

1) We present a new design framework for nonlinear systems with multiplicative measurement noise. Unlike the design for linear systems in [9], where the control gain and the noise intensity are coupled in a linear matrix inequality, we develop a step-by-step gain design for nonlinear systems, which clearly shows what the control gains are. Different from the design for nonlinear systems in [10], where time-invariant controllers are designed to achieve asymptotic mean-square stability, our design is characterized by a time-varying function that grows unbounded toward the terminal time, which can drive the system to be prescribed-time mean-square stable.

2) The existence of multiplicative noise makes stochastic prescribed-time designs in [22], [23], [24], and [25] inapplicable. In order to handle the multiplicative noise, we propose a new postulated controller whose gains are designed step by step. Different from the designs in [22], [23], [24], and [25], where the controller is designed recursively, in our design, the feedback is inserted into the system, which leads to the process noise intensity actually being nonzero, depending on the feedback gains, and even going to infinity at the terminal time. How to select the control gains to prescribed-time stabilize the system in the presence of the nonlinearities is a hard problem.

3) We propose a new stochastic inverse optimal control scheme for nonlinear systems. Unlike the stochastic inverse optimal designs in [31], [32], [33], and [34], where the noise intensities do not influence the control design, with the effect of the multiplicative noise, our postulated design makes the noise intensities not only include the control gains, but also contain the unbounded time-varying function. In our scheme, we choose optimal control gains to minimize the cost functional characterized by these gains, which is different from [31], [32], [33], and [34], whose cost functionals are characterized by controllers.

4) We propose a new criterion to determine whether the system has a strong solution at the prescribed time interval. This criterion is a prescribed-time version of the well-known criterion in [35, Th. 3.5], where the linear growth condition of the differential operator of the Lyapunov function guarantees the existence of a global solution on $[t_0, +\infty)$. However, due to the effect of the unbounded time-varying scaling function, the differential operator of the Lyapunov function in this article does not satisfy the linear growth condition, which makes the criterion in [35] fail.

The rest of this article is organized as follows. Section II is on problem formulation. Section III is focused on the prescribed control of systems with small linear vanishing noise. Section IV is devoted to the prescribed-time control of systems with arbitrary large noise but vanishing faster than linear. Section V gives two examples to illustrate the theoretical results. Section VI concludes this article. Finally, some useful lemmas and the proof of a crucial technical lemma are provided in the appendices.

II. Problem Formulation

Consider a class of nonlinear systems described by

$$\dot{x} = f_i(t, x), \quad i = 1, \ldots, n - 1$$

$$\dot{x}_n = u + f_n(t, x)$$

where $x = (x_1, \ldots, x_n) \in R^n$ and $u \in R$ are the system state and control input. The function $f_i : R^+ \times R^n \to R$ is piecewise continuous in $t$, locally bounded and locally Lipschitz continuous in $x$ uniformly in $t \in R^+$, $f_i(t, 0) = 0$, $i = 1, \ldots, n$.

We observe the state $x_i$ as $y_i$, which is described by

$$y_i = x_i(1 + g_i(t)\omega_i), \quad i = 1, \ldots, n$$

or

$$y_i dt = x_i dt + g_i(t) x_i d\omega_i, \quad i = 1, \ldots, n$$

where $g_1(t), \ldots, g_n(t)$ are continuous functions and $\omega_1, \ldots, \omega_n$ are scalar independent standard Wiener processes defined on the complete filtered probability space $($Ω, $\mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $P$-null sets).

We introduce the following scaling function:

$$\mu(t) = \left(\frac{T}{t_0 + T - t}\right)^2 \forall t \in [t_0, t_0 + T]$$

where, as shown in [14], the settling time $T > 0$ can be preassigned as needed within any physically allowable range.

Obviously, $\mu(t)$ is a monotonically increasing function on $[t_0, t_0 + T]$ with $\mu(t_0) = 1$ and $\lim_{t \to t_0 + T} \mu(t) = +\infty$ (In this article, $\lim_{t \to t_0 + T}$ means $t$ approaches $t_0 + T$ “from the left” or “from below”).
To facilitate the controller design and stability analysis, we will drop the arguments of some functions when there is no confusion. For example, we use \( \mu \) to denote \( \mu(t) \), and use \( g_i \) to denote \( g_i(t) \).

We design a new controller as

\[
    u = -k_1 \mu^n y_1 - k_2 \mu^{n-1} y_2 - \ldots - k_n \mu y_n
\]

where \( k_1, \ldots, k_n \) are positive control gains to be designed later. Substituting (3), (4), and (6) into (1) and (2), systems (1) and (2) can be written as

\[
    dx_i = (x_{i+1} + f_i(t, x))dt, \quad i = 1, \ldots, n - 1
\]

\[
    dx_n = \left( -\sum_{i=1}^{n} k_i \mu^{n+1-i} x_i + f_n(t, x) \right) dt + G(x)d\omega
\]

where \( \omega = (\omega_1, \ldots, \omega_n)^T \) and

\[
    G(x) = (-k_1 g_1 \mu^n x_1, \ldots, -k_n g_n \mu x_n).
\]

For systems (1) and (2), we need the following assumption.

**Assumption 1:** For \( i = 1, \ldots, n \), there exists a nonnegative constant \( c_i \) such that

\[
    |f_i(t, x)| \leq c_i (|x_1| + \ldots + |x_i|).
\]

In this article, we study the prescribed-time mean-square stabilization problem of systems (7)–(9) in two cases.

**Case 1.** Systems with small linearly vanishing noise: The noise intensity \( g_i(t) \) satisfies the following linearly vanishing condition.

**Assumption 2:** For \( i = 1, \ldots, n \), there exists a positive constant \( \delta_i \) such that

\[
    |g_i(t)| \leq \delta_i \left( 1 - \frac{t-t_0}{T} \right) \quad \forall t \in [t_0, t_0 + T).
\]

In this case, with Assumptions 1 and 2, when the noise power \( \delta_i \) is sufficient size, we aim to design the control gains \( k_1, \ldots, k_n \) to make systems (7)–(9) achieve prescribed-time mean-square stable with \( \lim_{t \to t_0 + T} E|x|^2 = 0 \). In addition, we redesign the control gains, which are not only optimal with respect to a meaningful cost functional but also globally stabilizes the closed-loop system in the prescribed-time.

**Case 2.** Systems with arbitrary large noise but vanishing faster than linearly: In this case, the noise intensity \( g_i(t) \) satisfies the following assumption.

**Assumption 3:** For \( i = 1, \ldots, n \), there exists a positive constant \( \delta_i \) such that

\[
    |g_i(t)| \leq \delta_i \left( 1 - \frac{t-t_0}{T} \right)^{1+\sigma_i} \quad \forall t \in [t_0, t_0 + T).
\]

where \( \sigma_i \) is an arbitrary positive constant.

In this case, with Assumptions 1 and 3, for the noise with arbitrary large power \( \delta_i \) but vanishing faster than linearly, we aim to solve the prescribed-time mean-square stabilization problem of systems (7)–(9) with \( \lim_{t \to t_0 + T} E|x|^2 = 0 \).

**Remark 1:** In cases 1 and 2, the noise is required to be vanishing in Assumptions 2 and 3. These two assumptions seem to be restrictive. However, Assumptions 2 and 3 are necessary even for a simple scalar linear system to achieve prescribed-time mean-square stabilization. Next, we give the detailed proof.

Consider a scalar system

\[
    \dot{x} = u.
\]

We observe the state \( x \) as \( y \), which is described by

\[
    y = x(1 + g(t)\dot{\omega})
\]

or

\[
    ydt = xdt + g(t)xd\omega
\]

where \( g(t) \) is the noise intensity, which is a continuous function. \( \omega \) is a scalar standard Wiener processes defined on the complete filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\).

To ensure the prescribed-time convergence, we design the controller as

\[
    u = -k \mu y
\]

where \( k \) is a positive control gain to be designed later. By (15) and (16), system (13) can be written as

\[
    dx = -k \mu x dt - k \mu g x d\omega.
\]

The solution of (17) is

\[
    x(t) = e^{-\int_{t_0}^{t} (k + k^2 \mu^2 g^2) ds - \int_{t_0}^{t} (k \mu g) d\omega} x(t_0).
\]

Thus, we have

\[
    x^2(t) = e^{-2 \int_{t_0}^{t} (k + k^2 \mu^2 g^2) ds - 2 \int_{t_0}^{t} (k \mu g) d\omega} x^2(t_0)
\]

\[
    = e^{\int_{t_0}^{t} (-2 k^2 \mu^2 g^2) ds - \int_{t_0}^{t} (2 k \mu g) d\omega} x^2(t_0)
\]

\[
    \cdot e^{\int_{t_0}^{t} (-2 k + 2 k^2 \mu^2 g^2) ds}
\]

Let

\[
    \xi = e^{\int_{t_0}^{t} (-2 k^2 \mu^2 g^2) ds - \int_{t_0}^{t} (2 k \mu g) d\omega} x^2(t_0)
\]

which is equivalent to

\[
    d\xi = 2 k \mu g \xi d\omega
\]

with \( \xi(t_0) = x^2(t_0) \).

By (21), we get

\[
    \xi = x^2(t_0) + 2 \int_{t_0}^{t} k \mu g \xi d\omega.
\]

It follows from (22) and [41, Th. 4.3.5, p. 48] that:

\[
    E \xi = x^2(t_0).
\]

From (19) and (23), we can prove that

\[
    E x^2 = x^2(t_0) e^{\int_{t_0}^{t} (-2 k + 2 k^2 \mu^2 g^2) ds}
\]

holds for \( \forall t \in [t_0, t_0 + T) \).

For any positive constants \( a_1 \) and \( a_2 \), from (5), we obtain

\[
    \lim_{t \to t_0 + T} e^{\int_{t_0}^{t} (-a_1 \mu + a_2 \mu^2 \chi) ds}
\]
From (24) and (25), we get the following conclusions.

System (17) is prescribed-time mean-square stable with \( \lim_{t \to t_0 + T} E[x]^2 = 0 \) if and only if

\[
|g(t)| \leq \delta_0 \left(1 - \frac{t - t_0}{T}\right) \quad \forall t \in [t_0, t_0 + T) \tag{26}
\]

\[
0 < \delta_0 < \sqrt{\frac{2}{k}} \tag{27}
\]

or

\[
|g(t)| \leq \delta_0 \left(1 - \frac{t - t_0}{T}\right)^{1+\sigma_0} \quad \forall t \in [t_0, t_0 + T) \tag{28}
\]

where \( \delta_0 \) and \( \sigma_0 \) are arbitrary positive constants.

It is obvious that (26) is Assumption 2 (Case 1) and (28) is Assumption 3 (Case 2). From (27), the noise power \( \delta_0 \) in Case 1 should be sufficiently small. By (28), the noise power \( \delta_0 \) in case 2 is allowed to be arbitrarily large.

From the above analysis, it is concluded that Assumptions 2 and 3 are necessary in ensuring the prescribed-time mean-square stabilization of systems (13)–(16).

**Remark 2:** How can the requirement that the noise vanish no later than the time \( t_0 + T \) (Assumptions 2 and 3) be motivated?

In real applications, an important scenario is that the noise is suddenly vanishing at some time \( t_0 + T_{\text{max}} \), the time \( T_{\text{max}} > 0 \) is known, and we are able to pick any time \( T \) for prescribed stabilization, with \( T \) strictly greater than \( T_{\text{max}} \). In this case, Assumption 3 (Assumption 2 is similar) is modified as

**Assumption 3′:** Let \( T > T_{\text{max}} > 0 \). There exists a nonnegative constant \( \delta_i \) such that

\[
|g_i(t)| \leq \delta_i \quad \forall t_0 \leq t < t_0 + T_{\text{max}} \tag{29}
\]

and

\[
|g_i(t)| = 0 \quad \forall t_0 + T_{\text{max}} \leq t < t_0 + T. \tag{30}
\]

Next, we prove Assumption 3′ is a special case of Assumption 3. Choosing

\[
\delta_i = \delta_i T^{1+\sigma_i} / (T - T_{\text{max}})^{1+\sigma_i} \tag{31}
\]

we have

\[
|g_i(t)| \leq \delta_i \left(1 - \frac{t - t_0}{T}\right)^{1+\sigma_i} \quad \forall t \in [t_0, t_0 + T) \tag{32}
\]

which shows that Assumption 3 can be satisfied easily by satisfying the simple, practically reasonable, albeit more conservative Assumption 3′.

Next, we will study the above two cases in the following two sections, respectively.
\[ + (c_2 + \alpha_1 \mu) |\xi_2| \]
\[ \leq \left( c_1 \alpha_1 + c_2 + \frac{2}{T} \alpha_1 + c_2 \alpha_1 + \alpha_1^2 \right) \mu^2 |\xi_1| \]
\[ + (c_2 + \alpha_1 \mu) |\xi_2|. \]  

By (43) and Lemma A.2, we get
\[ \frac{1}{\mu^2} \xi_2 \left( f_2 + \frac{2}{T} \mu^{3/2} \alpha_1 \xi_1 + \mu \alpha_1 (x_2 + f_1) \right) \]
\[ \leq \frac{1}{\mu} (c_2 + \alpha_1) \xi_2^2 + \left( c_1 \alpha_1 + c_2 + \frac{2}{T} \alpha_1 + c_2 \alpha_1 + \alpha_1^2 \right) |\xi_1||\xi_2| \]
\[ \leq \frac{1}{\mu} (c_2 + \alpha_1) \xi_2^2 + \left( c_1 \alpha_1 + c_2 + \frac{2}{T} \alpha_1 + c_2 \alpha_1 + \alpha_1^2 \right) |\xi_1||\xi_2| \]
\[ \leq \xi_1^2 \left( 1 + 2 \alpha_2 + 2 \alpha_1 + c_1 \alpha_1 + c_2 + \frac{2}{T} \alpha_1 + c_2 \alpha_1 + \alpha_1^2 \right) \xi_2^2. \]

Substituting (42) and (44) into (41) yields
\[ \mathcal{L} V_2 \leq -(|n - 1|) \mu \xi_c^2 + \frac{1}{\mu^2} \xi_2 (x_3 - x_3^*) + \frac{1}{\mu^2} \xi_2 x_3^* \]
\[ + \frac{1}{\mu} \left( 1 + 2 \alpha_2 + 2 \alpha_1 + c_1 \alpha_1 + c_2 + \frac{2}{T} \alpha_1 + c_2 \alpha_1 + \alpha_1^2 \right) \xi_2^2. \]

If we choose
\[ \alpha_2 = n - 1 + \frac{1}{2} \left( 1 + 2 \alpha_2 + 2 \alpha_1 + c_1 \alpha_1 + c_2 + \frac{2}{T} \alpha_1 + c_2 \alpha_1 + \alpha_1^2 \right) \]
\[ x_3^* = -\alpha_2 \mu \xi_2 \]

then we have
\[ \mathcal{L} V_2 \leq -(|n - 1|) \mu \xi_c^2 - (n - 1) \frac{\xi_2^2}{\mu} + \frac{1}{\mu^2} \xi_2 (x_3 - x_3^*). \]

**Deductive step:** In this step, we aim to design \( x_{k+2}^* \).

Assume that at step \( k \), there are a set of virtual controllers \( x_2^*, \ldots, x_{k+1}^* \) defined by
\[ x_2^* = -\alpha_1 \mu \xi_1, \quad \xi_1 = x_1 \]
\[ x_3^* = -\alpha_2 \mu \xi_2, \quad \xi_2 = x_2 - x_2^* \]
\[ \vdots \]
\[ x_{k+1}^* = -\alpha_k \mu \xi_k, \quad \xi_k = x_k - x_k^* \]
\[ \xi_{k+1} = x_{k+1} - x_{k+1}^* \]

such that
\[ \mathcal{L} V_k \leq - \sum_{i=1}^{k} (n - k + 1) \frac{\xi_i^2}{\mu^2 - 3} + \frac{1}{\mu^{2k-2}} \xi_k (x_{k+1} - x_{k+1}^*) \]

where \( \alpha_1, \ldots, \alpha_k \) are positive constants and
\[ V_k = \sum_{i=1}^{k} \frac{1}{2 \mu^{2i-2}} \xi_i^2. \]

To complete the induction, at the \( k + 1 \)th step, we consider the \( \xi_{k+1} \) system.

Let \( \xi_{k+1} = x_{k+1} - x_{k+1}^* \), substituting \( x_2^*, x_3^*, \ldots, x_{k+1}^* \) defined in (49)–(51) into \( \xi_2, \xi_3, \ldots, \xi_{k+1} \), respectively, step by step, we get
\[ \xi_{k+1} = x_{k+1} + \sum_{s=1}^{k} \alpha_s \ldots \alpha_k \mu^{k+1-s} x_s. \]

By (7) and (54), we get
\[ d \xi_{k+1} = \left( x_{k+2} + f_{k+1} + \sum_{s=1}^{k} \alpha_s \ldots \alpha_k \mu^{k+1-s} (x_{s+1} + f_s) \right) \]
\[ + \frac{2}{T} \sum_{s=1}^{k} (k + 1 - s) \alpha_s \ldots \alpha_k \mu^{k-s+3/2} x_s \]
\[ = \frac{d}{dt} \left( \sum_{s=1}^{k} \alpha_s \ldots \alpha_k \mu^{k-s+3/2} x_s \right). \]

We choose the new Lyapunov function
\[ V_{k+1} = V_k + \frac{1}{2 \mu^2} \xi_{k+1}^2. \]

It follows from (52), (55), and (56) and Itô’s formula that
\[ \mathcal{L} V_{k+1} \leq - \sum_{i=1}^{k} (n - k + 1) \frac{\xi_i^2}{\mu^2 - 3} + \frac{1}{\mu^{2k-2}} \xi_k (x_{k+1} - x_{k+1}^*) \]
\[ - \frac{2}{T} k \mu^{2k-2} \xi_{k+1}^2 + \frac{1}{\mu^2} \xi_k \sum_{s=1}^{k} \alpha_s \ldots \alpha_k \mu^{k-s+3/2} x_s \]
\[ - \frac{2}{T} \sum_{s=1}^{k} (k + 1 - s) \alpha_s \ldots \alpha_k \mu^{k-s+3/2} x_s \]
\[ \leq - \sum_{s=1}^{k} (n - k + 1) \frac{\xi_i^2}{\mu^2 - 3} + \frac{1}{\mu^{2k-2}} \xi_k (x_{k+1} - x_{k+1}^*) \]
\[ - \sum_{s=1}^{k} \alpha_s \ldots \alpha_k \mu^{k+1-s} (c_s |x_s| + \ldots + |x_{s+1}|) \]
\[ \leq - \sum_{s=1}^{k} \sum_{j=1}^{s} c_s \alpha_s \ldots \alpha_k \mu^{k+1-s} |x_j| \]
\[ + \sum_{s=1}^{k} \alpha_s \ldots \alpha_k \mu^{k+1-s} |x_{s+1}| \]

From Lemma A.2, we get
\[ \frac{1}{\mu^{2k-2}} \xi k \xi_{k+1} \leq \frac{1}{2 \mu^{2k-2}} \xi_k^2 + \frac{1}{2 \mu^{2k-1}} \xi_{k+1}^2. \]

By (10), we obtain
\[ \sum_{s=1}^{k} \alpha_s \ldots \alpha_k \mu^{k+1-s} (c_s |x_s| + \ldots + |x_{s+1}|) \]
\[ \leq \sum_{s=1}^{k} \sum_{j=1}^{s} c_s \alpha_s \ldots \alpha_k \mu^{k+1-s} |x_j| \]
\[ + \sum_{s=1}^{k} \alpha_s \ldots \alpha_k \mu^{k+1-s} |x_{s+1}| \]
\[
\begin{aligned}
&= \sum_{j=1}^{k} \left( \sum_{s=j}^{k} c_s \alpha_s \ldots \alpha_k \mu^{k+1-s} \right) |x_j| \\
&+ \sum_{j=2}^{k} \alpha_{j-1} \ldots \alpha_k \mu^{k+2-j} |x_j| + \alpha_k \mu |x_{k+1}| \\
&= \left( \sum_{s=1}^{k} c_s \alpha_s \ldots \alpha_k \mu^{k+1-s} \right) |x_1| + \alpha_k \mu |x_{k+1}| \\
&+ \sum_{j=2}^{k} \left( \alpha_{j-1} \ldots \alpha_k \mu^{k+2-j} + \sum_{s=j}^{k} c_s \alpha_s \ldots \alpha_k \right. \\
&\left. \cdot \mu^{k+1-s} \right) |x_j|.
\end{aligned}
\]

\[\text{(66)}\]

Denoting
\[\Delta_{k+1,1} = c_{k+1} + \sum_{s=1}^{k} c_s \alpha_s \ldots \alpha_k + \frac{2}{T} k \alpha_1 \ldots \alpha_k\]
\[\text{(60)}\]

\[\Delta_{k+1,j} = c_{k+1} + \alpha_{j-1} \ldots \alpha_k + \sum_{s=j}^{k} c_s \alpha_s \ldots \alpha_k \]
\[\text{(61)}\]

\[\Delta_{k+1,k+1} = c_{k+1} + \alpha_k\]
\[\text{(62)}\]

we have
\[f_{k+1} + \sum_{s=1}^{k} \alpha_s \ldots \alpha_k \mu^{k+1-s} (x_{s+1} + f_s)\]
\[= \frac{2}{T} \sum_{s=1}^{k} (k + 1 - j) \alpha_j \ldots \alpha_k \mu^{k-s+3/2} x_s\]
\[\leq \Delta_{k+1,1} \mu^{k+1} |x_1| + \sum_{j=2}^{k} \Delta_{k+1,j} \mu^{k+2-j} |x_j|\]
\[+ \Delta_{k+1,k+1} \mu |x_{k+1}|\]
\[\leq \Delta_{k+1,1} \mu^{k+1} |\xi_1| + \sum_{j=2}^{k} \Delta_{k+1,j} \mu^{k+2-j} |\xi_j|\]
\[+ \sum_{j=2}^{k} \Delta_{k+1,j} \mu^{k+3-j} \alpha_{j-1} |\xi_{j-1}|\]
\[+ \Delta_{k+1,k+1} \mu |\xi_{k+1}| + \Delta_{k+1,k+1} \mu^2 \alpha_k |\xi_k|\]
\[= \sum_{j=1}^{k} \left( \Delta_{k+1,j} + \Delta_{k+1,j+1} \alpha_j \right) \mu^{k+2-j} |\xi_j|\]
\[+ \Delta_{k+1,k+1} \mu |\xi_{k+1}|.
\]
\[\text{(63)}\]

By Lemma A.2, we get
\[\frac{1}{\mu^2} |\xi_{k+1}| \sum_{j=1}^{k-1} \left( \Delta_{k+1,j} + \Delta_{k+1,j+1} \alpha_j \right) \mu^{k+2-j} |\xi_j|\]
\[= \sum_{j=1}^{k-1} \left( \Delta_{k+1,j} + \Delta_{k+1,j+1} \alpha_j \right) \mu^{k+2-j} |\xi_j|\]
\[+ \frac{1}{\mu^2} \xi_{k+1} \sum_{j=1}^{k-1} \left( \Delta_{k+1,j} + \Delta_{k+1,j+1} \alpha_j \right) \mu^{k+2-j} |\xi_j|\]
\[= \sum_{j=1}^{k-1} \left( \Delta_{k+1,j} + \Delta_{k+1,j+1} \alpha_j \right) \mu^{k+2-j} |\xi_j|\]
\[+ \Delta_{k+1,k+1} \mu |\xi_{k+1}|.
\]
\[\text{(64)}\]

It can be inferred from (63) to (65) that
\[\frac{1}{\mu^2} |\xi_{k+1}| \sum_{j=1}^{k} \alpha_j \ldots \alpha_k \mu^{k+1-s} (x_{s+1} + f_s)\]
\[+ \frac{2}{T} \sum_{s=1}^{k} (k + 1 - s) \alpha_s \ldots \alpha_k \mu^{k-s+3/2} x_s\]
\[\leq \sum_{j=1}^{k-1} \frac{1}{\mu^2} |\xi_{k+1}| \sum_{j=1}^{k-1} \left( \Delta_{k+1,j} + \Delta_{k+1,j+1} \alpha_j \right) \mu^{k+2-j} |\xi_j|\]
\[+ \Delta_{k+1,k+1} \mu |\xi_{k+1}|.
\]
\[\text{(65)}\]

Substituting (58) and (66) into (57) yields
\[\mathcal{L} V_{k+1} \leq - \sum_{i=1}^{k} \left( n - k \right) \frac{1}{\mu^{2i+3}} \| \xi_i \|^2 + \frac{1}{\mu^2} \xi_{k+1} (x_{k+2} - x_{k+2})\]
\[+ \frac{1}{4 \mu^{2k-1}} \left( 2 + 2 (\Delta_{k+1,k} + \Delta_{k+1,k+1} \alpha_k) \right)\]
\[+ \sum_{j=1}^{k-1} \left( \Delta_{k+1,j} + \Delta_{k+1,j+1} \alpha_j \right) \mu^{k+2-j} |\xi_j|\]
\[+ \Delta_{k+1,k+1} \mu |\xi_{k+1}|.
\]
\[\text{(67)}\]

Choosing the virtual controller
\[\alpha_{k+1} = n - k + \frac{1}{4} \left( 2 + 2 (\Delta_{k+1,k} + \Delta_{k+1,k+1} \alpha_k) \right)\]
\[x_{k+2}' = - \alpha_{k+1} \mu \xi_{k+1}\]

then we have

\[LV_{k+1} \leq - \sum_{i=1}^{k+1} (n - k) \frac{1}{\mu^{2i-3}} \xi_i^2 + \frac{1}{\mu^2} \xi_{k+1} (x_{k+2} - x_{k+1}').\]  

(68)

It follows from (49)–(51), (69), (71), and (76) that

\[x_{n+1}' = - \sum_{i=1}^{n} \alpha_i \cdots \alpha_n \mu^{n-1} x_i'.\]  

(78)

If we choose

\[k_i = \prod_{s=i}^{n} \alpha_s\]  

(79)

from (77) to (79), we get

\[LV_n \leq - \sum_{i=1}^{n} \frac{1}{\mu^{2i-3}} \xi_i^2 + \frac{1}{2} \sum_{i=1}^{n} k_i^2 g_i^2 \mu^{-2i} x_i^2.\]  

(80)

**Remark 3:** From (35), (46), (68), (75), and (79), we observe that \(\alpha_1, \ldots, \alpha_n\) are completely determined by the growth rates \(c_1, \ldots, c_n\) described in Assumption 1. In other words, once systems (1) and (2) are given, we can immediately get \(\alpha_1, \ldots, \alpha_n\). Then, by (79), we can easily obtain the control gains \(k_1, \ldots, k_n\).

**B. Stability Analysis**

We are now ready to state the main stability results on systems (1) and (2).

**Theorem 1:** Consider the plant consisting of (1)–(3) and (6).

If Assumptions 1 and 2 hold and the noise power \(\delta_i\) satisfies

\[0 < \delta_i < \frac{1}{\alpha_{i-1} \alpha_i}\]  

(81)

\[0 < \delta_i < \min \left\{ \frac{\sqrt{1 - (\prod_{s=i+1}^{n} \alpha_s)^2 \delta_{s+1}^2 \alpha_i^2}}{\prod_{s=i}^{n} \alpha_s}, \frac{1}{\prod_{s=i-1}^{n} \alpha_s} \right\}; \quad 1 \leq i \leq n - 1\]  

(82)

where \(\alpha_0 = 1\), then the following conclusions hold.

1. The plant has an almost surely unique solution on \([t_0, t_0 + T]\).
2. The plant is prescribed-time mean-square stabilized with \(\lim_{t \to t_0 + T} E|x|^2 = 0\). Specifically, \(\forall t \in [t_0, t_0 + T]\), we have

\[E|x|^2 \leq 2 \mu^{2n}(1 + \alpha) e^{-c_0 T^2/(t_0 + T - t)^{\frac{1}{2}}}(x_1^2(t_0) + \sum_{k=2}^{n} \left( x_k(t_0) + \sum_{s=1}^{k-1} \prod_{j=s}^{k} \alpha_j x_s(t_0) \right)^2).\]  

(83)

3. The controller is bounded in the sense that

\[\sup_{t \in [t_0, t_0 + T]} E \left| \int_{t_0}^{t} u(s) ds \right| < +\infty.\]  

(84)
Proof: By (5) and (11), we obtain
\[
\frac{1}{2} \sum_{i=1}^{n} k_i^2 \sum_{j} g_{ij}^2 \mu^{4-2i} x_i^2 \leq \frac{1}{2} \sum_{i=1}^{n} k_i^2 \sum_{j} g_{ij}^2 \mu^{3-2i} x_i^2.
\]

From (49) to (51), we have
\[
x_i^2 = \xi_i^2,
\]
\[
x_i^2 \leq 2\epsilon_i + 2\mu^2 \xi_{i-1}^2 \xi_i^2, 2 \leq i \leq n.
\]

It follows from (85) to (87) that:
\[
\frac{1}{2} \sum_{i=1}^{n} k_i^2 \sum_{j} g_{ij}^2 \mu^{4-2i} x_i^2 \leq \frac{1}{2} \sum_{i=1}^{n} k_i^2 \sum_{j} g_{ij}^2 \mu^{3-2i} x_i^2
\]
\[
\leq \frac{1}{2} k_i^2 \mu \xi_i^2 + \sum_{i=2}^{n} k_i^2 \sum_{j} g_{ij}^2 \mu^{3-2i} (\xi_i^2 + \mu^2 \alpha_{i-1}^2 \xi_i^2) = \left( \frac{1}{2} k_i^2 \delta_i^2 + k_i^2 \sum_{j} g_{ij}^2 \mu^{3-2i} \right) \mu \xi_i^2 + k_i^2 \sum_{j} g_{ij}^2 \mu^{3-2i} \alpha_{i-1}^2 \xi_i^2
\]
\[
+ \sum_{i=2}^{n-1} (k_i^2 \sum_{j} g_{ij}^2 + k_{i+1}^2 \alpha_{i+1}^2 \xi_i^2) \mu^{3-2i} \alpha_{i-1}^2 \xi_i^2.
\]

Substituting (88) into (80) yields
\[
\mathcal{L}V_n \leq \left( 1 - \frac{1}{2} k_i^2 \delta_i^2 - k_i^2 \sum_{j} g_{ij}^2 \alpha_{i-1}^2 \right) \mu \xi_i^2
\]
\[
- \sum_{i=2}^{n-1} (1 - k_i^2 \delta_i^2 - k_{i+1}^2 \delta_{i+1}^2) \frac{1}{\mu^{2i-3}} \xi_i^2 - (1 - k_i^2 \delta_i^2) \frac{1}{\mu^{2i-3}} \xi_i^2.
\]

By (81) and (82), we have
\[
c_0 = 2 \min_{2 \leq i \leq n-1} \left\{ 1 - \frac{1}{2} k_i^2 \delta_i^2 - k_i^2 \sum_{j} g_{ij}^2 \alpha_{i-1}^2, 1 - k_i^2 \delta_i^2 \right\} - k_i^2 \delta_{i+1}^2 \alpha_{i-1}^2, 1 - k_i^2 \delta_{i+1}^2 \alpha_{i-1}^2 > 0.
\]

From (73), (89), and (90), we obtain
\[
\mathcal{L}V_n \leq -c_0 \sum_{i=1}^{n} \frac{1}{2 \mu^{2i-3}} \xi_i^2 \leq -c_0 \mu V_n.
\]

It follows from (54), (71), and (73) that:
\[
\lim_{|x| \to +\infty} \inf_{t \in [t_0, T]} V_n = +\infty \forall t \in (t_0, t_0 + T).
\]

By (91) and (92), the conditions (A.3) and (A.4) of Lemma A.1 are satisfied. From Lemma A.1 we get conclusion 1) and
\[
EV_n \leq e^{-c_0 T^2} (1 + \frac{1}{\mu T}) V_n(t_0) \forall t \in [t_0, t_0 + T).
\]

It can be inferred from (73), (86), and (87) that
\[
E|x|^2 \leq \xi_1^2 + 2 \sum_{i=2}^{n} \xi_i^2 + 2\mu^2 \sum_{i=2}^{n} \alpha_{i-1}^2 \xi_i^2 - 4\mu^2 (1 + \alpha) EV_n.
\]

where
\[
\alpha = \max_{1 \leq i \leq n} \alpha_i^2.
\]

By (93) and (94), we get (83). From (83), we have
\[
\lim_{t \to t_0^+} E|x|^2 = 0.
\]

Next, we prove (84).

From (83), there exists a positive constant \( M \) such that
\[
E|x|^2 \leq M^2 \forall t \in [t_0, t_0 + T)
\]
from which we get
\[
E|x| \leq \sqrt{E|x|^2} \leq M \forall t \in [t_0, t_0 + T).
\]

By (2), we have
\[
\int_{t_0}^{t} u(s) ds = x_n(t) - x_n(t_0) - \int_{t_0}^{t} f_n(s, x) ds
\]
which, along with (10), shows that
\[
\left| \int_{t_0}^{t} u(s) ds \right| \leq |x_n(t) - x_n(t_0)| + \int_{t_0}^{t} f_n(s, x) ds
\]
\[
\leq |x_n(t)| + |x_n(t_0)| + \int_{t_0}^{t} f_n(s, x) ds
\]
\[
\leq |x_n(t)| + |x_n(t_0)| + c_n \int_{t_0}^{t} |x_n| ds
\]
\[
\leq |x_n(t)| + |x_n(t_0)| + c_n \sqrt{n} \int_{t_0}^{t} |x| ds.
\]

It follows from (97) and (99) that:
\[
E \left| \int_{t_0}^{t} u(s) ds \right| \leq |x_n(t)| + E|x_n(t)| + c_n \sqrt{n} E \left( \int_{t_0}^{t} |x| ds \right)
\]
\[
\leq |x_n(t)| + E|\xi_1^2| + c_n \sqrt{n} \int_{t_0}^{t} |x| ds
\]
\[
\leq |x_n(t)| + M + c_n \sqrt{n} M (t - t_0)
\]
\[
< +\infty \forall t \in [t_0, t_0 + T).
\]

By (100), we get (84).

Thus, the theorem is true.

Remark 4: From (81) and (82), the noise powers are determined in the following sequence: \( \delta_n, \delta_{n-1}, \ldots, \delta_1 \). These powers not only ensures that \( c_0 \) in (90) is positive, but also guarantees the inequality (82) is well defined. Specifically, they make the radicand in the square root of (82) nonnegative.

Remark 5: In Theorem 1, we prove that the controller is bounded in the sense that \( \sup_{t \in [t_0, t_0 + T]} E \left| \int_{t_0}^{t} u(s) ds \right| < +\infty \), which shows that the controller is bounded in the conventional sense. In fact, the controller (6) cannot be bounded in
the conventional sense. This is mainly because the noise goes directly into the controller. Specifically, we study systems with measurement noise. From (3) and (6), the noise \( \dot{\omega}_i \) is a term of the control \( u \). Since it is meaningless to discuss the property of noise at any fixed time \( t \), it is meaningless to discuss the statistical property of \( u \) at the fixed \( t \). However, it is significant to study the statistical property of the integral of the noise. Therefore, it is significant and important to study the statistical property of the integral of control \( u \), i.e., \( E[ \int_{t_0}^t u \, ds] \) in this article. And this would be the case even for linear systems with multiplicative, measurement, noise, being controlled over infinite time in [9].

**Remark 6:** In this section, we propose a new postulated feedback (6) to stabilize systems (1) and (2) in the prescribed-time. The existence of multiplicative noise makes all the existing stochastic prescribed-time designs in [22], [23], [24], and [25] inapplicable. In order to handle the multiplicative noise, the feedback is already inserted into (8). As shown in [12], [13], [14], [15], [16], [17], [18], [19], [20], and [21], even \( G = 0 \), how to design the feedback gains \( k_1, \ldots, k_n \) is nontrivial. More importantly, note that, the perturbation \( G \) is actually nonzero, contains the feedback gains \( k_1, \ldots, k_n \) and even goes to infinity as \( t \rightarrow t_0 + T \). How to design \( k_1, \ldots, k_n \) to prescribed-time stabilize the system in the presence of the nonlinearities \( f_i \) is a hard problem.

**Remark 7:** Different from the existing stochastic prescribed-time designs where scaling-free quartic Lyapunov functions [22], scaled quartic Lyapunov functions [23] and [25], or scaled quadratic Lyapunov function [24] are used, the design is based on a new scaled quadratic Lyapunov function \( \sum_{i=1}^{n} \frac{1}{2n^2} \xi_i^2 \) where the power of \( \mu \) is lower than that in [24]. The advantage of this Lyapunov function is that it simplifies the design process, which yields a relatively simple controller.

### C. Inverse Optimal Selection of Gains in Postulated Controller

In this subsection, we study the inverse optimal gain selection for systems (7) and (8).

First, we rewrite (7) and (8) as

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} x_2 + f_1(t, x) \\ \vdots \\ x_n + f_{n-1}(t, x) \\ f_n(t, x) \end{bmatrix} dt + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ - \sum_{i=1}^{n} k_i \mu^{n+1-i} x_i \end{bmatrix} dt \\
&+ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ G(t, x) \end{bmatrix} d\omega.
\end{align*}
\]

From the design and analysis in the above two subsections, the control gain

\[ k_i = \prod_{s=1}^{n} \alpha_s, \quad 1 \leq i \leq n \tag{102} \]

leads to

\[ \mathcal{L} V_n(\tau) \leq -c_0 \mu V_n \tag{103} \]

where the definitions of \( \alpha_1, \ldots, \alpha_n \) and \( V_n \) can be found in Section A.

Denote

\[ F(t, x) = \begin{bmatrix} x_2 + f_1(t, x) \\ \vdots \\ x_n + f_{n-1}(t, x) \\ f_n(t, x) \end{bmatrix}. \tag{104} \]

In the following theorem, we state the main results on prescribed-time inverse optimal assignment for system (101).

**Theorem 2:** Suppose that Assumptions 1 and 2 hold for system (101) and the noise power \( \delta_i \) satisfies

\[ 0 < \delta_i < \frac{1}{\beta \alpha_i} \tag{105} \]

\[ 0 < \delta_1 < \min \left\{ \sqrt{1 - \left( \prod_{s=i}^{n} \alpha_s \right)^2 \frac{\delta_1^2}{\beta \prod_{s=i}^{n} \alpha_s ^2}}, 1 \leq i \leq n - 1 \right\} \tag{106} \]

where \( \alpha_0 = 1 \) and \( \beta \geq 2 \) is a constant.

Then the control gains

\[ k_i^* = \beta \prod_{s=1}^{n} \alpha_s \quad \forall \ 1 \leq i \leq n \tag{107} \]

solves the prescribed-time inverse optimal mean-square stabilization problem for system (101) by minimizing the cost functional

\[ J(k) = \lim_{r \to \infty} \sup_{T \to \infty} \mathbb{E} \left[ 2 \beta V_n(\tau_r, x(\tau_r)) + \int_{t_0}^{\tau_r} \left( l(t, x(t)) \right) dt \right] \tag{108} \]

where \( k = (k_1, \ldots, k_n)^T, \ r \in Z^+, \ \tau_r = (t_0 + T) \wedge \inf \{ t : t \geq t_0, |x(t)| \geq r \} \) and

\[ l(t, x) = 2\beta \left( \frac{1}{\mu^{2n-3}} \alpha_n x_n^2 - \frac{\partial V_n}{\partial t} - \frac{\partial V_n}{\partial x} F \\
- \frac{1}{2\mu^{2n-2}} |G|^2 \right) + \beta(\beta - 2) \alpha_n \frac{1}{\mu^{2n-3}} x_n^2 \tag{109} \]

is positive definite and radially unbounded but not necessarily decrescent.

**Proof:** From (76), (78), (79), (103), (104), and (109), we obtain

\[ l(t, x) \geq 2\beta c_0 \mu V_n + \beta(\beta - 2) \alpha_n \frac{1}{\mu^{2n-3}} x_n^2. \tag{110} \]
From (49) to (51), we get
\[
x = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\
-\alpha_1 \mu & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & -\alpha_{n-1} \mu & 1 
\end{bmatrix} \xi
\]  
which shows that
\[
|x|^2 \leq \left( n + \sum_{i=1}^{n-1} \alpha_i^2 \mu^2 \right) |\xi|^2
\]
where \(x = (x_1, \ldots, x_n)^T\) and \(\xi = (\xi_1, \ldots, \xi_n)^T\).

By (73) and (112), we obtain
\[
V_n \geq \frac{1}{2} \left( n + \sum_{i=1}^{n-1} \alpha_i^2 \mu^2 \right) |x|^2.
\]
Noting \(\beta \geq 2\), from (110) and (113), we have
\[
l(t, x) \geq 2\beta \sigma_0 \mu V_n
\]
\[
\geq \frac{\beta \sigma_0}{n + \sum_{i=1}^{n-1} \alpha_i^2 \mu^2} |x|^2.
\]
Since \(\frac{\beta \sigma_0}{n + \sum_{i=1}^{n-1} \alpha_i^2 \mu^2} \mu^{2n-3}\) is a positive scaling function and \(|x|^2\) is a positive definite function of \(x, l(t, x)\) is well defined. Thus, \(J(k)\) is a meaningful cost functional.

Before proving the gain (107) minimizes (108), we first prove the gain (107) can make system (101) achieve prescribed-time mean-square stabilization.

From (102), (103), (105), and (106), noting \(\beta \geq 2\), we have
\[
\mathcal{L}V_n|_{(101)} = -\beta \frac{1}{\mu^{2n-3}} \alpha_n \sigma_n^2 + \frac{\partial V_n}{\partial t} + \frac{\partial V_n}{\partial x} F
\]
\[
+ \frac{\beta^2}{2 \mu^{2n-2}} \sum_{i=1}^{n} \left( \prod_{s=i}^{n-1} \alpha_s \sigma_i \mu^{n+1-i} x_i \right)^2
\]
\[
\leq -\frac{1}{\mu^{2n-3}} \alpha_n \sigma_n^2 + \frac{\partial V_n}{\partial t} + \frac{\partial V_n}{\partial x} F
\]
\[
+ \frac{\beta^2}{2 \mu^{2n-2}} \sum_{i=1}^{n} \left( \prod_{s=i}^{n-1} \alpha_s \sigma_i \mu^{n+1-i} x_i \right)^2
\]
\[
\leq -\sigma_0 \mu V_n.
\]
By (115) and Theorem 1, the gain (107) can make system (101) achieve prescribed-time mean-square stabilizable.

Now, we prove optimality. By Dynkin’s formula in Lemma A.3, we get
\[
E \left\{ V_n(\tau_r, x(\tau_r)) - V_n(t_0, x(t_0)) + \int_{t_0}^{\tau_r} \mathcal{L}V_n|_{(101)} ds \right\} = 0.
\]  
From (108), (109), and (116), we have
\[
J(k) = \limsup_{r \to \infty} \mathbb{E} [2 \beta V_n(\tau_r, x(\tau_r)) + \int_{t_0}^{\tau_r} l(t, x) dt]
\]
\[
+ \frac{1}{\alpha_n} \left( \sum_{i=1}^{n} (k_i \mu^{3/2-i} x_i(t))^2 \right) dt\]
\[
= \limsup_{r \to \infty} \mathbb{E} [2 \beta V_n(t_0, x(t_0)) + \int_{t_0}^{\tau_r} (\beta^2 \alpha_n \mu^{3/2-i} x_i(t))^2 dt]
\]
\[
+ \frac{1}{\alpha_n} \left( \sum_{i=1}^{n} (k_i \mu^{3/2-i} x_i(t))^2 \right) dt
\]
\[
- 2 \beta \sum_{i=1}^{n} k_i \mu^{-n+3-i} x_i \xi_n dt.
\]
By using Lemma A.4 with \(\gamma = r^2\), we obtain
\[
2 \beta \sum_{i=1}^{n} k_i \mu^{-n+3-i} x_i \xi_n
\]
\[
= \left( \beta \sqrt{\alpha_n \mu^{-n+3/2} \xi_n} \right) \left( \frac{2}{\sqrt{\alpha_n}} \sum_{i=1}^{n} k_i \mu^{3/2-i} x_i \right)
\]
\[
\leq \beta^2 \alpha_n \mu^{-n+3/2} \xi_n^2 + \frac{1}{\alpha_n} \left( \sum_{i=1}^{n} (k_i \mu^{3/2-i} x_i)^2 \right).\]
The equality in (118) holds when
\[
k_i = 2 \prod_{s=i}^{n} \alpha_s, \quad 1 \leq i \leq n.
\]
Therefore, the minimum of (108) is obtained with \(k_i = k_i^*\) in (119), and
\[
\min_k J(k) = 2 \beta V_n(t_0, x(t_0)).
\]
Thus, the theorem is proved.

Remark 8: A new stochastic prescribed-time inverse optimal design is proposed in this section. Different from the existing designs [22], [23], [31], [32], [33], and [34], where the inverse optimal controllers are designed step by step, this section proposes a new postulated inverse optimal controller (6) with the gains (107) whose merit is that it can effectively deal with the multiplicative noise, where the noise intensity including the gains (107) themselves.
IV. SYSTEMS WITH ARBITRARILY LARGE NOISE BUT VANISHING FASTER THAN LINEARLY

In this section, when the noise power is arbitrary large but vanishing faster than linearly, we aim to solve the prescribed-time mean-square stabilization problem of systems (7)–(9).

The control gain design is the same as that in Section II. Specifically, using the control (6) with the gains (79), from (80) we get

\[ \mathcal{L}V_n \leq -\sum_{i=1}^{n} \frac{1}{\mu_i^{2i-3}} \xi_i^2 + \frac{1}{2} \sum_{i=1}^{n} k_i^2 \delta_i^2 \mu^{4-2i} \xi_i^2. \]  

(121)

In Section III, when the noise power \( \delta_i \) is small enough [as shown in (81) and (82)], from (91) we obtain that \( \mathcal{L}V_n \) can be made negative definite. Then the prescribed-time mean-square stable follows from Lemma A.1. In this section, we consider noise with an arbitrary large power \( \delta_i \), which makes (91) not hold. Specifically, the arbitrary large noise \( \delta_i \) yields \( \mathcal{L}V_n \leq \hat{c}_i \mu \xi_i \), with \( \hat{c} \) being a positive constant. In this case, Lemma A.1 fails since (A.4) does not hold. How to prove the plant has an almost surely unique solution on \( [t_0, t_0 + T] \) and how to prove the plant is prescribed-time mean-square stable is a challenge work. New tools should be developed.

Next, we first give a lemma, which proposes new tools to show that the plant has an almost surely unique strong solution on \( [t_0, t_0 + T] \) when \( \mathcal{L}V_n \leq \hat{c}_i \mu \xi_i \).

**Lemma 1:** Consider the system (A.1). If there exists a non-negative function \( U(t, x) \in C^{1,2}([t_0, t_0 + T) \times R^n; R^+) \), and positive constants \( c_0 \) such that

\[ \lim_{|x| \to +\infty} \inf_{t \in [t_0, T]} U(t, x) = +\infty \quad \forall T \in (t_0, t_0 + T) \]  

(122)

\[ \mathcal{L}U(t, x) \leq c_0 \mu U \quad \forall t \in [t_0, t_0 + T) \]  

(123)

then system (A.1) has an almost surely unique strong solution on \( [t_0, t_0 + T] \) for any \( x_0 \in R^n \).

**Proof:** The proof is given in Appendix B.

**Remark 9:** Lemma 1 is a prescribed-time version of [35, Th. 3.5] where the linear growth condition of \( \mathcal{L}U(t, x) \) guarantees the existence of a global solution on \( [t_0, +\infty) \). However, [35, Th. 3.5] fails to cover Lemma 1 since \( \mathcal{L}U(t, x) \) does not satisfy the linear growth condition with \( \lim_{t \to t_0 + T} \mu(t) = +\infty \).

In addition, the condition (123) fails to satisfy \( \mathcal{L}U(t, x) \leq -c_0 \mu U + \mu M_\theta \) required by [22, Lemma A.1]. Thus, Lemma 1 is a new result that is completely different from [35, Th. 3.5] and [22, Lemma A.1].

We now state the main stability results in this section.

**Theorem 3:** Consider the plant consisting of (1)–(3) and (6). If Assumptions 1 and 3 hold, then the following conclusions hold.

1. The plant has an almost surely unique solution on \( [t_0, t_0 + T) \).
2. The plant is prescribed-time mean-square stabilized with \( \lim_{t \to t_0 + T} E[x]^2 = 0 \).

**Proof:** We first prove that the plant has an almost surely unique solution on \( [t_0, t_0 + T) \).

It can be inferred from (5) and (12) that

\[ \frac{1}{2} \sum_{i=1}^{n} k_i^2 \delta_i^2 \mu^{4-2i} \xi_i^2 \leq \frac{1}{2} \sum_{i=1}^{n} k_i^2 \delta_i^2 \mu^{3-2i-\sigma_i} \xi_i^2. \]  

(124)

By (86), (87), and (124), we obtain

\[ \frac{1}{2} \sum_{i=1}^{n} k_i^2 \delta_i^2 \mu^{4-2i} \xi_i^2 \leq \frac{1}{2} \sum_{i=1}^{n} k_i^2 \delta_i^2 \mu^{3-2i-\sigma_i} \xi_i^2 + \frac{1}{2} \sum_{i=1}^{n} (1 - k_i^2 \delta_i^2 \mu^{3-2i-\sigma_i}) \xi_i^2. \]  

(125)

Substituting (125) into (121), we get

\[ \mathcal{L}V_n \leq \left( -\left( 1 - k_i^2 \delta_i^2 \mu^{3-2i-\sigma_i} \right) \mu \xi_i^2 - \sum_{i=2}^{n-1} \left( 1 - k_i^2 \delta_i^2 \mu^{3-2i-\sigma_i} \right) \xi_i^2 \right) \xi_1^2 \]  

(126)

Noting \( \mu \geq 1 \) and \( \sigma_i > 0 \), from (126), there exists a positive constant \( \bar{c}_0 \) such that

\[ \mathcal{L}V_n \leq \bar{c}_0 \mu V_n. \]  

(127)

By (127) and Lemma 1, the plant has an almost surely unique solution on \( [t_0, t_0 + T) \), which shows that conclusion 1 holds.

We then prove that the plant is prescribed-time mean-square stabilized with \( \lim_{t \to t_0 + T} E[x]^2 = 0 \).

Since \( \sigma_i > 0 \) and \( \lim_{t \to t_0 + T} \mu(t) = +\infty \), there exists a positive \( t^* \in (t_0, t_0 + T) \) such that \( \forall t \in [t^*, t_0 + T) \) and \( \forall 1 \leq i \leq n - 1 \),

\[ 1 - k_i^2 \delta_i^2 \mu^{3-2i-\sigma_i} \geq \frac{1}{2} \]  

(128)

\[ 1 - k_i^2 \delta_i^2 \mu^{3-2i-\sigma_i} \geq \frac{1}{2}. \]  

(129)

From (73), (126), (128), and (129), we have

\[ \mathcal{L}V_n \leq -\mu V_n \quad \forall t \in [t^*, t_0 + T). \]  

(130)

Choosing

\[ V = e^{H(t^*)} \mu(s) ds V_n. \]  

(131)

By (130) and (131), we obtain

\[ \mathcal{L}V = e^{H(t^*)} \mu(s) ds (\mathcal{L}V_n + \mu(t)V_n) \leq 0. \]  

(132)

Let \( k \) be a positive integer. Define the stopping time

\[ \sigma_k = \inf \{ t : t^* \leq t < t_0 + T, |x| + |\dot{x}| \geq k \}. \]  

(133)
From conclusion 1), the plant has an almost surely unique solution on \([t_0, t_0 + T]\). Thus, \(\sigma_k \to +\infty\) almost surely as \(k \to +\infty\).

Let \(t_k = \sigma_k \wedge t\) for any \(t \in [t^*, t_0 + T]\). Noting \(V(t^*, x(t^*)) = V_n(t^*)\) and using (132) and (133), by Lemma A.3 we obtain

\[
EV(t_k, x(t_k)) = V_n(t^*) + E \left\{ \int_{t_k}^{t_k} \mathcal{L} V(x(\tau), \tau) d\tau \right\} \\
\leq V_n(t^*) \quad \forall t \in [t^*, t_0 + T].
\]

(134)

For \(t \in [t^*, t_0 + T]\), by (131) and (134), we get

\[
E \left\{ e^{f_{\tau_0}^{t_2} \mu(s) ds} V_n(t_k, x(t_k)) \right\} \leq V_n(t^*).
\]

(135)

letting \(k \to +\infty\), by Fatou Lemma and (135), we get

\[
E \left\{ e^{f_{\tau_0}^{t_2} \mu(s) ds} V_n(t, x(t)) \right\} \leq V_n(t^*).
\]

(136)

By (136), we get

\[
E \{ V_n(t, x(t)) \} \\
\leq e^{-f_{\tau_0}^{t_2} \mu(s) ds} V_n(t^*) \\
\leq e^{-T^2 \left( \frac{1}{t^* - t_0} + \frac{1}{t^* - t} \right)} V_n(t^*) \quad \forall t \in [t^*, t_0 + T].
\]

(137)

From (94) and (137), for \(t \in [t^*, t_0 + T]\), we get

\[
E|x|^2 \leq 4\mu^2 (1 + \alpha) e^{-T^2 \left( \frac{1}{t^* - t_0} + \frac{1}{t^* - t} \right)} V_n(t^*)
\]

(138)

which shows that

\[
\lim_{t \to t_0 + T} E|x|^2 = 0.
\]

(139)

Thus, the theorem is true.

V. TWO SIMULATION EXAMPLES

In this section, we give two simulation examples to show the effectiveness of control schemes developed in the last two sections. Specifically, we first consider small linearly vanishing noise in Example 1, then consider arbitrary large noise vanishing faster than linearly in Example 2.

**Example 1:** Consider the mass-spring mechanical system shown in Fig. 1, where a mass \(m\) is attached to a wall through a spring and sliding on a horizontal surface. The mass is driven by an external force which serves as a control variable. Let \(y\) be the displacement from a reference position. By Newton’s law of motion, the system is described as [36]

\[
m\ddot{y} + F_f + F_{sp} = u
\]

(140)

where \(F_f\) is a resistive force due to friction and \(F_{sp}\) is the restoring force of the spring. We assume that the displacement is relative small, and thus, \(F_{sp}\) can be written as \(F_{sp} = ky\), where \(k\) is a spring parameter. Meantime, we assume the resistive force is linear viscous friction and write \(F_f = c\dot{y}\), where \(c\) is a friction parameter.

To obtain a state model for the mass-spring mechanical system, take the state variables as \(x_1 = y\) and \(x_2 = \dot{y}\). Then, from (140) we get the state-space form as

\[
\dot{x}_1 = x_2,
\]

(141)

\[
\dot{x}_2 = \frac{u}{m} - \frac{k}{m} x_1 - \frac{c}{m} x_2.
\]

(142)

Choosing \(m = 1\), \(k = 0.2\), \(c = 0.1\), Assumption 1 is satisfied with \(c_1 = 0\) and \(c_2 = 0.2\).

Let \(t_0 = 0\) and \(T = 1\). (5) can be rewritten as

\[
\mu(t) = \left( \frac{1}{1 - t} \right)^2 \quad \forall t \in [0, 1).
\]

(133)

We observe the state \(x_1\) as \(y_1\), which is described by

\[
y_1 = x_1 (1 + 0.007(1 - t)) \omega_1
\]

(144)

\[
y_2 = x_2 (1 + 0.01(1 - t)) \omega_2.
\]

(145)

From the design in Section III, we get \(\alpha_1 = 2\) and \(\alpha_2 = 40.68\). From (81) and (82), we have

\[
0 < \delta_1 < 0.0072 \quad \text{(146)}
\]

\[
0 < \delta_2 < 0.012. \quad \text{(147)}
\]

From (144) to (147), Assumption 2 holds.

By following the design procedure developed in Section III, we obtain the controller as

\[
u = -81.36\mu^2 y_1 - 40.68 \mu y_2.
\]

(148)

For simulation, we randomly set the initial conditions as \(x_1(0) = -1\), \(x_2(0) = 2\). Fig. 2 gives the responses of the controller and states, which shows that \(\lim_{t \to \infty} E|x|^2 = 0\). Therefore, the effectiveness of the control scheme developed in Section III is demonstrated.

**Example 2:** Consider the following system:

\[
\dot{x}_1 = x_2 + 0.5 \ln(1 + \sin^2 x_1)
\]

(149)
\[ \dot{x}_2 = u + x_2 \cos x_1. \]  
\[(150)\]

Obviously, Assumption 1 is satisfied with \( c_1 = 0.5 \) and \( c_2 = 1 \). Let \( t_0 = 0 \) and \( T = 2 \). (5) can be rewritten as

\[ \mu(t) = \left( \frac{2}{2 - t} \right)^2 \forall t \in [0, 2). \]
\[(151)\]

We observe the state \( x_i \) as \( y_i \), which is described by

\[ y_1 = x_1(1 + 5(1 - 0.5t)^{1.1} \omega_1) \]
\[(152)\]

\[ y_2 = x_2(1 + 2(1 - 0.5t)^{1.2} \omega_2). \]
\[(153)\]

From (152) and (153), Assumption 3 holds with \( \delta_1 = 5 \), \( \delta_2 = 2 \), \( \sigma_1 = 0.1 \) and \( \sigma_2 = 0.2 \).

From the design in Section IV, we get \( \alpha_1 = 2.5 \) and \( \alpha_2 = 94.1 \) and the controller as

\[ u = -235.25 \mu^2 y_1 - 94.1 \mu y_2. \]
\[(154)\]

For simulation, we randomly set the initial conditions as \( x_1(0) = -0.2 \), \( x_2(0) = 2 \). Fig. 3 gives the responses of the controller and states, which shows that \( \lim_{t \to T} E |x|^2 = 0. \) Therefore, the effectiveness of the control scheme developed in Section IV is demonstrated.

VI. CONCLUSION

In this article, we have addressed the prescribed-time designs for strict-feedback nonlinear systems with multiplicative measurement noise. When the noise is small and linearly vanishing, we first propose a new postulated feedback to solve the prescribed-time mean-square stabilization problem, then redesign the control gains to solve the prescribed-time stochastic inverse optimal assignment problem. When the noise is arbitrary large but vanishing faster than linearly, we have developed a new control scheme to make the system achieve prescribed-time mean-square stabilization. In order to handle the multiplicative noise, in our new designs, the feedback is inserted into the system, which leads to that the noise intensity is actually nonzero, contains the feedback gains and even goes to infinity in the terminal time, how to design the control gains to prescribed-time stabilize the system in the presence of the nonlinearities is a hard problem.

For the prescribed-time designs with multiplicative measurement noise, many important issues are still open and worth investigating, such as output-feedback control, prescribed-time control for more general systems with unknown control gain [42], etc.

APPENDIX A

The following notation will be used throughout the article. \( R^n \) denotes the real \( n \)-dimensional space. For a given vector or matrix \( X, X^T \) denotes its transpose, \( \text{Tr}\{X\} \) denotes its trace when \( X \) is square, and \( |X| \) is the Euclidean norm of a vector \( X \).

Defining \( |A| = \left( \sum_{i=1}^n \sum_{j=1}^m a_{ij}^2 \right)^{\frac{1}{2}} \) for a matrix \( A_{n \times m} \).

Consider the following stochastic nonlinear system:

\[ dx = f(t, x)dt + g(t, x)dw \quad \forall x_0 \in R^n \]
\[(A.1)\]

where \( x \in R^n \) is the system state. The functions \( f : R^+ \times R^n \to R^n \) and \( g : R^+ \times R^n \to \mathbb{R}^{m \times n} \) are continuous of their arguments and are locally Lipschitz in \( x, \omega \) is an \( m \)-dimensional independent standard Wiener process defined on the complete filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \).

The following definition and lemmas are used in the controller design and stability analysis. Definition A.1 ([22]): For stochastic system (A.1) with \( f(t, 0) = 0 \) and \( g(t, 0) = 0 \), the equilibrium \( x(t) = 0 \) is prescribed-time mean-square stable if there exist positive constants \( k_1 (1 \leq i \leq 4) \) such that

\[ E|\int_0^t x(s)ds|^2 \leq k_1 \int_0^t |x(t)|^2 (1 + \mu^2(t)) e^{-k_3\mu^2(t)} \]
\[(A.2)\]

holds for \( \forall t \in [t_0, t_0 + T). \)

Lemma A.1 ([22]): Consider the system (A.1). If there exist a nonnegative function \( U(t, x) \in C^{1,2}([t_0, t_0 + T) \times R^n; R^+) \), and positive constants \( c_0 \) and \( M_0 \) such that

\[ \lim_{|x| \to \infty} \inf_{t \in [t_0, t_0 + T]} U(t, x) = +\infty \quad \forall T_1 \in [t_0, t_0 + T). \]
\[(A.3)\]

\[ LU(t, x) \leq -c_0 \mu U + \mu M_0 \quad \forall t \in [t_0, t_0 + T) \]
\[(A.4)\]

then the following conclusions hold.

1) System (A.1) has an almost surely unique strong solution on \([t_0, t_0 + T)\) for any \( x_0 \in R^n \).

2) The function \( U(t, x) \) satisfies

\[ EU(t, x(t)) \leq e^{-c_0 \int_{t_0}^t \mu(s)ds} U(t_0, x_0) + \frac{M_0}{c_0} \quad \forall t \in [t_0, t_0 + T). \]
\[(A.5)\]

Lemma A.2 ([40]): For \((x, y) \in R^2\), the following Young’s inequality holds:

\[ xy \leq \frac{p^p}{p} |x|^p + \frac{1}{q^q} |y|^q \]
\[(A.6)\]

where \( \nu > 0 \), the constants \( p > 1 \) and \( \rho > 1 \) satisfy \( (p - 1)(q - 1) = 1 \).

Lemma A.3 ([38]): Let \( V \in C^{1,2}(R^+ \times R^n; R^+) \) and \( \tau_1, \tau_2 \) be bounded stopping times such that \( 0 \leq \tau_1 \leq \tau_2 \) a.s. If \( V(t, x) \) and \( LV(t, x) \) are bounded on \( t \in [\tau_1, \tau_2) \) a.s., then

\[ E[V(\tau_2, x) - V(\tau_1, x)] = E \left\{ \int_{\tau_1}^{\tau_2} LV(t, x)dt \right\}. \]
\[(A.7)\]
**Lemma A.4 ([39]):** For any two vectors \( x \) and \( y \), the following holds:

\[
x^Ty \leq \gamma(|x|) + \ell_\gamma(|y|)
\]  

(\( A.8 \))

and the equality is achieved if and only if

\[
y = \gamma(|x|) \frac{x}{|x|}
\]  

(\( A.9 \))

where \( \ell_\gamma(r) = r(\gamma)^{-1}(r) - \gamma((\gamma)^{-1}(r)) \), \( \gamma \) and its derivative \( \dot{\gamma} \) are both \( \mathcal{K}_\infty \) functions.

**APPENDIX B**

**PROOF OF LEMMA 1**

By [37, Th. 3.15], system (A.1) has an almost surely unique solution \( x(t) \) on \([t_0, \rho_\infty)\) with \( \rho_\infty = (t_0 + T) \land \lim_{T \to +\infty} \inf \{t_0 \leq t < t_0 + T : |x(t)| \geq \gamma \}. \) We need to prove \( \rho_\infty = t_0 + T \) a.s. If this is not true, we can find positive constants \( \varepsilon \) and \( T_2 \) (\( 0 < T_2 < T \)) such that

\[
P \{ \rho_\infty \leq t_0 + T_2 \} > 2\varepsilon.
\]

(\( B.1 \))

For each integer \( k > 0 \), define

\[
\rho_k = (t_0 + T) \land \inf \{ t : t_0 \leq t < t_0 + T, |x(t)| \geq k \}.
\]

(\( B.2 \))

Since \( \rho_k \to \rho_\infty \) a.s., there exists a sufficiently large integer \( k_0 \) such that

\[
P \{ \rho_k \leq t_0 + T_2 \} > \varepsilon \quad \forall k \geq k_0.
\]

(\( B.3 \))

Choosing

\[
\bar{U} = e^{-c_0 \int_{t_0}^t \mu(s)ds} U,
\]

(\( B.4 \))

From (123) and (B.4), we have

\[
L\bar{U} = e^{-c_0 \int_{t_0}^t \mu(s)ds} (LU - c_0 U) \
\]

\[ \leq 0. \tag{B.5} \]

Fix \( k \geq k_0 \). For any \( t_0 \leq t \leq t_0 + T_2 \), by (B.4) and (B.5) and Lemma A.3, we have

\[
\begin{align*}
E \bar{U}(t \land \rho_k, x(t \land \rho_k)) & = U(t_0, x_0) + E \left\{ \int_{t_0}^{t \land \rho_k} \bar{L}(x(\tau), \tau)d\tau \right\} \\
& \leq U(t_0, x_0).
\end{align*}
\]

(\( B.6 \))

By (B.6) we get

\[
E \bar{U}((t_0 + T_2) \land \rho_k, x((t_0 + T_2) \land \rho_k)) \leq U(t_0, x_0).
\]

(\( B.7 \))

It follows from (B.7) that:

\[
E \chi_{\rho_k \leq t_0 + T_2} \bar{U}(\rho_k, x(\rho_k)) \leq U(t_0, x_0) < +\infty.
\]

(\( B.8 \))

Define

\[
b_k = \inf \{ \bar{U}(t, x) : |x| \geq k, t \in [t_0, t_0 + T_2] \}.
\]

(\( B.9 \))

By (122) and (B.9), we get

\[
\lim_{k \to +\infty} b_k = +\infty.
\]

(\( B.10 \))

From (B.8), we obtain

\[
U(t_0, x_0) \geq b_k P \{ \rho_k \leq t_0 + T_2 \} > \varepsilon b_k.
\]

(\( B.11 \))

Letting \( k \to +\infty \) in both sides of (B.11), from (B.10), we obtain

\[
U(t_0, x_0) = +\infty
\]

(\( B.12 \))

which is a contradiction with (B.8). Thus, we have \( \rho_\infty = t_0 + T \). This completes the proof of Lemma 1.

**REFERENCES**


LI AND KRSTIC: PRESCRIBED-TIME CONTROL OF NONLINEAR SYSTEMS WITH LINEARLY VANISHING


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