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Stability analysis and predictor feedback control for systems with unbounded delays[☆]

Xiang Xu[a], Lu Liu[a,*], Miroslav Krstic[b], Gang Feng[a]

[a] Department of Biomedical Engineering, City University of Hong Kong, Kowloon, Hong Kong Special Administrative Region
[b] Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA

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ABSTRACT

This paper proposes a novel general framework with new Lyapunov stability theorems for control problems of systems with unbounded delays. Under this framework, the delayed input can be captured in the stability estimate by augmenting it as a state with initial conditions. Then by applying the novel framework, we consider the stabilization problems of linear systems with distributed unbounded input delays via the so-called predictor feedback controllers. Both time-invariant and time-varying cases are considered. It is shown that under the new framework, the previous works on low-gain based truncated predictor feedback controllers can be improved by removing an implicit assumption on initial conditions. It is also shown that the predictor feedback controllers can exponentially stabilize both time-invariant and time-varying linear systems with distributed unbounded input delays. Several examples are provided to illustrate the effectiveness of the predictor feedback controllers.

1. Introduction

In practical systems, time delay is often one of the main causes of instability and poor performance. As a result, analysis and control of systems with time delays has long been a research topic of considerable attraction, see, for example, (Fridman, 2014; Gu et al., 2003; Gu & Niculescu, 2003; Hale & Lunel, 1993; Niculescu, 2001; Richard, 2003) and references therein. Time-delayed systems are usually very challenging to deal with mainly because they involve infinite dimensional spaces. Many mathematical tools have been developed to deal with time-delayed systems. These tools can be typically classified as frequency domain methods and time domain methods. Frequency domain methods study the roots of characteristic equations of time-delayed systems, and is thus usually applicable to time-invariant linear systems, see, for example, Chen and Latchman (1995), Chen et al. (2008), Li et al. (2017), Zhong (2004, 2005). Time domain methods usually focus on constructing an appropriate Lyapunov function or functional satisfying specific conditions, and is thus in general more powerful than the frequency domain method. Lyapunov Krasovskii functional theorem and Lyapunov Razumikhin function theorem are two commonly-used theorems of time domain methods, and have been widely used in analysis and control of time-delayed systems, see, for example, Bekiaris-Liberis and Krstic (2011, 2016), Chakrabarty et al. (2018), Chen et al. (2017), Jankovic (2010), Krstic (2010), Lin and Fang (2007), Yoon and Lin (2015), Zhou et al. (2012).

However, all of these aforementioned works only focus on bounded delays. Unbounded delays, which are more general but more challenging to deal with, are rarely considered. Unbounded delays are also often called infinite delays in some existing literature. Both expressions will be used in the rest of this paper and systems with infinite or unbounded delays will be simply denoted as infinite-delayed systems for convenience. Unbounded delays do exist or are needed to describe the behaviors of some practical systems. Applications of systems with unbounded delays arise in the fields of biology (Culshaw et al., 2003; Djema et al., 2018; Josić et al., 2011), mechanics (Atay, 2003; Roesch & Roth, 2005), social science (Kuang & Smith, 1993), networked control (Gopalsamy & He, 1994; Jessop & Campbell, 2010; Michiels et al., 2009; Sipahi et al., 2007) and so on. In our previous works (Xu et al., 2018, 2019, 2020b), three challenges in handling systems with unbounded delays in comparison to systems with bounded delays were revealed. These challenges arise mainly because the states of systems with unbounded delays always contain a part of their initial conditions. In 1970s–1990s, some mathematicians developed some fundamental theories and stability results on systems

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* Corresponding author.
E-mail addresses: xiangxu5-c@my.cityu.edu.hk (X. Xu), luliu45@cityu.edu.hk (L. Liu), krstic@ucsd.edu (M. Krstic), megfeng@cityu.edu.hk (G. Feng).

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with infinite delays, see, for example, Hale (1974), Hale and Kato (1978), Hino (1983), Hino et al. (1990), Kato (1978), Sawano (1979), Zhang (1990, 2002). Results on stability of infinite-delay systems are also obtained by Kolmanovskii and Myshkis (1999) and Solomon and Fridman (2013). Kolmanovskii and Myshkis (1999) provide a Lyapunov theorem for asymptotic stability of nonlinear infinite-delay systems. Solomon and Fridman (2013) study stability of linear time-invariant systems with unbounded delays. More recently, we proposed several more general Lyapunov theorems for studying stabilities of nonlinear systems with infinite delays in Xu et al. (2020b).

Input delays, as one of the most important topics concerned with time-delayed systems, have been well studied. Artstein (1982) proposed a reduction method to handle systems with input delays, which can be seen as a pioneering work in this topic. More recently, another important milestone on this topic was developed in a series of works (Bekiaris-Liberis et al., 2013; Bekiaris-Liberis & Krstic, 2011, 2016; Krstic, 2010), where more rigorous stability analysis results for linear systems with bounded input delays were provided. A key idea of these works is to introduce a model where the delayed control input is dealt with as a state, i.e., the controller state. The controllers given in these works are called the predictor feedback controllers. However, these works only focused on bounded input delays though (Artstein, 1982) indeed mentioned the potential of the reduction method for handling systems with unbounded input delays without any rigorous theoretical authentication or stability analysis. In other words, how to deal with systems with unbounded input delays via predictor feedback controllers remains to be an open problem.

In this paper, we propose a novel general framework for control problems of systems with unbounded delays and applies it to design of predictor feedback controllers for linear systems with distributed unbounded input delays. The contributions of this paper in comparison with those existing relevant works can be summarized as follows.

First, we propose a novel general framework with new Lyapunov stability theorems for control problems of systems with unbounded delays. Under this new framework, the delayed input can be captured in the stability analysis by augmenting it as a state, i.e., the controller state, with initial conditions. The idea of the controller state originates from (Krstic & Smyshlyaev, 2008) and is further developed by Bekiaris-Liberis and Krstic (2011), Krstic (2010), but with transformations that do not extend to unbounded delays. In this work, we use a different approach for the infinite-delay systems. Our new framework provides a more general system model compared with that in our previous work (Xu et al., 2020b). Our Lyapunov theorems are also more general compared with those by Kolmanovskii and Myshkis (1999) and Solomon and Fridman (2013) as our results can be applied to verify asymptotic stability, global asymptotic stability and exponential stability of nonlinear infinite-delay systems. Moreover, the model under this framework is also more general when it reduces to the case of bounded delays in comparison with the model in those existing works on bounded delays (Bekiaris-Liberis et al., 2013; Bekiaris-Liberis & Krstic, 2011, 2016; Krstic, 2010).

Second, we apply the framework to stabilization problems of linear systems with distributed unbounded input delays. Predictor feedback controllers are designed and analyzed for both time-invariant and time-varying cases. We first show that under our new framework, the results obtained in Xu et al. (2020b), where a low-gain based truncated predictor feedback controller is designed, can be improved by removing an implicit assumption on initial conditions. Next, we further analyze the stabilities of time-invariant linear systems with distributed unbounded input delays under the predictor feedback controller. It is shown that the predictor feedback controller can exponentially stabilize the time-invariant linear systems with distributed unbounded input delays. A distinctive advantage of the predictor feedback controller, compared with the low-gain controller in Xu et al. (2020b), is that it can be applied to the case where the open loop dynamics are exponentially unstable. Last but not least, it is shown that our framework can also be used to analyze the stabilities of time-varying linear systems with distributed unbounded input delays under predictor feedback controllers.

Recently, we have also obtained some results on predictor feedback and integrator backstepping for infinite-delayed systems in Xu et al. (2021) where time-invariant linear systems with an infinite-delayed integrator are considered by applying stability theorems in our previous work (Xu et al., 2020b). In contrast, the present paper develops a novel general framework for control problems of systems with unbounded delays under which predictor feedback control for both time-invariant and time-varying linear systems with distributed unbounded input delays can be dealt with.

The rest of this paper is organized as follows. In Section 2, a novel framework for control problems of systems with unbounded delays, including a system model, a stability definition, and Lyapunov theorems, is given. In Section 3, we apply the novel framework to design of predictor feedback controllers for both time-invariant and time-varying linear systems with distributed infinite input delays and the stability analysis of the resulting closed loop control system. Simulation examples are given in Section 4 and conclusions are drawn in Section 5.

Notations: Throughout this paper, the following notations are used. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $|\cdot|$ represents the absolute value of real numbers, the module of complex numbers, the l^2 norm of vectors or the induced 2-norm of matrices. For a symmetric and positive definite matrix P, the notation $P^{\frac{1}{2}}$ denotes its unique positive definite square root. For any two symmetric matrices P_1 and P_2, the notation $P_1 \geq P_2$ means that $P_1 - P_2$ is positive semi definite. A continuous function $\alpha : [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{KL}$ if it is continuous and for each fixed r, the mapping $\beta(\cdot, r)$ is of class $\mathcal{K}$ and for each fixed s, the mapping $\beta(s, \cdot)$ is decreasing and $\lim_{r\to+\infty} \beta(s, r) = 0$. Let $\mathbb{B}$ be a normed vector space. Then a functional $f : \mathbb{B} \to \mathbb{R}^n$ is said to be completely continuous if it is continuous and maps any bounded set in $\mathbb{B}$ into a bounded set in $\mathbb{R}^n$. Let x be a continuous function of time variable t. Then we use $\dot{x}$ to denote its right hand derivative.

2. A novel general framework for control problems of systems with unbounded delays

In this section, we introduce a novel general framework for control problems of systems with unbounded delays. Under such a framework, many control problems concerned with infinite-delayed systems can be addressed.

2.1. System model

Suppose $+\infty \geq r \geq 0$ and $[-r, 0] = (-\infty, 0]$ if $r = +\infty$. Let $\mathbb{B}_x$ and $\mathbb{B}_u$ be two vector spaces equipped with semi norms $\|\cdot\|_{\mathbb{B}_x}$ and $\|\cdot\|_{\mathbb{B}_u}$, respectively. Let $x_t \in \mathbb{B}_x$ and $u_t \in \mathbb{B}_u$ be defined by $x_t(\theta) = x(t+\theta), \theta \in [-r, 0]$ and $u_t(\theta) = u(t+\theta), \theta \in [-r, 0]$, respectively. Consider the following system,

$$\dot{x}(t) = f(t, x_t, u_t), t \geq t_0, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $x_t \in \mathbb{B}_x$, $u_t \in \mathbb{B}_u$, $f : \mathbb{R} \times \mathbb{B}_x \times \mathbb{B}_u \to \mathbb{R}^n$ is completely continuous on $\mathbb{R} \times \mathbb{B}_x \times \mathbb{B}_u$ and $f(t, 0, 0) = 0$. In system

(1), x and u are respectively plant state and controller state with $\mathbb{B}_x$ and $\mathbb{B}_u$ being plant state space and controller state space, respectively. The initial conditions x_{t_0} and u_{t_0} are determined by spaces $\mathbb{B}_x$ and $\mathbb{B}_u$, and they satisfy $\|x_{t_0}\|_{B_x} < +\infty$ and $\|u_{t_0}\|_{B_u} < +\infty$. Furthermore, we consider the following controller,

$$u(t) = g(t, x_t, u_t), t \geq t_0, \tag{2}$$

where $g : \mathbb{R} \times \mathbb{B}_x \times \mathbb{B}_u \to \mathbb{R}^n$ is completely continuous on $\mathbb{R} \times \mathbb{B}_x \times \mathbb{B}_u$ and $g(t, 0, 0) = 0$. Then the closed loop system can be written as

$$\begin{aligned} \dot{x}(t) &= f(t, x_t, u_t), \\ u(t) &= g(t, x_t, u_t), t \geq t_0. \end{aligned} \tag{3}$$

It is noted that controller (2) is more general than that in Xu et al. (2020b), which is recalled as follows,

$$u_t = \bar{g}(t, x_t), t \geq t_0, \tag{4}$$

where $\bar{g} : \mathbb{R} \times \mathbb{B}_x \to \mathbb{B}_u$ is continuous on $\mathbb{R} \times \mathbb{B}_x$, $\bar{g}(t, 0) = 0$ and maps $\mathbb{R} \times$ (bounded set in $\mathbb{B}_x$) into a bounded set in $\mathbb{B}_u$. Controller (4) is more restrictive than controller (2) in many aspects. First, controller (4) does not allow the feedback of the delayed control input u. Second, it implies an implicit property of initial conditions, that is, $u(t_0 - \eta) = \bar{g}(t_0, x_{t_0})(-\eta), \forall \eta \geq 0$. This is a rather restrictive property as it requires the initial conditions of u to be determined by the initial conditions of x on the whole history. On the contrary, this property is not required in controller (2). In fact, controller (2) can include controller (4) as its special case. Consider the following controller,

$$u(t) = \bar{g}(t, x_t)(0) \triangleq g(t, x_t), t \geq t_0, \tag{5}$$

which is in the form of controller (2). If its initial condition satisfies $u(t_0 - \eta) = \bar{g}(t_0, x_{t_0})(-\eta), \forall \eta \geq 0$, then it reduces to controller (4), which implies that controller (4) is a special case of controller (2).

2.2. Hypothesizes for system model

For systems with bounded delays, the choice of phase spaces $\mathbb{B}_x$ and $\mathbb{B}_u$ is not critical from the view point of the qualitative theory (Hale & Kato, 1978; Hale & Lunel, 1993). However, it is critical for systems with unbounded delays mainly because the states of systems with unbounded delays always contain a part of their initial conditions (Hale & Kato, 1978). The phase spaces $\mathbb{B}_x$ and $\mathbb{B}_u$ in this paper are supposed to satisfy the following hypothesis.

Hypothesis 2.1. (Kato, 1978) Let $\mathbb{B}$ be a vector space equipped with semi norm $\|\cdot\|_{B}$. For any $A > \sigma \geq 0$, if $x(t)$ is well defined on $[-r, A)$ and continuous on $[\sigma, A)$ and $x_\sigma \in \mathbb{B}$, then for any $t \in [\sigma, A)$,

(1) $x_t \in \mathbb{B}$;

(2) x_t is continuous in t with respect to $\|\cdot\|_B$;

(3) There exist constants $M_0 > 0$, $K > 0$, and a nonnegative and continuous function $M(t)$ such that $\lim_{t\to+\infty} M(t) = 0$ and

$$\begin{aligned} &|x(t)| \leq M_0\|x_t\|_B, \\ &\|x_t\|_B \leq K \sup_{\sigma \leq s \leq t} |x(s)| + M(t-\sigma)\|x_\sigma\|_B. \end{aligned}$$

Hypothesis 2.1 has been commonly adopted in the study of systems with infinite delays since it was first proposed in 1970s (Hale & Kato, 1978; Hino, 1983; Kato, 1978; Sawano, 1979). It was also adopted in our recent work (Xu et al., 2020b) where new stability and stabilization results on infinite-delayed systems were given. It can be verified that the spaces $\mathbb{R}^n$ and $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$, which are frequently applied in the study of non-delayed and bounded-delay systems, satisfy Hypothesis 2.1. Moreover, there are also many spaces for infinite-delay systems satisfying this hypothesis, which can be found in Hale and Kato (1978), Hino (1983), Kato (1978), Sawano (1979), Xu et al. (2020b).

In addition, the following hypothesis is required for our new framework.

Hypothesis 2.2. The controller state $u(t)$ in controller (2) is continuous for $t \geq t_0$ and its right hand derivative satisfies

$$\dot{u}(t) = h(t, x_t, u_t), t \geq t_0, \tag{6}$$

where $h : \mathbb{R} \times \mathbb{B}_x \times \mathbb{B}_u \to \mathbb{R}^n$ is completely continuous on $\mathbb{R} \times \mathbb{B}_x \times \mathbb{B}_u$ and $h(t, 0, 0) = 0$.

Remark 2.1. Hypothesis 2.2 implies a property of system (3), that is, compatibility of initial conditions. In system (3), the initial condition of the controller state is said to be compatible with the control law if

$$u(t_0) = g(t_0, x_{t_0}, u_{t_0}). \tag{7}$$

The compatibility of initial conditions is also assumed in some existing works on predictor feedback controllers for systems with bounded delays (Bekiaris-Liberis & Krstic, 2016). It can be seen that the initial condition of the controller state is compatible if and only if it is continuous at initial time t_0.

Remark 2.2. It follows from the results by Hale and Kato (1978, Theorem 2.1) that the existence of solutions to the considered system (3) can be ensured under Hypotheses 2.1 and 2.2. Moreover, the uniqueness of solutions to (3) can be ensured if Lipschitz or Lipschitz-like conditions are additionally satisfied, see Hale and Kato (1978, Theorem 2.2) and our previous work Xu et al. (2020a, Theorem 3.9) for reference.

2.3. Stability definition and Lyapunov theorems

In this subsection, we provide the following new definition on stabilities of system (3).

Definition 2.1. Define $\Gamma(t) = \|x_t\|_{B_x} + \|u_t\|_{B_u}$. The zero solution of system (3) is said to be

(1) uniformly stable if there exist a class $\mathcal{K}$ function α and a positive constant δ_0, independent of t_0, such that

$$\Gamma(t_0) < \delta_0 \text{ implies } \Gamma(t) \leq \alpha(\Gamma(t_0)), \forall t \geq t_0;$$

(2) uniformly asymptotically stable if there exist a class $\mathcal{KL}$ function β and a positive constant $\delta_0 > 0$, independent of t_0, such that

$$\Gamma(t_0) < \delta_0 \text{ implies } \Gamma(t) \leq \beta(\Gamma(t_0), t - t_0), \forall t \geq t_0;$$

(3) globally uniformly asymptotically stable if there exists a class $\mathcal{KL}$ function β such that

$$\Gamma(t) \leq \beta(\Gamma(t_0), t - t_0), \forall t \geq t_0;$$

(4) exponentially stable if there exists $\alpha > 0, \beta > 0, \delta_0 > 0$ such that for any $t_0 \geq 0$,

$$\Gamma(t_0) < \delta_0 \text{ implies } \Gamma(t) \leq \alpha e^{-\beta(t-t_0)}\Gamma(t_0), \forall t \geq t_0;$$

(5) globally exponentially stable if there exist $\alpha > 0$ and $\beta > 0$ such that for any $t_0 \geq 0$, any $x_{t_0} \in \mathbb{B}_x$ and any $u_{t_0} \in \mathbb{B}_u$,

$$\Gamma(t) \leq \alpha e^{-\beta(t-t_0)}\Gamma(t_0), \forall t \geq t_0.$$

The stabilities defined in Definition 2.1 can be also given via the $\varepsilon-\delta$ language, which is adopted in Xu et al. (2020b). However, by following the similar proof of Lemma 4.5 in Khalil (1996), one can see that these two types of definitions are equivalent.

System (3) and its stability definition 2.1 are novel in the sense that the delayed input can be captured in the stability analysis. Such an idea originated in Bekiaris-Liberis and Krstic (2011), Krstic (2010) for systems with bounded input delays, and was widely adopted since then (Bekiaris-Liberis & Krstic, 2016; Cai et al., 2019). This idea is now extended to the case of infinite-delayed systems in this work. It should be noted that in the existing works on bounded input delays such as (Bekiaris-Liberis & Krstic, 2011, 2016; Cai et al., 2019; Krstic, 2010), the systems are described by an ODE–PDE cascade. In contrary, the system description in this work does not rely on PDEs. Moreover, system (3) is also more general when it reduces to the case of systems with bounded delays as it allows more general choices of plant state space $\mathbb{B}_x$, while the plant state space is always chosen as the Euclidean space $\mathbb{R}^n$ in the existing works (Bekiaris-Liberis & Krstic, 2011, 2016; Cai et al., 2019; Krstic, 2010).

Then based on the stability results in our previous work (Xu et al., 2020b), we have the following Lyapunov theorems for system (3).

Theorem 2.1. *Consider system (3) under Hypotheses 2.1–2.2. Assume that f and h map $\mathbb{R}\times$ (bounded set in $\mathbb{B}_x\times\mathbb{B}_u$) into a bounded set in $\mathbb{R}^n$. If there exists a continuous functional $V:\mathbb{R}\times\mathbb{B}_x\times\mathbb{B}_u\to\mathbb{R}$ such that $\forall x_t\in\mathbb{B}_x, u_t\in\mathbb{B}_u$,*

(1) $a_1(|x(t)|)+a_2(|u(t)|)\le V(t,x_t,u_t)$;

(2) $V(t,x_t,u_t)\le b_1(\|x_t\|_{B_x})+b_2(\|u_t\|_{B_u})$;

(3) $\dot V(t,x_t,u_t)\le -c_1(|x(t)|)-c_2(|u(t)|)$;

where a_1,a_2,b_1,b_2,c_1,c_2 are class $\mathcal{K}$ functions, then system (3) is uniformly asymptotically stable in the sense of Definition 2.1. Moreover, if a_1,a_2,b_1,b_2 are of class $\mathcal{K}_\infty$, then system (3) is globally uniformly asymptotically stable in the sense of Definition 2.1.

Theorem 2.2. *Consider system (3) under Hypotheses 2.1–2.2. Assume that there exist $L_f>0$, $L_g>0$ and $r>0$ such that for all $\|\phi\|_{B_x}<r$ and $\|\psi\|_{B_u}<r$,*

$$\begin{aligned}|f(t,\phi,\psi)|\le&L_f(\|\phi\|_{B_x}+\|\psi\|_{B_u}),\\ |h(t,\phi,\psi)|\le&L_g(\|\phi\|_{B_x}+\|\psi\|_{B_u}).\end{aligned}\tag{8}$$

If there exists a continuous functional $V:\mathbb{R}\times\mathbb{B}_x\times\mathbb{B}_u\to\mathbb{R}$ such that for all $x_t\in\mathbb{B}_x, u_t\in\mathbb{B}_u$,

(1) $a_1|x(t)|^\alpha+a_2|u(t)|^\alpha\le V(t,x_t,u_t)$;

(2) $V(t,x_t,u_t)\le b_1\|x_t\|^\alpha_{B_x}+b_2\|u_t\|^\alpha_{B_u}$;

(3) $\dot V(t,x_t,u_t)\le -c_1|x(t)|^\alpha-c_2|u(t)|^\alpha$;

where $a_1,a_2,b_1,b_2,c_1,c_2,\alpha$ are positive constants, then system (3) is exponentially stable in the sense of Definition 2.1. Moreover, if condition (8) holds for all $\phi\in\mathbb{B}_x,\psi\in\mathbb{B}_u$, then system (3) is globally exponentially stable in the sense of Definition 2.1.

Both theorems can be proved by applying the results in Xu et al. (2020b) and considering the augmented system composed of (3) and (6). We only provide the proof of Theorem 2.2. The proof of Theorem 2.1 is similar and thus omitted.

Proof of Theorem 2.2. Define $Z(t)=col(x(t),u(t))\in\mathbb{R}^{n+m}$ and $Z_t=col(x_t,u_t)\in\mathbb{B}_x\times\mathbb{B}_u$. Define the norms $|Z(t)|=|x(t)|+|u(t)|$ and $\|Z_t\|_z=\|x_t\|_{B_x}+\|u_t\|_{B_u}$. It follows from system (3) and Hypothesis 2.2 that

$$\dot Z=\bar f(t,Z_t),\tag{9}$$

where $\bar f(t,Z_t)=col(f(t,x_t,u_t),h(t,x_t,u_t))$. It further follows from the conditions that

(1) $\frac{\min\{a_1,a_2\}}{2^\alpha}|Z(t)|^\alpha\le V(t,Z_t)\le 2\max\{b_1,b_2\}\|Z_t\|_z^\alpha$;

(2) $\dot V(t,Z_t)\le-\frac{\min\{c_1,c_2\}}{2^\alpha}|Z(t)|^\alpha$;

where $V(t,Z_t)=V(t,x_t,u_t)$. Therefore, by Theorem 3.5 in Xu et al. (2020b), there exist $\alpha_z,\beta_z>0$ such that

$$\|Z_t\|_z\le\alpha_z e^{-\beta_z(t-t_0)}\|Z_{t_0}\|_z,\tag{10}$$

which implies that

$$\begin{aligned}&\|x_t\|_{B_x}+\|u_t\|_{B_u}\\ \le&\alpha_z e^{-\beta_z(t-t_0)}(\|x_{t_0}\|_{B_x}+\|u_{t_0}\|_{B_u}).\end{aligned}\tag{11}$$

The global exponential stability is established and the theorem is thus proved. □

3. Applications of the framework: Predictor feedback controllers for systems with distributed unbounded input delays

The novel framework proposed in Section 2 will be applied to solving the stabilization problems of systems with distributed unbounded input delays via predictor feedback controllers in this section. Both time-invariant and time-varying linear systems with distributed unbounded input delays will be considered.

3.1. Truncated predictor feedback controllers

In Xu et al. (2018) and Xu et al. (2020b), we propose a class of low-gain controllers for handling stabilization problems of systems with distributed unbounded input delays. Such controllers are also called truncated predictor feedback controllers in some existing works on systems with bounded input delays (Zhou, 2014; Zhou et al., 2012). Consider the following system,

$$\begin{aligned}\dot x=&Ax+\int_0^{+\infty}B_0(\eta)u(t-\eta)d\eta\\ &+\sum_{i=1}^k B_iu(t-\tau_i),\end{aligned}\tag{12}$$

where $0\le\tau_i\le\tau$, $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$ and $B_0(\eta)$ is a matrix function whose elements are all Lebesgue integrable on $[0,+\infty)$. Define $\mathfrak{B}=\int_0^{+\infty}e^{-A\eta}B_0(\eta)d\eta+\sum_{i=1}^k e^{-A\tau_i}B_i$ and the following controller can be proposed,

$$u=-\mathfrak{B}^TPx,\tag{13}$$

where P is a positive definite matrix designed through the low-gain approach. The exponential stabilities of the closed loop system composed of (12) and (13) are proved in Xu et al. (2018) and Xu et al. (2020b). However, these proofs are established based on an implicit assumption of initial conditions, that is, $u(-\eta)=-\mathfrak{B}^TPx(-\eta),\eta\le 0$. By using the new framework proposed in Section 2, this implicit assumption can be removed. Therefore, in this subsection, we further analyze the stabilities of the closed loop system composed of plant (12) and truncated predictor feedback controller (13).

In this case, the following assumptions are made.

Assumption 3.1. There exists a non-increasing and positive function $p(\eta)$, where $\int_0^{+\infty}p(\eta)d\eta<+\infty$ and $p(u+v)\le p(u)p(v)$, $\forall u,v\ge 0$, such that the following condition holds,

$$|B_0(\eta)|\le p(\eta).\tag{14}$$

Assumption 3.2. The matrix A has all its eigenvalues on the imaginary axis and the matrix pair $(A,\mathfrak{B})$ is controllable.

In this section, the plant state space can be chosen as the Euclidean space $\mathbb{R}^n$. The controller state space $\mathbb{B}_u$ is chosen as a space of functions mapping from $(-\infty, 0]$ to $\mathbb{R}^n$. The space $\mathbb{B}_u$ and the norm $\|\cdot\|_{\mathbb{B}_u}$ can be defined as follows,

$$\mathbb{B}_u = \{\psi : (-\infty, 0] \to \mathbb{R}^n : \|\psi\|_{\mathbb{B}_u} < +\infty\},$$
$$\|\psi\|_{\mathbb{B}_u} = (\sup_{-\tau\le s\le 0} |\psi(s)|^q + \int_0^{+\infty} p(\eta)|\psi(-\eta)|^q d\eta)^{\frac{1}{q}},$$

where $+\infty > q \ge 1$. It follows from Hale and Kato (1978, Example 1.2) that Hypothesis 2.1 holds. In fact, one can verify that

$$\begin{aligned}
&\|x_t\|_{B_q}\\
=&\Big(\sup_{-\tau\le s\le 0} |x(t+s)|^q + \int_0^{+\infty} p(\eta)|x(t-\eta)|^q d\eta\Big)^{\frac{1}{q}}\\
\le& \sup_{t-\tau\le s\le t} |x(s)| + \Big(\int_0^{+\infty} p(\eta)|x(t-\eta)|^q d\eta\Big)^{\frac{1}{q}}\\
=& \sup_{t-\tau\le s\le t} |x(s)| +\\
&\Big(\int_{t-\sigma}^{+\infty} p(\eta)|x(t-\eta)|^q d\eta + \int_0^{t-\sigma} p(\eta)|x(t-\eta)|^q d\eta\Big)^{\frac{1}{q}}\\
\le& \sup_{t-\tau\le s\le t} |x(s)| + \Big(\int_0^{+\infty} p(t-\sigma+\eta)|x(\sigma-\eta)|^q d\eta\\
&+ \int_0^{t-\sigma} p(\eta)d\eta \sup_{\sigma\le s\le t} |x(s)|^q\Big)^{\frac{1}{q}}\\
\le& \sup_{t-\tau\le s\le t} |x(s)| + \Big(p(t-\sigma)\int_0^{+\infty} p(\eta)|x(\sigma-\eta)|^q d\eta\\
&+ \int_0^{+\infty} p(\eta)d\eta \sup_{\sigma\le s\le t} |x(s)|^q\Big)^{\frac{1}{q}}\\
\le& \sup_{\sigma\le s\le t} |x(s)| + e^{-(t-\tau-\sigma)} \sup_{\sigma-\tau\le s\le\sigma} |x(s)|\\
&+ p(t-\sigma)^{\frac{1}{q}}\|x_\sigma\|_{B_q} + \Big(\int_0^{+\infty} p(\eta)d\eta\Big)^{\frac{1}{q}} \sup_{\sigma\le s\le t} |x(s)|\\
\triangleq& K \sup_{\sigma\le s\le t} |x(s)| + M(t-\sigma)\|x_\sigma\|_{B_q},
\end{aligned}$$

where $K = 1+(\int_0^{+\infty} p(\eta)d\eta)^{\frac{1}{q}}$ and $M(s) = e^{-(s-\tau)}+p(s)^{\frac{1}{q}}$, $s \ge 0$. It is further noted that $\lim_{s\to+\infty} M(s) = 0$ and thus Hypothesis 2.1 is implied. Moreover, we have that

$$\dot{u} = -\mathfrak{B}^T P\dot{x} = -\mathfrak{B}^T P\Big(Ax + \int_0^{+\infty} B_0(\eta)u(t-\eta)d\eta + \sum_{i=1}^k B_i u(t-\tau_i)\Big), \tag{15}$$

which implies that Hypothesis 2.2 holds. Then similar to the results in our previous work (Xu et al., 2020b), we can obtain the following theorem.

Theorem 3.1. *Let Assumptions 3.1 and 3.2 be satisfied. There exists $\gamma^* > 0$ such that system (12) under truncated predictor feedback controller (13) is globally exponentially stable in the sense of Definition 2.1 for all $\gamma \in (0, \gamma^*]$, where P is the unique positive definite solution to the following algebraic Riccati equation,*

$$A^T P + PA - P\mathfrak{B}\mathfrak{B}^T P = -\gamma P. \tag{16}$$

Proof. Define $\phi = \int_{-\infty}^t \int_{t-s}^{+\infty} e^{A(t-s-\eta)}B_0(\eta)u(s)d\eta ds + \sum_{i=1}^k \int_{t-\tau_i}^t e^{A(t-s-\tau_i)}B_i u(s)ds$ and $z = x + \phi$. Then one has that

$$\dot{z} = Az + \mathfrak{B}u = (A - \mathfrak{B}\mathfrak{B}^T P)z + \mathfrak{B}\mathfrak{B}^T P\phi. \tag{17}$$

It can be verified that condition (8) of Theorem 2.2 is satisfied. Furthermore, we define the following functionals,

$$\begin{aligned}
V_1 =& |P^{\frac{1}{2}}z|,\\
V_{21} =& \int_0^\tau \int_{t-\eta}^t |u(\theta)|d\theta d\eta,\\
V_{20} =& \int_{-\infty}^t |u(\eta)| \int_{t-\eta}^{+\infty} p(\theta)d\theta d\eta,\\
V_3 =& |P^{\frac{1}{2}}x|,\\
V_4 =& \sum_{i=1}^k \int_{t-\tau_i}^t |u(s)|ds.
\end{aligned}$$

With the similar proof as in Xu et al. (2020b), we can obtain the following results concerned with their derivatives,

$$\begin{aligned}
\dot{V}_1 \le& -\frac{\gamma}{2}|P^{\frac{1}{2}}z| + |P|^{\frac{3}{2}}|\mathfrak{B}|^2|\phi|\\
\le& -\frac{\gamma}{2}|P^{\frac{1}{2}}x| + \rho_1(\gamma)\int_{t-\tau}^t |u(\eta)|d\eta\\
&+ \rho_0(\gamma)\int_0^{+\infty} p(\eta)|u(t-\eta)|d\eta,\\
\dot{V}_{21} =& \tau|u| - \int_{t-\tau}^t |u(\eta)|d\eta,\\
\dot{V}_{20} =& \int_0^{+\infty} p(\eta)d\eta|u| - \int_0^{+\infty} p(\eta)|u(t-\eta)|d\eta,\\
\dot{V}_3 \le& \sigma_1(\gamma)|P^{\frac{1}{2}}x| + \sigma_2(\gamma)\sum_{i=1}^k |u(t-\tau_i)|\\
&+ |P^{\frac{1}{2}}|\int_0^{+\infty} p(\eta)|u(t-\eta)|d\eta,\\
\dot{V}_4 \le& k|u| - \sum_{i=1}^k |u(t-\tau_i)|,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\rho_1(\gamma) =& \Big(\frac{\gamma}{2}|P^{\frac{1}{2}}| + |\mathfrak{B}|^2|P^{\frac{1}{2}}|^3\Big)\sum_{i=1}^k |B_i| \sup_{0\le s\le\tau_i} |e^{-As}|,\\
\rho_0(\gamma) =& \Big(\frac{\gamma}{2}|P^{\frac{1}{2}}| + |\mathfrak{B}|^2|P^{\frac{1}{2}}|^3\Big)\int_0^{+\infty} |e^{-A\eta}|p(\eta)d\eta,\\
\sigma_1(\gamma) =& -\frac{\gamma}{2} + \frac{1}{2}|\mathfrak{B}|^2|P|,\\
\sigma_2(\gamma) =& |P^{\frac{1}{2}}| \sup_{i=1,\ldots,k} |B_i|.
\end{aligned} \tag{19}$$

Furthermore, define the candidate Lyapunov functional,

$$\begin{aligned}
V =& V_1 + \rho_0(\gamma)V_{20} + \rho_1(\gamma)V_{21}\\
&+ o(\gamma)(V_3 + |P^{\frac{1}{2}}|V_{20} + \sigma_2(\gamma)V_4),
\end{aligned} \tag{20}$$

where $o(\gamma)$ is to be determined. Via the fact that $|u| \le |\mathfrak{B}||P^{\frac{1}{2}}|\,|P^{\frac{1}{2}}x|$, one has

$$\begin{aligned}
V \ge& o(\gamma)|P^{\frac{1}{2}}x|\\
\ge& \frac{o(\gamma)}{2}|P^{\frac{1}{2}}x| + \frac{o(\gamma)}{2|\mathfrak{B}||P^{\frac{1}{2}}|}|u|,
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
V \le& (1 + o(\gamma))|P^{\frac{1}{2}}||x|\\
&+ b(\gamma)\int_0^{+\infty} p(\eta)|u(t-\eta)|d\eta
\end{aligned}$$

$$\leq (1+o(\gamma))|P^{\frac{1}{2}}||x| + b(\gamma)\Big(\int_0^{+\infty} p(\eta)d\eta\Big)^{1-\frac{1}{q}}\|u_t\|_{B_u}, \tag{22}$$

where $b(\gamma) = |P^{\frac{1}{2}}|\int_0^{+\infty}|e^{-A\eta}|p(\eta)d\eta + \Big(\rho_0(\gamma)+o(\gamma)|P^{\frac{1}{2}}|\Big)\int_0^{+\infty}p(\eta)d\eta + \frac{1}{p(\tau)}\Big(|P^{\frac{1}{2}}|\sum_{i=1}^k|B_i|\sup_{0\leq s\leq\tau_i}|e^{-As}| + \rho_1(\gamma)\tau + o(\gamma)k\sigma_2(\gamma)\Big)$. Moreover, it can be derived that

$$\begin{aligned}\dot V \leq & -\frac{\gamma}{2}|P^{\frac{1}{2}}x| + \big(\rho_1(\gamma)\tau+\rho_0(\gamma)\times \\ & \int_0^{+\infty}p(\eta)d\eta\big)|u| + o(\gamma)\big(\sigma_1(\gamma)|P^{\frac{1}{2}}x| \\ & + |P^{\frac{1}{2}}|\int_0^{+\infty}p(\eta)d\eta|u| + \sigma_2(\gamma)k|u|\big) \\ \leq & -(\frac{\gamma}{2}-\rho'(\gamma))|P^{\frac{1}{2}}x|,\end{aligned} \tag{23}$$

where $\rho'(\gamma) = |\mathfrak{B}||P^{\frac{1}{2}}|\Big(\rho_1(\gamma)\tau+\rho_0(\gamma)\int_0^{+\infty}p(\eta)d\eta\Big) + o(\gamma)\Big(|\mathfrak{B}||P|\int_0^{+\infty}p(\eta)d\eta + \sigma_2(\gamma)k|\mathfrak{B}||P^{\frac{1}{2}}| + \sigma_1(\gamma)\Big)$. In Xu et al. (2018) and Xu et al. (2020b), we have already shown that $\lim_{\gamma\to 0^+}\frac{|\mathfrak{B}||P^{\frac{1}{2}}|}{\gamma^2}(\rho_1(\gamma)\tau+\rho_0(\gamma)\int_0^{+\infty}p(\eta)d\eta) < +\infty$. Then we can choose $\gamma^*>0$ and $o(\gamma)$ such that for any $\gamma\in(0,\gamma^*]$,

$$\frac{\gamma}{4} \geq |\mathfrak{B}||P^{\frac{1}{2}}|(\rho_1(\gamma)\tau+\rho_0(\gamma)\int_0^{+\infty}p(\eta)d\eta),$$
$$\frac{\gamma}{8} \geq o(\gamma)\Big(|\mathfrak{B}||P|\int_0^{+\infty}p(\eta)d\eta + \sigma_2(\gamma)k|\mathfrak{B}||P^{\frac{1}{2}}| + \sigma_1(\gamma)\Big),$$

which leads to that

$$\begin{aligned}\dot V \leq & -\frac{\gamma}{8}|P^{\frac{1}{2}}x| \\ \leq & -\frac{\gamma}{16}|P^{\frac{1}{2}}x| - \frac{\gamma|u|}{16|\mathfrak{B}||P^{\frac{1}{2}}|}.\end{aligned} \tag{24}$$

It then follows from Theorem 2.2 that system (12) with controller (13) is globally exponentially stable in the sense of Definition 2.1. The theorem is thus proved. □

Next, under the new framework proposed in this work, we can further consider a special case where the following assumption is satisfied.

Assumption 3.2′ Define $\mathfrak{B}' = \int_0^{+\infty}B_0(\eta)d\eta + \sum_{i=1}^k B_i$. The matrix A has all its eigenvalues at zero and the matrix pair $(A,\mathfrak{B}')$ is controllable.

In this case, we propose the following controller,

$$u = -\mathfrak{B}'^T Px. \tag{25}$$

The case where A has all its eigenvalues at zero is also discussed in Xu et al. (2018) and Xu et al. (2020b). However, the proofs are also established on an implicit assumption that $u(-\eta) = -\mathfrak{B}'^T Px(-\eta)$, $\eta \leq 0$. By applying the new framework in this paper, this implicit assumption can be also removed. The result is summarized in the following theorem.

Theorem 3.2. *Let Assumptions 3.1 and 3.2′ be satisfied. There exists $\gamma^*>0$ such that system (12) under truncated predictor feedback controller (25) is globally exponentially stable in the sense of Definition 2.1 for all $\gamma\in(0,\gamma^*]$, where P is the unique positive definite solution to the following algebraic Riccati equation,*

$$A^TP + PA - P\mathfrak{B}'\mathfrak{B}'^TP = -\gamma P. \tag{26}$$

Proof. The proof follows the similar arguments as in those for Theorem 3.1 in this paper and Theorem 4.2 in Xu et al. (2020b), and is thus omitted. □

The algebraic Riccati Eqs. (16) and (26) originate from (Zhou et al., 2008) and Zhou and Duan (2009). Since then, they have been widely applied on stabilization of systems with bounded input delays. In our previous works (Xu et al., 2018, 2020b), we further show their potential in solving stabilization problems of systems with unbounded input delays.

Remark 3.1. Theorems 3.1 and 3.2 are more general than the results in Xu et al. (2018) and Xu et al. (2020b) as the implicit assumptions on initial conditions can be removed under the new framework. Moreover, if these implicit assumptions are additionally assumed to be satisfied, then Theorems 3.1 and 3.2 reduce to the results in Xu et al. (2018) and Xu et al. (2020b).

3.2. Predictor feedback controller: Time-invariant case

The truncated predictor feedback controller considered in the previous subsection is only applicable to those linear systems where their open loop dynamics are not exponentially unstable. However, many practical systems with unbounded input delays may have exponentially unstable open loop dynamics. By using the predictor feedback controller, such systems can be dealt with. Consider the following predictor feedback controller for system (12),

$$\begin{aligned} u =& Kz, \\ z =& x + \int_{-\infty}^t\int_{t-s}^{+\infty} e^{A(t-s-\eta)}B_0(\eta)u(s)d\eta ds \\ & + \sum_{i=1}^k\int_{t-\tau_i}^t e^{A(t-s-\tau_i)}B_iu(s)ds.\end{aligned} \tag{27}$$

One can first obtain that

$$\dot z = (A+\mathfrak{B}K)z, \tag{28}$$

where $\mathfrak{B} = \int_0^{+\infty}e^{-A\eta}B_0(\eta)d\eta + \sum_{i=1}^k e^{-A\tau_i}B_i$. To proceed, the following two assumptions are made.

Assumption 3.3. There exists a non-increasing and positive function $p(\eta)$, where $\int_0^{+\infty}p(\eta)d\eta < +\infty$ and $p(u+v)\leq p(u)p(v)$, $\forall u,v\geq 0$, such that the following conditions hold,

$$\begin{aligned}&|B_0(\eta)| \leq p(\eta), \\ &\int_0^{+\infty}|e^{-A\eta}p(\eta)|d\eta < +\infty.\end{aligned} \tag{29}$$

Assumption 3.4. The matrix pair $(A,\mathfrak{B})$ is stabilizable.

Assumption 3.4 is more general than Assumption 3.2. It allows exponentially unstable open loop dynamics, that is, the matrix A can have eigenvalues with positive real parts. Moreover, the matrix pair $(A,\mathfrak{B})$ is only assumed to be stabilizable in Assumption 3.4 instead of controllable in Assumption 3.2.

Choose the same plant state space and controller state space as those in the previous subsection and thus Hypothesis 2.1 is satisfied. Furthermore, one has that

$$\dot u = K(Az+\mathfrak{B}u). \tag{30}$$

Note that z is a functional of x and u_t. Thus Hypothesis 2.2 is satisfied. Then we can obtain the following theorem.

Theorem 3.3. *Choose K such that $A+\mathfrak{B}K$ is Hurwitz. Then system (12) with predictor feedback controller (27) is globally exponentially stable in the sense of Definition 2.1.*

Proof. It can be verified that condition (8) is satisfied. Since $A+\mathfrak{B}K$ is Hurwitz, then we can choose P such that

$$P(A+\mathfrak{B}K)+(A+\mathfrak{B}K)^TP=-I. \quad (31)$$

Define the following functionals,

$$\begin{aligned} V_1 =&|P^{\frac{1}{2}}z|,\\ V_2 =&|x|,\\ V_{30} =&\int_{-\infty}^{t}|u(s)|\int_{t-s}^{+\infty}p(\eta)d\eta ds,\\ V_{31} =&\int_0^{\tau}\int_{t-s}^{t}|u(\eta)|d\eta ds,\\ V_{32} =&\sum_{i=1}^{k}\int_{t-\tau_i}^{t}|u(s)|ds. \end{aligned} \quad (32)$$

For the functional V_1, it first follows from the fact $|u|\le|K||z|$ that

$$\begin{aligned} \dot V_1 =&\frac{z^TP\dot z}{|P^{\frac{1}{2}}z|}=-\frac{|z|^2}{2|P^{\frac{1}{2}}z|}\\ \le&-\frac{|z|}{2|P^{\frac{1}{2}}|}\le-\frac{|u|}{2|P^{\frac{1}{2}}||K|}, \end{aligned} \quad (33)$$

and

$$\begin{aligned} \dot V_1\le&-\frac{|z|}{2|P^{\frac{1}{2}}|}\le-\frac{1}{2|P^{\frac{1}{2}}|}(|x|-|z-x|)\\ \le&-\frac{|x|}{2|P^{\frac{1}{2}}|}+M_0\int_0^{+\infty}p(\eta)|u(t-\eta)|d\eta\\ &+M_1\int_{t-\tau}^{t}|u(s)|ds, \end{aligned} \quad (34)$$

where $M_0=\frac{1}{2|P^{\frac{1}{2}}|}\int_0^{+\infty}|e^{-A\eta}|p(\eta)d\eta$ and $M_1=\frac{1}{2|P^{\frac{1}{2}}|}\sum_{i=1}^{k}|B_i|\sup_{0\le s\le\tau_i}|e^{-As}|$. For the functional V_2, one has that

$$\begin{aligned} \dot V_2 =&\frac{x^T}{|x|}(Ax+\int_0^{+\infty}B(\eta)u(t-\eta)d\eta+\sum_{i=1}^{k}B_iu(t-\tau_i))\\ \le&|A||x|+\int_0^{+\infty}p(\eta)|u(t-\eta)|d\eta\\ &+\sup_{i=1,2,\ldots,k}|B_i|\sum_{i=1}^{k}|u(t-\tau_i)|. \end{aligned} \quad (35)$$

And one can further obtain the following results,

$$\begin{aligned} \dot V_{30} =&\int_0^{+\infty}p(\eta)d\eta|u|-\int_0^{+\infty}p(\eta)|u(t-\eta)|d\eta,\\ \dot V_{31} =&\tau|u|-\int_{t-\tau}^{t}|u(s)|ds,\\ \dot V_{31} =&k|u|-\sum_{i=1}^{k}|u(t-\tau_i)|. \end{aligned} \quad (36)$$

Then we can define the candidate Lyapunov functional as follows,

$$\begin{aligned} V =&(c_{11}+c_{12})V_1+V_2\\ &+c_{30}V_{30}+c_{31}V_{31}+c_{32}V_{32}, \end{aligned} \quad (37)$$

where $c_{11},c_{12},c_{30},c_{31},c_{32}$ are positive constants to be determined. Then it can be derived that

$$\begin{aligned} \dot V\le&-(\frac{c_{11}}{2|P^{\frac{1}{2}}|}-|A|)|x|+(c_{11}M_1-c_{31})\int_{t-\tau}^{t}|u(s)|ds\\ &+(c_{11}M_0+1-c_{30})\int_0^{+\infty}p(\eta)|u(t-\eta)|d\eta\\ &+(\sup_{i=1,2,\ldots,k}|B_i|-c_{32})\sum_{i=1}^{k}|u(t-\tau_i)|\\ &-(\frac{c_{12}}{2|P^{\frac{1}{2}}||K|}-c_{30}\int_0^{+\infty}p(\eta)d\eta-c_{31}\tau-c_{32}k)|u|. \end{aligned} \quad (38)$$

Choose $c_{11},c_{12},c_{30},c_{31},c_{32}$ such that

$$\begin{aligned} &c_{11}>2|P^{\frac{1}{2}}||A|,\quad c_{31}\ge c_{11}M_1,\\ &c_{30}\ge c_{11}M_0+1,\ \ c_{32}\ge\sup_{i=1,2,\ldots,k}|B_i|,\\ &c_{12}>2|P^{\frac{1}{2}}||K|(c_{30}\int_0^{+\infty}p(\eta)d\eta+c_{31}\tau+c_{32}k), \end{aligned} \quad (39)$$

which yields that

$$\dot V\le-\bar c_1|x|-\bar c_2|u|, \quad (40)$$

where $\bar c_1=\frac{c_{11}}{2|P^{\frac{1}{2}}|}-|A|>0$ and $\bar c_2=\frac{c_{12}}{2|P^{\frac{1}{2}}||K|}-c_{30}\int_0^{+\infty}p(\eta)d\eta-c_{31}\tau-c_{32}k>0$. It can be further verified that

$$V\ge(c_{11}+c_{12})\lambda_{min}(P^{\frac{1}{2}})\frac{|u|}{|K|}+|x|, \quad (41)$$

where $\lambda_{min}(P^{\frac{1}{2}})$ denotes the smallest eigenvalue of $P^{\frac{1}{2}}$ and

$$\begin{aligned} V\le&((c_{11}+c_{12})|P^{\frac{1}{2}}|+1)|x|\\ &+\bar M\int_0^{+\infty}p(\eta)|u(t-\eta)|d\eta\\ \le&((c_{11}+c_{12})|P^{\frac{1}{2}}|+1)|x|\\ &+\bar M(\int_0^{+\infty}p(\eta)|d\eta)^{1-\frac{1}{q}}\|u_t\|_{B_u}, \end{aligned} \quad (42)$$

where $\bar M=(c_{11}+c_{12})|P^{\frac{1}{2}}|(M_0+\frac{M_1}{p(\tau)})+c_{30}\int_0^{+\infty}p(\eta)d\eta+\frac{1}{p(\tau)}(c_{31}\tau+c_{32}k)$. It then follows from Theorem 2.2 that system (12) under controller (27) is globally exponentially stable in the sense of Definition 2.1. The theorem is thus proved. □

Remark 3.2. When the infinite distributed delay reduces to the special case of bounded distributed delay, by rewriting V_{30} in (32), one can obtain a valid Lyapunov functional for the special case. In this case, $V_{30}=\int_{t-\tau}^{t}|u(s)|\int_{t-s}^{\tau}p(\eta)d\eta ds$, where τ is the delay bound. However, for Lyapunov functionals for bounded delays, there exist many more choices which are not limited to the form of (32), as conditions (1)-(2) in Theorem 2.2 are often easy to be satisfied for bounded delays.

Compared with the predictor feedback controller, the integration term is truncated in controllers (13) and (25). Therefore, in some existing works (Zhou, 2014; Zhou et al., 2012), these two controllers are called the truncated predictor feedback controllers. The truncated predictor feedback controllers are simpler but have more limitations, compared with the predictor feedback controllers. One of the limitations is that the truncated predictor feedback controllers cannot handle delayed systems with arbitrarily large input delays when the open loop dynamics is exponentially unstable. Moreover, the truncated predictor feedback controllers are usually designed by the low-gain method, which would lead to slow convergence. Comparisons between the two types of controllers will be further illustrated in the simulation part.

Our predictor feedback controller for systems with unbounded delays can include that for systems with bounded delays (Bekiaris-Liberis & Krstic, 2011) as its special case. Moreover, different from (Bekiaris-Liberis & Krstic, 2011), where PDEs are applied to model and analyze the closed loop system, we use the novel framework proposed in our Theorems 2.1–2.2. In another relevant work (Mazenc et al., 2012), input-to-state stability of systems with bounded input delays under predictor feedback control law is studied. It should be noted that our results can include the undisturbed case in Mazenc et al. (2012), i.e., $\delta(t)=0$, as special cases. First, the Lyapunov functionals constructed in Mazenc et al. (2012) satisfy our Theorem 2.1 when $\delta(t)=0$. Second, the controllers (20) and (31) in Mazenc et al. (2012) are the special cases of our controller (27). One only needs to write controller (27) in the following form,

$$
\begin{aligned}
u =&\hat{K}\hat{z},\\
\hat{z} =&Mz,
\end{aligned}
\tag{43}
$$

where z is the same as that in (27) and M is an invertible matrix satisfying $MA=AM$. When the system reduces to the cases of one constant delay or bounded distributed delay, then our controller reduces to (20) and (31) in Mazenc et al. (2012) with $M=e^{A\tau}$ where τ is the delay bound.

More general distributed unbounded delays can be presented in the form of Stieltjes integral, see Fridman and Shaikhet (2019) for example. For systems with input delays given by Stieltjes integral, one can construct a predictor following the reduction approach by Artstein (1982). However, in this case, stability analysis will be more difficult as some new technical challenges arise. Assumptions 3.1–3.4 should also be modified with new phase spaces chosen.

3.3. Predictor feedback controller: Time-varying case

(Artstein, 1982) showed the potential of predictor feedback controllers to handle time-varying linear systems with distributed unbounded input delays. By applying our new framework, we can also consider the time-varying linear systems with distributed unbounded input delays under predictor feedback controllers but with rigorous theoretical analysis. Consider the following time-varying linear system,

$$
\dot{x}=A(t)x+\int_0^{+\infty}B(t,\eta)u(t-\eta)d\eta,
\tag{44}
$$

where $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^m$. Let $\Phi(\cdot,\cdot)$ denote the state transition matrix associated with $A(t)$. Design the following predictor feedback controller,

$$
\begin{aligned}
&u=K(t)z,\\
&z=x+\int_{-\infty}^{t}\int_{t-s}^{+\infty}\Phi(t,s+\eta)B(s+\eta,\eta)u(s)d\eta ds.
\end{aligned}
\tag{45}
$$

It can be calculated that

$$
\dot{z}=(A(t)+\hat{B}(t)K(t))z,
\tag{46}
$$

where $\hat{B}(t)=\int_0^{+\infty}\Phi(t,t+\eta)B(t+\eta,\eta)d\eta$. The following assumptions are made.

Assumption 3.5. There exist positive constants k_1, k_2, μ, and a continuous differentiable matrix function $K(t)\in\mathbb{R}^{m\times n}$ such that $|A(t)|\le\mu$, $|K(t)|\le k_1$, $|\dot{K}(t)|\le k_2$, and

$$
\dot{z}=(A(t)+\hat{B}(t)K(t))z
\tag{47}
$$

is globally exponentially stable, i.e.,

$$
|z(t)|\le\alpha e^{\beta(t-t_0)}|z(t_0)|,\ \forall t\ge t_0,
\tag{48}
$$

for some $\alpha>0$, and $\beta>0$.

Assumption 3.6. There exist a positive constant M and a positive nonincreasing and Lebesgue integrable function $p(\eta)$ such that

$$
\begin{aligned}
&|B(t,\eta)|\le p(\eta),\\
&p(u+v)\le p(u)p(v),\ \forall u,v\ge 0,\\
&\int_0^{+\infty}|\Phi(t,t+\eta)|p(\eta)d\eta\le M.
\end{aligned}
\tag{49}
$$

Note that time-varying linear systems studied in Xu et al. (2020b) also satisfy Assumptions 3.5–3.6 and thus can be included as special cases. In Xu et al. (2020b), low-gain based truncated predictor feedback controllers are designed for those time-varying linear systems. The predictor feedback controller in the form of (45) can be also designed for those systems. Moreover, time-varying linear systems considered in this work are much more general than those in Xu et al. (2020b).

Choose the same plant state space and controller state space as those in previous subsections and thus Hypothesises 2.1 is satisfied. Moreover, one has that

$$
\begin{aligned}
\dot{u} =&\dot{K}(t)z+K(t)\dot{z}\\
=&\dot{K}(t)z+K(t)(A(t)+\hat{B}(t)K(t))z.
\end{aligned}
\tag{50}
$$

Note that z is a functional of x and u_t. Thus Hypothesis 2.2 is satisfied. Then by applying the new framework proposed in this paper, we can obtain the following result.

Theorem 3.4. *Under Assumptions 3.5 and 3.6, system (44) with predictor feedback controller (45) is globally exponentially stable in the sense of Definition 2.1.*

Proof. First of all, it follows from Assumption 3.6 that

$$
\begin{aligned}
|z| \le&|x|+|\int_{-\infty}^{t}\int_{t-s}^{+\infty}\Phi(t,s+\eta)B(s+\eta,\eta)u(s)d\eta ds|\\
\le&|x|+\int_{-\infty}^{t}\int_{t-s}^{+\infty}|\Phi(t,s+\eta)|p(\eta)|u(s)|d\eta ds\\
=&|x|+\int_{-\infty}^{t}\int_{0}^{+\infty}|\Phi(t,t+\eta)|p(\eta+t-s)|u(s)|d\eta ds\\
\le&|x|+\int_0^{+\infty}p(s)|u(t-s)|ds\int_0^{+\infty}|\Phi(t,t+\eta)|p(\eta)d\eta\\
\le&|x|+M\int_0^{+\infty}p(s)|u(t-s)|ds,
\end{aligned}
\tag{51}
$$

and similarly,

$$
\begin{aligned}
|z| \ge&|x|-|\int_{-\infty}^{t}\int_{t-s}^{+\infty}\Phi(t,s+\eta)B(s+\eta,\eta)u(s)d\eta ds|\\
\ge&|x|-M\int_0^{+\infty}p(s)|u(t-s)|ds.
\end{aligned}
\tag{52}
$$

Therefore, we can verify that condition (8) is satisfied under Assumptions 3.5 and 3.6.

It further follows from Assumption 3.5 and Lemma A.1 that there exist a symmetric and continuous differentiable matrix function $P(t)$ and positive constants ρ_1 and ρ_2 such that

$$
\begin{aligned}
&\rho_1 I\le P(t)\le\rho_2 I,\\
&A_c^T(t)P(t)+P(t)A_c(t)+\dot{P}(t)=-I,
\end{aligned}
\tag{53}
$$

where $A_c(t)=A(t)+\hat{B}(t)K(t)$. Then we define the following functionals,

$$
\begin{aligned}
V_1 =&|P^{\frac{1}{2}}(t)z|,\\
V_2 =&|x|,
\end{aligned}
$$

$$V_3 = \int_{-\infty}^{t} |u(s)| \int_{t-s}^{+\infty} p(\eta) d\eta ds, \tag{54}$$

and choose the candidate Lyapunov functional as follows,

$$V = (c_1 + c_2)V_1 + V_2 + c_3 V_3, \tag{55}$$

where c_1, c_2, c_3 are positive constants to be determined. It can be derived that

$$\begin{aligned}
\dot{V} =& (c_1+c_2)\frac{z^T(A_c^T(t)P(t)+P(t)A_c(t)+\dot{P}(t))z}{2|P^{\frac{1}{2}}(t)z|} \\
&+ \frac{x^T(A(t)x+\int_0^{+\infty} B(t,\eta)u(t-\eta)d\eta)}{|x|} \\
&+ c_3(\int_0^{+\infty} p(\eta)d\eta|u| - \int_0^{+\infty} p(\eta)|u(t-\eta)|d\eta) \\
\leq& -(c_1+c_2)\frac{|z|^2}{2|P^{\frac{1}{2}}(t)||z|} + a|x| + \int_0^{+\infty} p(\eta)|u(t-\eta)|d\eta \\
&+ c_3(\int_0^{+\infty} p(\eta)d\eta|u| - \int_0^{+\infty} p(\eta)|u(t-\eta)|d\eta) \\
\leq& -\frac{c_1|u|}{2k_1\sqrt{\rho_2}} - \frac{c_2}{2\sqrt{\rho_2}}(|x| - M\int_0^{+\infty} p(s)|u(t-s)|ds) \\
&+ a|x| + c_3\int_0^{+\infty} p(\eta)d\eta|u| \\
&- (c_3-1)\int_0^{+\infty} p(\eta)|u(t-\eta)|d\eta \\
=& -(\frac{c_1}{2k_1\sqrt{\rho_2}} - c_3\int_0^{+\infty} p(\eta)d\eta)|u| - (\frac{c_2}{2\sqrt{\rho_2}} - a)|x| \\
&- (c_3 - 1 - \frac{c_2 M}{2\sqrt{\rho_2}})\int_0^{+\infty} p(s)|u(t-s)|ds.
\end{aligned} \tag{56}$$

Choose c_1, c_2, c_3 such that

$$\begin{aligned}
&c_2 > 2a\sqrt{\rho_2}, \\
&c_3 \geq 1 + \frac{c_2 M}{2\sqrt{\rho_2}}, \\
&c_1 > 2c_3 k_1\sqrt{\rho_2}\int_0^{+\infty} p(\eta)d\eta.
\end{aligned} \tag{57}$$

Then condition (3) of Theorem 2.2 is satisfied. Moreover, it can be further obtained that

$$\begin{aligned}
V \geq& (c_1+c_2)\sqrt{\rho_1}|z| + |x| \\
\geq& (c_1+c_2)\frac{\sqrt{\rho_1}}{k_1}|u| + |x|,
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
V \leq& (c_1+c_2)\sqrt{\rho_2}|z| + |x| \\
&+ c_3\int_0^{+\infty} p(\eta)d\eta \int_0^{+\infty} p(s)|u(t-s)|ds \\
\leq& (c_1+c_2)\sqrt{\rho_2}(|x| + M\int_0^{+\infty} p(\eta)|u(t-\eta)|d\eta) \\
&+ |x| + c_3\int_0^{+\infty} p(\eta)d\eta \int_0^{+\infty} p(s)|u(t-s)|ds \\
\leq& \big((c_1+c_2)\sqrt{\rho_2}+1\big)|x| + \bar{M}\|u_t\|_{B_u},
\end{aligned} \tag{59}$$

where $\bar{M} = \big((c_1+c_2)\sqrt{\rho_2}M + c_3\int_0^{+\infty} p(\eta)d\eta\big) \times \big(\int_0^{+\infty} p(\eta)d\eta\big)^{1-\frac{1}{q}}$. Thus conditions (1) and (2) of Theorem 2.2 are satisfied, which implies that the closed loop system is globally exponentially stable in the sense of Definition 2.1. The theorem is thus proved. □

Remark 3.3. The main idea of the predictor feedback controller is to reduce the delayed system to a non-delayed system. Therefore, the predictor feedback controller method is also called reduction method in Artstein (1982). It should be noted that although the controller design was proposed in Artstein (1982), no rigorous stability analysis is given. In Bekiaris-Liberis and Krstic (2011, 2016), Krstic (2010), the authors provided rigorous stability analysis for the cases of bounded input delays. We further provide rigorous stability analysis to the cases of distributed unbounded input delays in this work.

4. Simulation examples

In this section, we will provide three examples to illustrate the effectiveness of the predictor feedback controllers.

4.1. Example 1

The first example originates from (Zhou et al., 2014) and is also used in Xu et al. (2020b). Consider the following system, which can be used to describe a spacecraft rendezvous process,

$$\dot{x} = Ax + \int_0^{+\infty} B_0(\eta)u(t-\eta)d\eta + B_1 u(t-2), \tag{60}$$

where $A, B_0(\eta), B_1$ are given as follows,

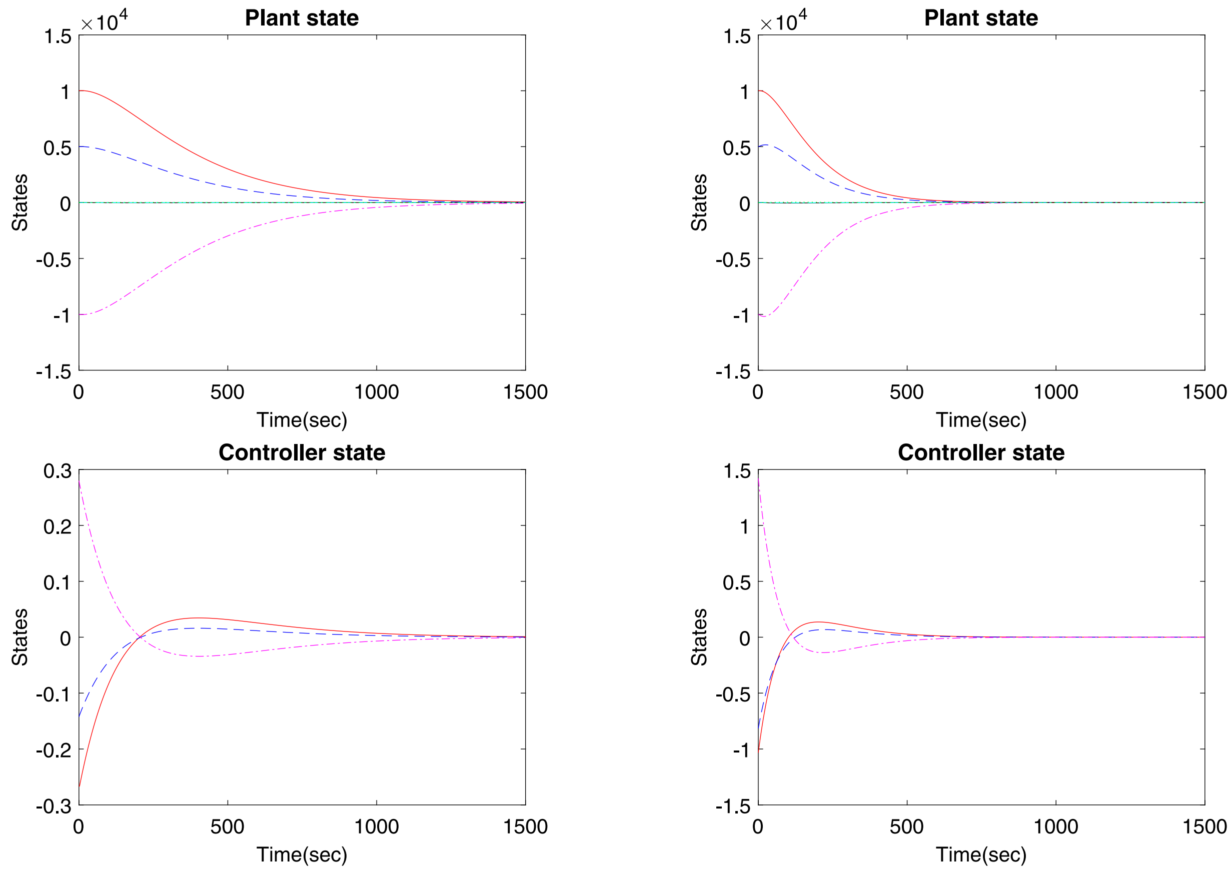
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & -2\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \end{bmatrix},$$

$$B_0(\eta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \eta e^{-\eta}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Choose $\omega = 7.2722 \times 10^{-5} rad/s$. Then we have that

$$\begin{aligned}
\mathfrak{B} =& \int_0^{+\infty} e^{-A\eta}B_0(\eta)d\eta + e^{-2A}B_1 \\
=& \begin{bmatrix} -2 & 4\times 10^{-4} & 0 \\ -3\times 10^{-4} & -2 & 0 \\ 0 & 0 & -2 \\ 1 & -3\times 10^{-4} & 0 \\ 3\times 10^{-4} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{61}$$

In this model, unbounded distributed input delays would occur when the control signals are sent through wireless networks, as pointed out in Roesch and Roth (2005). It can be verified that Assumptions 3.1–3.4 are all satisfied. Thus both the truncated predictor feedback controller in the form of (13) and the predictor feedback controller in the form of (27) can be designed for system (60). Figs. 1(a) and 1(b) show the state responses of the plant and controller for Example 1 under controller (13) and controller (27), respectively. It can be seen that both controllers are effective on stabilizing the systems with distributed unbounded input delays. Moreover, although the predictor feedback controller (27) is of more complicated form, it leads to faster convergence than the truncated predictor feedback controller (13).



(a) State responses for Example 1 under truncated predictor feedback controller.

(b) State responses for Example 1 under predictor feedback controller.

Fig. 1. State responses for Example 1 under two different controllers.

4.2. Example 2

In this example, we consider a system whose open loop dynamic is exponentially unstable. Consider the following time-delayed invented pendulum system borrowed from (Zhou & Wang, 2016),

$$\dot{x} = Ax + \int_0^{+\infty} B_0(\eta)u(t-\eta)d\eta, \tag{62}$$

where A and $B_0(\eta)$ are given as follows,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{140}{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{140}{17} & 0 & 0 & 0 \end{bmatrix}, B_0(\eta) = \begin{bmatrix} 0 \\ -\frac{15}{34}\Gamma_1(\eta) \\ 0 \\ \frac{14}{17}\Gamma_2(\eta) \end{bmatrix}, \tag{63}$$

with $\Gamma_1(\eta) = 3e^{-3\eta}$ and $\Gamma_2(\eta) = 9\eta e^{-3\eta}$. In this model, unbounded distributed input delays would occur due to viscoelasticity of materials, as pointed out in Kolmanovskii and Myshkis (1999). It is noted that A has an eigenvalue with positive real part and thus the low-gain based truncated predictor feedback controller (13) cannot be adopted. However, the predictor feedback controller (27) can be still applied here. It can be verified that Assumptions 3.3–3.4 are satisfied, by calculating

$$\mathfrak{B} = \int_0^{+\infty} e^{-A\eta}B_0(\eta)d\eta = \begin{bmatrix} -0.1203 \\ 0.3610 \\ -0.2816 \\ 0.0214 \end{bmatrix}. \tag{64}$$

Fig. 2 shows the state responses under the predictor feedback controller. It can be seen that the system can be stabilized under the predictor feedback controller.

4.3. Example 3

In this example, we further provide a simulation example on time-varying linear systems with distributed infinite input delays. Consider the following time-varying linear system adopted from (Sanz et al., 2019),

$$\dot{x} = \frac{2+t^2}{1+t^2}x + \int_0^{+\infty} \eta e^{-\eta+\arctan t}u(t-\eta)d\eta. \tag{65}$$

It can be obtained that

$$\begin{aligned} \Phi(t,\tau) &= e^{t-\tau+\arctan t-\arctan\tau}, \\ \hat{B}(t) &= \int_0^{+\infty} e^{-\eta+\arctan t-\arctan(t+\eta)} \\ &\quad \times e^{-\eta+\arctan(t+\eta)}\eta d\eta = \frac{1}{4}e^{\arctan t}. \end{aligned}$$

Furthermore, by choosing $K(t) = -8e^{-\arctan t}$, we can verify that Assumptions 3.5–3.6 are satisfied. Fig. 3 shows the state responses under the predictor feedback controller (45). It can be seen that the controller can stabilize the time-varying linear systems with distributed unbounded input delays.

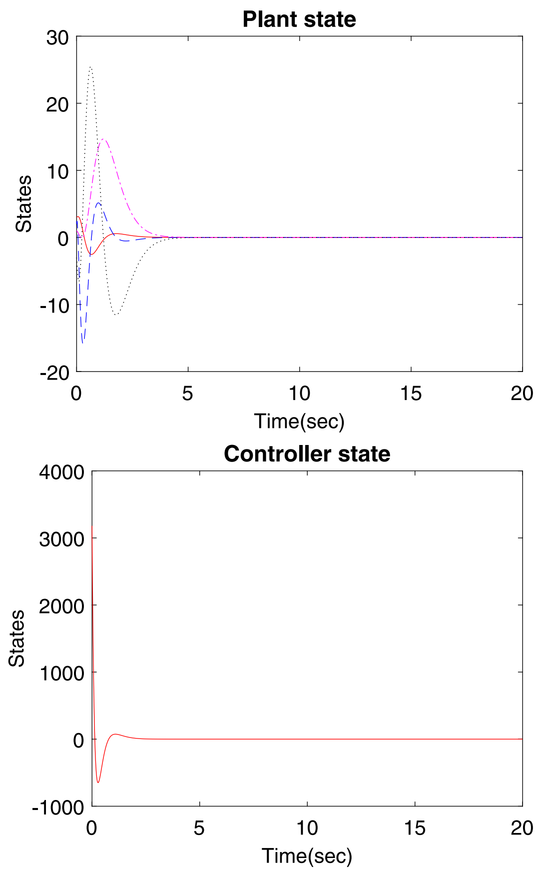


Fig. 2. State responses for Example 2 under predictor feedback controller.

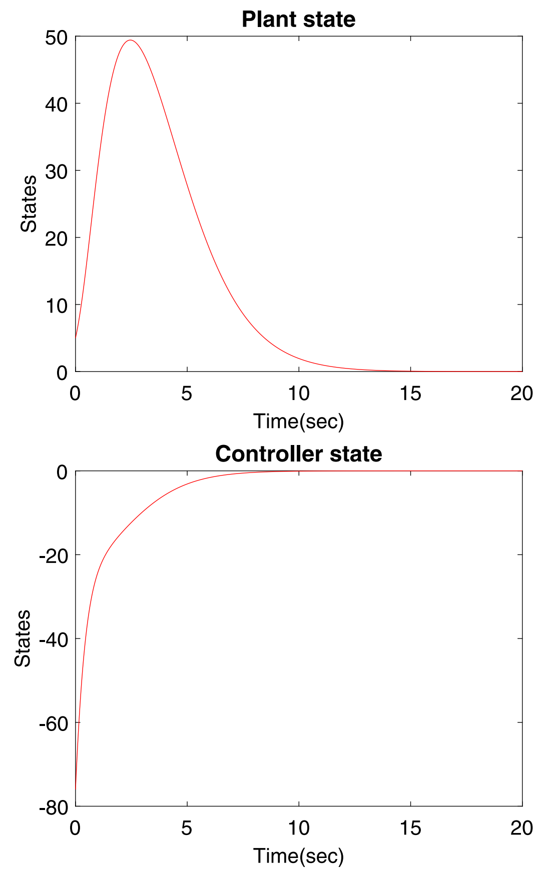


Fig. 3. State responses for Example 3 under predictor feedback controller.

5. Conclusion

In this paper, we first propose a novel general framework to handle control problems of systems with unbounded delays. This framework consists of a system model, stability definitions and corresponding Lyapunov stability theorems. The system model allows us to capture the delayed input in the stability via augmenting the delayed control input as a state, i.e., controller state, with initial conditions. Then we apply the novel framework to stabilization problems of linear systems with distributed unbounded input delays via predictor feedback controllers. Both time-invariant and time-varying linear systems with distributed unbounded input delays are considered. It is shown that under our new framework, our previous works on low-gain based truncated predictor feedback controllers can be improved by removing an implicit assumption on initial conditions. It is also shown that the predictor feedback controllers can exponentially stabilize both time-invariant and time-varying linear systems with distributed unbounded input delays. Several examples are provided to show the effectiveness of the predictor feedback controllers. In the future, we will investigate how to extend our novel framework to stabilization problems of nonlinear systems with infinite delays.

Appendix

Lemma A.1 (Rugh, 1996)**.** *Consider the following time-varying linear system,*

$$\dot{x} = A(t)x, \tag{A.1}$$

where $A(t) \in \mathbb{R}^{n\times n}$. Suppose that system (A.1) is globally exponentially stable, and there exists $a > 0$ such that $|A(t)| \le a$ for all t. Then there exist a symmetric and continuously differentiable matrix function $P(t)$ and two positive constants ρ_1 and ρ_2 such that

$$\begin{aligned} &\rho_1 I \le P(t) \le \rho_2 I, \\ &A^T(t)P(t) + P(t)A(t) + \dot{P}(t) = -I. \end{aligned} \tag{A.2}$$

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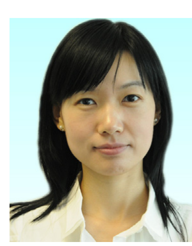
Xiang Xu received the Bachelor of Engineering degree from Nanjing University of Science and Technology, China in 2014 and the Ph.D. degree from City University of Hong Kong, Hong Kong in 2018. He is now a postdoctoral fellow in the Department of Biomedical Engineering, City University of Hong Kong. His research interests include multiagent systems and time delay systems.



Lu Liu received the Ph.D. degree from the Department of Mechanical and Automaton Engineering, Chinese University of Hong Kong, Hong Kong, in 2008.

From 2009 to 2012, she was an Assistant Professor with the University of Tokyo, Japan, and then a Lecturer with the University of Nottingham, U.K. After that, she joined City University of Hong Kong, Hong Kong, where she is currently an Associate Professor. Her current research interests include networked dynamical systems, control theory and applications, and biomedical devices.

Dr. Liu is an Associate Editor of the IEEE Transactions on Cybernetics, Control Theory and Technology, Transactions of the Institute of Measurement and Control, and Unmanned Systems.

Miroslav Krstc is Distinguished Professor of Mechanical and Aerospace Engineering, holds the Alspach endowed chair, and is the founding director of the Cymer Center for Control Systems and Dynamics at UC San Diego. He also serves as Senior Associate Vice Chancellor for Research at UCSD. As a graduate student, Krstc won the UC Santa Barbara best dissertation award and student best paper awards at CDC and ACC. Krstc has been elected Fellow of seven scientific societies – IEEE, IFAC, ASME, SIAM, AAAS, IET (UK), and AIAA (Assoc. Fellow) – and as a foreign member of the Serbian Academy of Sciences and Arts and of the Academy of Engineering of Serbia. He has received the Richard E. Bellman Control Heritage Award, SIAM Reid Prize, ASME Oldenburger Medal, Nyquist Lecture Prize, Paynter Outstanding Investigator Award, Ragazzini Education Award, IFAC Nonlinear Control Systems Award, Chestnut textbook prize, Control Systems Society Distinguished Member Award, the PECASE, NSF Career, and ONR Young Investigator awards, the Schuck ('96 and '19) and Axelby paper prizes, and the first UCSD Research Award given to an engineer. Krstc has also been awarded the Springer Visiting Professorship at UC Berkeley, the Distinguished Visiting Fellowship of the Royal Academy of Engineering, the Invitation Fellowship of the Japan Society for the Promotion of Science, and four honorary professorships outside of the United States. He serves as Editor-in-Chief of Systems & Control Letters and has been serving as Senior Editor in Automatica and IEEE Transactions on Automatic Control, as editor of two Springer book series, and has served as Vice President for Technical Activities of the IEEE Control Systems Society and as chair of the IEEE CSS Fellow Committee. Krstc has coauthored fifteen books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.

Gang Feng received the Ph.D. degree in Electrical Engineering from the University of Melbourne, Australia. He has been with City University of Hong Kong since 2000 after serving as lecturer/senior lecturer at School of Electrical Engineering, University of New South Wales, Australia, 1992–1999. He is now Chair Professor of Mechatronic Engineering. He has been awarded IEEE Computational Intelligence Society Fuzzy Systems Pioneer Award, an Alexander von Humboldt Fellowship, the IEEE Transactions on Fuzzy Systems Outstanding Paper Award, Changjiang chair professorship from Education Ministry of China, and CityU Outstanding Research Award. He is listed as a SCI highly cited researcher by Clarivate Analytics. His current research interests include multi-agent systems and control, intelligent systems and control, and networked systems and control.

Prof. Feng is an IEEE Fellow, an associate editor of *IEEE Trans. Fuzzy Systems* and *Journal of Systems Science and Complexity*, and was an associate editor of *IEEE Trans. Automatic Control, IEEE Trans. Systems, Man & Cybernetics, Part C, Mechatronics*, and *Journal of Control Theory and Applications*. He is on the Advisory Board of *Unmanned Systems*.