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Towards implementation of PDE control for Stefan system: Input-to-state stability and sampled-data design☆

Shumon Koga[a,*], Iasson Karafyllis[b], Miroslav Krstic[c]

[a] Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA
[b] Department of Mathematics, National Technical University of Athens, 15780 Athens, Greece
[c] Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA

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A B S T R A C T

The Stefan system is a representative model for a liquid–solid phase change which describes the dynamics of a material's temperature profile and the liquid–solid interface position. Our previous work designed a boundary feedback control to stabilize the phase interface position modeled by the Stefan system. This paper resolves two issues our previous work did not study, that are, the robustness analysis under the unknown heat loss and the digital control action. First, we introduce the one-phase Stefan problem with a heat loss by modeling a 1-D diffusion Partial Differential Equation (PDE) dynamics of the liquid temperature and the interface position governed by an Ordinary Differential Equation (ODE) with a time-varying disturbance. We focus on the closed-loop system under the control law proposed in our previous work, and show an estimate of L_2 norm in a sense of Input-to-State Stability (ISS) with respect to the unknown heat loss. Second, we consider the sampled-data control of the one-phase Stefan problem without the heat loss, by applying Zero-Order-Hold (ZOH) to the control law in our previous work. We prove that the closed-loop system under the sampled-data control law satisfies the global exponential stability in the spatial L_2 norm. Analogous ISS result for the two-phase Stefan problem which incorporates the dynamics of the solid phase is also obtained. Numerical simulation verifies our theoretical results for showing the robust performance under the heat loss and the digital control implemented to vary at each sampling time.

1. Introduction

1.1. Background

Liquid–solid phase transitions are physical phenomena which appear in various kinds of science and engineering processes. Representative applications include sea-ice melting and freezing (Koga & Krstic, 2020a), continuous casting of steel (Petrus, Bentsman, & Thomas, 2012), cancer treatment by cryosurgeries (Rabin & Shitzer, 1998), additive manufacturing for materials of both polymer (Koga, Straub, Diagne and Krstic, 2020) and metal (Koga, Krstic and Beaman, 2020), crystal growth (Conrad, Hilhorst, & Seidman, 1990), lithium-ion batteries (Koga, Camacho-Solorio, & Krstic, 2021), and thermal energy storage systems (Koga, Makihata, Chen, Krstic and Pisano, 2020). Physically, these processes are described by a temperature profile along a liquid–solid material, where the dynamics of the liquid–solid interface is influenced by the heat flux induced by melting or solidification. A mathematical model of such a physical process is called the Stefan problem (Gupta, 2003), which is formulated by a diffusion PDE defined on a time-varying spatial domain. The domain's length dynamics is described by an ODE dependent on the Neumann boundary value of the PDE state. Apart from the thermodynamical model, the Stefan problem has been employed to model several chemical, electrical, social, and financial dynamics such as tumor growth process (Friedman & Reitich, 1999), domain walls in ferroelectric thin films (McGilly, Yudin, Feigl, Tagantsev, & Setter, 2015), spreading of invasive species in ecology (Du & Lin, 2010), information diffusion on social networks (Lei, Lin, & Wang, 2013), and optimal exercise boundary of the American put option (Chen, Chadam, Jiang, & Zheng, 2008).

While the numerical analysis of the one-phase Stefan problem is broadly covered in the literature, their control related problems have been addressed relatively fewer. In addition to

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* Corresponding author.
E-mail addresses: skoga@eng.ucsd.edu (S. Koga), iasonkar@central.ntua.gr (I. Karafyllis), krstic@ucsd.edu (M. Krstic).

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it, most of the proposed control approaches are based on finite dimensional approximations with the assumption of an explicitly given moving boundary dynamics (Armaou & Christofides, 2001; Daraoui, Dufour, Hammouri, & Hottot, 2010). For control objectives, infinite-dimensional approaches have been used for stabilization of the temperature profile and the moving interface of a 1D Stefan problem, such as enthalpy-based feedback (Petrus et al., 2012) and geometric control (Maidi & Corriou, 2014). These works designed control laws ensuring the asymptotical stability of the closed-loop system in the L_2 norm. However, the results in Maidi and Corriou (2014) are established based on the assumptions on the liquid temperature being greater than the melting temperature, which must be ensured by showing the positivity of the boundary heat input.

Recently, boundary feedback controllers for the Stefan problem have been designed via a "backstepping transformation" (Krstic, 2009; Krstic & Smyshlyaev, 2008; Smyshlyaev & Krstic, 2004) which has been used for many other classes of infinite-dimensional systems. For instance, Koga, Diagne, and Krstic (2019) designed a state feedback control law, an observer design, and the associated output feedback control law by introducing a nonlinear backstepping transformation for moving boundary PDE, which achieved the exponentially stabilization of the closed-loop system in the $\mathcal{H}_1$ norm without imposing any *a priori* assumption, with ensuring the robustness with respect to the physical parameters' uncertainty. Koga, Bresch-Pietri and Krstic (2020) developed a delay-compensated control for the Stefan problem with proving the robustness to the delay mismatch.

All the aforementioned results have not focused on two issues arising in practical situation. First, they consider the one-phase Stefan problem which neglects the cooling heat caused by the solid phase, but any analysis on the system incorporating the cooling heat at the liquid–solid interface has not been established. The one-phase Stefan problem with a prescribed heat flux at the interface was studied in Sherman (1967) and the existence and uniqueness of the solution was proved. Regarding the added heat flux at the interface as the heat loss induced by the remaining other phase dynamics, it is reasonable to treat the prescribed heat flux as the disturbance of the system. The norm estimate of systems with a disturbance is often analyzed in terms of Input-to-State Stability (ISS) (Sontag, 2008), which serves as a criterion for the robustness of the controller or observer design (Arcak & Kokotovic, 2001; Freeman & Kokotovic, 2008). The characterizations of ISS have been investigated in Sontag and Wang (1995, 1996), which have been utilized for the derivation of small gain theorems (Jiang, Mareels, & Wang, 1996; Jiang, Teel, & Praly, 1994). Recently, the ISS for infinite dimensional systems with respect to the boundary disturbance was developed in Karafyllis and Krstic (2016, 2017b, 2019) using the spectral decomposition of the solution of linear parabolic PDEs in one dimensional spatial coordinate with Strum–Liouville operators. An analogous result for the diffusion equations with a radial coordinate in n-dimensional balls is shown in Camacho-Solorio, Moura, and Krstic (2018) with proposing an application to robust observer design for battery management systems (Moura, Chaturvedi, & Krstic, 2014).

Second, the aforementioned results assumed the control input to be varying continuously in time; however, in practical implementation of the control systems it is impossible to dynamically change the control input continuously in time due to limitations of the sensors, actuators, and software. Instead, the control input can be adjusted at each sampling time at which the measured states are obtained or the actuator is manipulated. One of the most fundamental and well known method to design such a "sampled-data" control is the so-called "emulation design" that applies "Zero-Order-Hold" (ZOH) to the nominal "continuous-time" control law. A general result for nonlinear ODEs to guarantee the global stability of the closed-loop system under such a ZOH-based sampled-data control was studied in Karafyllis and Kravaris (2009b), and the sampled-data observer design under discrete-time measurement is developed in Karafyllis and Kravaris (2009a) by introducing inter-sampled output predictor. As further extensions, the stability of the sampled-data control for general nonlinear ODEs under actuator delay is shown in Karafyllis and Krstic (2012, 2017a) by applying predictor-based feedback developed in Krstic (2009), and results for a linear parabolic PDE are given in Karafyllis and Krstic (2018) by employing Sturm–Liouville operator theory. The sampled-data control for parabolic PDEs has been intensively developed by Fridman and coworkers by utilizing linear matrix inequalities (Am & Fridman, 2014; Fridman, 2013; Fridman & Blighovsky, 2012; Selivanov & Fridman, 2016). However, none of the existing work on the sampled-data control has studied the class of the Stefan problem described by a parabolic PDE with state-dependent moving boundaries "(a nonlinear system)".

1.2. Contributions and results

The contributions of the paper are as follows:

- proving ISS of the closed-loop system of the one-phase Stefan problem under the control law proposed in Koga et al. (2019) with respect to unknown heat loss,
- ensuring the global exponential stability of the closed-loop system under the sampled-data control for the one-phase Stefan problem,
- and deriving analogous ISS result for the two-phase Stefan problem by incorporating the dynamics of the solid phase temperature with utilizing the control design in Koga and Krstic (2020b).

First, we revisit the result of our conference paper (Koga, Karafyllis, & Krstic, 2018) which proved ISS of the one-phase Stefan problem under the control law proposed in Koga et al. (2019) with respect to the heat loss at the interface. We consider a prescribed open-loop control of the one-phase Stefan problem in which the solution of the system is equivalent with the proposed closed-loop control. Using the result of Sherman (1967), the well-posedness of the solution and the positivity conditions are proved. Then, we apply the closed-loop control through the backstepping transformation as in Koga et al. (2019). The well-posedness of the closed-loop solution and the positivity conditions for the model to be valid are ensured by showing the differential equation of the control law. The associated target system has the disturbance at the interface dynamics due to the heat loss. An estimate of the L_2 norm of the closed-loop system is developed in the sense of ISS through Lyapunov analysis.

Second, we consider the sampled-data control of the one-phase Stefan problem by applying ZOH to the continuous-time control law developed in Koga et al. (2019). The approach employed in this paper is distinct from the methodology developed in literature. Namely, we solve the growth of the system's energy analytically in time under the proposed sampled-data feedback control that is in the form of an energy-shaping design. Then, a perturbation that is incorporated in the closed-loop system due to the error between the continuous-time design and the sampled-data design can be represented analytically, and the closed-loop stability is proven by using Lyapunov method. Finally, the similar procedure is performed to extend the ISS result to the two-phase Stefan problem by utilizing the control design in Koga and Krstic (2020b). We note that, in our other paper (Koga, Makihata et al., 2020), the sampled-data control is implemented in experiments of melting paraffin.

1.3. One-phase and two-phase results

We present results for both one-phase (Part I) and two-phase (Part II) Stefan systems. We do that for several reasons. First, for pedagogical reasons, we introduce novel concepts for the Stefan system, such as ISS and sampled-data control, first for the one-phase system. This brings clarity and lays the foundation for the two-phase system, so that the extension from one phase to two phases can be done in a minimum amount of space. Second, we present the one-phase and two-phase cases separately because the one-phase case is not simply a special case of the two-phase problem, particularly in the presence of heat loss.

1.4. Organization of the paper

This paper is organized as follows. The one-phase Stefan problem with heat loss is presented in Section 2. Section 3 introduces our result for the ISS with respect to the heat loss with its proof. Section 4 is devoted to the sampled-data control law for the one-phase Stefan problem and the stability proof of the closed-loop system. The extension of the ISS result to the two-phase Stefan problem is addressed in Section 5. Supportive numerical simulations are provided in Section 6. The paper ends with some final remarks in Section 7.

1.5. Notation and definition

Throughout this paper, partial derivatives and L_2-norm are denoted as $u_t(x,t) = \frac{\partial u}{\partial t}(x,t)$, $u_x(x,t) = \frac{\partial u}{\partial x}(x,t)$, and $\|u[t]\| = \sqrt{\int_0^{s(t)} u(x,t)^2 dx}$, where $u[t]$ is a function defined on $[0, s(t)]$ with real values defined by $(u[t])(x) = u(x,t)$ for all $x \in [0, s(t)]$. $\mathbb{R}_+ := [0, +\infty)$, and $\mathbb{Z}_+$ is the set of nonnegative integers $\{0, 1, 2, \ldots\}$. $C^0(U; \Omega)$ is the class of continuous mappings on $U \subseteq \mathbb{R}^n$, which takes values in $\Omega \subseteq \mathbb{R}$, and $C^k(U; \Omega)$, where $k \geq 1$ is the class of continuous functions on U, which have continuous derivatives of order k on U and takes values in Ω. A continuous function $\eta : [0, a) \to \mathbb{R}_+$ is said to belong to class-$\mathscr{K}$ if it is strictly increasing and $\eta(0) = 0$. A continuous function $\zeta : [0, a) \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class-$\mathscr{KL}$ if, for each fixed s, the mapping $\zeta(r, s)$ belongs to class-$\mathscr{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Part I: One-Phase Stefan System

2. One-phase Stefan problem with heat loss

Consider a physical model which describes the melting or solidification mechanism in a pure one-component material of length L in one dimension. In order to mathematically describe the position at which phase transition occurs, we divide the domain $[0, L]$ into two time-varying sub-domains, namely, the interval $[0, s(t)]$ which contains the liquid phase, and the interval $[s(t), L]$ that contains the solid phase. A heat flux enters the material through the boundary at $x = 0$ (the fixed boundary of the liquid phase) which affects the liquid–solid interface dynamics through heat propagation in liquid phase. In addition, due to the cooling effect from the solid phase, there is a heat loss at the interface position $x = s(t)$. As a consequence, the heat equation alone does not provide a complete description of the phase transition and must be coupled with the dynamics that describes the moving boundary. This configuration is shown in Fig. 1.

The energy conservation and heat conduction laws yield the heat equation of the liquid phase, the boundary conditions, and the initial values as follows

$$T_t(x,t) = \alpha T_{xx}(x,t), \quad \text{for} \quad t > 0, \quad 0 < x < s(t), \tag{1}$$

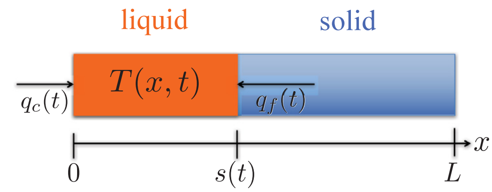


Fig. 1. Schematic of the one-phase Stefan problem.

$$-kT_x(0,t) = q_c(t), \quad \text{for} \quad t > 0, \tag{2}$$

$$T(s(t),t) = T_m, \quad \text{for} \quad t > 0, \tag{3}$$

$$s(0) = s_0, \text{ and } T(x,0) = T_0(x), \text{ for } x \in (0, s_0], \tag{4}$$

where $\alpha := \frac{k}{\rho C_p}$, and $T(x,t)$, $q_c(t)$, ρ, C_p, and k are the distributed temperature of the liquid phase, the manipulated heat flux, the liquid density, the liquid heat capacity, and the liquid heat conductivity, respectively.

The first problem we consider in this paper assumes that the heat loss at the interface caused by the solid phase temperature dynamics is unknown but described as a time-varying function, denoted as $q_f(t)$. Note that the heat loss is assumed not to be dependent on any of the heat input $q_c(t)$, the initial values (T_0, s_0), and the state variables (T, s). A full physical model for the heat loss is given in Part II of this paper as the two-phase Stefan problem. Then, the local energy balance at the liquid–solid interface $x = s(t)$ is given by

$$\rho \Delta H^* \dot{s}(t) = -kT_x(s(t),t) - q_f(t), \tag{5}$$

where ΔH^* represents the latent heat of fusion. In (5), the left hand side represents the latent heat, and the first and second term of the right hand side represent the heat flux by the liquid phase and the heat loss caused by the solid phase, respectively. As the governing equations (1)–(5) suffice to determine the dynamics of the states (T, s), the temperature in the solid phase is not needed to be modeled.

Remark 1. As the moving interface $s(t)$ depends on the temperature, the problem defined in (1)–(5) is nonlinear.

There are two underlying assumptions to validate the model (1)–(5). First, the liquid phase is not frozen to solid from the boundary $x = 0$. This condition is ensured if the liquid temperature $T(x,t)$ is greater than the melting temperature. Second, the material is not completely melt or frozen to single phase through the disappearance of the other phase. This condition is guaranteed if the interface position remains inside the material's domain. In addition, these conditions are also required for the well-posedness (existence and uniqueness) of the solution in this model. With these model validity conditions, we emphasize the following remark.

Remark 2. To maintain the model (1)–(5) to be physically validated, the following conditions must hold (Alexiades, 1992):

$$T(x,t) \geq T_m, \quad \forall x \in (0, s(t)), \quad \forall t > 0, \tag{6}$$

$$0 < s(t) < L, \quad \forall t > 0. \tag{7}$$

A mathematically rigorous definition of such a physically meaningful solution is given in Sherman (1967) and stated as follows.

Definition 1. Let $s_0 > 0$ be a given constant, $q_c, q_f : \mathbb{R}_+ \to \mathbb{R}_+$, $T_0 : [0, s_0] \to [T_m, \infty)$ with $T_0(s_0) = T_m$ be given functions. The initial–boundary value problem (1)–(5) has a solution if there exists a pair of functions $s \in C^0(\mathbb{R}_+; (0, L)) \cap C^1((0, +\infty); (0, L))$, T :

$D \to [T_m, \infty)$ with $D = \bigcup_{t\geq 0}(t \times [0, s(t)])$ such that (i) T, T_x are continuous for $t > 0, x \in [0, s(t)]$, (ii) T_t, T_{xx} are continuous for $t > 0, x \in (0, s(t))$, (iii) T is continuous for $t = 0, x \in (0, s_0]$ with $T_m \leq \liminf_{(x,t)\to(0,0)}(T(x, t)) \leq \limsup_{(x,t)\to(0,0)}(T(x, t)) < +\infty$, (iv) (1), (2), (3), (4), (5) hold.

Based on the above conditions, we impose the following assumption on the initial data and the heat loss.

Assumption 1. $s_0 > 0$, $T_0(x) \in C^1([0, s_0]; [T_m, +\infty))$ with $T_0(s_0) = T_m$.

Assumption 2. The heat loss $q_f(t)$ is a nonnegative and continuous function, i.e., $q_f \in C^0(\mathbb{R}_+; \mathbb{R}_+)$.

By the virtue of the theorem on page 55 and Lemma 1 on page 61 in Sherman (1967), we have the following result.

Lemma 3. *Let Assumptions 1–2 hold, $q_c(t)$ be a nonnegative continuous function $q_c \in C^0(\mathbb{R}_+; \mathbb{R}_+)$, and it holds*

$$\frac{ks_0}{\beta} + \int_0^t (q_c(\tau) - q_f(\tau))d\tau + \frac{k}{\alpha}\int_0^{s_0}(T_0(x) - T_m)dx > 0 \tag{8}$$

for all $t \geq 0$, where $\beta := \frac{k}{\rho \Delta H^*}$. Then there exists a unique pair of functions $s \in C^0(\mathbb{R}_+; (0, +\infty)) \cap C^1((0, +\infty); (0, +\infty))$, $T : D \to [T_m, +\infty)$ with $D = \bigcup_{t\geq 0}(t \times [0, s(t)])$ such that (i) T, T_x are continuous for $t > 0, x \in [0, s(t)]$, (ii) T_t, T_{xx} are continuous for $t > 0, x \in (0, s(t))$, (iii) T is continuous for $t = 0, x \in (0, s_0]$ with $T_m \leq \liminf_{(x,t)\to(0,0)}(T(x, t)) \leq \limsup_{(x,t)\to(0,0)}(T(x, t)) < +\infty$, (iv) (1), (2), (3), (4), (5) hold.

Lemma 3 implies that $T_x(s(t), t) \leq 0$ for all $t \geq 0$ (a direct consequence of the fact that $T(s(t), t) = T_m$ and $T : D \to [T_m, +\infty)$, i.e., $T(x, t) \geq T_m$ for all $t \geq 0, x \in [0, s(t)]$. Lemma 3 does not imply that there exists a solution to the initial–boundary value problem (1)–(5) since $s \in C^1(\mathbb{R}_+; (0, +\infty))$ and hence it does not guarantee that $s(t) < L$ for all $t \geq 0$.

3. ISS for one-phase Stefan problem

This section is devoted to the analysis of ISS with respect to the heat loss $q_f(t)$.

3.1. Problem setup

The steady-state solution $(T_{eq}(x), s_{eq})$ of the system (1)–(5) with $q_c(t) = 0$ and $q_f(t) = 0$ yields a uniform melting temperature $T_{eq}(x) = T_m$ and a constant interface position given by the initial data. In Koga et al. (2019), the authors developed the exponential stabilization of the interface position $s(t)$ at a desired reference setpoint s_r with zero heat loss $q_f(t) = 0$ through the following state feedback control design of $q_c(t)$:

$$q_c(t) = -c\left(\frac{k}{\alpha}\int_0^{s(t)}(T(x, t) - T_m)dx + \frac{k}{\beta}(s(t) - s_r)\right), \tag{9}$$

where $c > 0$ is the control gain which can be chosen by user. To maintain the positivity of the heat input $q_c(t)$ as stated in Lemma 3, the control law (9) at $t = 0$ must be positive, which leads to the necessity of the following assumption:

Assumption 4. The setpoint is chosen to verify

$$s_0 + \frac{\beta}{\alpha}\int_0^{s_0}(T_0(x) - T_m)dx < s_r < L. \tag{10}$$

The proof of $q_c(t) \geq 0$ for all $t \geq 0$ will be given later. Finally, we impose the following conditions.

Assumption 5. The total energy of the heat loss is bounded, i.e., there exists $M > 0$ such that

$$\int_0^\infty q_f(t)dt < M. \tag{11}$$

Assumption 6. $q_f : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded and the control gain c is chosen sufficiently large to satisfy $c > \frac{\beta}{ks_r}\bar{q}_f$, where $\bar{q}_f := \sup_{0\leq t\leq\infty}\{q_f(t)\}$.

We prove the ISS of the reference error system with the control design (9) with respect to the heat loss at the interface by studying the norm estimate.

3.2. Open-loop system and analysis

The well-posedness of the one-phase Stefan problem with heat flux at the interface was developed in Sherman (1967) with a prescribed open-loop heat input for $q_c(t)$. To apply the result, in this section we focus on an open-loop control which has an equivalent solution as the closed-loop control introduced later, and prove the well-posedness of the open-loop system.

Lemma 7. *Suppose that Assumptions 1–4 hold, and $q_c(t)$ is given by the following open-loop control:*

$$q_c(t) = q_0 e^{-ct} + c\int_0^t e^{-c(t-\tau)}q_f(\tau)d\tau, \quad \forall t \geq 0, \tag{12}$$

where

$$q_0 = -c\left(\frac{k}{\alpha}\int_0^{s_0}(T_0(x) - T_m)dx + \frac{k}{\beta}(s_0 - s_r)\right). \tag{13}$$

Then the initial–boundary value problem (1)–(5) has a unique solution.

Proof. Since Assumption 4 leads to $q_0 > 0$, the open-loop controller (12) remains positive and continuous for all $t > 0$ by Assumption 2. Taking the time derivative of (12) yields $\dot{q}_c(t) = -c(q_c(t) - q_f(t))$, which leads to

$$\int_0^t (q_c(\tau) - q_f(\tau))d\tau = -\frac{1}{c}(q_c(t) - q_0). \tag{14}$$

Let $E(t)$ be the left hand side of (8). Substituting (14) and (12), and noting that $E(0) = \frac{k}{\alpha}\int_0^{s_0}(T_0(x) - T_m)dx + \frac{k}{\beta}s_0 = -\frac{1}{c}q_0 + \frac{k}{\beta}s_r$, we get

$$E(t) = e^{-ct}\left[E(0) + \frac{ks_r}{\beta}(e^{ct} - 1) - \int_0^t e^{c\tau}q_f(\tau)d\tau\right]. \tag{15}$$

Let $f(t)$ be a function in time defined by

$$f(t) = E(0) + \frac{ks_r}{\beta}(e^{ct} - 1) - \int_0^t e^{c\tau}q_f(\tau)d\tau. \tag{16}$$

Since $E(t) = e^{-ct}f(t)$, we can see that $E(t) > 0$ for all $t \geq 0$ if and only if $f(t) > 0$ for all $t > 0$. By (16), we have $f(0) = E(0) > 0$. Taking the time derivative of (16) yields

$$\dot{f}(t) = e^{ct}\left(\frac{ks_r c}{\beta} - q_f(t)\right). \tag{17}$$

By Assumption 6, (17) leads to $\dot{f}(t) > 0$ for all $t \geq 0$. Therefore, $f(t) > 0$ for all $t \geq 0$, and we conclude $E(t) > 0$ for all $t \geq 0$, and hence condition (8) holds. By virtue of Lemma 3 there exists a unique pair of functions $s \in C^0(\mathbb{R}_+; (0, +\infty)) \cap C^1((0, +\infty); (0, +\infty))$, $T : D \to [T_m, +\infty)$ with $D = \bigcup_{t\geq 0}(t \times [0, s(t)])$ for which properties (i)–(iv), described in the statement of Lemma 3, hold.

In order to prove that this pair of functions $s \in C^0(\mathbb{R}_+; (0,+\infty)) \cap C^1((0,+\infty);(0,+\infty))$, $T: D \to [T_{\rm m}, +\infty)$ is a solution of the initial–boundary value problem (1)–(5), we have to show that $s(t) < L$ for all $t \geq 0$.

Integrating (5) and using the definition of β and (4), we get for all $t \geq 0$:

$$s(t) - s_0 = -\beta \int_0^t T_x(s(\tau),\tau)d\tau - \frac{\beta}{k}\int_0^t q_{\rm f}(\tau)d\tau \tag{18}$$

On the other hand, using (1) and (2), we get for all $t > 0$:

$$\begin{aligned} T_x(s(t),t) =& T_x(0,t) + \int_0^{s(t)} T_{xx}(\xi,t)d\xi \\ =& -\frac{1}{k}q_{\rm c}(t) + \frac{1}{\alpha}\int_0^{s(t)} T_t(\xi,t)d\xi. \end{aligned} \tag{19}$$

Eq. (3) implies that $\frac{d}{dt}\left(\int_0^{s(t)}(T(\xi,t)-T_{\rm m})d\xi\right) = \int_0^{s(t)} T_t(\xi,t)d\xi$ for all $t > 0$. Combining the previous equation with (4), (18), and (19), we get for all $t > 0$:

$$\begin{aligned} s(t) =& s_0 - \frac{\beta}{\alpha}\int_0^{s(t)}(T(x,t)-T_{\rm m})dx + \frac{\beta}{\alpha}\int_0^{s_0}(T_0(x)-T_{\rm m})dx \\ &+ \frac{\beta}{k}\int_0^t (q_{\rm c}(\tau) - q_{\rm f}(\tau))d\tau. \end{aligned} \tag{20}$$

Using the fact that $\dot{q}_{\rm c}(t) = -c(q_{\rm c}(t) - q_{\rm f}(t))$ for all $t > 0$, we obtain that $\int_0^t (q_{\rm c}(\tau) - q_{\rm f}(\tau))d\tau = -\frac{q_{\rm c}(t)-q_{\rm c}(0)}{c}$ for all $t \geq 0$. Consequently, we obtain from (20), (12) and (13):

$$s(t) = s_{\rm r} - \frac{\beta}{\alpha}\int_0^{s(t)}(T(x,t)-T_{\rm m})dx - \frac{\beta}{ck}q_{\rm c}(t). \tag{21}$$

The fact that $s(t) < L$ for all $t \geq 0$ is a direct consequence of (21), (10) and the fact that $q_{\rm c}(t) \geq 0$ for $t \geq 0$ and $T(x,t) \geq T_{\rm m}$ for $t \geq 0$ and $x \in [0, s(t)]$. The proof of Lemma 7 is complete.

3.3. Closed-loop analysis and ISS proof

While the open-loop input (12) ensures the well-posedness of system (1)–(5) with (12), the analysis does not enable to prove an ISS property. In addition, the open-loop design requires knowledge of the heat loss at the interface $q_{\rm f}(t)$, which cannot be done in practice. In this section, we show that the closed-loop solution with the control law proposed in Koga et al. (2019) is equivalent to the open-loop solution of system (1)–(5) with (12). The controller is using the liquid temperature profile and the interface position $(T(x,t), s(t))$. The heat loss $q_{\rm f}(t)$ is regarded as a disturbance, and the norm estimate of the reference error is derived in a sense of ISS.

Our first main result is stated in the following theorem.

Theorem 8. *Under Assumptions 1–6, the closed-loop system consisting of (1)–(5) with the control law (9) has a unique solution in the sense of Definition 1, and is ISS with respect to the heat loss $q_{\rm f}(t)$ at the interface, i.e., there exist a class-$\mathscr{KL}$ function ζ and a class-$\mathscr{K}$ function η such that the following estimate holds:*

$$\Psi(t) \leq \zeta(\Psi(0),t) + \eta\left(\sup_{\tau\in[0,t]} q_{\rm f}(\tau)\right), \tag{22}$$

for all $t \geq 0$, in the L_2 norm

$$\Psi(t) = \left(\|T[t]-T_{\rm m}\|^2 + (s(t)-s_{\rm r})^2\right)^{\frac{1}{2}}. \tag{23}$$

Moreover, there exist positive constants $M_1 > 0$ and $M_2 > 0$ such that the explicit functions of ζ and η are given by $\zeta(\Psi(0),t) = M_1\Psi(0)e^{-\lambda t}$, $\eta(\sup_{\tau\in[0,t]} q_{\rm f}(\tau)) = M_2 \sup_{\tau\in[0,t]} q_{\rm f}(\tau)$, where $\lambda = \frac{1}{32}\min\left\{\frac{\alpha}{s_{\rm r}^2}, c\right\}$, which ensures the exponentially ISS.

The proof of Theorem 8 is established in the remainder of this section.

3.3.1. Reference error system

Let $u(x,t)$ and $X(t)$ be reference error variables defined by $u(x,t) := T(x,t) - T_{\rm m}$, and $X(t) := s(t) - s_r$. Then, the system (1)–(5) is rewritten as

$$u_t(x,t) = \alpha u_{xx}(x,t), \tag{24}$$

$$u_x(0,t) = -q_{\rm c}(t)/k, \tag{25}$$

$$u(s(t),t) = 0, \tag{26}$$

$$\dot{X}(t) = -\beta u_x(s(t),t) - d(t), \tag{27}$$

where $d(t) = \frac{\beta}{k}q_{\rm f}(t)$. The controller is designed to stabilize (u,X)-system at the origin for $d(t) = 0$.

3.3.2. Backstepping transformation

Introduce the following backstepping transformation

$$\begin{aligned} w(x,t) =& u(x,t) - \frac{\beta}{\alpha}\int_x^{s(t)}\phi(x-y)u(y,t)dy \\ &- \phi(x-s(t))X(t), \end{aligned} \tag{28}$$

which maps into

$$\begin{aligned} w_t(x,t) =& \alpha w_{xx}(x,t) + \dot{s}(t)\phi'(x-s(t))X(t) \\ &+ \phi(x-s(t))d(t), \end{aligned} \tag{29}$$

$$w_x(0,t) = \frac{\beta}{\alpha}\phi(0)u(0,t), \tag{30}$$

$$w(s(t),t) = \varepsilon X(t), \tag{31}$$

$$\dot{X}(t) = -cX(t) - \beta w_x(s(t),t) - d(t). \tag{32}$$

The objective of the transformation (28) is to add a stabilizing term $-cX(t)$ in (32) of the target (w,X)-system which is easier to prove the stability for $d(t) = 0$ than (u,X)-system. By taking the derivative of (28) with respect to t and x respectively along the solution of (24)–(27), to satisfy (29), (31), (32), we derive the conditions on the gain kernel solution which yields the solution as

$$\phi(x) = c\beta^{-1}x - \varepsilon. \tag{33}$$

By matching the transformation (28) with the boundary condition (30), the control law is derived as

$$q_{\rm c}(t) = -c\left(\frac{k}{\alpha}\int_0^{s(t)} u(y,t)dy + \frac{k}{\beta}X(t)\right). \tag{34}$$

Rewriting (34) by $T(x,t)$ and $s(t)$ yields (9).

3.3.3. Inverse transformation

Suppose that the transformation from (w,X) to (u,X) can be formulated by

$$\begin{aligned} u(x,t) =& w(x,t) - \frac{\beta}{\alpha}\int_x^{s(t)}\psi(x-y)w(y,t)dy \\ &- \psi(x-s(t))X(t), \end{aligned} \tag{35}$$

where ψ is a gain kernel function to be determined. Taking the derivatives of (35) in x and t along (29)–(32), to match with (24)–(27), one can obtain the conditions for ψ to satisfy, which leads to the unique solution given by

$$\psi(x) = e^{\bar{\lambda}x}\left(p_1\sin(\omega x) + \varepsilon\cos(\omega x)\right), \tag{36}$$

where $\bar{\lambda} = \frac{\beta\varepsilon}{2\alpha}$, $\omega = \sqrt{\frac{4\alpha c-(\varepsilon\beta)^2}{4\alpha^2}}$, $p_1 = -\frac{1}{2\alpha\beta\omega}\left(2\alpha c - (\varepsilon\beta)^2\right)$, and $0 < \varepsilon < 2\frac{\sqrt{\alpha c}}{\beta}$ is to be chosen later. Thus, we deduce that the transformation (28) with (33) is invertible, where the inverse transformation is given by (35) with (36). Finally, using (35), the boundary condition (30) is rewritten as

$$w_x(0,t) = -\frac{\beta}{\alpha}\varepsilon\left[w(0,t) - \frac{\beta}{\alpha}\int_0^{s(t)}\psi(-y)w(y,t)\mathrm{d}y - \psi(-s(t))X(t)\right]. \tag{37}$$

In other words, the target (w, X)-system is described by (29), (31), (32), and (37). Note that the boundary condition (31) and the kernel function (33) are modified from the one in Koga et al. (2019), while the control design (34) is equivalent. The target system derived in Koga et al. (2019) requires $\mathscr{H}_1$-norm analysis for stability proof. However, with the unknown heat loss at the interface, $\mathscr{H}_1$-norm analysis fails to show the stability due to the non-monotonic moving boundary dynamics. The modification of the boundary condition (31) enables to prove the stability in L_2 norm as shown later.

3.3.4. Analysis of closed-loop system

Here, we prove that the closed-loop system has a unique solution in the sense of Definition 1, which satisfies some required physical conditions.

Lemma 9. *Under Assumptions 1–6, the closed-loop solution of (1)–(5) with the control law (9) is equivalent to the open-loop solution of (1)–(5) with the control law (12) for all $t \geq 0$, and has a unique solution in the sense of Definition 1 which satisfies*

$$q_c(t) > 0, \quad \forall t \geq 0, \tag{38}$$

$$u(x,t) \geq 0, \quad u_x(s(t),t) \leq 0, \quad \forall x \in (0, s(t)), \forall t \geq 0, \tag{39}$$

$$0 < s(t) < s_r, \quad \forall t \geq 0. \tag{40}$$

Proof. Taking the time derivative of the control law (9) along with the system (1)–(5) leads to the following differential equation

$$\dot{q}_c(t) = -cq_c(t) + cq_f(t), \tag{41}$$

which has the same explicit solution as the open-loop control (12). Hence, the closed-loop solution is equivalent to the open-loop solution with (12). Thus, $q_c(t) : \mathbb{R}_+ \to \mathbb{R}_+$, and by virtue of Lemma 7 the closed-loop system (1)–(5) with (9) has a unique solution which satisfies (38)–(40).

3.3.5. Stability analysis

To conclude the ISS of the original system, first we show the ISS of the target system (29), (31), (32), and (37) with respect to the disturbance $d(t)$. We consider

$$V(t) = \frac{1}{2\alpha}\|w\|^2 + \frac{\varepsilon}{2\beta}X(t)^2. \tag{42}$$

Then, as proven in Appendix, for a sufficiently small $\varepsilon > 0$, the following inequality is derived:

$$\dot{V}(t) \leq -bV(t) + \Gamma d(t)^2 + a|\dot{s}(t)|V(t), \tag{43}$$

where $a = \frac{2\beta\varepsilon}{\alpha}\max\left\{1, \frac{\alpha c^2 s_r}{2\beta^3\varepsilon^3}\right\}$, $b = \frac{1}{8}\min\left\{\frac{\alpha}{s_r^2}, c\right\}$, and $\Gamma = \frac{\varepsilon}{\beta c} + \frac{2s_r^3}{\alpha^2}\left(\frac{cs_r}{\beta} + \varepsilon\right)^2$. Let $z(t)$ be defined by

$$z(t) := s(t) + 2\int_0^t d(\tau)\mathrm{d}\tau. \tag{44}$$

By (11) and (40), we have

$$0 < z(t) < \bar{z} := s_r + \frac{2\beta M}{k} \tag{45}$$

The time derivative of (44) is given by

$$\dot{z}(t) = -\beta u_x(s(t),t) + d(t) \tag{46}$$

Since $\dot{s}(t) = -\beta u_x(s(t),t) - d(t)$ and recalling $u_x(s(t),t) < 0$ and $d(t) > 0$, the following inequality holds:

$$|\dot{s}(t)| \leq -\beta u_x(s(t),t) + d(t) = \dot{z}(t). \tag{47}$$

Applying (47) to (43) leads to

$$\dot{V}(t) \leq -bV(t) + \Gamma d(t)^2 + a\dot{z}(t)V(t). \tag{48}$$

Consider the following functional

$$W(t) = V(t)e^{-az(t)}. \tag{49}$$

Taking the time derivative of (49) with the help of (48) and applying (45), we deduce

$$\dot{W}(t) \leq -bW(t) + \Gamma d(t)^2. \tag{50}$$

Since (50) leads to the statement that either $\dot{W}(t) \leq -\frac{b}{2}W(t)$ or $W(t) \leq \frac{2}{b}\Gamma d(t)^2$ is true, following the procedure in Sontag (2008) (proof of Theorem 5 in Section 3.3), one can derive

$$W(t) \leq W(0)e^{-\frac{b}{2}t} + \frac{2}{b}\Gamma \sup_{\tau\in[0,t]} d(\tau)^2. \tag{51}$$

By (49), we have $V(t) = W(t)e^{az(t)}$. Applying (51), we get $V(t) \leq e^{az(t)}W(0)e^{-\frac{b}{2}t} + \frac{2}{b}\Gamma\sup_{\tau\in[0,t]} d(\tau)^2$. Again by (49), we have $W(0) = V(0)e^{-az(0)}$. Combining these two with the help of (45), finally we obtain the following estimate on the L_2 norm of the target system

$$V(t) \leq V(0)e^{a\bar{z}}e^{-\frac{b}{2}t} + \frac{2}{b}\Gamma e^{a\bar{z}}\sup_{\tau\in[0,t]} d(\tau)^2. \tag{52}$$

Due to the invertibility of the transformation from (u, X) to (w, X) together with the boundedness of the domain $0 < s(t) < s_r$, there exist positive constants $\underline{M} > 0$ and $\overline{M} > 0$ such that the following inequalities hold:

$$\underline{M}\Psi(t)^2 \leq V(t) \leq \overline{M}\Psi(t)^2, \tag{53}$$

where $\Psi(t)$ is the L_2 norm of the original system defined in (23). Finally, applying (53) to (52), one can derive the norm estimate on the original (T, s)-system as $\Psi(t) \leq \sqrt{\frac{\overline{M}e^{a\bar{z}}}{\underline{M}}}\Psi(0)e^{-\frac{b}{4}t} + \sqrt{\frac{2\Gamma e^{a\bar{z}}}{b\underline{M}}}\sup_{\tau\in[0,t]} d(\tau)$, which completes the proof of Theorem 8.

4. Sampled-data control for the one-phase Stefan problem

4.1. Problem statement and main result

In practical implementation, the actuation value cannot be changed continuously in time. Instead, by obtaining the measured value as signals discretely in time, the control value needs to be implemented at each sampling time. One of the most typical design for such a sampled-data control is the application of "Zero-Order-Hold" (ZOH) to the nominal continuous time control law. Through ZOH, during the time intervals between each sampling, the control maintains the value at the previous sampling time. Let t_j be the jth sampling time for $j \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and τ_j be defined by

$$\tau_j = t_{j+1} - t_j. \tag{54}$$

The application of ZOH to the nominal control law (9) leads to the following design for the sampled-data control

$$q_c(t) = -c\left(\frac{k}{\alpha}\int_0^{s(t_j)}(T(x,t_j) - T_m)\mathrm{d}x + \frac{k}{\beta}(s(t_j) - s_r)\right), \quad \forall t \in [t_j, t_{j+1}), \tag{55}$$

of which the right hand side is constant during the time interval $t \in [t_j, t_{j+1})$. Let us denote $q_j = q_c(t)$ for $t \in [t_j, t_{j+1})$. Hereafter, all the variables with subscript j denote the variables at $t = t_j$. Moreover, we neglect the heat loss in this setup, namely,

$$q_f(t) = 0, \quad \forall t \geq 0, \tag{56}$$

is assumed. First, we introduce the following assumption on the sampling scheduling.

Assumption 10. The sampling schedule has a finite upper diameter and a positive lower diameter, i.e., there exist constants $0 < r \leq R$ such that

$$\sup_{j \in \mathbb{Z}_+}\{\tau_j\} \leq R, \tag{57}$$

$$\inf_{j \in \mathbb{Z}_+}\{\tau_j\} \geq r. \tag{58}$$

Since the sampled-data control input is not a continuous function in time but a piecewise continuous function, we require an alternative definition of the solution.

Definition 2. Let an increasing sequence $\{t_j : j \in \mathbb{Z}_+\}$ with $t_0 = 0$ and $\lim_{j\to+\infty}(t_j) = +\infty$ be given. Let also a sequence $\{q_j : j \in \mathbb{Z}_+\}$ be given. Let $s_0 > 0$ be a given constant and $T_0 : [0, s_0] \to [T_m, +\infty)$ with $T_0(s_0) = T_m$ be a given function. The initial–boundary value problem (1)–(5), $q_c(t) = q_j$ for all $t \in [t_j, t_{j+1}), j \in \mathbb{Z}_+$ with $q_f(t) \equiv 0$ has a solution if there exists a pair of functions $s \in C^0(\mathbb{R}_+; (0, L)) \cap C^1(I; (0, L))$, $T : D \to [T_m, +\infty)$ with $I = \mathbb{R}_+ \setminus \{t_j : j \in \mathbb{Z}_+\}$, $D = \bigcup_{t\geq 0}(t \times [0, s(t)])$ such that (i) T, T_x are continuous for $t \in I, x \in [0, s(t)]$, (ii) $T[t] \in C^1((0, s(t)]) \cap C^2((0, s(t)))$ for all $t > 0$, (iii) $\lim_{t\to t_j}(T(x, t)) = T(x, t_j)$ for all $x \in (0, s(t)), j \in \mathbb{Z}_+$, (iv) T_t, T_{xx} are continuous for $t \in I, x \in (0, s(t))$, (v) (3) holds for all $t \geq 0$, (vi) $T_m \leq \liminf_{(x,t)\to(0,t_j)}(T(x, t)) \leq \limsup_{(x,t)\to(0,t_j)}(T(x, t)) < +\infty$ for $j \in \mathbb{Z}_+$, (vii) (1) holds for $t \in I$, $x \in (0, s(t))$, (viii) (2), (5) hold for $t \in I$, (ix) (4) holds for $x \in (0, s_0]$.

Our main theorem is given next.

Theorem 11. *Consider the closed-loop system (1)–(5), (55), (56) under Assumption 1, 4. Then for every $0 < r \leq R < 1/c$, there exists a constant $M := M(r)$ for which the following property holds: for every sequence $\{t_j \geq 0 : j = 0, 1, 2, \ldots\}$ with $t_0 = 0$ for which Assumption 10 holds, the initial–boundary value problem (1)–(5) with (55), (56) has a unique solution in the sense of Definition 2, which satisfies (6), (7), as well as the estimate $\Psi(t) \leq M\Psi(0)\exp(-bt)$, where $b = \frac{1}{8}\min\left\{\frac{\alpha}{s_r^2}, c\right\}$, for all $t \geq 0$, in the L_2 norm (23).*

The proof of Theorem 11 is established through several steps in the next sections. The positive constant M in (22) has a dependency on $r > 0$ as

$$M(r) = M_1 + \frac{M_2}{1 - (1 - cr)^2 e^{\frac{cr}{8}}}, \tag{59}$$

for some positive constants $M_1 > 0$ and $M_2 > 0$ that are not dependent on $r > 0$.

4.2. Some key properties of the closed-loop system

We first provide the following lemma.

Lemma 12. *The closed-loop system consisting of the plant (1)–(5) under the sampled-data control law (55) has a unique solution in the sense of Definition 2, which is equivalent to the open-loop solution of (1)–(5) with the control law of*

$$q_c(t) = q_j = q_0 \prod_{i=0}^{j-1}(1 - c\tau_i), \quad \forall t \in [t_j, t_{j+1}), \ \forall j \in \mathbb{Z}_+ \tag{60}$$

where $q_0 = -c\left(\frac{k}{\alpha}\int_0^{s_0}(T_0(x) - T_m)dx + \frac{k}{\beta}(s_0 - s_r)\right)$. Furthermore, the closed-loop solution satisfies

$$\dot{s}(t) > 0, \quad \forall t \geq 0, \tag{61}$$

$$s_0 < s(t) < s_r, \quad \forall t \geq 0. \tag{62}$$

Proof. There are three steps to prove Lemma 12.

(I) Equivalence with open-loop solution

We introduce the reference error states $u(x, t) = T(x, t) - T_m$ and $X(t) = s(t) - s_r$. The governing equations (1)–(5) are rewritten as the following reference error system

$$u_t(x, t) = \alpha u_{xx}(x, t), \tag{63}$$

$$u_x(0, t) = -q_c(t)/k, \tag{64}$$

$$u(s(t), t) = 0, \tag{65}$$

$$\dot{X}(t) = -\beta u_x(s(t), t). \tag{66}$$

Define the internal energy as follows:

$$\tilde{E}(t) = \frac{k}{\alpha}\int_0^{s(t)} u(x, t)dx + \frac{k}{\beta}X(t). \tag{67}$$

Taking the time derivative of (67) along the solution of (63)–(66) leads to the following energy conservation law

$$\frac{d}{dt}\tilde{E}(t) = q_c(t). \tag{68}$$

Noting that $q_c(t)$ is constant for $t \in [t_j, t_{j+1})$ as $q_c(t) = q_j$ under ZOH-based sampled-data control, taking the integration of (68) from $t = t_j$ to $t = t_{j+1}$ yields

$$\tilde{E}_{j+1} - \tilde{E}_j = \tau_j q_j, \tag{69}$$

where $\tilde{E}_j = \tilde{E}(t_j)$ and $\tau_j = t_{j+1} - t_j$. The sampled-data control (55) and the internal energy (67) at each sampling time satisfy the following relation:

$$q_j = -c\tilde{E}_j. \tag{70}$$

Substituting (70) into (69), we obtain $\tilde{E}_{j+1} = (1 - c\tau_j)\tilde{E}_j$, which leads to the explicit solution as follows:

$$\tilde{E}_j = \tilde{E}_0 \prod_{i=0}^{j-1}(1 - c\tau_i). \tag{71}$$

Substituting (71) into (70) gives (60). Therefore, the closed-loop system (1)–(5) with $q_f(t) \equiv 0$ under the sampled-data feedback control (55) is equivalent to the open-loop system with the control input (60). Moreover, under Assumption 4, 10, and the fact that $c < \frac{1}{R}$, the input (60) is positive, i.e.,

$$q_j > 0, \quad \forall j \in \mathbb{Z}_+ \tag{72}$$

(II) Existence of solution with $s(t)$ in $(0, +\infty)$

To show the existence of the closed-loop solution, first we consider a solution under $q_c(t) = q_0$ for all $t \geq 0$. By virtue of Lemma 3, there exists a unique pair of functions $s \in C^0(\mathbb{R}_+; (0, \infty)) \cap C^1((0, +\infty); (0, +\infty))$, $T : D \to [T_m, +\infty)$ with $D = \bigcup_{t\geq 0}(t \times [0, s(t)])$ such that (i) T, T_x are continuous for $t > 0, x \in [0, s(t)]$, (ii) T_t, T_{xx} are continuous for $t > 0, x \in (0, s(t))$, (iii) T is continuous for $t = 0, x \in (0, s_0]$ with $T_m \leq$

$\liminf_{(x,t)\to(0,0)}(T(x,t)) \le \limsup_{(x,t)\to(0,0)}(T(x,t)) < +\infty$, (iv) (1) holds for $t>0$, $x\in(0,s(t))$, (v) $-kT_x(0,t)=q_0$, (3), (5) hold for $t>0$, (vi) (4) holds.

Next, we consider the solution with $q_c(t)=q_1$ for all $t\ge 0$ with setting $s_1=s(t_1)$ and $T_1(x)=T(x,t_1)$ for all $x\in[0,s_1]$. By virtue of Lemma 3, there exists a unique $\tilde{s}\in C^0(\mathbb{R}_+;(0,\infty))\cap C^1((0,+\infty);(0,+\infty))$, $\tilde{T}:\tilde{D}\to[T_{\rm m},+\infty)$ with $\tilde{D}=\bigcup_{t\ge 0}(t\times[0,\tilde{s}(t)])$ such that (i) $\tilde{T},\tilde{T}_x$ are continuous for $t>0, x\in[0,\tilde{s}(t)]$, (ii) $\tilde{T}_t,\tilde{T}_{xx}$ are continuous for $t>0, x\in(0,\tilde{s}(t))$, (iii) $\tilde{T}$ is continuous for $t=0$, $x\in(0,s_0]$ with $T_{\rm m}\le\liminf_{(x,t)\to(0,0)}(\tilde{T}(x,t))\le\limsup_{(x,t)\to(0,0)}(\tilde{T}(x,t))<+\infty$, (iv) $\tilde{T}_t(x,t)=\alpha\tilde{T}_{xx}(x,t)$ for $t>0$, $x\in(0,\tilde{s}(t))$, (v) $-k\tilde{T}_x(0,t)=q_1$, $\tilde{T}(\tilde{s}(t),t)=T_{\rm m}$, $\rho\Delta H^*\dot{\tilde{s}}(t)=-k\tilde{T}_x(\tilde{s}(t),t)$ hold for $t>0$, (vi) $\tilde{s}(0)=s_1$ and $\tilde{T}(x,0)=T_1(x)$ for $x\in(0,s_1]$. We then set

$$s(t_1+p)=\tilde{s}(p),\quad T(x,t_1+p)=\tilde{T}(x,p), \tag{73}$$

for all $p\in(0,t_2-t_1]$, and for all $x\in(0,\tilde{s}(p))$. We then set $s_2=s(t_2)$ and $T_2(x)=T(x,t_2)$ for all $x\in[0,s_2]$. Repeating the above process ad infinitum, we construct a unique pair of functions $s\in C^0(\mathbb{R}_+;(0,+\infty))\cap C^1(I;(0,L))$, $T:D\to[T_{\rm m},+\infty)$ with $I=\mathbb{R}_+\setminus\{t_j: j\in\mathbb{Z}_+\}$, $D=\bigcup_{t\ge0}(t\times[0,s(t)])$ such that (i) T,T_x are continuous for $t\in I, x\in[0,s(t)]$, (ii) $T[t]\in C^1((0,s(t)])\cap C^2((0,s(t)))$ for all $t>0$, (iii) $\lim_{t\to t_j}(T(x,t))=T(x,t_j)$ for all $x\in(0,s(t))$, $j\in\mathbb{Z}_+$, (iv) T_t,T_{xx} are continuous for $t\in I, x\in(0,s(t))$, (v) (3) holds for all $t\ge 0$, (vi) $T_{\rm m}\le\liminf_{(x,t)\to(0,t_j)}(T(x,t))\le\limsup_{(x,t)\to(0,t_j)}(T(x,t))<+\infty$ for $j\in\mathbb{Z}_+$, (vii) (1) holds for $t\in I$, $x\in(0,s(t))$, (viii) (2), (5) hold for $t\in I$, (ix) (4) holds for $x\in(0,s_0]$, where

$$q_c(t)=q_j,\quad \forall t\in[t_j,t_{j+1}),\quad \forall j\in\mathbb{Z}_+ \tag{74}$$

(III) Existence of solution in the sense of Definition 2 (i.e., $s(t)\in(0,L)$)

As remarked in the paragraph below Lemma 3, the pair of functions above satisfies $T_x(s(t),t)\le 0$ for all $t\ge 0$. Hence, by (5) with $q_f(t)\equiv 0$, one can deduce (61), and $s_0<s(t)$ for all $t\ge 0$. We show $s(t)<s_r$ for all $t\ge 0$. Integrating (68) from $t=t_j$ to $t\in[t_j,t_{j+1})$ leads to

$$\widetilde{E}(t)-\widetilde{E}_j=(t-t_j)q_j,\quad \forall t\in[t_j,t_{j+1}). \tag{75}$$

With the help of (70) and (71), Eq. (75) yields

$$\widetilde{E}(t)=\left(1-c(t-t_j)\right)\widetilde{E}_j,\quad \forall t\in[t_j,t_{j+1}). \tag{76}$$

By Assumption 10 and since $c<\frac{1}{R}$, we have $0<c<\frac{1}{\tau_j}$ for all $j\in\mathbb{Z}_+$. In addition, for all $t\in[t_j,t_{j+1})$ and for all $j\in\mathbb{Z}_+$, it holds $t-t_j\le\tau_j$. Hence, we have $1-c(t-t_j)>0$, for all $t\in[t_j,t_{j+1})$ and for all $j\in\mathbb{Z}_+$. Applying this to (76) and noting that $\widetilde{E}_j<0$, for all $j\in\mathbb{Z}_+$, deduced from (71) and Assumption 4, one can obtain

$$\widetilde{E}(t)<0,\quad \forall t\ge 0. \tag{77}$$

Substituting (77) into (67) and applying $u(x,t)>0$ for all $x\in(0,s(t))$ and $t\ge 0$, we have $X(t)<0$ for all $t\ge 0$, which leads to $s(t)<s_r$ for all $t\ge 0$. Thus, (62) holds, which guarantees $s(t)\in(0,L)$ for the solution in the sense of Definition 2.

4.3. Stability analysis

To conclude Theorem 11, we follow similar procedure to Section 3.3, and prove the closed-loop stability.

4.3.1. State transformation

We apply the same backstepping transformation as in Section 3.3, namely, the direct transformation (28) with the gain kernel function (33), and the inverse transformation (35) with the kernel function (36). Here, we do not have an unknown heat loss, but instead we have a perturbation by error of continuous-time design and the sampled-data design. The perturbation is given in the spatial derivative of the transformation at $x=0$, which is

$$w_x(0,t)=-\frac{q_c(t)}{k}-\frac{\beta}{\alpha}\varepsilon u(0,t)-\frac{c}{\alpha}\int_0^{s(t)}u(y,t)\mathrm{d}y-\frac{c}{\beta}X(t). \tag{78}$$

Substituting the design of the sampled-data control $q_c(t)=q_j=-c\widetilde{E}_j$ for all $t\in[t_j,t_{j+1})$ and for all $j\in\mathbb{Z}_+$, and applying (67) and (76), the boundary condition (78) is given by

$$w_x(0,t)=\delta(t)-\frac{\beta}{\alpha}\varepsilon u(0,t), \tag{79}$$

$$\delta(t):=\frac{c^2}{k}\widetilde{E}_j\cdot(t-t_j),\quad \forall t\in[t_j,t_{j+1}),\quad j\in\mathbb{Z}_+. \tag{80}$$

Finally, applying the inverse transformation (35) to (79), and obtaining other equations similarly to Section 3.3, one can derive the following closed-form of the target system:

$$w_t(x,t)=\alpha w_{xx}(x,t)+\dot{s}(t)\phi'(x-s(t))X(t), \tag{81}$$

$$w_x(0,t)=\delta(t)-\frac{\beta}{\alpha}\varepsilon\left[w(0,t)-\frac{\beta}{\alpha}\int_0^{s(t)}\psi(-y)w(y,t)\mathrm{d}y\right.$$
$$\left.-\psi(-s(t))X(t)\right], \tag{82}$$

$$w(s(t),t)=\varepsilon X(t), \tag{83}$$

$$\dot{X}(t)=-cX(t)-\beta w_x(s(t),t). \tag{84}$$

4.3.2. Lyapunov method

First we show the stability of the target system (81)–(84). For a given $t\ge 0$, we define

$$n:=\{n\in\mathbb{Z}_+|t_n\le t<t_{n+1}\}, \tag{85}$$

and we firstly apply Lyapunov method for the time interval $t\in[t_j,t_{j+1})$ for all $j=0,1,\ldots,n-1$, and next for the interval from t_n to t. For both cases, we consider $V=\frac{1}{2\alpha}\|w\|^2+\frac{\varepsilon}{2\beta}X(t)^2$. As proven in Appendix, and applying (61), for a sufficiently small $\varepsilon>0$, the following inequality is derived:

$$\dot{V}\le -bV+2s_r\delta(t)^2+a\dot{s}(t)V, \tag{86}$$

where $b=\frac{1}{8}\min\left\{\frac{\alpha}{s_r^2},c\right\}$, $a=\frac{2\beta\varepsilon}{\alpha}\max\left\{1,\frac{\alpha c^2 s_r}{2\beta^3\varepsilon^3}\right\}$. Consider

$$W=Ve^{-as(t)}. \tag{87}$$

Taking the time derivative of (87) and applying (86) yields

$$\dot{W}\le -bW+2s_r\delta(t)^2. \tag{88}$$

(i) For $t\in[t_j,t_{j+1})$, for all $j=0,1,\ldots,n-1$,

Applying comparison principle to (88) for $t\in[t_j,t_{j+1})$ yields

$$W(t)\le W(t_j)e^{-b(t-t_j)}+2s_re^{-bt}\int_{t_j}^{t}e^{b\tau}\delta(\tau)^2\mathrm{d}\tau. \tag{89}$$

Setting $t=t_{j+1}$ and substituting (80), we get

$$W_{j+1}\le W_je^{-b\tau_j}+\frac{2c^4s_r}{k^2}e^{-b\tau_j}\widetilde{E}_j^2I_j, \tag{90}$$

where $W_j=W(t_j)$, and I_j is defined by $I_j:=\int_{t_j}^{t_{j+1}}e^{b(\tau-t_j)}(\tau-t_j)^2\mathrm{d}\tau$. Then, by introducing the variable $s=b(\tau-t_j)$ and integration by substitution, with the help of $b\tau_j<\frac{1}{8}c\tau_j<\frac{1}{8}$ for all $j\in\mathbb{Z}_+$ deduced from the definition of b, Assumption 10 and the fact that $c<\frac{1}{R}$, one can derive

$$I_j=\frac{1}{b^3}\int_0^{b\tau_j}e^ss^2\mathrm{d}s\le\frac{J}{b^3}, \tag{91}$$

where $J := \int_0^{\frac{1}{8}} e^s s^2 ds$. Applying (91) to (90) yields

$$W_{j+1} \le W_j e^{-b\tau_j} + B_j, \tag{92}$$

$$B_j := \frac{2Jc^4 s_r}{k^2 b^3} e^{-b\tau_j} \widetilde{E}_j^2. \tag{93}$$

Applying (92) from $j = n-1$ to $j = 0$ inductively, we get

$$W_n \le W_0 e^{-b\sum_{i=0}^{n-1} \tau_i} + B_{n-1} + \sum_{i=0}^{n-2} B_i e^{-b\sum_{j=i+1}^{n-1} \tau_j}. \tag{94}$$

By (93) and the solution of $\widetilde{E}_j$ given in (71), we have

$$\sum_{i=0}^{n-2} B_i e^{-b\sum_{j=i+1}^{n-1}\tau_j} \le \frac{2Jc^4 s_r \widetilde{E}_0^2 e^{-b\sum_{j=0}^{n-1}\tau_j}}{k^2 b^3} \times \left(1 + \sum_{i=1}^{n-2}\left(\prod_{k=0}^{i-1}(1-c\tau_k)^2 e^{b\tau_k}\right)\right). \tag{95}$$

Since $b = \frac{1}{8}\min\left\{\frac{\alpha}{s_r^2}, c\right\} < \frac{c}{8}$, by using $r = \inf_{j\in\mathbb{Z}_+}\{\tau_j\} > 0$ given in Assumption 10, the following inequality holds

$$(1-c\tau_i)^2 e^{b\tau_i} \le (1-cr)^2 e^{\frac{cr}{8}} := \delta < 1, \quad \forall j \in \mathbb{Z}_+. \tag{96}$$

Thus, the inequality (95) leads to

$$\sum_{i=0}^{n-2} B_i e^{-b\sum_{j=i+1}^{n-1}\tau_j} \le \frac{2Jc^4 s_r \widetilde{E}_0^2}{k^2 b^3 (1-\delta)} e^{-b\sum_{j=0}^{n-1}\tau_j}. \tag{97}$$

In the similar way, we get

$$B_{n-1} \le \frac{2Jc^4 s_r \widetilde{E}_0^2}{k^2 b^3(1-\delta)} e^{-b\sum_{j=0}^{n-1}\tau_j}. \tag{98}$$

Recalling that $\tau_j = t_{j+1} - t_j$ and $t_0 = 0$, we get $\sum_{j=0}^{n-1}\tau_j = t_n$. Applying (97) and (98) to (94), we arrive at

$$W_n \le (W_0 + A\widetilde{E}_0^2)e^{-bt_n}. \tag{99}$$

where $A = \frac{2Jc^4 s_r}{k^2 b^3(1-\delta)}$.

(ii) For $t \in [t_n, t_{n+1})$,

Applying comparison principle to (88) from t_n to $t \in [t_n, t_{n+1})$, we get

$$W(t) \le W_n e^{-b(t-t_n)} + A\widetilde{E}_0^2 e^{-bt}. \tag{100}$$

Finally, combining (99) and (100), we obtain

$$W(t) \le (W_0 + 2A\widetilde{E}_0^2)e^{-bt}. \tag{101}$$

Recalling the relation $W = Ve^{-as(t)}$ defined in (87), and applying $0 < s(t) < s_r$, the norm estimate for W in (101) leads to the following estimate for V:

$$V(t) \le e^{as_r}(V_0 + 2A\widetilde{E}_0^2)e^{-bt}. \tag{102}$$

As presented in Section 3, due to the invertibility of the transformations, there exist positive constants $\underline{M} > 0$ and $\overline{M} > 0$ such that (53) holds. Moreover, due to the definition of the reference energy $\widetilde{E}(t) = \frac{k}{\alpha}\int_0^{s(t)} u(x,t)dx + \frac{k}{\beta}X(t)$ given in (67), using Young's and Cauchy–Schwarz inequalities one can show $\widetilde{E}_0^2 \le K\Psi_0$, where $K = 2k^2\max\{\frac{s_r}{\alpha^2}, \frac{1}{\beta^2}\}$. Applying them to (102), we deduce that there exists a positive constant $M > 0$ such that $\Psi(t) \le M\Psi_0 e^{-bt}$ holds, which completes the proof of Theorem 11.

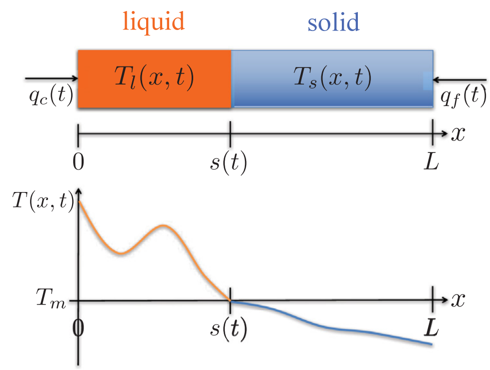


Fig. 2. Schematic of the two-phase Stefan problem.

Part II: Two-Phase Stefan System

5. Two-phase Stefan problem with heat loss

In this section, we extend the ISS result to the "two-phase" Stefan problem, where the heat loss at the interface is precisely modeled by the temperature dynamics in the solid phase, following the work in Koga and Krstic (2020b). An unknown heat loss is then accounted not at the interface, but at the boundary of the solid phase. This configuration is depicted in Fig. 2.

5.1. Problem statement

The governing equations are described by the following coupled PDE–ODE–PDE system:

$$\frac{\partial T_l}{\partial t}(x,t) = \alpha_l \frac{\partial^2 T_l}{\partial x^2}(x,t), \quad \text{for} \quad t > 0, \quad 0 < x < s(t), \tag{103}$$

$$\frac{\partial T_l}{\partial x}(0,t) = -q_c(t)/k_l, \quad T_l(s(t),t) = T_m, \ \text{for } t > 0, \tag{104}$$

$$\frac{\partial T_s}{\partial t}(x,t) = \alpha_s \frac{\partial^2 T_s}{\partial x^2}(x,t), \quad \text{for} \quad t > 0, \quad s(t) < x < L, \tag{105}$$

$$\frac{\partial T_s}{\partial x}(L,t) = -q_f(t)/k_s, \quad T_s(s(t),t) = T_m, \ \text{for } t > 0, \tag{106}$$

$$\gamma\dot{s}(t) = -k_l\frac{\partial T_l}{\partial x}(s(t),t) + k_s\frac{\partial T_s}{\partial x}(s(t),t), \tag{107}$$

where $\gamma = \rho_l \Delta H^*$, and all the variables denote the same physical value with the subscript "l" for the liquid phase and "s" for the solid phase, respectively. The solid phase temperature must be lower than the melting temperature, which serves as one of the conditions for the model validity, as stated in the following.

Remark 3. To keep the physical state of each phase meaningful, the following conditions must be maintained:

$$T_l(x,t) \ge T_m, \quad \forall x \in (0, s(t)), \quad \forall t > 0, \tag{108}$$

$$T_s(x,t) \le T_m, \quad \forall x \in (s(t), L), \quad \forall t > 0, \tag{109}$$

$$0 < s(t) < L, \quad \forall t > 0. \tag{110}$$

The definition of the solution to the two-phase Stefan problem is given below.

Definition 3. Let $s_0 \in (0, L)$ be a given constant, $q_c, q_f : \mathbb{R}_+ \to \mathbb{R}_+$, $T_{l,0} : [0, s_0] \to [T_m, \infty)$ with $T_{l,0}(s_0) = T_m$, $T_{s,0} : [s_0, L] \to (-\infty, T_m]$ with $T_{s,0}(s_0) = T_m$ be given functions. The initial–boundary value problem (103)–(107) has a solution if there exists a pair of functions $s \in C^0(\mathbb{R}_+; (0,L)) \cap C^1((0,+\infty); (0,L))$, $T_l : D_l \to [T_m, \infty)$ with $D_l = \bigcup_{t\ge 0}(t \times [0, s(t)])$, $T_s : D_s \to$

$(-\infty, T_m]$ with $D_s = \bigcup_{t\geq 0}(t \times [s(t), L])$ such that (i) T_l, $\frac{\partial T_l}{\partial x}$ are continuous for $t > 0, x \in [0, s(t)]$, and T_s, $\frac{\partial T_s}{\partial x}$ are continuous for $t > 0, x \in [s(t), L]$, (ii) $\frac{\partial T_l}{\partial t}$, $\frac{\partial^2 T_l}{\partial x^2}$ are continuous for $t > 0, x \in (0, s(t))$, and $\frac{\partial T_s}{\partial t}$, $\frac{\partial^2 T_s}{\partial x^2}$ are continuous for $t > 0, x \in (s(t), L)$ (iii) T_l is continuous for $t = 0, x \in (0, s_0]$ with $T_m \leq \liminf_{(x,t)\to(0,0)}(T_l(x,t)) \leq \limsup_{(x,t)\to(0,0)}(T_l(x,t)) < +\infty$, and T_s is continuous for $t = 0, x \in [s_0, L)$ with $-\infty \leq \liminf_{(x,t)\to(L,0)}(T_s(x,t)) \leq \limsup_{(x,t)\to(L,0)}(T_s(x,t)) < T_m$ (iv) (103), (104), (105), (106), (107) hold, (v) $s(0) = s_0$, $T_l(x,0) = T_{l,0}(x)$, for $x \in (0, s_0]$, and $T_s(x,0) = T_{s,0}(x)$, for $x \in [s_0, L)$.

The following assumption on the initial data $(T_{l,0}(x), T_{s,0}(x), s_0) := (T_l(x,0), T_s(x,0), s(0))$ is imposed.

Assumption 13. $0 < s_0 < L$, $T_{l,0}(x) \in C^0([0, s_0]; [T_m, +\infty))$, $T_{s,0}(x) \in C^0([s_0, L]; (-\infty, T_m])$, and $T_{l,0}(s_0) = T_{s,0}(s_0) = T_m$. Also, there exists constants $\bar{T}_l, \bar{T}_s, \eta_l, \eta_s > 0$ such that $0 \leq T_{l,0}(x) - T_m \leq \bar{T}_l\left(1 - \exp L\eta_l\alpha_l^{-1}(x - s_0)\right)$ for $x \in [0, s_0]$ and $-\bar{T}_s\left(1 - \exp L\eta_s\alpha_s^{-1}(x - s_0)\right) \leq T_{s,0}(x) - T_m \leq 0$ for $x \in [s_0, L]$.

The following lemma is provided to ensure the conditions of the model validity.

Lemma 14. *Let Assumption 13 hold, $q_c(t)$ and $q_f(t)$ be bounded nonnegative continuous functions $q_c \in C^0(\mathbb{R}_+; [0, \bar{q}_c))$, $q_f \in C^0(\mathbb{R}_+; [0, \bar{q}_f))$ for some $\bar{q}_c, \bar{q}_f > 0$, and*

$$\max\left\{\frac{k_l\varepsilon_l}{\alpha_l}\left(1 + \frac{\alpha_l}{L^2\eta_l}\right), \frac{k_s\varepsilon_s}{\alpha_s}\left(1 + \frac{\alpha_s}{L^2\eta_s}\right)\right\} < \frac{\gamma}{4}, \tag{111}$$

hold, where $\varepsilon_l := \max\left\{\bar{T}_l, \bar{q}_c L k_l^{-1}\right\}$, $\varepsilon_s := \max\left\{\bar{T}_s, \bar{q}_f L k_s^{-1}\right\}$. Furthermore, suppose it holds

$$0 < \gamma s_\infty + \int_0^t (q_c(s) - q_f(s))ds < \gamma L, \tag{112}$$

for all $t \geq 0$, where $s_\infty := s_0 + \frac{k_l}{\alpha_l\gamma}\int_0^{s_0}(T_{l,0}(x) - T_m)dx + \frac{k_s}{\alpha_s\gamma}\int_{s_0}^L (T_{s,0}(x) - T_m)dx$. Then the initial–boundary value problem (103)–(107) has a unique solution in the sense of Definition 3.

Lemma 14 is proven in Cannon and Primicerio (1971) (Theorem 1 in p.4 and Theorem 4 in p.8) by employing the maximum principle. The variable s_∞ is the final interface position $s_\infty = \lim_{t\to\infty} s(t)$ under the zero heat input $q_c(t) \equiv q_f(t) \equiv 0$ for all $t \geq 0$. For (112) to hold for all $t \geq 0$, we at least require it to hold at $t = 0$, which leads to the following assumption.

Assumption 15. The variable s_∞ given by initial values satisfies $0 < s_\infty < L$.

Furthermore, we impose Assumption 6, and the restriction for the setpoint is given as follows.

Assumption 16. The setpoint s_r satisfies $s_\infty < s_r < L$.

Physically, Assumption 15 states that neither phase disappears under $q_c(t) \equiv q_f(t) \equiv 0$, and Assumption 16 states that the choice of the setpoint for the melting is far beyond s_∞ from the heat input. A graphical illustration of the assumptions can be found in Koga and Krstic (2020b).

5.2. ISS for two-phase Stefan problem

We apply the boundary feedback control developed in (Koga & Krstic, 2020b)

$$q_c(t) = -c\left(\frac{k_l}{\alpha_l}\int_0^{s(t)}(T_l(x,t) - T_m)dx + \frac{k_s}{\alpha_s}\int_{s(t)}^L (T_s(x,t) - T_m)dx + \gamma(s(t) - s_r)\right), \tag{113}$$

to the two-phase Stefan problem with the unknown heat loss $q_f(t)$ governed by (103)–(107). To satisfy the inequality (111), we impose the following assumption.

Assumption 17. $q_f(t) \in C^0(\mathbb{R}_+; [0, \bar{q}_f))$ for some $\bar{q}_f > 0$, and the inequality (111) holds with $\bar{q}_c = q_0 + \bar{q}_f$, where

$$q_0 = -c\left(\frac{k_l}{\alpha_l}\int_0^{s_0}(T_{l,0}(x) - T_m)dx + \frac{k_s}{\alpha_s}\int_{s_0}^L (T_{s,0}(x) - T_m)dx + \gamma(s_0 - s_r)\right).$$

As in the previous procedure, the equivalence of the closed-loop system under the control law (113) with the system under an open-loop input is presented in the following lemma.

Lemma 18. *Under Assumptions 6, and 13–17, the closed-loop system consisting of (103)–(107) with the control law (113) has a unique solution in the sense of Definition 3 satisfying (108)–(110) for all $t \geq 0$, which is equivalent to the open-loop solution of (103)–(107) with $q_c(t) = q_0 e^{-ct} + c\int_0^t e^{-c(t-\tau)}q_f(\tau)d\tau$, for all $t \geq 0$.*

The proof of Lemma 18 is established by following the same procedure as Lemmas 7, 9 with the help of Lemma 14. We omit the proof in this paper. Note that the open-loop control is bounded by $q_c(t) \leq q_0 e^{-ct} + \bar{q}_f c\int_0^t e^{-c(t-\tau)}d\tau = q_0 e^{-ct} + \bar{q}_f(1 - e^{-ct}) \leq q_0 + \bar{q}_f$. We present ISS result for the closed-loop system in the following theorem.

Theorem 19. *Under Assumptions 6 and 13–17, the closed-loop system consisting of (103)–(107) with the control law (113) has a unique solution in the sense of Definition 3, and is ISS with respect to the heat loss $q_f(t)$ at the boundary of the solid phase. Moreover, there exist positive constants $M_1 > 0$ and $M_2 > 0$ such that the following estimate holds:*

$$\Psi(t) \leq M_1\Psi(0)e^{-\lambda t} + M_2 \sup_{\tau\in[0,t]} q_f(\tau), \tag{114}$$

for all $t \geq 0$, where $\lambda = \frac{1}{32}\min\left\{\frac{\alpha_l}{L^2}, \frac{2\alpha_s}{L^2}, c\right\}$, in the L_2 norm $\Psi(t) = \left(\int_0^{s(t)}(T_l[t] - T_m)^2dx + \int_{s(t)}^L (T_s[t] - T_m)^2dx + (s(t) - s_r)^2\right)^{\frac{1}{2}}$.

Proof. The conditions (108)–(110) are ensured by Lemma 18. To prove ISS, by following the procedure in Koga and Krstic (2020b), first we introduce the reference error states as $u(x,t) := T_l(x,t) - T_m$, $v(x,t) := T_s(x,t) - T_m$, and

$$X(t) := s(t) - s_r + \frac{\beta_s}{\alpha_s}\int_{s(t)}^L (T_s(x,t) - T_m)dx. \tag{115}$$

Then, the total PDE–ODE–PDE system given in (103)–(107) is reduced to the following PDE–ODE system

$$u_t(x,t) = \alpha_l u_{xx}(x,t), \quad 0 < x < s(t), \tag{116}$$

$$u_x(0,t) = -q_c(t)/k_l, \quad u(s(t),t) = 0, \tag{117}$$

$$\dot{X}(t) = -\beta_l u_x(s(t),t) - d(t), \tag{118}$$

where $d(t) = \frac{\beta_s}{k_s}q_f(t)$. Note that the formulation of the above system is equivalent to (24)–(27), and hence essentially we follow similar procedure to Section 3.3. Therefore, applying the same transformation $(u, X) \Rightarrow (w, X)$ by (28) and (33), and the inverse transformation $(w, X) \Rightarrow (u, X)$ by (35) and (36), the target (w, X)-system is given by (29), (31), (32), and (37). Then, considering the Lyapunov function $V(t) = \frac{1}{2\alpha_l}\|w\|^2 + \frac{\varepsilon}{2\beta_l}X(t)^2$, as proven in Appendix, for a sufficiently small $\varepsilon > 0$, the inequality (43) is derived.

However, the dynamics of the interface is distinct in the two-phase Stefan problem from the one-phase Stefan problem, and hence the definition of $z(t)$ given by (44) in Section 3.3 should be modified. We introduce

$$z(t) := s(t) + \frac{2\beta_s}{\alpha_s} \int_{s(t)}^{L} v(x,t)dx + 2\int_0^t d(\tau)d\tau. \quad (119)$$

The time derivative of (119) is given by

$$\dot{z}(t) = -\beta_l u_x(s(t),t) - \beta_s v_x(s(t),t) \geq 0, \quad (120)$$

where the positivity follows from $u_x(s(t),t) < 0$ and $v_x(s(t),t) > 0$ derived from Hopf's lemma. Since $\dot{s}(t) = -\beta_l u_x(s(t),t) + \beta_s v_x(s(t),t)$, it holds that $|\dot{s}(t)| \leq -\beta_l u_x(s(t),t) - \beta_s v_x(s(t),t) = \dot{z}(t)$. Applying this inequality to (43) leads to

$$\dot{V}(t) \leq -bV(t) + \Gamma d(t)^2 + a\dot{z}(t)V(t). \quad (121)$$

By (11), (110), Assumption 13, and (120), we have $0 < z(t) - z(0) < \bar{z} := L + \frac{2\beta M}{k} - \frac{2\beta_s}{\alpha_s}\int_{s_0}^{L} v_0(x)dx$. By this and (121), applying the same procedure for the derivation from (48) to (52) in Section 3, we derive that there exist positive constants $N_1 > 0$ and $N_2 > 0$ such that the following norm estimate holds:

$$\Phi(t) \leq N_1\Phi(0)e^{-\frac{b}{4}t} + N_2 \sup_{\tau\in[0,t]} q_f(\tau), \quad (122)$$

where $\Phi(t) = \left(\int_0^{s(t)} u(x,t)^2dx + X(t)^2\right)^{\frac{1}{2}}$, $b = \frac{1}{8}\min\left\{\frac{\alpha_l}{L^2}, c\right\}$. Let us define the following threefunctionals $V_1(t) = \int_0^{s(t)}(T_l(x,t) - T_m)^2dx$, $V_2(t) = \int_{s(t)}^{L}(T_s(x,t) - T_m)^2dx$, and $V_3(t) = (s(t) - s_r)^2$. Taking the time derivative of V_2 along with the solid phase dynamics (105) and (106), and applying Young's, Cauchy–Schwarz, Poincare's and Agmon's inequalities, we get

$$\dot{V}_2(t) \leq -\frac{\alpha_s}{4L^2}V_2(t) + \frac{4L\alpha_s}{k_s^2}q_f(t)^2. \quad (123)$$

Applying the same procedure as the derivation from (50) to (51), and taking the square root, one can derive

$$\sqrt{V_2(t)} \leq \sqrt{V_2(0)}e^{-\frac{\alpha_s}{16L^2}t} + 4L\sqrt{2L}k_s^{-1}\sup_{\tau\in[0,t]} q_f(\tau). \quad (124)$$

Taking the square of (115), and applying Young's and Cauchy–Schwarz inequalities with $0 < s(t) < L$ leads to

$$X(t)^2 \leq 2V_3(t) + 2L\beta_s^2\alpha_s^{-2}V_2(t). \quad (125)$$

Applying the same manner to $s(t) - s_r = X(t) - \frac{\beta_s}{\alpha_s}\int_{s(t)}^{L}(T_s(x,t) - T_m)dx$ obtained by (115), one can also derive

$$V_3(t) \leq 2X(t)^2 + 2L\beta_s^2\alpha_s^{-2}V_2(t). \quad (126)$$

Combining (122), (124), (125), and (126) using the definitions of V_1, V_2, V_3, and noting that $\Phi(t)^2 = V_1(t) + X(t)^2$, $\Psi(t)^2 = V_1(t) + V_2(t) + V_3(t)$, it holds that $\Psi(t)^2 \leq V_1 + \left(1 + \frac{2L\beta_s^2}{\alpha_s^2}\right)V_2 + 2X(t)^2 \leq 2\Phi(t)^2 + \left(1 + \frac{2L\beta_s^2}{\alpha_s^2}\right)V_2$, which yields $\Psi(t) \leq \sqrt{2}\left(N_1\Phi(0)e^{-\frac{b}{4}t} + N_2\sup_{\tau\in[0,t]} q_f(\tau)\right) + \sqrt{\left(1 + \frac{2L\beta_s^2}{\alpha_s^2}\right)}\left(\sqrt{V_2(0)}e^{-\frac{\alpha_s}{16L^2}t} + \frac{4L\sqrt{2L}}{k_s}\sup_{\tau\in[0,t]} q_f(\tau)\right)$. Thus, the inequality (114) is derived for some $M_1 > 0$ and $M_2 > 0$, which completes the proof of Theorem 19.

Remark 4. The extension of the sampled-data result in Section 4 to the two-phase Stefan problem is not trivial. The challenge lies in the proof of the well-posedness of the solution which requires to guarantee the inequality (111) to hold at every sampling time $t = t_j$ for all $j \in \mathbb{Z}_+$ under the closed-loop system (103)–(107) with the control law applying ZOH to (113).

Table 1
Physical properties of paraffin (liquid).

Description	Symbol	Value
Density	ρ	790 kg m^{-3}
Latent heat of fusion	ΔH^*	210 J g^{-1}
Heat capacity	C_p	2.38 J g^{-1} °C^{-1}
Melting temperature	T_m	37.0 °C
Thermal conductivity	k	0.220 W m^{-1}

6. Numerical simulation

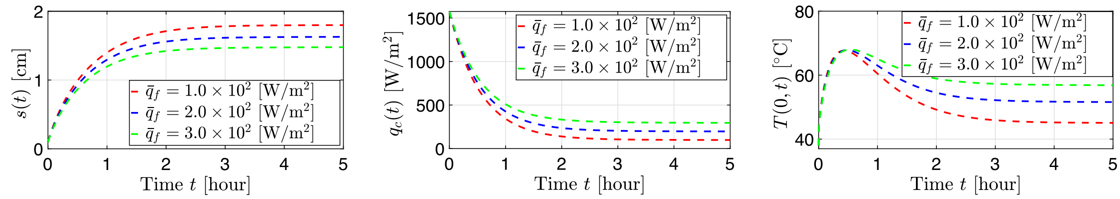
Simulation results are performed for the one-phase Stefan problem by considering a cylinder of paraffin whose physical properties are given in Table 1. Here, we use the well known boundary immobilization method combined with finite difference semi-discretization (Kutluay, Bahadir, & Özdes, 1997). The setpoint and the initial values are chosen as $s_r = 2$ [cm], $s_0 = 0.1$ [cm], and $T_0(x) - T_m = \bar{T}_0(1 - x/s_0)$ with $\bar{T}_0 = 1$ [°C]. Then, the setpoint restriction (10) is satisfied.

First, we incorporate the heat loss at the interface and apply the continuous-time control law (9). The control gain is set as $c = 5.0\times 10^{-3}$ [/s], and the heat loss at the interface is set as $q_f(t) = \bar{q}_f e^{-Kt}$, where $K = 5.0 \times 10^{-6}$ [/s]. The closed-loop responses for $\bar{q}_f = 1.0 \times 10^2$ [W/m^2] (red), 2.0×10^2 [W/m^2] (blue), and 3.0×10^3 [W/m^2] (green) are implemented as depicted in Fig. 3. Fig. 3(a) shows the dynamics of the interface, which illustrates the convergence to the setpoint with an error due to the unknown heat loss at the interface. This error becomes larger as $\bar{q}_f$ gets larger, which is consistent with the ISS result. In addition, the property $0 < s(t) < s_r$ is observed, which is consistent with Lemma 9. Fig. 3(b) shows the dynamics of the proposed closed-loop control law, and Fig. 3(c) shows the dynamics of the boundary temperature $T(0,t)$. These figures illustrate the other conditions proved in Lemma 9. Hence, we can observe that the simulation results are consistent with the theoretical result we prove for the model validity conditions and ISS.

Second, we neglect the heat loss (i.e., $q_f(t) = 0$) and apply the sampled-data control law (55). We consider periodic sampling with period given by $\tau_j = R = 10$ [min], for all $j \in \mathbb{Z}$. The control gain is set as $c = 5.0\times 10^{-3}$/s, by which the requirement $R < \frac{1}{c}$ is satisfied. The time responses of the interface position, the control input, and the boundary temperature under the closed-loop system are depicted in Fig. 4(a)–(c), respectively. Fig. 4(a) illustrates that the interface position $s(t)$ converges to the setpoint s_r monotonically and smoothly without overshooting, i.e., $\dot{s}(t) > 0$ and $s_0 < s(t) < s_r$ hold for all $t \geq 0$. Fig. 4(b) shows that the proposed sampled-data control law maintains constant positive value for every sampling period and is monotonically decreasing to zero. Fig. 4(c) illustrates that the boundary temperature $T(0,t)$ keeps greater than the melting temperature T_m with accompanying "spikes" at every sampling time $t = \tau_j$ up to 2 h. Such spikes are caused by the large drop of the control input $q_c(t)$ at sampling time observed from Fig. 4(b), which affects the boundary temperature directly as given in the boundary condition (2). Therefore, the numerical results are consistent with the theoretical results we have established in Lemmas 12 for the required properties and in Theorem 11 for the closed-loop stability.

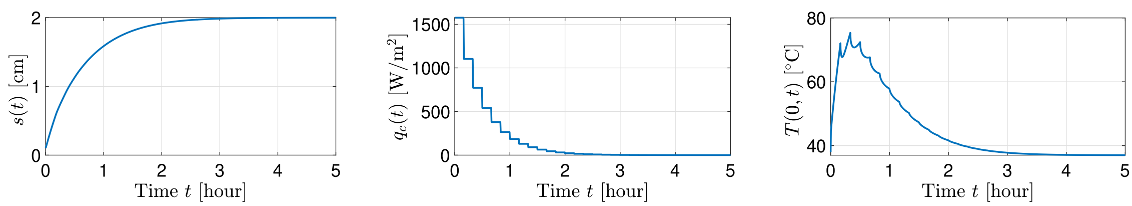
7. Conclusions

In this paper we develop two new results for the control of the Stefan system, that are, ISS and the sampled-data control. First we revisit the one-phase Stefan problem with heat loss at the interface studied in Koga et al. (2018), and show that the closed-loop



(a) Convergence of the interface is observed with offsets from the setpoint depending on the magnitude of the heat loss.
(b) Positivity of the closed-loop controller maintains.
(c) The model validity of the boundary liquid temperature holds, i.e., $T(0,t) > T_m$.

Fig. 3. The responses of the system (1)–(5) with the heat loss $q_f(t) = \bar{q}_f e^{-Kt}$ under the continuous-time feedback control (9). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(a) Convergence of the interface to the setpoint is observed without the overshoot.
(b) The sampled-data controller maintains positive value.
(c) The boundary temperature accompanies "spikes" at every sampling time.

Fig. 4. The responses of the system (1)–(5) without the heat loss (i.e., $q_f(t) \equiv 0$) under the ZOH-based sampled-data control (55).

system under the control law in Koga et al. (2019) is well-posed and satisfies an estimate of the L_2 norm in the sense of ISS with respect to the heat loss. Second, we consider the sampled-data control of the one-phase Stefan problem by applying ZOH to the continuous-time control law developed in Koga et al. (2019), and show the global exponential stability of the closed-loop system without the heat loss by analyzing the growth of the system's energy. Finally, the similar procedure is performed to extend the ISS result to the two-phase Stefan problem by utilizing the control design in Koga and Krstic (2020b).

For the practical implementation it is significant to design an observer-based output feedback control with only the boundary temperature measured at each sampling time, which will be considered as one future work. Another interesting direction is "quantized control" which has a finite or regularly distributed discrete sets of the input value in addition to the sampling time as a *digital* nature (Hayakawa, Ishii, & Tsumura, 2009; Selivanov & Fridman, 2016).

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Appendix. Stability analysis of target system

We prove the following lemma.

Lemma 20. *Consider the target system governed by*

$$w_t(x,t) = \alpha w_{xx}(x,t) + \frac{c}{\beta}\dot{s}(t)X(t) + \phi(x - s(t))d(t), \tag{A.1}$$

$$w_x(0,t) = \delta(t) - \frac{\beta}{\alpha}\varepsilon\left[w(0,t) - \frac{\beta}{\alpha}\int_0^{s(t)} \psi(-y)w(y,t)\mathrm{d}y \right.$$
$$\left. - \psi(-s(t))X(t)\right], \tag{A.2}$$

$$w(s(t),t) = \varepsilon X(t), \tag{A.3}$$

$$\dot{X}(t) = -cX(t) - \beta w_x(s(t),t) - d(t), \tag{A.4}$$

where $\phi(x)$ is defined by (33), and $\psi(x)$ is defined by (36). Let $V(t)$ be a Lyapunov function defined by

$$V(t) = \frac{1}{2\alpha}\|w\|^2 + \frac{\varepsilon}{2\beta}X(t)^2. \tag{A.5}$$

Then, there exists a positive constant $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ the following inequality holds:

$$\dot{V}(t) \le -bV(t) + \Gamma d(t)^2 + 2s_r\delta(t)^2 + a|\dot{s}(t)|V(t), \tag{A.6}$$

where $a = \frac{2\beta\varepsilon}{\alpha}\max\left\{1, \frac{\alpha c^2 s_r}{2\beta^3\varepsilon^3}\right\}$, $b = \frac{1}{8}\min\left\{\frac{\alpha}{s_r^2}, c\right\}$, and $\Gamma = \frac{\varepsilon}{\beta c} + \frac{2s_r^3}{\alpha^2}\left(\frac{cs_r}{\beta} + \varepsilon\right)^2$.

Proof. Note that Poincare's and Agmon's inequalities for the system (A.1)–(A.3) with $0 < s(t) < s_r$ lead to

$$\|w\|^2 \le 2s_r\varepsilon^2 X(t)^2 + 4s_r^2\|w_x\|^2, \tag{A.7}$$

$$w(0,t)^2 \le 2\varepsilon^2 X(t)^2 + 4s_r\|w_x\|^2. \tag{A.8}$$

Taking the time derivative of (A.5) along with the solution of (A.1)–(A.4), we have

$$\dot{V}(t) = -\|w_x\|^2 - \frac{\varepsilon}{\beta}cX(t)^2 + \frac{\beta}{\alpha}\varepsilon w(0,t)^2 - w(0,t)\delta(t)$$
$$- \frac{\beta}{\alpha}\varepsilon w(0,t)\left[\frac{\beta}{\alpha}\int_0^{s(t)} \psi(-y)w(y,t)\mathrm{d}y\right.$$
$$\left. + \psi(-s(t))X(t)\right]$$
$$- \frac{\varepsilon}{\beta}X(t)d(t) + \frac{1}{\alpha}\int_0^{s(t)} \phi(x - s(t))w(x,t)\mathrm{d}x d(t)$$
$$+ \frac{\dot{s}(t)}{\alpha}\left(\frac{\varepsilon^2}{2}X(t)^2 + \frac{c}{\beta}\int_0^{s(t)} w(x,t)\mathrm{d}x X(t)\right). \tag{A.9}$$

Applying Young's inequality to the last term in the first line, the second, third, and fourth lines of (A.9), we obtain

$$-w(0,t)f(t) \leq \frac{1}{8s_r}w(0,t)^2 + 2s_r\delta(t)^2, \tag{A.10}$$

$$\begin{aligned} &- w(0,t)\left[\frac{\beta}{\alpha}\int_0^{s(t)}\psi(-y)w(y,t)dy + \psi(-s(t))X(t)\right], \\ \leq &\frac{\gamma_1}{2}w(0,t)^2 + \frac{\beta^2}{\alpha^2\gamma_1}\left(\int_0^{s(t)}\psi(-y)w(y,t)\mathrm{d}y\right)^2 \\ &+ \frac{1}{\gamma_1}\left(\psi(-s(t))X(t)\right)^2, \end{aligned} \tag{A.11}$$

$$\begin{aligned} &- \frac{\varepsilon}{\beta}X(t)d(t) + \frac{1}{\alpha}\int_0^{s(t)}\phi(x-s(t))w(x,t)\mathrm{d}x d(t) \\ \leq &\frac{1}{2\gamma_2}\left(\frac{\varepsilon}{\beta}X(t)\right)^2 + \frac{(\gamma_2+\gamma_3)}{2}d(t)^2 \\ &+ \frac{1}{2\alpha^2\gamma_3}\left(\int_0^{s(t)}\phi(x-s(t))w(x,t)\mathrm{d}x\right)^2, \end{aligned} \tag{A.12}$$

where $\gamma_i > 0$ for $i = \{1,2,3\}$. Applying (A.10)–(A.12) and Cauchy–Schwarz, Poincare, and Agmon's inequalities to (A.9) with choosing $\gamma_1 = 16e^{\frac{\varepsilon s_r}{\alpha}}$, $\gamma_2 = \frac{2\varepsilon}{\beta c}$, and $\gamma_3 = \frac{8(c^2s_r^2+\varepsilon^2)s_r^3}{\alpha^2\beta^2}$, and setting $\varepsilon < \frac{\ln(2)\alpha}{s_r}$, we have

$$\begin{aligned} \dot{V}(t) \leq &-\left(\frac{1}{4} - \frac{\beta s_r}{\alpha}\left(38 + \frac{cs_r^2}{\alpha}\right)\varepsilon\right)\|w_x\|^2 \\ &- \varepsilon\left(\frac{c}{4\beta} + \frac{\varepsilon}{8s_r} + g(\varepsilon)\right)X(t)^2 + \Gamma d(t)^2 + 2s_r\delta(t)^2 \\ &+ \frac{|\dot{s}(t)|}{2\alpha}\left(\varepsilon^2X(t)^2 + \frac{2c}{\beta}\left|\int_0^{s(t)}w(x,t)\mathrm{d}xX(t)\right|\right), \end{aligned} \tag{A.13}$$

where $\Gamma = \frac{(\gamma_2+\gamma_3)}{2}$, and $g(\varepsilon) = \frac{c}{4\beta} - \frac{\varepsilon}{4s_r} - \frac{\beta}{\alpha}\left(36 + \frac{cs_r^2}{2\alpha}\right)\varepsilon^2$. Since $g(0) > 0$ and $g'(\varepsilon) < 0$ for all $\varepsilon > 0$, there exists ε^* such that $g(\varepsilon) > 0$ for all $\varepsilon \in (0, \varepsilon^*)$ and $g(\varepsilon^*) = 0$. Thus, setting $\varepsilon < \min\left\{\varepsilon^*, \alpha\left(8\beta s_r\left(38 + \frac{cs_r^2}{\alpha}\right)\right)^{-1}\right\}$, the inequality (A.13) leads to

$$\begin{aligned} \dot{V}(t) \leq &-bV(t) + \Gamma d(t)^2 + 2s_r\delta(t)^2 \\ &+ \frac{|\dot{s}(t)|}{2\alpha}\left(\varepsilon^2X(t)^2 + \frac{2c}{\beta}\left|\int_0^{s(t)}w(x,t)\mathrm{d}xX(t)\right|\right), \end{aligned} \tag{A.14}$$

where $b = \frac{1}{2}\min\left\{\frac{\alpha}{8s_r^2}, c\right\}$. Applying (47) and Young's inequality to (A.14), the inequality (A.6) is derived.

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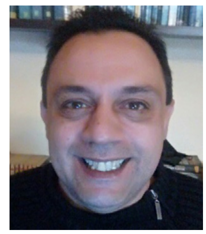
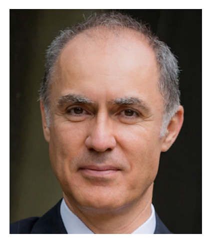
Shumon Koga received the B.S. degree in Applied Physics from Keio University (Japan) in 2014, and the M.S. and Ph.D. degrees in Mechanical and Aerospace Engineering from the University of California, San Diego (USA) in 2016 and 2020, respectively. He is currently a Postdoctoral Researcher at Existential Robotics Laboratory in Electrical and Computer Engineering at the University of California, San Diego.

He was an intern at NASA Jet Propulsion Laboratory (USA) and Mitsubishi Electric Research Laboratories (USA), during the fall of 2017 and the summer of 2018, respectively. He received the Robert E. Skelton Systems and Control Dissertation Award from UC San Diego Center for Control Systems and Dynamics in 2020, the O. Hugo Schuck Best Paper Award from American Automatic Control Council in 2019, and the Outstanding Graduate Student Award in Mechanical and Aerospace Engineering from UC San Diego in 2018, respectively.

His doctoral research interests include distributed parameter systems, optimization by extremum seeking, and their applications to additive manufacturing, battery management, thermal management in buildings, transportation systems, and global climate systems. He is currently focused on optimization and machine learning for robotics, in particular, Simultaneous Localization and Mapping (SLAM), path planning, and safety-critical systems.



Iasson Karafyllis is an Associate Professor in the Department of Mathematics, NTUA, Greece. He is a coauthor (with Z.-P. Jiang) of the book Stability and Stabilization of Nonlinear Systems, Springer-Verlag London, 2011 and a coauthor (with M. Krstic) of the books Predictor Feedback for Delay Systems: Implementations and Approximations, Birkhäuser, Boston 2017 and Input-to-State Stability for PDEs, Springer-Verlag London, 2019. Since 2013 he is an Associate Editor for the International Journal of Control and for the IMA Journal of Mathematical Control and Information. Since 2019 he is an Associate Editor for Systems & Control Letters and Mathematics of Control, Signals and Systems. His research interests include mathematical control theory and nonlinear systems theory.



Miroslav Krstic is Distinguished Professor of Mechanical and Aerospace Engineering, holds the Alspach endowed chair, and is the founding director of the Cymer Center for Control Systems and Dynamics at UC San Diego. He also serves as Senior Associate Vice Chancellor for Research at UCSD. As a graduate student, Krstic won the UC Santa Barbara best dissertation award and student best paper awards at CDC and ACC. Krstic has been elected Fellow of seven scientific societies — IEEE, IFAC, ASME, SIAM, AAAS, IET (UK), and AIAA (Assoc. Fellow) - and as a foreign member of the Serbian Academy of Sciences and Arts and of the Academy of Engineering of Serbia. He has received the SIAM Reid Prize, ASME Oldenburger Medal, Nyquist Lecture Prize, Paynter Outstanding Investigator Award, Ragazzini Education Award, IFAC Nonlinear Control Systems Award, Chestnut textbook prize, Control Systems Society Distinguished Member Award, the PECASE, NSF Career, and ONR Young Investigator awards, the Schuck ('96 and '19) and Axelby paper prizes, and the first UCSD Research Award given to an engineer. Krstic has also been awarded the Springer Visiting Professorship at UC Berkeley, the Distinguished Visiting Fellowship of the Royal Academy of Engineering, and the Invitation Fellowship of the Japan Society for the Promotion of Science. He serves as Editor-in-Chief of Systems & Control Letters and has been serving as Senior Editor in Automatica and IEEE Transactions on Automatic Control, as editor of two Springer book series, and has served as Vice President for Technical Activities of the IEEE Control Systems Society and as chair of the IEEE CSS Fellow Committee. Krstic has coauthored thirteen books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.