IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 67, NO. 3, MARCH 2022

Stochastic Nonlinear Prescribed-Time Stabilization and Inverse Optimality

Wuquan Li, *Senior Member, IEEE*, and Miroslav Krstic, *Fellow, IEEE*

***Abstract*—We solve the prescribed-time mean-square stabilization and inverse optimality control problems for stochastic strict-feedback nonlinear systems by developing a new nonscaling backstepping design scheme. A key novel design ingredient is that the time-varying function is not used to scale the coordinate transformations and is only suitably introduced into the virtual controllers. The advantage of this approach is that a simpler controller results and the control effort is reduced. By using this method, we design a new controller to guarantee that the equilibrium at the origin of the closed-loop system is prescribed-time mean-square stable. Then, we redesign the controller and solve the prescribed-time inverse optimal mean-square stabilization problem, with an infinite gain margin. Specifically, the designed controller is not only optimal with respect to a meaningful cost functional but also globally stabilizes the closed-loop system in the prescribed-time. Finally, two simulation examples are given to illustrate the stochastic nonlinear prescribed-time control design.**

***Index Terms*—Inverse optimality, nonscaling design, prescribed-time stabilization, stochastic nonlinear systems.**

I. Introduction

RESEARCH on stochastic stability of systems modeled by stochastic differential equations has attracted much attention in the past two decades [1]–[4]. For the stochastic nonlinear control, there are mainly two Lyapunov-based controller design approaches: quartic Lyapunov functions-based design [5], [6] and weighted quadratic Lyapunov functions-based design [7], which are further developed by [8]–[12]. In recent years, attempts have been made toward studying stochastic nonlinear finite-time control. Specifically, [13] and [14] establish the Lyapunov criterion of stochastic finite-time stability. By designing state-feedback controllers, [15]–[17] solve the finite-time stabilization problems for stochastic nonlinear strict-feedback systems; [18] discusses the finite-time stability of homogeneous stochastic nonlinear systems whose coefficients have negative degrees of homogeneity; [19] relaxes the constraint on the differential operator and gives a more general stochastic finite-time stability criteria. It should be noted that all the above mentioned results [13]–[19] achieve stochastic finite-time stabilization within some stochastic settling time, which typically depends on initial conditions and is often unknown (only almost sure finiteness can be ensured). However, the unknown and stochastic character of the settling time makes the results in [13]–[19] difficult to use in many real applications. In several real-world applications, discussed in [20], [25], and [26], stabilization is required within a known finite time to meet the control objectives, motivating the study of prescribed-time control.

In the prescribed-time control, the user can prescribe a known specific convergence time, irrespective of initial conditions. In this direction, [20] develops a scaling design method to solve the prescribed-time regulation problem of nonlinear systems in normal form, in which the system state is scaled by a time-varying function that grows unbounded toward the terminal time; [21] demonstrates the differences between the prescribed-time control [20] and the traditional finite-time control [22], [23]; [24] presents the prescribed-time consensus design for networked first-order multiagent systems; [25] solves the prescribed-time estimation problem for linear systems in the observer canonical form. By leveraging the prescribed-time state-feedback control [20] and the prescribed-time observer [25], [26] designs a prescribed-time output feedback controller for linear time-invariant systems in controllable canonical form; [27] and [28] focus on the prescribed-time stabilization of nonlinear strict-feedback-like systems; [29] and [30] study the prescribed-time output-feedback stabilization problems for reaction–diffusion equations. It should be emphasized that all the above-mentioned results [20], [30] on prescribed-time control are focused on deterministic systems. However, as demonstrated by [1], [4], [31]–[33], and [42], the perturbations and unmodeled dynamics in practical systems are often described by noise entering the model; thus, the research of stochastic control has drawn considerable attention and is gaining importance in econometrics, biology, environmental science, and other areas. Therefore, from

Manuscript received March 9, 2020; revised October 9, 2020; accepted February 7, 2021. Date of publication February 23, 2021; date of current version February 28, 2022. The work of Wuquan Li was supported in part by National Natural Science Foundation of China under Grant 61973150, in part by the Young Taishan Scholars Program of Shandong Province of China under Grant tsqn20161043, in part by the Shandong Provincial Natural Science Foundation for Distinguished Young Scholars under Grant ZR2019JQ22, and in part by the Shandong Province Higher Educational Excellent Youth Innovation Team under Grant 2019KJN017. Recommended by Associate Editor V. Andrieu. *(Corresponding author: Wuquan Li.)*

Wuquan Li is with the School of Mathematics and Statistics Science, Ludong University, Yantai 264025, China, and also with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093 USA (e-mail: sea81@126.com).

Miroslav Krstic is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla 92093, CA USA (e-mail: krstic@ucsd.edu).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2021.3061646.

Digital Object Identifier 10.1109/TAC.2021.3061646

both practical and theoretical points of view, it is imperative to study the prescribed-time control of stochastic nonlinear systems.

The inverse optimal control problem has traditionally been studied as a differential game problem, which circumvents the task of solving a Hamilton–Jacobi equation and results in a controller optimal with respect to a meaningful cost functional. This approach requires the knowledge of a control Lyapunov function and a stabilizing control law of a particular form. In this direction, [34]–[36] focus on the inverse optimal control of deterministic systems. For stochastic nonlinear systems, [37] is the first paper to study the inverse optimality. Subsequently, [38] solves the inverse optimal gain assignment problem for systems driven by noise of unknown covariance. The work [39] studies the inverse optimal control for high-order stochastic nonlinear systems whose Jacobian linearization is neither controllable nor feedback linearizable. The work [40] addresses the inverse optimal stabilization problem for stochastic nonholonomic systems. However, the controllers in [37]–[40] can only ensure global asymptotic stability in probability. To the authors' knowledge, there is no related results about prescribed-time inverse optimal control of stochastic nonlinear systems.

Motivated by the above observations, we study the prescribed-time mean-square stabilization and prescribed-time inverse optimality control for stochastic strict-feedback nonlinear systems. The contributions of this article are fourfold.

1) For prescribed-time control, we consider a general system model in addition to those in the existing results [5]–[21] and [24]–[30]. Different from the time-invariant systems [5]–[19], we focus on designing controllers characterized by a time-varying function that grows unbounded toward the terminal time, which makes the studied system essentially time-varying. Unlike the deterministic systems [20], [21] and [24]–[30], the systems studied in this article are perturbed by stochastic noise. In this article, the inherent time-varying character of the closed-loop system and the complexity of the stochastic process involved makes the controller design and the stability analysis much more difficult.
2) We present a new nonscaling design framework for stochastic nonlinear systems in this article. Different from the scaling design in [20], [21] and [24]–[26] where the time-varying function is used to scale the states in all the transformations, our approach does not use the scaling function in the coordinate transformations. To achieve prescribed-time stabilization, the time-varying scaling function is suitably used to design virtual controllers. In this way, a simpler controller can be designed since the computation burden for the derivative of the time-varying scaling function can be largely reduced with nonscaling transformations. Therefore, the control effort can be saved. This advantage is especially obvious when the system order is high. It should be emphasized that even for the deterministic nonlinear systems, the nonscaling design scheme proposed in this article is new.
3) Compared with the stochastic finite-time stability results [13]–[19] where the settling time is stochastic, unknown, and heavily relies on the initial conditions, the prescribed-time control developed in this article has a clear advantage that the settling time is deterministic, known, and irrespective of initial conditions, which allows the user to prescribe the convergence time *a priori*. Therefore, our control schemes are more practical in real applications.
4) We propose a new prescribed-time inverse optimal control design in this article. In [37]–[40], asymptotic stabilization controllers are constructed to minimize cost functionals which are characterized by time-invariant value functions. In this article, we design a prescribed-time mean-square stable controller to minimize a cost functional equipped with time-varying value functions.

The remainder of this article is organized as follows. Section II is on preliminaries. Section III is focused on nonscaling controller design and stability analysis. Section IV is devoted to prescribed-time inverse optimal stabilization. Section V gives two examples to illustrate the theoretical results. Section VI includes concluding remarks. Finally, some useful lemmas and the proof of a crucial technical lemma are provided in the appendices.

II. Preliminaries

The following notation will be used throughout the article. R^+, R^n, Z, and Z^+ denote the set of nonnegative real numbers, the real n-dimensional space, the set of integers, and the set of positive integers, respectively. For a given vector or matrix X, X^T denotes its transpose, $\mathrm{Tr}\{X\}$ denotes its trace when X is square, and $|X|$ is the Euclidean norm of a vector X. Defining $|A| = \left(\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2\right)^{\frac{1}{2}}$ for a matrix $A_{n\times m}$. For any $a, b \in R$, let $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. $\chi_A(\cdot)$ denotes the indicator function of A. Let $C^{1,2}(R^+ \times R^n; R^+)$ denote all nonnegative functions $V(t,x)$ on $R^+ \times R^n$ which are $\mathcal{C}^1$ in t and $\mathcal{C}^2$ in x. $\mathcal{K}$ denotes the set of all functions: $R^+ \to R^+$, which are continuous, strictly increasing, and vanishing at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded. For a $\mathcal{K}_\infty$ function γ whose derivative exists and is also a $\mathcal{K}_\infty$ function, ℓ_γ denotes the transform $\ell_\gamma(r) = r(\dot{\gamma})^{-1}(r) - \gamma((\dot{\gamma})^{-1}(r))$, where $(\dot{\gamma})^{-1}(r)$ stands for the inverse function of $\frac{d\gamma(r)}{dr}$.

We introduce the following scaling functions:

$$\mu_1(t) = \frac{T}{t_0 + T - t} \tag{1}$$

$$\mu(t) = \left(\frac{T}{t_0 + T - t}\right)^m = \mu_1^m(t) \forall\, t \in [t_0, t_0 + T) \tag{2}$$

where $m \geq 2$ is a integer and $T > 0$ is the freely prescribed time.

Obviously, $\mu(t)$ is a monotonically increasing function on $[t_0, t_0 + T)$ with $\mu(t_0) = 1$ and $\lim_{t\to t_0+T} \mu(t) = +\infty$ (in this article, $\lim_{t\to t_0+T}$ means t approaches $t_0 + T$ "from the left" or "from below").

Consider the following Itô stochastic nonlinear system:

$$dx = f(t, x, u(t, x))dt + g^T(t, x)d\omega \quad \forall\, x_0 \in R^n \tag{3}$$

where $x \in R^n$ and $u(t, x) \in R$ are the system state and control input. ω is an m-dimensional independent standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). The functions $f : R^+ \times R^n \times R \to R^n$ and $g : R^+ \times R^n \to R^{m\times n}$ are continuous of their arguments and are locally Lipschitz in x (i.e., for every real number T_1 satisfying $0 < T_1 < T$ and integer $k \geq 1$, there exists a positive constant $K_{T_1,k}$ such that $|f(t, x, u(t, x)) - f(t, y, u(t, y))| \vee |g(t, x)) - g(t, y)| \leq K_{T_1,k}|x - y|$ holds for all $t \in [t_0, t_0 + T_1]$ and all $x, y \in R^n$ with $|x| \vee |y| \leq k$). By [2, Th. 3.15], system (3) has an almost surely unique strong solution $x(t)$ on $[t_0, \rho_\infty)$, where ρ_∞ is the finite escape time.

For any given $W(t, x) \in C^{1,2}$ associated with Itô stochastic system (3), the differential operator $\mathcal{L}$ is defined as $\mathcal{L}W \triangleq \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f(t, x, u(t, x)) + \frac{1}{2}\mathrm{Tr}\{g(t, x)\frac{\partial^2 W}{\partial x^2} g^T(t, x)\}$.

Next, we give the definitions of prescribed-time mean-square stable and prescribed-time inverse optimal mean-square stabilization.

Definition 1: For stochastic system (3) with $f(t, 0, 0) = 0$ and $g(t, 0) = 0$, the equilibrium $x(t) = 0$ is prescribed-time mean-square stable if there exist positive constants k_i $(1 \leq i \leq 4)$ such that

$$E|x(t)|^2 \leq k_1|x(t_0)|^2(1 + \mu_1^{k_2}(t))e^{-k_3\mu_1^{k_4}(t)} \quad \forall\, t \in [t_0, t_0 + T). \tag{4}$$

Definition 2: The problem of prescribed-time inverse optimal mean-square stabilization for system (3) is solvable if there exist a $\mathcal{K}_\infty$ function $\gamma(x)$ whose derivative is also a $\mathcal{K}_\infty$ function, a matrix-valued function $R(x)$ such that $R(x) = R^T(x) > 0$ for all x, a nonnegative function $l(t, x)$ which is positive definite on x for fixed t, a nonnegative function $S(t, x)$ which is positive definite and radially unbounded on x for fixed t, and a continuous feedback control law $u = \alpha(t, x)$, which makes system (3) prescribed-time mean-square stable and minimizes the cost functional

$$J(u) = \limsup_{r\to\infty} E\left[S(\tau_r, x(\tau_r)) + \int_{t_0}^{\tau_r} (l(t, x(t)) + \mu^{k_5}(t)\gamma(|Ru(t)|))dt\right] \tag{5}$$

where $r \in Z^+$, $k_5 \in Z$ and $\tau_r = (t_0 + T) \wedge \inf\{t : t \geq t_0,\ |x(t)| \geq r\}$.

Remark 1: In Definition 1, by (1), we have $\lim_{t\to t_0+T} E|x|^2 = 0$. Besides, denoting

$$\varrho(t) = k_1|x(t_0)|^2(1 + \mu_1^{k_2}(t))e^{-k_3\mu_1^{k_4}(t)} \tag{6}$$

then we have

$$\begin{aligned}\frac{d\varrho}{dt} &= k_1\frac{k_2}{T}|x(t_0)|^2 e^{-k_3\mu_1^{k_4}}\mu_1^{k_2+1} \cdot \left(1 - \frac{k_3k_4}{k_2}(\mu_1^{k_4-k_2} + \mu_1^{k_4})\right)\\ &\leq k_1\frac{k_2}{T}|x(t_0)|^2 e^{-k_3\mu_1^{k_4}}\mu_1^{k_2+1}\left(1 - \frac{k_3k_4}{k_2}\mu_1^{k_4}\right).\end{aligned} \tag{7}$$

It can be deduced from (7) that $E|x|^2$ is a strictly decreasing function in $[T^*, t_0 + T)$, where

$$T^* = \max\left\{t_0, t_0 + T - T(\frac{k_3k_4}{k_2})^{1/k_4}\right\}. \tag{8}$$

From (8), it is obvious that $t_0 \leq T^* < t_0 + T$.

Remark 2: In Definition 1, the prescribed-time mean-square stability is characterized by four positive constants k_i $(1 \leq i \leq 4)$. Since k_1, k_2, k_3, and k_4 are free and independent from each other, it is difficult to decide whether this stability is uniform or not. However, for some special cases, we can get the uniform stability. For example, if $\frac{k_3k_4}{k_2} \geq \frac{1}{2}$ and $k_4 \geq k_2$, from the first equality in (7), we get $\frac{d\varrho}{dt} \leq 0$ on $[t_0, t_0 + T)$. Subsequently, it follows from (4) in Definition 1 that $E|x(t)|^2 \leq 2k_1|x(t_0)|^2e^{-k_3}$ $\forall\, t \in [t_0, t_0 + T)$, which implies that the mean-square stable is uniform. Therefore, with (4), we conclude that the equilibrium $x(t) = 0$ of stochastic system (3) is uniform prescribed-time mean-square stable in this case.

Remark 3: Although the prescribed-time inverse optimal control in Definition 2 is motivated by [36]–[39], Definition 2 has the following two novel features.

1) The inverse optimal controller $u = \alpha(t, x)$ can make system (3) achieve prescribed-time mean-square stable while the controllers in [36]–[39] can only ensure global asymptotic stability in probability.
2) The value functions $l(t, x)$ and $S(t, x)$ are time-varying, while they are time-invariant in [36]–[39]. Besides, the cost functional (5) is scaled with a time-varying function $\mu^{k_5}(t)$, which can be viewed as a penalty factor on the controller.

In the following lemma, under some mild conditions, we prove that the finite escape time ρ_∞ of the strong solution to system (3) is as least $t_0 + T$. Besides, we provide a basic tool for analyzing the prescribed-time mean-square stability. The proof of this lemma is given in Appendix B.

Lemma 1: Consider the system (3). If there exist a nonnegative function $U(t, x) \in C^{1,2}([t_0, t_0 + T) \times R^n; R^+)$ and positive constants c_0 and M_0 such that

$$\lim_{|x|\to+\infty} \inf_{t\in[t_0,T_1]} U(t, x) = +\infty \ \forall T_1 \in (t_0, t_0 + T) \tag{9}$$

$$\mathcal{L}U(t, x) \leq -c_0\mu U + \mu M_0 \quad \forall\, t \in [t_0, t_0 + T) \tag{10}$$

then the following conclusions hold.

1) System (3) has an almost surely unique strong solution on $[t_0, t_0 + T)$ for any $x_0 \in R^n$.
2) The function $U(t, x)$ satisfies

$$EU(t, x(t)) \leq e^{-c_0\int_{t_0}^t \mu(s)ds}U(t_0, x_0) + \frac{M_0}{c_0} \quad \forall\, t \in [t_0, t_0 + T). \tag{11}$$

Next, we give an example to illustrate and verify the inequality (11) in Lemma 1.

Example 1: Consider the system

$$dx = -\mu(t)xdt + d\omega \quad \forall\, t \in [0,1) \tag{12}$$

where $\mu(t) = \frac{1}{(1-t)^2}$, $t \in [0,1)$, and ω is a scalar standard Wiener process.

The unique solution of (12) is

$$x(t) = e^{1-\frac{1}{1-t}}\left(x(0) + \int_0^t e^{\frac{1}{1-s}-1}d\omega(s)\right) \quad \forall t \in [0,1) \tag{13}$$

where $x(0)$ is the deterministic initial condition.

Noting

$$E\left\{\int_0^t e^{\frac{2}{1-s}-2}ds\right\} \le te^{\frac{2}{1-t}-2} < +\infty \;\; \forall t \in [0,1) \tag{14}$$

we have

$$E\left\{\int_0^t e^{\frac{1}{1-s}-1}d\omega(s)\right\} = 0 \tag{15}$$

$$E\left(\int_0^t e^{\frac{1}{1-s}-1}d\omega(s)\right)^2 = \int_0^t e^{\frac{2}{1-s}-2}ds. \tag{16}$$

It follows from (13), (15), and (16) that

$$E\left\{x^2(t)\right\} = e^{2-\frac{2}{1-t}}x^2(0) + e^{2-\frac{2}{1-t}}\int_0^t e^{\frac{2}{1-s}-2}ds \tag{17}$$

holds for $\forall t \in [0,1)$.

Choosing $U(t,x) = (t+1)x^2$, it is obvious that

$$\lim_{|x|\to+\infty}\inf_{t\in[0,T_1]} U(t,x) = \lim_{|x|\to+\infty} x^2 = +\infty \tag{18}$$

holds for $\forall T_1 \in (0,1)$.

In addition, from (12), we get

$$\begin{aligned}\mathcal{L}U(t,x) &= x^2 - 2(t+1)\mu x^2 + t + 1\\ &= \frac{1}{t+1}U - 2\mu U + t + 1\\ &\le \mu U - 2\mu U + \mu\\ &= -\mu U + \mu \;\forall t \in [0,1).\end{aligned} \tag{19}$$

With (18) and (19), from (11) in Lemma 1, we have

$$EU(t,x(t)) \le e^{1-\frac{1}{1-t}}x^2(0) + 1 \;\; \forall t \in [0,1). \tag{20}$$

Next, we verify whether (20) holds for system (12).

By (17), we obtain

$$\begin{aligned}EU &= (t+1)e^{2-\frac{2}{1-t}}x^2(0) + (t+1)e^{2-\frac{2}{1-t}}\\ &\cdot\int_0^t e^{\frac{2}{1-s}-2}ds \;\; \forall t \in [0,1).\end{aligned} \tag{21}$$

Let

$$h_1(t) = (t+1)e^{1-\frac{1}{1-t}} \;\; \forall t \in [0,1). \tag{22}$$

By (22), we have

$$\frac{dh_1(t)}{dt} = e^{1-\frac{1}{1-t}}\left(1 - \frac{t+1}{(1-t)^2}\right) \le 0 \;\forall t \in [0,1). \tag{23}$$

By (22) and (23), we get

$$h_1(t) \le h_1(0) = 1 \;\; \forall t \in [0,1). \tag{24}$$

From (22) and (24), we have

$$(t+1)e^{2-\frac{2}{1-t}}x^2(0) \le e^{1-\frac{1}{1-t}}x^2(0) \;\; \forall t \in [0,1). \tag{25}$$

Let

$$h_2(t) = \frac{1}{2}e^{\frac{2}{1-t}} - \int_0^t e^{\frac{2}{1-s}}ds. \tag{26}$$

From (26), we get

$$\frac{dh_2(t)}{dt} = e^{\frac{2}{1-t}}\left(\frac{1}{(1-t)^2} - 1\right) \ge 0 \;\; \forall t \in [0,1) \tag{27}$$

which means that

$$h_2(t) \ge h_2(0) = \frac{1}{2}e^2 > 0 \;\; \forall t \in [0,1). \tag{28}$$

It follows from (26) and (28) that

$$\frac{1}{2}e^{\frac{2}{1-t}} \ge \int_0^t e^{\frac{2}{1-s}}ds \;\; \forall t \in [0,1) \tag{29}$$

which implies that

$$(t+1)e^{2-\frac{2}{1-t}}\int_0^t e^{\frac{2}{1-s}-2}ds \le \frac{1}{2}(t+1) \le 1 \forall t \in [0,1). \tag{30}$$

From (21), (25), and (30), we get (20). Thus, the inequality (11) in Lemma 1 is verified.

III. Nonscaling Controller Design and Prescribed-Time Stabilization for Stochastic Strict Feedback Dominated Systems

A. Problem Formulation

Consider a class of stochastic nonlinear systems described by

$$dx_i = (x_{i+1} + f_i(t,x))dt + g_i^T(t,x)d\omega, \quad i = 1,\ldots,n-1, \tag{31}$$

$$dx_n = (u + f_n(t,x))dt + g_n^T(t,x)d\omega \tag{32}$$

where $x = (x_1,\ldots,x_n)^T \in R^n$ and $u \in R$ are the system state and control input. The functions $f_i : R^+ \times R^n \to R$ and $g_i : R^+ \times R^n \to R^m$ are continuous of their arguments and are locally Lipschitz in x (i.e., for every real number T_1 satisfying $0 < T_1 < T$ and integer $k \ge 1$, there exists a positive constant $\bar{K}_{T_1,k}$ such that $|f_i(t,x) - f_i(t,y)| \vee |g_i(t,x) - g_i(t,y)| \le \bar{K}_{T_1,k}|x-y|$ holds for all $t \in [t_0, t_0+T_1]$ and all $x,y \in R^n$ with $|x| \vee |y| \le k$), $f_i(t,0) = 0$, $g_i(t,0) = 0$, $i = 1,\ldots,n$. ω is an m-dimensional independent standard Wiener process whose definition can be found in system (3).

To proceed further, we need the following assumption.

Assumption 1: For $i = 1,\ldots,n$, there exist positive constants c_{i1} and c_{i2} such that

$$|f_i(t,x)| \le c_{i1}(|x_1| + \cdots + |x_i|) \tag{33}$$

$$|g_i(t,x)| \le c_{i2}(|x_1| + \cdots + |x_i|). \tag{34}$$

In this section, for system (31)–(32) with Assumption 1, we first develop a novel nonscaling design scheme, by which a new time-varying controller is designed; then we analyze the prescribed-time mean-square stability of the closed-loop system.

B. Controller Design

Next, we design a time-varying controller for system (31)–(32) step by step.

Step 1. Define $V_1 = \frac{1}{4}\xi_1^4, \xi_1 = x_1$, from (31), (33)–(34), and the definition of $\mathcal{L}$, we have

$$\mathcal{L}V_1(\xi_1) = \xi_1^3 x_2 + \xi_1^3 f_1 + \frac{3}{2}\xi_1^2|g_1|^2$$
$$\leq \xi_1^3(x_2 - x_2^*) + \xi_1^3 x_2^* + \xi_1^4\left(c_{11} + \frac{3}{2}c_{12}^2\right). \quad (35)$$

Choosing

$$x_2^* = -\mu^{\delta_1}\left(c_1 + c_{11} + \frac{3}{2}c_{12}^2\right)\xi_1 \triangleq -\mu^{\delta_1}\alpha_1\xi_1 \quad (36)$$

which substitutes into (35) yields

$$\mathcal{L}V_1(\xi_1) \leq -c_1\mu^{\delta_1}\xi_1^4 + \xi_1^3(x_2 - x_2^*) \quad (37)$$

where $\delta_1 = 1$, $c_1 > 0$ is a design parameter and $\alpha_1 = c_1 + c_{11} + \frac{3}{2}c_{12}^2$.

Deductive Step: Assume that at step $k-1$, there are set of virtual controllers $x_2^*, \dots, x_k^*$ defined by

$$x_2^* = -\mu^{\delta_1}\alpha_1\xi_1, \qquad \xi_1 = x_1, \quad (38)$$
$$x_3^* = -\mu^{\delta_2}\alpha_2\xi_2, \qquad \xi_2 = x_2 - x_2^*, \quad (39)$$
$$\vdots \qquad\qquad \vdots$$
$$x_k^* = -\mu^{\delta_{k-1}}\alpha_{k-1}\xi_{k-1}, \quad \xi_{k-1} = x_{k-1} - x_{k-1}^* \quad (40)$$

such that

$$\mathcal{L}V_{k-1}(\bar{\xi}_{k-1}) \leq -\sum_{i=1}^{k-1}(c_i - a_{k-1,i})\mu^{\delta_i}\xi_i^4 + \xi_{k-1}^3(x_k - x_k^*) \quad (41)$$

where $\alpha_1, \dots, \alpha_{k-1}$ are positive constants, $c_i > 0$ is a design parameter, $a_{k-1,1}, \dots, a_{k-1,k-2}$ are arbitrary positive constants, $a_{k-1,k-1} = 0$, $\bar{\xi}_{k-1} = (\xi_1, \dots, \xi_{k-1})^T$, $V_{k-1}(\bar{\xi}_{k-1}) = \frac{1}{4}\sum_{i=1}^{k-1}\xi_i^4$ and

$$\delta_1 = 1, \quad \delta_i = 3\cdot 5^{i-2}, \quad 2 \leq i \leq k-1. \quad (42)$$

To complete the induction, at the kth step, we consider the ξ_k-system.

Define $\xi_k = x_k - x_k^*$, and from (38)–(40), we obtain

$$\xi_k = x_k + \sum_{i=1}^{k-1}\beta_i(t)x_i \quad (43)$$

$$\beta_i(t) = \prod_{j=i}^{k-1}\mu^{\delta_j}\alpha_j. \quad (44)$$

Noting that $\frac{\partial^2(\beta_i x_i)}{\partial x_i \partial x_j} = 0$, by (31), (43), and Itô's formula, we get

$$d\xi_k = \left(x_{k+1} + f_k + \sum_{i=1}^{k-1}\dot{\beta}_i x_i + \sum_{i=1}^{k-1}\beta_i(x_{i+1} + f_i)\right)dt + \left(g_k^T + \sum_{i=1}^{k-1}\beta_i g_i^T\right)d\omega. \quad (45)$$

We choose the Lyapunov function

$$V_k(\bar{\xi}_k) = V_{k-1}(\bar{\xi}_{k-1}) + \frac{1}{4}\xi_k^4. \quad (46)$$

It follows from (41), (45)–(46), and the definition of $\mathcal{L}$ that

$$\mathcal{L}V_k(\bar{\xi}_k) \leq -\sum_{i=1}^{k-1}(c_i - a_{k-1,i})\mu^{\delta_i}\xi_i^4 + \xi_{k-1}^3\xi_k + \xi_k^3 x_{k+1} + \xi_k^3 f_k + \xi_k^3\sum_{i=1}^{k-1}\beta_i(x_{i+1} + f_i) + \xi_k^3\sum_{i=1}^{k-1}\dot{\beta}_i x_i + \frac{3}{2}\xi_k^2\left|g_k^T + \sum_{i=1}^{k-1}\beta_i g_i^T\right|^2. \quad (47)$$

For $i = 1, \dots, k$, by (33)–(34) and (38)–(40), we get

$$|f_i(t,x)| \leq \hat{c}_{i1}(\mu^{\delta_1}|\xi_1| + \cdots + \mu^{\delta_{i-1}}|\xi_{i-1}| + |\xi_i|) \quad (48)$$
$$|g_i(t,x)| \leq \hat{c}_{i2}(\mu^{\delta_1}|\xi_1| + \cdots + \mu^{\delta_{i-1}}|\xi_{i-1}| + |\xi_i|) \quad (49)$$

where $\hat{c}_{i1}$ and $\hat{c}_{i2}$ are positive constants.

By (42), (48), and Lemma A.2, we have

$$\xi_{k-1}^3\xi_k \leq a_{k,k-1,1}\mu^{\delta_{k-1}}\xi_{k-1}^4 + \frac{1}{4}\left(\frac{4}{3}a_{k,k-1,1}\right)^{-3}\xi_k^4, \quad (50)$$

$$\xi_k^3 f_k \leq \hat{c}_{k1}|\xi_k|^3(\mu^{\delta_1}|\xi_1| + \cdots + \mu^{\delta_{k-1}}|\xi_{k-1}| + |\xi_k|)$$
$$\leq \sum_{i=1}^{k-1}a_{k,i,2}\mu^{\delta_i}\xi_i^4 + \mu^{\delta_{k-1}}\left(\hat{c}_{k1} + \frac{3}{4}\hat{c}_{k1}^{4/3}\cdot\sum_{i=1}^{k-1}(4a_{k,i,2})^{-1/3}\right)\xi_k^4 \quad (51)$$

where $a_{k,k-1,1}$ and $a_{k,i,2}$ are arbitrary positive constants.

From (44) and the definition of δ_k, we have

$$\beta_i = \mu^{\delta_i + \cdots + \delta_{k-1}}\prod_{j=i}^{k-1}\alpha_j, \quad (52)$$

$$|\dot{\beta}_i| \leq \left(\frac{m}{T}\sum_{j=i}^{k-1}\delta_j\right)\mu^{\delta_i + \cdots + \delta_{k-1} + 1}\prod_{j=i}^{k-1}\alpha_j. \quad (53)$$

By (38)–(40), (42), (52), and Lemma A.2, we obtain

$$\xi_k^3\sum_{i=1}^{k-1}\beta_i x_{i+1} \leq |\xi_k|^3\sum_{i=1}^{k-1}\mu^{\delta_i + \cdots + \delta_{k-1}}\cdot\left(\prod_{j=i}^{k-1}\alpha_j\right)\left(|\xi_{i+1}| + \mu^{\delta_i}\alpha_i|\xi_i|\right)$$
$$\leq \mu^{\delta_{k-1}}\alpha_{k-1}\xi_k^4 + \sum_{i=1}^{k-1}\mu^{2\delta_i + \delta_{i+1} + \cdots + \delta_{k-1}}\cdot(\alpha_i^2 + 1)\left(\prod_{j=i-1}^{k-1}\alpha_j\right)|\xi_i||\xi_k|^3$$
$$\leq \sum_{i=1}^{k-1}a_{k,i,3}\mu^{\delta_i}\xi_i^4 + \mu^{7\delta_{k-1}/3}\Big(\alpha_{k-1}$$

$$+\frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,3})^{-1/3}(\alpha_i^2+1)^{4/3}\cdot\left(\prod_{j=i-1}^{k-1}\alpha_j\right)^{4/3}\Bigg)\xi_k^4 s \tag{54}$$

where $a_{k,i,3}$ is an arbitrary positive constant and $\alpha_0=1$.

It follows from (42), (48), (52), and Lemma A.2 that

$$\begin{aligned}\xi_k^3\sum_{i=1}^{k-1}\beta_i f_i \le{}& \sum_{i=1}^{k-1}(\mu^{\delta_1}|\xi_1|+\cdots+\mu^{\delta_{i-1}}|\xi_{i-1}|+|\xi_i|)\cdot\left(\hat{c}_{i1}\mu^{\delta_i+\cdots+\delta_{k-1}}\prod_{j=i}^{k-1}\alpha_j|\xi_k|^3\right)\\ \le{}& (\mu^{\delta_1}|\xi_1|+\cdots+\mu^{\delta_{k-2}}|\xi_{k-2}|+|\xi_{k-1}|)\cdot\left(\bar{c}_{k-1,1}(k-1)\mu^{\delta_1+\cdots+\delta_{k-1}}\prod_{j=1}^{k-1}\alpha_j|\xi_k|^3\right)\\ \le{}& \sum_{i=1}^{k-1}a_{k,i,4}\mu^{\delta_i}\xi_i^4+\frac{3}{4}\mu^{2\delta_{k-1}}\Bigg(\sum_{i=1}^{k-1}(4a_{k,i,4})^{-1/3}\cdot\left(\bar{c}_{k-1,1}(k-1)\prod_{j=1}^{k-1}\alpha_j\right)^{4/3}\Bigg)\xi_k^4\end{aligned} \tag{55}$$

where $a_{k,i,4}$ is an arbitrary positive constant and $\bar{c}_{k-1,1}=\max\{\hat{c}_{11},\hat{c}_{21},\ldots,\hat{c}_{k-1,1}\}$.

By Lemma A.2, it can be deduced from (40), (42), and (53) that

$$\begin{aligned}\xi_k^3\sum_{i=1}^{k-1}\dot{\beta}_i x_i \le{}& |\xi_k|^3\sum_{i=1}^{k-1}\left(\prod_{j=i}^{k-1}\alpha_j\right)\left(\frac{m}{T}\sum_{j=i}^{k-1}\delta_j\right)\cdot\mu^{\delta_i+\cdots+\delta_{k-1}+1}\left(|\xi_i|+\mu^{\delta_{i-1}}\alpha_{i-1}|\xi_{i-1}|\right)\\ \le{}& \sum_{i=1}^{k-1}\left(\prod_{j=i}^{k-1}\alpha_j\right)\left(\frac{2\,m}{T}\sum_{j=i}^{k-1}\delta_j\right)\cdot\mu^{\delta_i+\cdots+\delta_{k-1}+1}|\xi_i||\xi_k|^3\\ \le{}& \sum_{i=1}^{k-1}a_{k,i,5}\mu^{\delta_i}\xi_i^4+\frac{3}{4}\mu^{2\delta_{k-1}}\sum_{i=1}^{k-1}(4a_{k,i,5})^{-1/3}\cdot\left(\prod_{j=i}^{k-1}\alpha_j\right)^{4/3}\left(\frac{2\,m}{T}\sum_{j=i}^{k-1}\delta_j\right)^{4/3}\xi_k^4\end{aligned} \tag{56}$$

where $a_{k,i,5}$ is an arbitrary positive constant and $\alpha_0=\xi_0=0$.

By (49) and (52), we obtain

$$\begin{aligned}\frac{3}{2}\xi_k^2\left|g_k^T+\sum_{i=1}^{k-1}\beta_i g_i^T\right|^2 \le{}& \frac{3}{2}\xi_k^2\Bigg(\hat{c}_{k2}(\mu^{\delta_1}|\xi_1|+\cdots+\mu^{\delta_{k-1}}|\xi_{k-1}|+|\xi_k|)\\ &+\sum_{i=1}^{k-1}\mu^{\delta_i+\cdots+\delta_{k-1}}\prod_{j=i}^{k-1}\alpha_j\hat{c}_{i2}(\mu^{\delta_1}|\xi_1|+\cdots+\mu^{\delta_{i-1}}|\xi_{i-1}|+|\xi_i|)\Bigg)^2\\ \le{}& \frac{3}{2}\xi_k^2\Bigg(\hat{c}_{k2}|\xi_k|+\sum_{i=1}^{k-1}\mu^{\delta_i+\cdots+\delta_{k-1}}\cdot\left(\hat{c}_{k2}+\sum_{s=i}^{k-1}\prod_{j=s}^{k-1}\alpha_s\hat{c}_{s2}\right)|\xi_i|\Bigg)^2\\ \le{}& \frac{3}{2}k\hat{c}_{k2}^2\xi_k^4+\frac{3}{2}k\sum_{i=1}^{k-1}\mu^{2\delta_i+\cdots+2\delta_{k-1}}\cdot\left(\hat{c}_{k2}+\sum_{s=i}^{k-1}\prod_{j=s}^{k-1}\alpha_s\hat{c}_{s2}\right)^2\xi_i^2\xi_k^2.\end{aligned} \tag{57}$$

From (42), we have

$$3\delta_i+4(\delta_{i+1}+\cdots+\delta_{k-1}) =\begin{cases}5\delta_{k-1}, & \text{if } i=1,\\ 5\delta_{k-1}-2\delta_i, & \text{if } 2\le i\le k-1.\end{cases} \tag{58}$$

By (58) and Lemma A.2, (57) can be written as

$$\begin{aligned}\frac{3}{2}\xi_k^2\left|g_k^T+\sum_{i=1}^{k-1}\beta_i g_i^T\right|^2 \le{}& \sum_{i=1}^{k-1}a_{k,i,6}\mu^{\delta_i}\xi_i^4+\mu^{5\delta_{k-1}}\Bigg(\frac{3}{2}k\hat{c}_{k2}^2+\frac{9}{16}k^2\cdot\sum_{i=1}^{k-1}a_{k,i,6}^{-1}\left(\hat{c}_{k2}+\sum_{s=i}^{k-1}\prod_{j=s}^{k-1}\alpha_s\hat{c}_{s2}\right)^4\Bigg)\xi_k^4\end{aligned} \tag{59}$$

where $a_{k,i,6}$ is an arbitrary positive constant.

From (42), (50)–(51), (54)–(56), and (59), we can choose

$$\begin{aligned}\delta_k ={}& \max\left\{\delta_{k-1},2\delta_{k-1},\frac{7}{3}\delta_{k-1},5\delta_{k-1}\right\}\\ ={}& 5\delta_{k-1}=3\cdot5^{k-2}.\end{aligned} \tag{60}$$

With (60), substituting (50)–(51), (54)–(56), and (59) into (47) yields

$$\begin{aligned}\mathcal{L}V_k(\bar{\xi}_k) \le{}& -\sum_{i=1}^{k-1}(c_i-a_{k,i})\mu^{\delta_i}\xi_i^4+\xi_k^3(x_{k+1}-x_{k+1}^*)\\ &+\xi_k^3x_{k+1}^*+\mu^{\delta_k}\xi_k^4\Bigg(\hat{c}_{k1}+\alpha_{k-1}+\frac{3}{2}k\hat{c}_{k2}^2\\ &+\frac{3}{4}\hat{c}_{k1}^{4/3}\sum_{i=1}^{k-1}(4a_{k,i,2})^{-1/3}\\ &+\frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,3})^{-1/3}(\alpha_i^2+1)^{4/3}\left(\prod_{j=i-1}^{k-1}\alpha_j\right)^{4/3}\end{aligned}$$

$$+\frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,4})^{-1/3}\left(\bar{c}_{k-1,1}(k-1)\prod_{j=1}^{k-1}\alpha_j\right)^{4/3}$$
$$+\frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,5})^{-1/3}\left(\prod_{j=i}^{k-1}\alpha_j\right)^{4/3}$$
$$\cdot\left(\frac{2\,m}{T}\sum_{j=i}^{k-1}\delta_j\right)^{4/3}+\frac{9}{16}k^2\sum_{i=1}^{k-1}a_{k,i,6}^{-1}\Big(\hat{c}_{k2}$$
$$+\sum_{s=i}^{k-1}\prod_{j=s}^{k-1}\alpha_s\hat{c}_{s2}\Big)^4+\frac{1}{4}\left(\frac{4}{3}a_{k,k-1,1}\right)^{-3}\Bigg) \quad (61)$$

where

$$a_{k,i}=a_{k-1,i}+\sum_{j=2}^{6}a_{k,i,j},\quad i=1,\ldots,k-2 \quad (62)$$

$$a_{k,k-1}=a_{k-1,k-1}+\sum_{j=1}^{6}a_{k,i,j}. \quad (63)$$

Choosing the virtual controller

$$x_{k+1}^*=-\mu^{\delta_k}\xi_k\Bigg(c_k+\hat{c}_{k1}+\alpha_{k-1}+\frac{3}{2}k\hat{c}_{k2}^2$$
$$+\frac{1}{4}(\frac{4}{3}a_{k,k-1,1})^{-3}+\frac{3}{4}\hat{c}_{k1}^{4/3}\sum_{i=1}^{k-1}(4a_{k,i,2})^{-1/3}$$
$$+\frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,3})^{-1/3}(\alpha_i^2+1)^{4/3}\left(\prod_{j=i-1}^{k-1}\alpha_j\right)^{4/3}$$
$$+\frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,4})^{-1/3}\left(\bar{c}_{k-1,1}(k-1)\prod_{j=1}^{k-1}\alpha_j\right)^{4/3}$$
$$+\frac{3}{4}\sum_{i=1}^{k-1}(4a_{k,i,5})^{-1/3}\left(\prod_{j=i}^{k-1}\alpha_j\right)^{4/3}\left(\frac{2\,m}{T}\sum_{j=i}^{k-1}\delta_j\right)^{4/3}$$
$$+\frac{9}{16}k^2\sum_{i=1}^{k-1}a_{k,i,6}^{-1}\left(\hat{c}_{k2}+\sum_{s=i}^{k-1}\prod_{j=s}^{k-1}\alpha_s\hat{c}_{s2}\right)^4\Bigg)$$
$$\triangleq-\mu^{\delta_k}\alpha_k\xi_k \quad (64)$$

with which (61) can be rewritten as

$$\mathcal{L}V_k(\bar{\xi}_k)\le-\sum_{i=1}^{k}(c_i-a_{k,i})\mu^{\delta_i}\xi_i^4+\xi_k^3(x_{k+1}-x_{k+1}^*) \quad (65)$$

where $c_k>0$ is a design parameter and $a_{k,k}=0$.

Step n **.** Similar to (64)–(65), by choosing the actual control law

$$u=-\mu^{\delta_n}\alpha_n\xi_n \quad (66)$$

we have

$$\mathcal{L}V_n(\bar{\xi}_n)\le-\sum_{i=1}^{n}(c_i-a_{n,i})\mu^{\delta_i}\xi_i^4 \quad (67)$$

where $c_n>0$ is a design parameter, $\delta_n=3\cdot5^{n-2}$, α_n is a positive constant, $a_{n,n}=0, a_{n,1},\ldots,a_{n,n-1}$ are positive constants, $\xi_n=x_n-x_n^*$ and $V_n(\bar{\xi}_n)=\frac{1}{4}\sum_{i=1}^{n}\xi_i^4$.

Choosing the design parameters as

$$c_i>a_{n,i},\quad i=1,\ldots,n-1 \quad (68)$$
$$c_n>0 \quad (69)$$

from (42) and (67)–(69), we have

$$\mathcal{L}V_n(\bar{\xi}_n)\le-\frac{1}{4}c\sum_{i=1}^{n}\mu^{\delta_i}\xi_i^4\le-c\mu V_n \quad (70)$$

where $c=4\min_{1\le i\le n-1}\{c_i-a_{n,i},c_n\}$.

Remark 4: From the design process, it can be observed that the order of μ in the controller is suitably constructed so that the negative term $-\mu^{\delta_i}\xi_i^4$ dominates the nonnegative terms produced by Itô's formula. For example, the order of μ in the virtual controller (64) is carefully chosen as $\delta_k=3\cdot5^{k-2}$. On the one hand, if $\delta_k<3\cdot5^{k-2}$, from (57)–(59), we conclude that some nonlinear terms like $\mu^p\xi_1^4$ $(p>1)$ appear, which cannot be dominated by $-\mu^{\delta_1}\xi_1^4$, losing the guarantee of stability. On the other hand, if $\delta_k>3\cdot5^{k-2}$, although the stochastic prescribed-time stability is achieved, the control effort will be larger. Therefore, a good choice of δ_k is nontrivial. In fact, it can be deduced from (47)–(60) that the minimum suitable value of δ_k is mainly decided by the Hessian term $\frac{3}{2}\xi_k^2\left|g_k^T+\sum_{i=1}^{k-1}\beta_ig_i^T\right|^2$ [more details are found in (57)–(60)].

C. Stability Analysis

In the following theorem, we give the main stability results on system (31)–(32).

Theorem 1: Consider the plant consisting of (31)–(32), (66), and (68)–(69). If Assumption 1 holds, then the following conclusions hold.

1) The plant has an almost surely unique strong solution on $[t_0,t_0+T)$.
2) The equilibrium at the origin of the plant is prescribed-time mean-square stable with $\lim_{t\to t_0+T}E|x|^2=\lim_{t\to t_0+T}Eu^2=0$. Moreover, for $\forall\, t\in[t_0,t_0+T)$, the following estimates hold:

$$E|x|^2\le\sqrt{n}\left(n+\sum_{i=1}^{n-1}\alpha_i^2\mu^{2\delta_i}\right)$$
$$\cdot\left(x_1^4(t_0)+\sum_{k=2}^{n}\left(x_k(t_0)+\sum_{i=1}^{k-1}\prod_{j=i}^{k-1}\alpha_jx_i(t_0)\right)^4\right)^{1/2}$$
$$\cdot e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}}-\frac{1}{T^{m-1}}\right)} \quad (71)$$

$$Eu^2\le\sqrt{n}\alpha_n^2\mu^{2\delta_n}\Bigg(x_1^4(t_0)$$
$$+\sum_{k=2}^{n}\left(x_k(t_0)+\sum_{i=1}^{k-1}\prod_{j=i}^{k-1}\alpha_jx_i(t_0)\right)^4\Bigg)^{1/2}$$
$$\cdot e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}}-\frac{1}{T^{m-1}}\right)}. \quad (72)$$

Proof: From (66), for every real number T_1 satisfying $0 < T_1 < T$ and integer $k \geq 1$, there exists a positive constant $\tilde{K}_{T_1,k}$ such that

$$|u(t,x) - u(t,y)| \leq \tilde{K}_{T_1,k}|x-y| \tag{73}$$

holds for all $t \in [t_0, t_0 + T_1]$ and all $x, y \in R^n$ with $|x| \vee |y| \leq k$. Besides, $f_i(t,x)$ and $g_i(t,x)$ are locally Lipschitz in x. Thus, the plant satisfies the local Lipschitz condition.

From (38)–(40), we have

$$x = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_1\mu^{\delta_1} & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\alpha_{n-1}\mu^{\delta_{n-1}} & 1 \end{bmatrix} \xi \triangleq H_1(\mu)\xi. \tag{74}$$

By (74), we get

$$|x| \leq \left(n + \sum_{i=1}^{n-1} \alpha_i^2 \mu^{2\delta_i}\right)^{1/2} |\xi| \tag{75}$$

which means that

$$|\xi| \geq \left(n + \sum_{i=1}^{n-1} \alpha_i^2 \mu^{2\delta_i}\right)^{-1/2} |x|. \tag{76}$$

Noting that $V_n(\bar{\xi}_n) = \frac{1}{4}\sum_{i=1}^n \xi_i^4$, by (70) and (76), the conditions (9) and (10) in Lemma 1 hold.

Therefore, by Lemma 1, conclusion 1) holds and

$$EV_n(t,x) \leq e^{-c\int_{t_0}^t \mu(s)ds} V_n(t_0, x_0) \quad \forall\, t \in [t_0, t_0 + T). \tag{77}$$

By (77) and Schwarz inequality, we obtain

$$\begin{aligned} E|\xi|^2 &\leq \left\{E|\xi|^4\right\}^{1/2} \\ &\leq 2\sqrt{n}\left\{EV_n\right\}^{1/2} \\ &\leq 2\sqrt{n} e^{-\frac{c}{2}\int_{t_0}^t \mu(s)ds} V_n^{1/2}(t_0, x_0) \forall\, t \in [t_0, t_0 + T). \end{aligned} \tag{78}$$

By (2) and (78), we get

$$\begin{aligned} E|\xi|^2 \leq\, & 2\sqrt{n} e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)} \\ & \cdot V_n^{1/2}(t_0, x_0), \quad \forall\, t \in [t_0, t_0 + T). \end{aligned} \tag{79}$$

From (43)–(44), (75), and (79), we have

$$\begin{aligned} E|x|^2 \leq\, & \sqrt{n}\left(n + \sum_{i=1}^{n-1}\alpha_i^2\mu^{2\delta_i}\right) \\ & \cdot \left(x_1^4(t_0) + \sum_{k=2}^n \left(x_k(t_0) + \sum_{i=1}^{k-1}\prod_{j=i}^{k-1}\alpha_j x_i(t_0)\right)^4\right)^{1/2} \\ & \cdot e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)} \forall\, t \in [t_0, t_0 + T). \end{aligned} \tag{80}$$

Noting that $c > 0$, $T > 0$, and $m \geq 2$, by (2), we obtain

$$\lim_{t \to t_0 + T} \mu^k e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)} = 0 \tag{81}$$

holds for any $k \in R$.

By (80) and (81), we get

$$\lim_{t \to t_0 + T} E|x|^2 = 0. \tag{82}$$

Similar to (80) and (82), it follows from (66) and (79) that

$$\begin{aligned} Eu^2 \leq\, & \sqrt{n}\alpha_n^2\mu^{2\delta_n}\Bigg(x_1^4(t_0) \\ & + \sum_{k=2}^n \left(x_k(t_0) + \sum_{i=1}^{k-1}\prod_{j=i}^{k-1}\alpha_j x_i(t_0)\right)^4\Bigg)^{1/2} \\ & \cdot e^{-\frac{cT^m}{2(m-1)}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)} \forall\, t \in [t_0, t_0 + T), \end{aligned} \tag{83}$$

$$\lim_{t \to t_0 + T} E|u|^2 = 0. \tag{84}$$

Remark 5: In this section, we propose a new nonscaling backstepping design scheme for stochastic nonlinear system (31)–(32) to achieve prescribed-time mean-square stable. The merit of this design is not using the time-varying μ to scale the coordinate transformations $\xi_i = x_i - x_i^*$, $i = 1, \ldots, n$, and μ is suitably introduced into the virtual controller x_i^*. This approach is essentially different from the scaling method developed in [20], [21], and [24]–[26] where the scaled transformation $\xi_i = \mu^k(x_i - x_i^*)$ is used for the controller design at every step. The main advantage of our approach is that a simpler controller is designed and the computation burden arising from the derivative of μ is largely reduced with nonscaling transformations. Therefore, the control effort can be saved.

It should be emphasized that, even when there is no noise in system (31)–(32), the design method developed in this article is also new for deterministic nonlinear systems. For the prescribed-time control of deterministic nonlinear systems, to the best of the authors' knowledge, only the scaling design method is studied in [20], [21], and [24]–[26]. Next, we use a scalar example to demonstrated the differences between the two methods.

Consider the scalar system

$$\dot{x} = u + \varphi(t,x) \tag{85}$$

where $\varphi(t,x)$ is a known continuous function.

The scaling method in [20], [21], and [24]–[26]: With the scaling transformation $z = \mu x$ and the Lyapunov function $U_1 = \frac{1}{2}z^2$, we have

$$\begin{aligned} u = & -\varphi(t,x) - k_0\left(\frac{T}{t_0 + T - t}\right)^m x \\ & - \frac{m}{T}\left(\frac{T}{t_0+T-t}\right)x \end{aligned} \tag{86}$$

$$\begin{aligned} |x(t)| = & \left(1 - \frac{t-t_0}{T}\right)^m e^{-\frac{k_0T^m}{m-1}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)} \\ & \cdot |x(t_0)| \qquad \forall\, t \in [t_0, t_0 + T) \end{aligned} \tag{87}$$

where k_0 is a positive constant.

The nonscaling method in this article: Using the nonscaling transformation $z = x$ and the Lyapunov function $U_2 = \frac{1}{2}z^2$, we get

$$u = -\varphi(t,x) - k_0\left(\frac{T}{t_0+T-t}\right)^m x \tag{88}$$

$$\begin{aligned} |x(t)| = & e^{-\frac{k_0T^m}{m-1}\left(\frac{1}{(t_0+T-t)^{m-1}} - \frac{1}{T^{m-1}}\right)} \\ & \cdot |x(t_0)| \qquad \forall\, t \in [t_0, t_0 + T). \end{aligned} \tag{89}$$

From (86)–(89), the above two approaches can both make system (85) achieve prescribed-time stable with $\lim_{t\to t_0+T}|x| = \lim_{t\to t_0+T}|u| = 0$. However, compared with the controller (86), our controller (88) is simpler, and, thus, it is easier for implementation. With the order of the system increasing, this advantage becomes more obvious. From (87) and (89), we find that the cost for a simpler controller is the relatively slower converge speed.

Remark 6: From the controller design and the stability analysis developed above, it can be observed that the linear growth condition in Assumption 1 guarantees the existence of the moments of x and u. This is a reasonable assumption since if the f_i's and g_i's grow too fast, the moments of x and u will go to infinity in very short time [2]. In addition, as shown by [33, Th. 5.4, Pg. 133], [41, Th. 10.7.2, Pg. 210], and [42, Th. 4.4 in Pg. 61], the linear growth condition in Assumption 1 is crucial to make stochastic systems achieve mean-square stable.

In this article, if Assumption 1 is not satisfied, by taking a similar design procedure as that in Section III, it is easily concluded that $\alpha_1,\ldots,\alpha_{k-1}$ in (38)–(40), α_k in (64), and α_n in (66) will be not constants but abstract functions (similar to [5, (3.23)], it is nearly impossible to express them as an explicit form of $x_1,\ldots,x_n$ for system (31)–(32)]. Next, we explain what prevents to generalize the design in this article if Assumption 1 fails.

1) If the drift terms $f_1(t,x),\ldots,f_{n-1}(t,x)$ and the diffusion terms $g_1(t,x),\ldots,g_{n-1}(t,x)$ do not satisfy Assumption 1, it follows from the design process that α_{n-1} will be an abstract nonnegative smooth function of $x_1,\ldots,x_{n-1}$, which implies that $H_1(\mu)$ in (74) will be an abstract function of $x_1,\ldots,x_{n-1}$. Although we can clearly describe the structure μ in $H_1(\mu,x)$, it is nearly impossible to tell the concrete structure of $x=(x_1,\ldots,x_n)^T$ in $H_1(\mu,x)$, which means that we cannot analyze the mean-square property of x since it is difficult to analyze $E\{H_1(\mu,x)\}$. Thus, even we can get prescribed-time mean-square stable for $\xi_1,\ldots,\xi_n$ from (70), it is difficult to achieve prescribed-time mean-square stable for the states $x_1,\ldots,x_n$.
2) If the drift terms $f_n(t,x)$ and the diffusion terms $g_n(t,x)$ do not satisfy Assumption 1, it follows from the design process that α_n in (66) will be an abstract nonnegative smooth function of $x_1,\ldots,x_n$. Therefore, it is difficult to get the mean-square property of the controller u since it is difficult to analyze $E\{\alpha_n\}$.

Nevertheless, in some special cases, the requirement of the linear growth condition on f_i's and g_i's can be slightly relaxed. For example, if $f_i(t,x)$ and $g_i(t,x)$ $(1\le i\le n-1)$ satisfy Assumption 1, but $f_n(t,x)$ and $g_n(t,x)$ do not satisfy Assumption 1, by following the design and analysis method developed in Section III, we can still solve the prescribed-time mean-square stabilization problem to get (71). However, for the controller u, we can only get a weaker result than (72) since α_n in the controller (66) may no longer be a constant but a function of x. In the simulation section, we use Example 4 to further demonstrate this point.

IV. Prescribed-Time Inverse Optimal Stabilization

In this section, we redesign the controller to solve the prescribed-time inverse optimal mean-square stabilization problem for system (31)–(32).

First, we rewrite (31)–(32) as

$$\begin{aligned} dx &= \begin{bmatrix} x_2+f_1(t,x)\\ \vdots\\ x_n+f_{n-1}(t,x)\\ f_n(t,x)\end{bmatrix}dt + \begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}udt \\ &\quad + \begin{bmatrix} g_1^T(t,x)\\ \vdots\\ g_{n-1}^T(t,x)\\ g_n^T(t,x)\end{bmatrix}d\omega \\ &= F(t,x)dt + G_1udt + G_2(t,x)d\omega. \end{aligned} \tag{90}$$

From the design in Section III, the controller

$$u = -\mu^{\delta_n}\alpha_n\xi_n \tag{91}$$

with $c_i = a_{n,i}+\frac{1}{4}$ and $c_n=\frac{1}{4}$ leads to

$$\mathcal{L}V_n|_{(90)} \le -\mu V_n \tag{92}$$

where the definitions of δ_{n+1}, α_n, ξ_n, and V_n can be found in Section III.

In the following theorem, we give the main results on prescribed-time inverse optimal mean-square stabilization for system (31)–(32).

Theorem 2: If Assumption 1 holds for system (31)–(32), then the control law

$$u^*(t,x) = -\frac{2}{3}\beta\mu^{\delta_n}\alpha_n\xi_n,\quad \beta\ge 2\quad \forall\, t\in[t_0,t_0+T) \tag{93}$$

solves the prescribed-time inverse optimal mean-square stabilization problem for system (31)–(32) by minimizing the cost functional

$$\begin{aligned} J(u) &= \limsup_{r\to\infty} E\left[2\beta V_n(\tau_r,x(\tau_r))\right.\\ &\quad \left.+\int_{t_0}^{\tau_r}\left(l(t,x(t))+\frac{27}{16\beta^2\mu^{3\delta_n}(t)\alpha_n^3}u^4(t)\right)dt\right] \end{aligned} \tag{94}$$

where

$$\begin{aligned} l(t,x) &= 2\beta\Bigg(\mu^{\delta_n}\alpha_n\xi_n^4 - \frac{\partial V_n}{\partial t} - \frac{\partial V_n}{\partial x}F \\ &\quad -\frac{1}{2}\mathrm{Tr}\left\{G_2^T\frac{\partial^2 V_n}{\partial x^2}G_2\right\}\Bigg) + \beta(\beta-2)\mu^{\delta_n}\alpha_n\xi_n^4 \end{aligned} \tag{95}$$

is positive definite and radially unbounded but not necessarily decrescent.

Proof: From (92) and (95), we obtain

$$l(t,x) \ge 2\beta\mu V_n + \beta(\beta-2)\mu^{\delta_n}\alpha_n\xi_n^4. \tag{96}$$

By (2), (76), and the definition of V_n, we get

$$V_n \ge \frac{1}{4n}|\xi|^4 \ge \frac{1}{4n\left(n+\sum_{i=1}^{n-1}\alpha_i^2\mu^{2\delta_i}\right)^2}|x|^4. \tag{97}$$

Noting $\beta\ge 2$, from (96) and (97), we have

$$l(t,x) \ge 2\beta\mu V_n \ge \frac{\beta\mu}{2n\left(n+\sum_{i=1}^{n-1}\alpha_i^2\mu^{2\delta_i}\right)^2}|x|^4. \quad (98)$$

Since $\frac{\beta\mu}{2n(n+\sum_{i=1}^{n-1}\alpha_i^2\mu^{2\delta_i})^2}$ is a positive scaling function and $|x|^4$ is a positive definite function of x, $l(t,x)$ is well defined. Thus, $J(u)$ is a meaningful cost functional.

Before proving the controller (93) minimizes (94), we first prove it is a stabilizing controller for system (90). From (91)–(92), noting $\beta \ge 2$, we have

$$\begin{aligned}\mathcal{L}V_n|_{(90)} =& -\frac{2}{3}\beta\mu^{\delta_n}\alpha_n\xi_n^4 + \frac{\partial V_n}{\partial t} + \frac{\partial V_n}{\partial x}F \\ &+ \frac{1}{2}\mathrm{Tr}\left\{G_2^T\frac{\partial^2 V_n}{\partial x^2}G_2\right\} \\ \le& -\mu^{\delta_n}\alpha_n\xi_n^4 + \frac{\partial V_n}{\partial t} + \frac{\partial V_n}{\partial x}F \\ &+ \frac{1}{2}\mathrm{Tr}\left\{G_2^T\frac{\partial^2 V_n}{\partial x^2}G_2\right\} \\ \le& -\mu V_n. \end{aligned} \quad (99)$$

By (99) and Theorem 1, the controller (93) can make system (90) achieve prescribed-time mean-square stable.

Now, we prove optimality. By Dynkin's formula in Lemma A.1, we get

$$E\left\{V_n(\tau_r, x(\tau_r)) - V_n(t_0, x(t_0)) - \int_{t_0}^{\tau_r}\mathcal{L}V_n|_{(90)}ds\right\} = 0. \quad (100)$$

From (94)–(95) and (100), we have

$$\begin{aligned}J(u) =& \limsup_{r\to\infty} E\left[2\beta V_n(\tau_r, x(\tau_r)) + \int_{t_0}^{\tau_r}\left(l(t,x) + \frac{27}{16\beta^2\mu^{3\delta_n}\alpha_n^3}u^4\right)dt\right] \\ =& \limsup_{r\to\infty} E\left[2\beta V_n(t_0, x(t_0)) + \int_{t_0}^{\tau_r}\left(2\beta\mathcal{L}V_n|_{(90)} + l(t,x) + \frac{27}{16\beta^2\mu^{3\delta_n}\alpha_n^3}u^4\right)dt\right] \\ =& \limsup_{r\to\infty} E\left[2\beta V_n(t_0, x(t_0)) + \int_{t_0}^{\tau_r}\left(2\beta\xi_n^3 u + \beta^2\mu^{\delta_n}\alpha_n\xi_n^4 + \frac{27}{16\beta^2\mu^{3\delta_n}\alpha_n^3}u^4\right)dt\right]. \end{aligned} \quad (101)$$

By using Lemma A.3 with $\gamma = r^{4/3}$, we obtain

$$\begin{aligned}-2\beta\xi_n^3 u =& \beta^2\left(-\xi_n^3\mu^{3\delta_n/4}\alpha_n^{3/4}\right)\left(\frac{2}{\beta}\mu^{-3\delta_n/4}\alpha_n^{-3/4}u\right) \\ \le& \beta^2\mu^{\delta_n}\alpha_n\xi_n^4 + \frac{27}{16\beta^2\mu^{3\delta_n}\alpha_n^3}u^4. \end{aligned} \quad (102)$$

The equality in (102) holds when

$$u^*(t,x) = -\frac{2}{3}\beta\mu^{\delta_n}\alpha_n\xi_n. \quad (103)$$

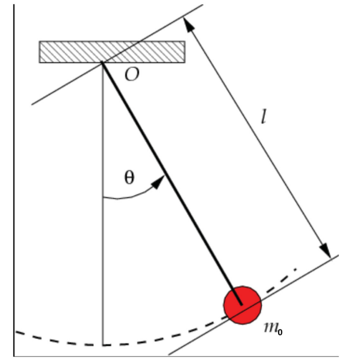


Fig. 1. Pendulum system.

Therefore, the minimum of (94) is obtained with $u(t,x) = u^*(t,x)$ in (103), and

$$\min_u J(u) = 2\beta V_n(t_0, x(t_0)). \quad (104)$$

Thus, the theorem is proved.

Remark 7: Even though not explicit in the statement of Theorem 2, $V_n(t,x)$ solves the following family of Hamilton–Jacobi–Bellman equations parameterized by $\beta \in [2, +\infty)$

$$\frac{\partial V_n}{\partial t} + \frac{\partial V_n}{\partial x}F + \frac{1}{2}\mathrm{Tr}\left\{G_2^T\frac{\partial^2 V_n}{\partial x^2}G_2\right\} - \frac{\beta}{2}\mu^{\delta_n}\alpha_n\xi_n^4 + \frac{l(t,x)}{2\beta} = 0. \quad (105)$$

Remark 8: By choosing $k_5 = -3\delta_n$, $S(t,x) = 2\beta V_n(t,x)$, $l(t,x)$ in (95), $\gamma(r) = r^4$, and $R(x) = (\frac{27}{16\beta^2\alpha_n^3})^{1/4}$, Theorem 2 solves the prescribed-time inverse optimal mean-square stabilization problem described in Definition 2.

V. Two Simulation Examples

In this section, we give two simulation examples to show the effectiveness of the prescribed-time control schemes developed in this article.

Example 2: Consider the pendulum system shown in Fig. 1. By Newton's law of motion, the system is described as [43]

$$m_0 l\ddot{\theta} = -m_0 g\sin\theta - kl\dot{\theta} + \frac{1}{l}\bar{T} \quad (106)$$

where l denotes the length of the rod, m_0 denotes the mass of the bob, and g denotes the acceleration due to gravity. Assume the rod is rigid and has zero mass. Let θ denote the angle subtended by the rod and the vertical axis through the pivot point. The pendulum is free to swing in the vertical plane. The bob of the pendulum moves in a circle of radius l. There is also a frictional force resisting the motion, which we assume to be proportional to the speed of the bob with a coefficient of friction k. $\bar{T}$ is the torque applied to the pendulum.

Suppose we want to stabilize the pendulum at an angle $\theta = \theta_0$. First of all, we need to apply a torque $\bar{T}_0$ to make θ_0 an equilibrium

$$\bar{T}_0 - m_0 gl\sin\theta_0 = 0. \quad (107)$$

Motivated by [44], we consider the coefficient of friction $k(t)$ with a nominal value k_0 and $k(t) \in (k_0 - 0.8, k_0 + 0.8)$. Let $\Delta(t) = k(t) - k_0$. $\Delta(t)$ is the Gaussian white noise process with zero mean and $E(\Delta(t))^2 = \sigma^2$. We can then choose the value of parameter σ such that $k(t)$ obeys the bound $-0.8 \le k(t) - k_0 \le 0.8$ with a sufficiently high probability. For example, for $\sigma = 0.08$, by Chebyshev's inequality in [2], we can get

$$P(|k(t) - k_0|^2 > 0.64) \le \frac{\sigma^2}{0.64} = 0.01 \tag{108}$$

which means that

$$\begin{aligned} P(|k(t) - k_0| \le 0.8) &= 1 - P(|k(t) - k_0| > 0.8) \\ &= 1 - P(|k(t) - k_0|^2 > 0.64) \\ &\ge 0.99. \end{aligned} \tag{109}$$

Obviously, $P(|k(t) - k_0| \le 0.8)$ is increasing as $\sigma^2 \downarrow 0$.

To obtain a state model for the pendulum system, choose the state variables $x_1 = \theta - \theta_0$ and $x_2 = \dot{\theta}$, and the control $u = \bar{T} - \bar{T}_0$. Then, from (106), we get the state-space form as

$$dx_1 = x_2 dt \tag{110}$$

$$\begin{aligned} dx_2 = &\left(\frac{1}{m_0 l^2} u - \frac{g}{l}\sin(x_1 + \theta_0) - \frac{k_0}{m_0} x_2 + \frac{1}{m_0 l^2}\bar{T}_0\right) dt \\ &- \frac{\sigma}{m_0} x_2 d\omega. \end{aligned} \tag{111}$$

Choosing $\theta_0 = \frac{\pi}{6}$, $l = g$ and $m_0 = k_0 = \sigma = \frac{1}{g^2}$, by (107), we have $\bar{T}_0 = \frac{1}{2}$. Then (110)–(111) can be written as

$$dx_1 = x_2 dt \tag{112}$$

$$dx_2 = \left(u - \sin(x_1 + \frac{\pi}{6}) - x_2 + \frac{1}{2}\right) dt - x_2 d\omega. \tag{113}$$

Noting that

$$\begin{aligned} &\frac{1}{2} - \sin(x_1 + \frac{\pi}{6}) - x_2 \\ &= \sin\frac{\pi}{6} - \sin(x_1 + \frac{\pi}{6}) - x_2 \\ &= -2\cos\left(\frac{1}{2}x_1 + \frac{\pi}{6}\right)\sin\frac{x_1}{2} + x_2 \end{aligned} \tag{114}$$

we have

$$\left|\frac{1}{2} - \sin(x_1 + \frac{\pi}{6}) - x_2\right| \le |x_1| + |x_2| \tag{115}$$

which means that Assumption 1 is satisfied.

By following the design procedure developed in Section III, we design the controller as

$$\begin{aligned} u = &-\mu^3\left(11 + c_1 + c_2 + 9c_1^4 + \frac{3}{4}(1 + 3c_1 + c_1^2)^{4/3}\right) \\ &\cdot (x_2 + c_1\mu x_1). \end{aligned} \tag{116}$$

For simulation, we select $t_0 = 0$, $T = 1$, $m = 2$, the parameters $c_1 = 1$, $c_2 = \frac{1}{2}$, and randomly set the initial conditions as $x_1(0) = -2$, $x_2(0) = 0.6$. Fig. 2 gives the response of the closed-loop system (112), (113), and (116). From Fig. 2, we find that $\lim_{t\to 1} E|x|^2 = 0$, which means that prescribed-time mean-square stabilization is achieved. Therefore, the effectiveness of the controller design developed in Section III is demonstrated.

In Remark 6, we claim that even when Assumption 1 is not satisfied, by following the controller design developed in this article, we still solve the prescribed-time mean-square stabilization problem when the nonlinear terms are in some special forms. In the following example, we give the detailed design procedure to demonstrate this point.

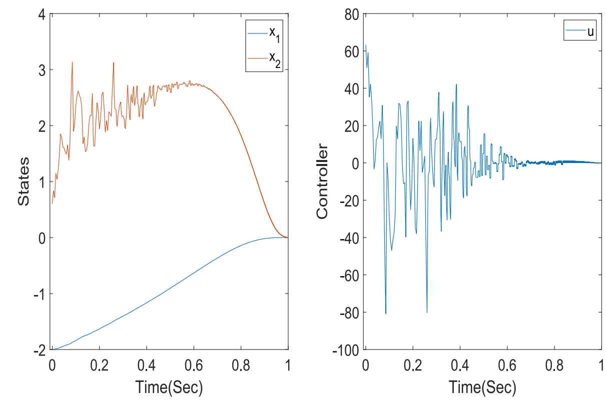


Fig. 2. Response of the closed-loop system (112), (113), and (116).

Example 3: Consider the following system:

$$dx_1 = x_2 dt + x_1 \sin x_1 d\omega \tag{117}$$

$$dx_2 = (u + x_1 x_2)dt + x_1^{5/3} d\omega. \tag{118}$$

Since the drift term $x_1 x_2$ and the diffusion term $x_1^{5/3}$ do not satisfy the linear growth condition required in Assumption 1, we give the detailed controller design process for the system (117)–(118).

Step 1. Defining $\xi_1 = x_1$ and considering $V_1(\xi_1) = \frac{1}{4}\xi_1^4$, we have

$$\mathcal{L}V_1 \le \xi_1^3(x_2 - x_2^*) + \xi_1^3 x_2^* + \frac{3}{2}\xi_1^4. \tag{119}$$

Choosing the virtual controller

$$x_2^* = -\left(c_1 + \frac{3}{2}\right)\mu\xi_1 \tag{120}$$

(119) can be rewritten as

$$\mathcal{L}V_1 \le -c_1\mu\xi_1^4 + \xi_1^3(x_2 - x_2^*) \tag{121}$$

where $c_1 > \frac{5}{4}$ is a design parameter.

Step 2. Choosing $\xi_2 = x_2 - x_2^*$, from (118) and (120), we have

$$\begin{aligned} d\xi_2 = &\left(u + x_1 x_2 + \frac{m}{T}\left(c_1 + \frac{3}{2}\right)\mu^{(m+1)/m} x_1 \right. \\ &\left. + \left(c_1 + \frac{3}{2}\right)\mu x_2\right) dt \\ &+ \left(x_1^{5/3} + \left(c_1 + \frac{3}{2}\right)\mu x_1 \sin x_1\right) d\omega. \end{aligned} \tag{122}$$

With the Lyapunov function $V_2 = V_1 + \frac{1}{4}\xi_2^4$, by (121) and (122), we obtain

$$\begin{aligned} \mathcal{L}V_2 \le &-c_1\mu\xi_1^4 + \xi_1^3\xi_2 + \xi_2^3 u + \xi_2^3\left(x_1 x_2 \right. \\ &\left. + \frac{m}{T}\left(c_1 + \frac{3}{2}\right)\mu^{(m+1)/m} x_1 + \left(c_1 + \frac{3}{2}\right)\mu x_2\right) \\ &+ \frac{3}{2}\xi_2^2\left(x_1^{5/3} + \left(c_1 + \frac{3}{2}\right)\mu x_1 \sin x_1\right)^2. \end{aligned} \tag{123}$$

By using Lemma A.2, we get the following estimates:

$$\xi_1^3\xi_2 \le \frac{1}{4}\mu\xi_1^4 + \frac{27}{4}\xi_2^4, \quad (124)$$

$$\xi_2^3 x_1 x_2 \le |\xi_1||\xi_2|^3|x_2| \le \frac{1}{4}\mu\xi_1^4 + \frac{3}{4}x_2^{4/3}\xi_2^4, \quad (125)$$

$$\frac{m}{T}\left(c_1+\frac{3}{2}\right)\mu^{(m+1)/m}x_1\xi_2^3 \le \frac{1}{4}\mu\xi_1^4 + \frac{3}{4}\mu^2\left(\frac{m}{T}\right)^{4/3}\left(c_1+\frac{3}{2}\right)^{4/3}\xi_2^4, \quad (126)$$

$$\left(c_1+\frac{3}{2}\right)\mu x_2\xi_2^3 \le \frac{1}{4}\mu\xi_1^4 + \mu^{7/3}\left(c_1+\frac{3}{2}+\frac{3}{4}\left(c_1+\frac{3}{2}\right)^{8/3}\right)\xi_2^4, \quad (127)$$

$$\frac{3}{2}\xi_2^2\left(x_1^{5/3}+\left(c_1+\frac{3}{2}\right)\mu x_1\sin x_1\right)^2 \le \frac{1}{4}\mu\xi_1^4 + \frac{9}{4}\mu^3\left(c_1+\frac{3}{2}+x_1^{2/3}\right)^4\xi_2^4. \quad (128)$$

Noting that $\mu^3 \ge \mu^{7/3} \ge \mu^2$, substituting (124)–(128) into (123), we get

$$\mathcal{L}V_2 \le -\left(c_1-\frac{5}{4}\right)\mu\xi_1^4 + \xi_2^3 u + \mu^3\xi_2^4\left(c_1+\frac{33}{4} + \frac{3}{4}x_2^{4/3} + \frac{3}{4}\left(\frac{m}{T}\right)^{4/3}\left(c_1+\frac{3}{2}\right)^{4/3} + \frac{3}{4}\left(c_1+\frac{3}{2}\right)^{8/3} + \frac{9}{4}\left(c_1+\frac{3}{2}+x_1^{2/3}\right)^4\right). \quad (129)$$

Choosing the actual control law as

$$u = -\left(c_1+c_2+\frac{33}{4}+\frac{3}{4}x_2^{4/3}+\frac{3}{4}\left(\frac{m}{T}\right)^{4/3}\cdot\left(c_1+\frac{3}{2}\right)^{4/3}+\frac{3}{4}\left(c_1+\frac{3}{2}\right)^{8/3}+\frac{9}{4}\left(c_1+\frac{3}{2}+x_1^{2/3}\right)^4\right)\mu^3\xi_2 \quad (130)$$

which substituting into (129) yields

$$\mathcal{L}V_2 \le -\left(c_1-\frac{5}{4}\right)\mu\xi_1^4 - c_2\mu^3\xi_2^4 \le -\tilde{c}_0\mu V_2 \quad (131)$$

where $c_2 > 0$ is a design parameter and $\tilde{c}_0 = 4\min\{c_1-\frac{5}{4}, c_2\}$.

For simulation, we select $t_0 = 0$, $T = 2$, $m = 2$, the parameters $c_1 = \frac{3}{2}$, $c_2 = \frac{1}{2}$, and randomly set the initial conditions as $x_1(0) = -0.5$, $x_2(0) = 2$. Similar to (71), we obtain

$$E|x|^2 \le \left(1+\frac{72}{(2-t)^4}\right)e^{-\frac{2}{2-t}+1}\ \forall\, t\in[0,2) \quad (132)$$

which means that

$$E|x|^2 \text{ is bounded on } [0,2) \text{ and } \lim_{t\to 2} E|x|^2 = 0. \quad (133)$$

From (130), using Lemma A.2, we obtain

$$E|u| \le M\mu^{13/3}\left(E\left\{|\xi_1|^{7/3}\right\} + E\left\{|\xi_1|^{11/3}\right\} + E|\xi_2| + E\left\{|\xi_2|^{7/3}\right\} + E\left\{|\xi_2|^{11/3}\right\}\right)\ \forall\, t\in[0,2) \quad (134)$$

where M is a positive constant.

By (131), similar to (77), we obtain

$$E(\xi_1^4+\xi_2^4) \le \frac{1}{8}e^{-\frac{4}{2-t}+2}\quad \forall\, t\in[0,2). \quad (135)$$

From the Lyapunov inequality, we know

$$\left(E\left\{|\xi_1|^{7/3}\right\}\right)^{3/7} \le \left(E\left\{|\xi_1|^{11/3}\right\}\right)^{3/11} \le \left(E\left\{|\xi_1|^4\right\}\right)^{1/4} \quad (136)$$

$$E\left\{|\xi_2|\right\} \le \left(E\left\{|\xi_1|^{7/3}\right\}\right)^{3/7} \le \left(E\left\{|\xi_2|^{11/3}\right\}\right)^{3/11} \le \left(E\left\{|\xi_2|^4\right\}\right)^{1/4}. \quad (137)$$

It follows from (134)–(137) that

$$E|u| \text{ is bounded on } [0,2) \text{ and } \lim_{t\to 2} E|u| = 0. \quad (138)$$

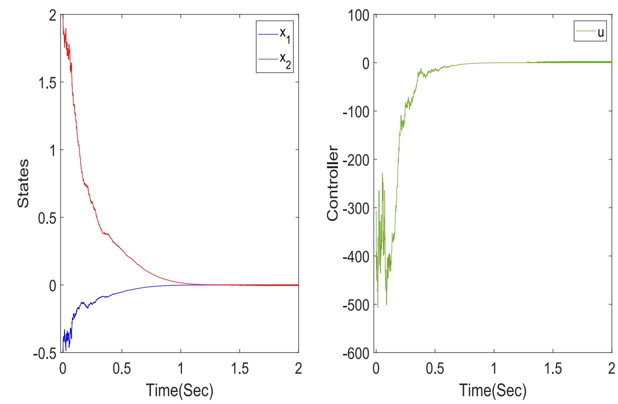


Fig. 3. Response of the closed-loop system (117), (118), and (130).

Fig. 3 gives the response of the closed-loop system (117), (118), and (130). From Fig. 3, we can find that $\lim_{t\to 2} E|x|^2 = \lim_{t\to 2} E|u| = 0$, which verifies the properties in (133) and (138). In other words, the prescribed-time mean-square stabilization can be achieved. Therefore, the effectiveness of the controller design developed in (119)–(130) is demonstrated.

VI. Conclusion

In this article, we have addressed the prescribed-time mean-square stabilization and inverse optimality control design for stochastic strict-feedback nonlinear systems. By developing a new nonscaling backstepping design method, a new controller is designed to guarantee that the equilibrium at the origin of the closed-loop system is prescribed-time mean-square stable. In addition, we redesign the controller to achieve an infinite gain margin and prescribed-time inverse optimal mean-square stabilization. Specifically, the optimal controller minimizes a meaningful cost functional and stabilizes the closed-loop system in prescribed-time simultaneously.

For the stochastic nonlinear systems, many open issues are worth investigating, such as prescribed-time output-feedback, control for more general systems, etc.

Appendix

A. Useful Lemmas

Lemma A.1 [1]: Let $V \in C^{1,2}(R^+ \times R^n; R^+)$ and τ_1, τ_2 be bounded stopping times such that $0 \le \tau_1 \le \tau_2$ $a.s.$ If $V(t,x)$ and $\mathcal{L}V(t,x)$ are bounded on $t \in [\tau_1,\ \tau_2]$ a.s., then

$$E[V(\tau_2,x) - V(\tau_1,x)] = E\left\{\int_{\tau_1}^{\tau_2} \mathcal{L}V(t,x)dt\right\}. \quad \text{(A.1)}$$

Lemma A.2 [45]: Let x, y be real variables; then for any positive real numbers a, m, and n, we have

$$ax^m y^n \le b|x|^{m+n} + \frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-\frac{m}{n}} a^{\frac{m+n}{n}} b^{-\frac{m}{n}} |y|^{m+n} \quad \text{(A.2)}$$

where $b > 0$ is any real number.

Lemma A.3 [46]: For any two vectors x and y, the following holds:

$$x^T y \le \gamma(|x|) + \ell_\gamma(|y|) \quad \text{(A.3)}$$

and the equality is achieved if and only if

$$y = \dot{\gamma}(|x|)\tfrac{x}{|x|} \quad \text{(A.4)}$$

where γ and its derivative $\dot{\gamma}$ are both $\mathcal{K}_\infty$ functions.

B. Proof of Lemma 1

Step 1. We first show that system (3) has an almost surely unique solution on $[t_0, t_0+T)$ for any initial condition $x_0 \in R^n$.

Since system (3) has an almost surely unique solution $x(t)$ on $[t_0, \rho_\infty)$ with $\rho_\infty = (t_0+T) \wedge \lim_{\tau\to+\infty} \inf\{t_0 \le t < t_0 + T : |x(t)| \ge \tau\}$, we need to prove $\rho_\infty = t_0 + T$ a.s. If this is not true, we can find positive constants ε and T_2 $(0 < T_2 < T)$ such that

$$P\{\rho_\infty \le t_0 + T_2\} > 2\varepsilon. \quad \text{(B.1)}$$

For each integer $k > 0$, define

$$\rho_k = (t_0+T) \wedge \inf\{t : t_0 \le t < t_0+T,\ |x(t)| \ge k\}. \quad \text{(B.2)}$$

Since $\rho_k \to \rho_\infty$ a.s., there exists a sufficiently large integer k_0 such that

$$P\{\rho_k \le t_0 + T_2\} > \varepsilon \quad \forall k \ge k_0. \quad \text{(B.3)}$$

Choosing

$$\bar{U} = e^{c_0\int_{t_0}^t \mu(s)ds} U. \quad \text{(B.4)}$$

From (10) and $(B.4)$, we have

$$\begin{aligned}\mathcal{L}\bar{U} &= e^{c_0\int_{t_0}^t \mu(s)ds}(\mathcal{L}U + c_0\mu U)\\ &\le M_0\mu e^{c_0\int_{t_0}^t \mu(s)ds}.\end{aligned} \quad \text{(B.5)}$$

Fix $k \ge k_0$. For any $t_0 \le t \le t_0 + T_2$, by $(B.5)$ and Lemma A.1, we have

$$\begin{aligned}
&E\bar{U}(t\wedge\rho_k, x(t\wedge\rho_k))\\
&= U(t_0,x_0) + E\left\{\int_{t_0}^{t\wedge\rho_k} \mathcal{L}\bar{U}(x(\tau),\tau)d\tau\right\}\\
&\le U(t_0,x_0) + M_0 E\left\{\int_{t_0}^{t\wedge\rho_k} \mu(\tau)e^{c_0\int_{t_0}^{\tau}\mu(s)ds}d\tau\right\}\\
&= U(t_0,x_0) + M_0 E\left\{\int_{t_0}^{t\wedge\rho_k} e^{c_0\int_{t_0}^{\tau}\mu(s)ds} \cdot d\left(\int_{t_0}^{\tau}\mu(s)ds\right)\right\}\\
&= U(t_0,x_0) + \frac{M_0}{c_0}E\left\{e^{c_0\int_{t_0}^{\tau}\mu(s)ds}\right\}\Big|_{\tau=t_0}^{\tau=t\wedge\rho_k}\\
&\le U(t_0,x_0) + \frac{M_0}{c_0}e^{c_0\int_{t_0}^{t}\mu(s)ds} - \frac{M_0}{c_0}.
\end{aligned} \quad \text{(B.6)}$$

By $(B.6)$, we get

$$\begin{aligned}&E\bar{U}((t_0+T_2)\wedge\rho_k, x((t_0+T_2)\wedge\rho_k))\\ &\quad\le U(t_0,x_0) + \tfrac{M_0}{c_0}e^{c_0\int_{t_0}^{t_0+T_2}\mu(s)ds}.\end{aligned} \quad \text{(B.7)}$$

It follows from (2) and $(B.7)$ that

$$\begin{aligned}
&E\chi_{\rho_k\le t_0+T_2}\bar{U}(\rho_k, x(\rho_k))\\
&\le U(t_0,x_0) + \frac{M_0}{c_0}e^{c_0\int_{t_0}^{t_0+T_2}\mu(s)ds}\\
&= U(t_0,x_0) + \frac{M_0}{c_0}e^{\frac{c_0T^m}{m-1}\left(\frac{1}{(T-T_2)^{m-1}} - \frac{1}{T^{m-1}}\right)}\\
&< +\infty. \quad \text{(B.8)}
\end{aligned}$$

Define

$$b_k = \inf\left\{\bar{U}(t,x) : |x| \ge k, t \in [t_0, t_0+T_2]\right\}. \quad \text{(B.9)}$$

By (9) and $(B.4)$, we get

$$\lim_{k\to+\infty} b_k = +\infty. \quad \text{(B.10)}$$

From $(B.8)$, we obtain

$$\begin{aligned}&U(t_0,x_0) + \tfrac{M_0}{c_0}e^{\frac{c_0T^m}{m-1}\left(\frac{1}{(T-T_2)^{m-1}} - \frac{1}{T^{m-1}}\right)}\\ &\quad\ge b_k P\{\rho_k \le t_0+T_2\} > \varepsilon b_k.\end{aligned} \quad \text{(B.11)}$$

Letting $k \to +\infty$ in both sides of $(B.11)$, from $(B.10)$, we obtain

$$U(t_0,x_0) + \tfrac{M_0}{c_0}e^{\frac{c_0T^m}{m-1}\left(\frac{1}{(T-T_2)^{m-1}} - \frac{1}{T^{m-1}}\right)} = +\infty \quad \text{(B.12)}$$

which is a contradiction with $(B.8)$. Thus, we have $\rho_\infty = t_0 + T$.

Step 2. We then prove the function $U(t, x(t))$ satisfies (11).

Let k be a positive integer. Define the stopping time

$$\sigma_k = \inf\{t : t_0 \le t < t_0+T,\ |x(t)| \ge k\}. \quad \text{(B.13)}$$

From *Step 1*, system (3) has an almost surely unique solution on $[t_0, t_0+T)$. Thus, $\sigma_k \to +\infty$ almost surely as $k \to +\infty$.

Let $t_k = \sigma_k \wedge t$ for any $t \in [t_0, t_0+T)$. Noting $\bar{U}(t_0,x_0) = U(t_0,x_0)$ and using Itô's formula on the interval $[t_0, t_k]$, we get

$$\begin{aligned}\bar{U}(t_k, x(t_k)) &= U(t_0,x_0) + \textstyle\int_{t_0}^{t_k}\mathcal{L}\bar{U}(x(\tau),\tau)d\tau\\ &\quad + \textstyle\int_{t_0}^{t_k}\frac{\partial\bar{U}}{\partial x}g^T(\tau,x(\tau))d\omega(\tau).\end{aligned} \quad \text{(B.14)}$$

With the definition of t_k, using $(B.5)$, taking expectation on both sides of $(B.14)$, we obtain

$$\begin{aligned}
&E\bar{U}(t_k, x(t_k))\\
&= U(t_0, x_0) + E\left\{\int_{t_0}^{t_k} \mathcal{L}\bar{U}(x(\tau), \tau)d\tau\right\}\\
&\le U(t_0, x_0) + M_0 E\left\{\int_{t_0}^{t_k} \mu(\tau)e^{c_0\int_{t_0}^{\tau}\mu(s)ds}d\tau\right\}\\
&= U(t_0, x_0) + M_0 E\bigg\{\int_{t_0}^{t_k} e^{c_0\int_{t_0}^{\tau}\mu(s)ds}\\
&\quad \cdot d\bigg(\int_{t_0}^{\tau}\mu(s)ds\bigg)\bigg\}\\
&= U(t_0, x_0) + \frac{M_0}{c_0}E\left\{e^{c_0\int_{t_0}^{\tau}\mu(s)ds}\right\}\bigg|_{\tau=t_0}^{\tau=t_k}\\
&\le U(t_0, x_0) + \frac{M_0}{c_0}\left(e^{c_0\int_{t_0}^{t}\mu(s)ds} - 1\right). \qquad \text{(B.15)}
\end{aligned}$$

By $(B.4)$ and $(B.15)$, we get

$$\begin{aligned}
&E\left\{e^{c_0\int_{t_0}^{t_k}\mu(s)ds}U(t_k, x(t_k))\right\} \le U(t_0, x_0)\\
&+ \frac{M_0}{c_0}\left(e^{c_0\int_{t_0}^{t}\mu(s)ds} - 1\right) \quad \forall\, t \in [t_0, t_0+T). \qquad \text{(B.16)}
\end{aligned}$$

Letting $k \to +\infty$, using Fatou Lemma, $(B.16)$ can be rewritten as

$$\begin{aligned}
&E\left\{e^{c_0\int_{t_0}^{t}\mu(s)ds}U(t, x(t))\right\} \le U(t_0, x_0)\\
&+ \frac{M_0}{c_0}\left(e^{c_0\int_{t_0}^{t}\mu(s)ds} - 1\right) \quad \forall\, t \in [t_0, t_0+T). \qquad \text{(B.17)}
\end{aligned}$$

By $(B.17)$, we get

$$\begin{aligned}
EU(t, x) &\le e^{-c_0\int_{t_0}^{t}\mu(s)ds}U(t_0, x_0)\\
&\qquad + \frac{M_0}{c_0}\left(1 - e^{-c_0\int_{t_0}^{t}\mu(s)ds}\right)\\
&\le e^{-c_0\int_{t_0}^{t}\mu(s)ds}U(t_0, x_0)\\
&\qquad + \frac{M_0}{c_0} \quad \forall\, t \in [t_0, t_0+T). \qquad \text{(B.18)}
\end{aligned}$$

This completes the proof of Lemma 1.

References

[1] H. J. Kushner, *Stochastic Stability and Control*. New York, NY, USA: Academic, 1967.

[2] X. Mao and C. Yuan, *Stochastic Differential Equations With Markovian Switching*. London, U.K.: Imperial College Press, 2006.

[3] M. Krstić and H. Deng, *Stabilization of Uncertain Nonlinear Systems*. New York, NY, USA: Springer, 1998.

[4] R. Khasminskii, *Stochastic Stability of Differential Equations*, 2nd ed. London, U.K.: Springer, 2012.

[5] H. Deng and M. Krstić, "Stochastic nonlinear stabilization, Part I: A backstepping design," *Syst. Control Lett.*, vol. 32, no. 3, pp. 143–150, 1997.

[6] H. Deng and M. Krstić, "Output-feedback stochastic nonlinear stabilization," *IEEE Trans. Autom. Control*, vol. 44, no. 2, pp. 328–333, Feb. 1999.

[7] Z. G. Pan and T. Basar, "Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criterion," *SIAM J. Control Optim.*, vol. 37, no. 3, pp. 957–995, Feb. 1999.

[8] Y. G. Liu and J. F. Zhang, "Practical output-feedback risk-sensitive control for stochastic nonlinear systems with stable zero-dynamics," *SIAM J. Control Optim.*, vol. 45, no. 3, pp. 885–926, 2006.

[9] W. Q. Li, X. J. Xie, and S. Y. Zhang, "Output-feedback stabilization of stochastic high-order nonlinear systems under weaker conditions," *SIAM J. Control Optim.*, vol. 49, no. 3, pp. 1262–1282, 2011.

[10] Y. M. Li, L. Liu, and G. Feng, "Robust adaptive output feedback control to a class of non-triangular stochastic nonlinear systems," *Automatica*, vol. 89, pp. 325–332, 2018.

[11] W. Q. Li, X. X. Yao, and M. Krstic, "Adaptive-gain observer-based stabilization of stochastic strict-feedback systems with sensor uncertainty," *Automatica*, vol. 120, 2020, Art. no. 109112.

[12] W. Q. Li and M. Krstic, "Stochastic adaptive nonlinear control with filterless least-squares," *IEEE Trans. Autom. Control*, to be published, doi: 10.1109/TAC.2020.3027650.

[13] W. Chen and L. C. Jiao, "Finite-time stability theorem of stochastic nonlinear systems," *Automatica*, vol. 46, no. 12, pp. 2105–2108, 2010.

[14] J. L. Yin, S. Y. Khoo, Z. H. Man, and X. H. Yu, "Finite-time stability and instability of stochastic nonlinear systems," *Automatica*, vol. 47, no. 12, pp. 2671–2677, 2011.

[15] S. Y. Khoo, J. L. Yin, Z. H. Man, and X. H. Yu, "Finite-time stabilization of stochastic nonlinear systems in strict-feedback form," *Automatica*, vol. 49, no. 5, pp. 1403–1410, 2013.

[16] H. Wang and Q. X. Zhu, "Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form," *Automatica*, vol. 54, pp. 284–291, 2015.

[17] M. M. Jiang, X. J. Xie, and K. M. Zhang, "Finite-time stabilization of stochastic high-order nonlinear systems with FT-SISS inverse dynamics," *IEEE Trans. Autom. Control*, vol. 64, no. 1, pp. 313–320, Jan. 2019.

[18] J. L. Yin, S. Y. Khoo, and Z. H. Man, "Finite-time stability theorems of homogeneous stochastic nonlinear systems," *Syst. Control Lett.*, vol. 100, pp. 6–13, 2017.

[19] X. Yu, J. L. Yin, and S. Y. Khoo, "Generalized Lyapunov criteria on finite-time stability of stochastic nonlinear systems," *Automatica*, vol. 107, pp. 183–189, 2019.

[20] Y. D. Song, Y. J. Wang, J. C. Holloway, and M. Krstic, "Time-varying feedback for robust regulation of normal-form nonlinear systems in prescribed finite time," *Automatica*, vol. 83, pp. 243–251, 2017.

[21] Y. D. Song, Y. J. Wang, and M. Krstic, "Time-varying feedback for stabilization in prescribed finite time," *Int. J. Robust Nonlinear Control*, vol. 29, no. 3, pp. 618–633, 2019.

[22] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[23] E. Moulay and W. Perruquetti, "Finite time stability and stabilization of a class of continuous systems," *J. Math. Anal. Appl.*, vol. 323, no. 2, pp. 1430–1443, 2006.

[24] Y. J. Wang, Y. D. Song, D. J. Hill, and M. Krstic, "Prescribed finite time consensus and containment control of networked multi-agent systems," *IEEE Trans. Cybern.*, vol. 49, no. 4, pp. 1138–1147, Apr. 2019.

[25] J. Holloway and M. Krstic, "Prescribed-time observers for linear systems in observer canonical form," *IEEE Trans. Autom. Control*, vol. 64, no. 9, pp. 3905–3912, Sep. 2019.

[26] J. Holloway and M. Krstic, "Prescribed-time output feedback for linear systems in controllable canonical form," *Automatica*, vol. 107, pp. 77–85, 2019.

[27] P. Krishnamurthy, F. Khorrami, and M. Krstic, "Adynamic high-gain design for prescribed-time regulation of nonlinear systems," *Automatica*, vol. 115, 2020, Art. no. 108860.

[28] P. Krishnamurthy, F. Khorrami, and M. Krstic, "Robust adaptive prescribed-time stabilization via output feedback for uncertain nonlinear strict-feedback-like systems," *Eur. J. Control*, vol. 55, pp. 14–23, 2020.

[29] D. Steeves, M. Krstic, and R. Vazquez, "Prescribed-time estimation and output regulation of the linearized Schrödinger equation by backstepping," *Eur. J. Control*, vol. 55, pp. 3–13, 2020.

[30] D. Steeves, M. Krstic, and R. Vazquez, "Prescribed-time H^1-stabilization of reaction-diffusion equations by means of output feedback," in *Proc. Eur. Control Conf.*, 2019, pp. 1932–1937.

[31] H. M. Taylor and S. Karlin, *An Introduction to Stochastic Modeling*. New York, NY, USA: Academic Press, 1984.

[32] C. W. Gardiner, *Handbook of Stochastic Methods for Physics Chemistry and the Natural Sciences*. Berlin, Germany: Springer, 1993.

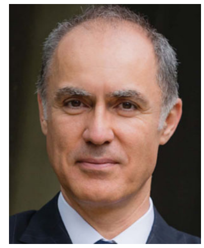
[33] F. C. Klebaner, *Introduction to Stochastic Calculus With Applications*, 2nd ed. London, U.K.: Imperial College Press, 2005.

[34] R. A. Freeman and P. V. Kokotović, "Optimality of robust nonlinear feedback control," Univ. California, Santa Barbara, CA, USA, Tech. Rep. CCEC11-09-93, 1993.
[35] M. Krstic and P. Tsiotras, "Inverse optimality results for the attitude motion of a rigid spacecraft," *IEEE Trans. Autom. Control*, vol. 44, no. 5, pp. 1042–1049, May 1999.
[36] M. Krstic and Z. H. Li, "Inverse optimal design of input-to-state stabilizing nonlinear controllers," *IEEE Trans. Autom. Control*, vol. 43, no. 3, pp. 336–351, Mar. 1998.
[37] H. Deng and M. Krstic, "Stochastic nonlinear stabilization-Part II: Inverse optimality," *Syst. Control Lett.*, vol. 32, no. 3, pp. 151–159, 1997.
[38] H. Deng, M. Krstic, and R. Williams, "Stabilization of stochastic nonlinear systems driven by noise of unknown covariance," *IEEE Trans. Autom. Control*, vol. 46, no. 8, pp. 1237–1253, Aug. 2001.
[39] W. Q. Li, Y. W. Jing, and S. Y. Zhang, "State-feedback control of a class of high-order stochastic nonlinear systems," *Int. J. Robust Nonlinear Control*, vol. 22, no. 14, pp. 1583–1601, 2012.
[40] K. D. Do, "Global inverse optimal stabilization of stochastic nonholonomic systems," *Syst. Control Lett.*, vol. 75, pp. 41–55, 2015.
[41] H. Kuo, *Introduction to Stochastic Integration*. New York, NY, USA: Universitext-Springer, 2006.
[42] X. Mao, *Stochastic Differential Equations and Applications*, 2nd ed. Chichester, U.K.: Horwood, 2008.
[43] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Englewood Cliffs, NJ, USA: Prentice Hall, 2002.
[44] V. A. Ugrinovskii, "Robust H_∞ infinity control in the presence of stochastic uncertainty," *Int. J. Control*, vol. 71, no. 2, pp. 219–237, 1998.
[45] B. Yang and W. Lin, "Homogeneous observers, iterative design and global stabilization of high order nonlinear systems by smooth output feedback," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1069–1080, Jul. 2004.
[46] G. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 1989.

Wuquan Li (Senior Member, IEEE) received the Ph.D. degree in control theory and control engineering from College of Information Science and Engineering, Northeastern University, Shenyang, China, in 2011.

He carried out his postdoctoral research with Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, from 2012 to 2014. From 2018 to 2019, he was a Visiting Scholar with University of California, San Diego, CA, USA. Since 2011, he has been with School of Mathematics and Statistics Science, Ludong University, Yantai, China, where he is currently a Professor. His research interests include stochastic nonlinear systems control and identification of nonlinear systems.

Dr. Li is Young Taishan Scholar and Shandong Provincial Distinguished Young Scholar in China. He currently serves as an Associate Editor for two international journals: *Systems & Control Letters* and *Asian Journal of Control*.



Miroslav Krstic (Fellow, IEEE) received the undergraduate degree (*summa cum laude*) in electrical engineering from the University of Belgrade, Belgrade, Serbia, in 1989, and the M.S. and Ph.D. degrees from the University of California Santa Barbara, Santa Barbara, CA, USA, in 1992 and 1994, respectively.

He is a Distinguished Professor of mechanical and aerospace engineering, holds the Alspach Endowed Chair, and is the founding Director of the Cymer Center for Control Systems and Dynamics, University of California San Diego (UCSD), San Diego, CA, USA. He also serves as Senior Associate Vice-Chancellor for Research with UCSD. He has coauthored 15 books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.

Dr. Krstic, as a graduate student, won the University of California Santa Barbara Best Dissertation Award and Student Best Paper awards at CDC and ACC. He has been elected as Fellow of seven scientific societies—IEEE, IFAC, ASME, SIAM, AAAS, IET (U.K.), and AIAA (Associate Fellow)—and as a Foreign Member of the Serbian Academy of Sciences and Arts and of the Academy of Engineering of Serbia. He is the recipient of the SIAM Reid Prize, ASME Oldenburger Medal, Nyquist Lecture Prize, Paynter Outstanding Investigator Award, Ragazzini Education Award, IFAC Nonlinear Control Systems Award, Chestnut Textbook Prize, Control Systems Society Distinguished Member Award, the PECASE, NSF Career, and ONR Young Investigator awards, the Schuck (1996 and 2019) and Axelby paper prizes, and the first UCSD Research Award given to an engineer. He has also been awarded the Springer Visiting Professorship at University of California Berkeley, Berkeley, CA, USA, the Distinguished Visiting Fellowship of the Royal Academy of Engineering, and the Invitation Fellowship of the Japan Society for the Promotion of Science. He serves as Editor for two Springer book series. He serves as Editor-in-Chief for *Systems & Control Letters*, has been serving as Senior Editor for *Automatica* and IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and has served as Vice-President for Technical Activities of the IEEE Control Systems Society and as Chair of the IEEE CSS Fellow Committee.