Nonsmooth Extremum Seeking Control With User-Prescribed Fixed-Time Convergence

Jorge I. Poveda and Miroslav Krstić, *Fellow, IEEE*

***Abstract*—This article introduces a new class of nonsmooth extremum seeking controllers (ESCs) with convergence bounds given by class-$\mathcal{KL}$ functions that have a uniformly bounded settling time. These ESCs are characterized by nominal average systems that render *uniformly globally fixed-time stable* (UGFxTS), the set of minimizers of the response map of a stable nonlinear plant. Given that, under suitable tuning of the parameters of the controllers, the ESCs inherit the convergence properties of their average systems, the proposed dynamics can achieve a better transient performance compared to the traditional ESCs based on gradient descent or Newton flows. Moreover, for the case when the plant is a static map, the convergence time of the proposed algorithms can be prescribed *a priori* by the users for all initial conditions without the need of retuning the gain of the learning dynamics of the ESC. Since autonomous feedback controllers with fixed-time convergence properties are necessarily non-Lipschitz continuous, standard averaging and singular perturbation tools, traditionally used in ESC, are not applicable anymore. We address this issue by using averaging and singular perturbation tools for nonsmooth and set-valued systems, which further allows us to consider ESCs modeled by discontinuous vector fields that are typical in fixed-time and finite-time optimization problems.**

***Index Terms*—Adaptive control, extremum seeking, optimization.**

I. Introduction

Extremum seeking control (ESC) has shown to be a powerful technique for the solution of model-free optimization problems in dynamical systems [1]–[4]. Stability, convergence, and robustness guarantees for different types of constrained and unconstrained smooth ESCs have been extensively studied in the literature [5]–[8]. Recently, ESC has also been extended to nonsmooth and hybrid settings that are able to overcome some of the intrinsic limitations of smooth feedback controllers [9], [10]. An early use of nonsmooth ESC can also be found in [11, Sec. 5].

In this article, we extend and generalize some of these results by introducing a new class of nonsmooth ESCs that have stability properties characterized by class-$\mathcal{KL}$ functions[1] β with the "fixed-time convergence property," namely, there exists a continuous settling time function $r \mapsto T(r)$ and a positive number $T^* > 0$ such that $\lim_{s\to T(r)} \beta(r,s) = 0$ and $T(r) < T^*$, for all $r \geq 0$. Functions with this attribute are also said to be of class-$\mathcal{KL}_{\mathcal{T}}$ [12], and they characterize the convergence properties of systems whose solutions converge to a particular set in a finite time $T(r)$ that can be upper bounded by a constant T^* that is *independent* of the distance r of the initial conditions to the set. This powerful property has motivated the development of new algorithms in the context of regulation, optimization, and estimation problems, see [13]–[16] and [17]. Nevertheless, ESCs with fixed-time convergence properties remain completely unexplored in the literature, and, as we will show in this article, they have the potential of inducing dramatical improvements in the transient performance of the closed-loop system for certain classes of plants having response maps with strong monotonicity properties.

In order to design ESCs with $\mathcal{KL}_{\mathcal{T}}$ convergence bounds, which we call fixed-time extremum seeking controllers (FxTESCs), our starting point is the averaging-based paradigm considered in [3], [5], [18], and [19], which, due to its modular approach, can accommodate different types of optimization algorithms, making a direct connection between the bounds that characterize the convergence of the trajectories of the nominal average system and the actual control signal generated by the ESCs. However, since most of the existing results in the literature rely on averaging and singular perturbation tools for Lipschitz continuous systems, they are not suitable for the design and analysis of ESCs with $\mathcal{KL}_{\mathcal{T}}$ convergence bounds. Instead, in this article, we use averaging and singular perturbation tools for nonsmooth and set-valued systems [9], [20]–[22], which allows us to additionally consider ESCs based on differential inclusions that may be related to discontinuous optimization algorithms. Moreover, we also develop Newton-like fixed-time ESCs that remove from the convergence bound T^* the dependence on the unknown parameters of the Hessian matrix of the response map. To the best of our knowledge, the ESCs for dynamical systems presented in this article are the first that have convergence bounds characterized by class-$\mathcal{KL}_{\mathcal{T}}$ functions. For *static* maps and *specific* continuous ESC algorithms, preliminary results with sketches of the proofs were presented in the conference papers [1] and [2]. In contrast to these works, this article addresses the general ESC problem in *dynamic* plants, considers a general family of possibly *discontinuous* ESCs (which subsume those considered in [1] and [2]), derives tighter convergence bounds for the algorithms, new auxiliary averaging results for nonsmooth systems, and also presents the complete stability analysis. Moreover, in contrast to [1] and [2], the results of this article are also applicable to ESC algorithms with nominal average systems having only *finite-time* convergence properties, thus, addressing another existing gap in the literature of ESC.

The rest of this article is organized as follows. Section II introduces the notation and preliminaries. Section III presents the problem statement. Section IV presents the main results for gradient-based ESCs,

Manuscript received August 9, 2020; revised January 27, 2021; accepted February 17, 2021. Date of publication March 3, 2021; date of current version December 3, 2021. This work was supported in part by CU Boulder ASIRT Seed Grant and NSF Grant 1823983 and Grant 1711373. This paper was presented in part at the 21st IFAC World Congress, July 2020, and American Control Conference, Denver, CO, USA, July 2020. Recommended by Associate Editor N. Van De Wouw.
(Corresponding author: Jorge I. Poveda.)

Jorge I. Poveda is with the Department of Electrical Computer and Energy Engineering, University of Colorado, Boulder, CO 80302 USA (e-mail: jorge.poveda@colorado.edu).

Miroslav Krstić is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA 92093-0411 USA (e-mail: krstic@ucsd.edu).
Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2021.3063700.

Digital Object Identifier 10.1109/TAC.2021.3063700

[1]A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \beta(r,s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r,s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$.

Section V studies Newton-like algorithms, and finally Section VI gives some conclusions.

II. Notation and Preliminaries

Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define $|x|_{\mathcal{A}} := \min_{s\in\mathcal{A}} \|x - s\|_2$. We use $\mathbb{S}^1 := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, and $r\mathbb{B}$ to denote a closed ball of radius $r > 0$ in the Euclidean space of appropriate dimension, and centered at the origin. We define $\mathcal{D} \in \mathbb{R}^{n\times 2n}$ as the matrix that maps a vector $x = [x_1, x_2, \ldots, x_{2n}]^\top \in \mathbb{R}^{2n}$ to a vector $\tilde{x} := [x_1, x_3, \ldots, x_{2n-1}]^\top$ that has only the odd components of x. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is: (a) outer semicontinuous (OSC) at x, if for each sequence $\{x_i, y_i\} \to (x, y) \in \mathbb{R}^n \times \mathbb{R}^p$ satisfying $y_i \in M(x_i)$ for all $i \in \mathbb{Z}_{\geq 0}$, we have $y \in M(x)$; (b) locally bounded (LB) at x, if there exists an open neighborhood $N_x \subset \mathbb{R}^n$ of x such that $M(N_x)$ is bounded.

In this article, we consider systems of the form

$$x \in C, \quad \dot{x} \in F(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $C \subset \mathrm{dom}(F)$ is a closed set (called the flow set), and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping that is said to satisfy the *Basic Conditions* if it is OSC, LB, and convex-valued with respect to C. A solution to system (1) is an absolutely continuous function $x : \mathrm{dom}(x) \to \mathbb{R}^n$ that satisfies: a) $x(0) \in C$; b) $x(t) \in C$ for all $t \in \mathrm{dom}(x)$; and c) $\dot{x}(t) \in F(x(t))$ for almost all $t \in \mathrm{dom}(x)$. A solution x is said to be complete if $\mathrm{dom}(x) = [0, \infty)$. System (1) is said to render a compact set $\mathcal{A} \subset C$ uniformly globally asymptotically stable (UGAS) if there exists a class $\mathcal{KL}$ function β such that every solution of (1) satisfies $|x(t)|_{\mathcal{A}} \leq \beta(|x(0)|_{\mathcal{A}}, t)$, for all $t \in \mathrm{dom}(x)$. If, additionally, the function β is of class $\mathcal{KL}_{\mathcal{T}}$, we say that system (1) renders the set $\mathcal{A}$ UGFxTS. We also consider ε-parameterized systems, given by $x \in C, \dot{x} \in F_\varepsilon(x)$, where C is compact. In this case, a compact set $\mathcal{A} \subset C$ is said to be globally practically asymptotically stable (GPAS) as $\varepsilon \to 0^+$ if there exists a class $\mathcal{KL}$ function β such that for each $\nu > 0$ there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ every solution x_ε satisfies $|x_\varepsilon(t)|_{\mathcal{A}} \leq \beta(|x_\varepsilon(0)|_{\mathcal{A}}, t) + \nu$, for all $t \in \mathrm{dom}(x_\varepsilon)$. The notion of GPAS can be extended to systems that depend on multiple parameters $\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell]^\top$. In this case, and with some abuse of notation, we say that the system renders the set $\mathcal{A}$ GPAS as $(\varepsilon_\ell, \ldots, \varepsilon_2, \varepsilon_1) \to 0^+$, where in general the parameter ε_k depends on ε_{k-1}, for each $k \in \mathbb{Z}_{\geq 2}$.

The algorithms considered in this article make use of sinusoidal excitation signals that we model as solutions of n uncoupled linear oscillators evolving on the n-torus $\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \subset \mathbb{R}^{2n}$. The state of the oscillators is $\mu \in \mathbb{R}^{2n}$, and their dynamics are

$$\mu \in \mathbb{T}^n, \quad \varepsilon_1 \dot{\mu} = -2\pi \mathcal{R}_\theta \mu, \quad \varepsilon_1 > 0 \tag{2}$$

where the matrix $\mathcal{R}_\theta \in \mathbb{R}^{2n\times 2n}$ is defined as a block diagonal matrix parametrized by a vector of gains $\theta = [\theta_1, \theta_2, \ldots, \theta_n]^\top$. The ith diagonal block of $\mathcal{R}_\theta$ is defined as the skew symmetric matrix $\mathcal{R}_i := [0, -\theta_i; \theta_i, 0] \in \mathbb{R}^{2\times 2}$, where θ_i is a positive rational number that satisfies $\theta_i \neq \theta_j$, $\theta_i \neq 2\theta_j$, and $\theta_i \neq 3\theta_j$ for all $i \neq j$. The odd entries μ_i of the solutions μ of system (2) can be explicitly computed as $\mu_i(t) = \mu_i(0)\cos(\frac{2\pi}{\varepsilon_1}\theta_i t) + \mu_{i+1}(0)\sin(\frac{2\pi}{\varepsilon_1}\theta_i t)$, with $\mu_i(0)^2 + \mu_{i+1}(0)^2 = 1$, for all $i \in \{1, 3, 5, \ldots, n-1\}$. The following lemma, corresponding to [23, Lemma 2], will also be instrumental for our results.

Lemma 1: Consider the dynamics (1) with $C = \mathbb{R}^n$ and F being singled-valued with unique equilibrium point at the origin, and suppose $\exists a, b > 0$ and $\gamma_1 = 1 - \frac{1}{2\alpha}$, $\gamma_2 = 1 + \frac{1}{2\alpha}$, with $\alpha > 1$, and a positive definite, radially unbounded and smooth Lyapunov function V satisfying $\dot{V}(x) \leq -aV(x)^{\gamma_1} - bV(x)^{\gamma_2}$, for all $x \in \mathbb{R}^n$. Then, the origin $x^* = 0$ is UGFxTS for (1) with $T^* = \frac{\alpha\pi}{\sqrt{ab}}$. □

III. Problem Statement and Motivation

Consider a nonlinear dynamic plant with input $u \in \mathbb{R}^n$, output $y \in \mathbb{R}$, and state $x \in \Xi \subset \mathbb{R}^p$, modeled by the equations

$$\dot{x} = f(x, u), \quad y = h(x, u) \tag{3}$$

where f is locally Lipschitz, and h is continuously differentiable. In this article, we assume that the operational space Ξ for the states of the plant (3) is closed and bounded. In practice, boundedness of Ξ can be related to the physical limitations of the plant, or to operational sets that are rendered forward invariant by using internal feedback controllers that implement mechanisms such as Lipschitz projections or barrier functions. Compactness of Ξ is also guaranteed when the dynamics (3) have the bounded-input bounded-state property and u is uniformly bounded (as will be the case in our results).

In order to have a well-defined ESC problem, we also make the following standard stability assumption on system (3).

Assumption 1: There exists a continuous function $\ell_x : \mathbb{R}^n \to \mathbb{R}^p$, such that for each compact set $K_u \subset \mathbb{R}^n$ the constrained dynamical system $(x, u) \in \Xi \times K_u, \dot{x} = f(x, u), \dot{u} = 0$, renders UGAS the compact set $M_{K_u} := \{(x, u) \in \Xi \times K_u : x = \ell_x(u)\}$. □

The existence of a UGAS quasi-steady-state manifold ℓ_x for the plant (3) is a standard assumption in ESC, see [3], [5], and [6].

The *response map* of the plant (3) is defined as $\phi(u) := h(\ell_x(u), u)$, which is assumed to satisfy the following.

Assumption 2: The function $u \mapsto \phi(u)$ is twice continuously differentiable, and the set $\mathcal{A}_\phi := \mathrm{argmin}_{u\in\mathbb{R}^n} \phi(u)$ is nonempty and compact. □

Based on these assumptions, the *ESC problem* that we study in this article consists on regulating the input u of system (3) toward the set $\mathcal{A}_\phi$, by using only output measurements of y, without any knowledge of the mathematical form of f, h, or ϕ.

A. Transient Limitations of Gradient-Descent-Based ES

To motivate the FxTESCs considered in this article, we first review the convergence properties of the standard gradient descent-based extremum seeking controller (GDESC) studied in [3], [6], and [5]. This controller is characterized by the feedback law and the dynamics

$$u := \hat{u} + a\mathcal{D}\mu, \quad \dot{\hat{u}} = -k_1 \xi \tag{4}$$

where $k_1 := \varepsilon_0 k$ and (k, ε_0, a) are positive tunable parameters. The auxiliary states (ξ, μ) of (4) have dynamics

$$\dot{\xi} = -k_2 \left(\xi - F_G(y, \mu)\right), \quad \dot{\mu} = -k_3 \mathcal{R}_\theta \mu, \quad \mu \in \mathbb{T}^n \tag{5}$$

where $k_2 := \varepsilon_0/\varepsilon_2$, $k_3 := 2\pi\varepsilon_0/\varepsilon_1$, and

$$0 < \varepsilon_0 \ll \varepsilon_1 \ll \varepsilon_2 \ll 1/k. \tag{6}$$

The mapping F_G in (5) is defined as $F_G(y, \mu) := yM(\mu)$, with $M(\mu) := \frac{2}{a}\mathcal{D}\mu$. We study the transient performance of this controller under the following additional assumption on the response map ϕ.

Assumption 3: There exists $\kappa > 0$ such that $|\nabla\phi(u)|^2 \geq 2\kappa(\phi(u) - \phi(u^*))$ for all $u \in \mathbb{R}^n$. Moreover, $\nabla\phi$ is L-globally Lipschitz, and $\mathcal{A}_\phi = \{u^*\}$. □

Typical examples of response maps that satisfy the inequality of Assumption 3 include strongly convex functions, such as positive definite quadratic functions, which are ubiquitous in the literature of ESC, e.g., [3], [8], and [24]. However, Assumptions 1–3 do not necessarily ask for convexity of h or linearity of (3). Indeed, response maps ϕ satisfying Assumptions 2 and 3 can be generated by different classes of plants (3) having linear/nonlinear dynamics f and nonlinear/linear outputs h. Nevertheless, it is important to note that in the context of ESC, the goal is to achieve model-free optimization; thus, the mathematical forms of the functions (f, h) are generally unknown.

To guarantee convergence to $\mathcal{A}_\phi$, a time-scale separation is induced between the dynamics of the plant (3) and the dynamics of the controller (4), (5). In particular, by introducing the new time scale $\tau := t\varepsilon_0$, and by using the definition of $\mathcal{D}$, the dynamics of the closed-loop system can be written as follows:

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}\tau} = -kF_u(\hat{u},\xi), \quad \frac{\mathrm{d}\xi}{\mathrm{d}\tau} = -\frac{1}{\varepsilon_2}\left(\xi - F_G(y,\mu)\right) \tag{7a}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}\tau} = -\frac{2\pi}{\varepsilon_1}\mathcal{R}_\theta\mu, \quad \varepsilon_0\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x,\hat{u}+a\mathcal{D}\mu) \tag{7b}$$

where according to (4), the *learning dynamics* of the GDESC are characterized by the mapping $F_u(\hat{u},\xi)=\xi$. When ε_0 is sufficiently small, system (7) is a singularly perturbed system with fast "boundary layer dynamics" corresponding to the dynamics of the plant, and slow "reduced dynamics" corresponding to the dynamics of the states $(\hat{u},\xi,\mu)$. By Assumption 1, the plant has a well-defined quasi-steady-state manifold $\ell_x(\cdot)$. Thus, the reduced dynamics are as follows:

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}\tau} = -kF_u(\hat{u},\xi), \quad \frac{\mathrm{d}\xi}{\mathrm{d}\tau} = -\frac{1}{\varepsilon_2}\left(\xi - F_G(\phi(\hat{u}+a\mathcal{D}\mu),\mu)\right)$$

$$\varepsilon_1\frac{\mathrm{d}\mu}{\mathrm{d}\tau} = -2\pi\mathcal{R}_\theta\mu, \quad \mu\in\mathbb{T}^n$$

where we used the definitions of ϕ and F_G. Let $\tilde{\mu}:=\mathcal{D}\mu$. For small values of a, we can perform a Taylor expansion of $\phi(\hat{u}+a\tilde{\mu})$ around the point $\hat{u}$, leading to $\phi(\hat{u}+a\tilde{\mu})=\phi(\hat{u})+a\tilde{\mu}^\top\nabla\phi(\hat{u})+O(a^2)$. By using the definitions of $M(\cdot)$, the fact that the solutions of the oscillator are given by sinusoids with unitary amplitude, and [2, Lemma 6], we can average the dynamics of the states $(\hat{u},\xi)$ along the trajectories $\tilde{\mu}$. Since F_u is independent of $\tilde{\mu}$, the resulting *average system* is

$$\frac{\mathrm{d}\hat{u}^A}{\mathrm{d}\tau} = -kF_u(\hat{u}^A,\xi^A), \quad \varepsilon_2\frac{\mathrm{d}\xi^A}{\mathrm{d}\tau} = -\xi^A + \tilde{F}_G(\hat{u}^A,a) \tag{8}$$

where the function $\tilde{F}_G$ is given by

$$\tilde{F}_G(\hat{u}^A,a) := \nabla\phi(\hat{u}^A) + \mathcal{O}(a). \tag{9}$$

For each $\varepsilon_2>0$, system (8) is a $\mathcal{O}(a)$-perturbed version of a nominal average system with $\mathcal{O}(a)=0$. In turn, since $\varepsilon_2 \ll 1/k$, this nominal average system is also a singularly perturbed system with exponentially stable boundary layer ξ^A-dynamics with equilibrium point $\xi^{A*}=\nabla\phi(\hat{u}^A)$, and *reduced nominal average* dynamics with state $\hat{u}_r$, given by

$$\frac{\mathrm{d}\hat{u}_r}{\mathrm{d}\tau} = -kF_u(\hat{u}_r,\nabla\phi(\hat{u}_r)) = -k\nabla\phi(\hat{u}_r) \tag{10}$$

which is a gradient descent (GD) flow with gain $k>0$. Therefore, by using the Lyapunov function $V(\hat{u}_r)=\phi(\hat{u}_r)-\phi(u^*)$, it can be shown that under Assumption 3, all solutions of the GD flow (10) satisfy the bound $|\hat{u}_r(\tau)|_{\mathcal{A}_\phi} \le \sqrt{\frac{L}{\kappa}}|\hat{u}_r(0)|_{\mathcal{A}_\phi}e^{-k\kappa\tau}$, for all $\tau\ge 0$. This establishes a UGAS result for system (10) with an exponential class-$\mathcal{KL}$ function $\beta(r,s)=\sqrt{\frac{L}{\kappa}}re^{-k\kappa s}$. After some manipulations (see proof of Th. 1), we can now repeatedly apply singular perturbation and averaging theory [18, Th. 1], as well as structural robustness results for smooth ODEs [25, Lemma 7.20], to conclude that for each pair $\Delta>\nu>0$ there exists $\varepsilon_2^*>0$ such that for each $\varepsilon_2\in(0,\varepsilon_2)$ there exists $a^*>0$, such that for each $a\in(0,a^*)$ there exists $\varepsilon_1^*>0$, such that for each $\varepsilon_1\in(0,\varepsilon_1^*)$ there exists $\varepsilon_0^*>0$, such that for each $\varepsilon_0\in(0,\varepsilon_0^*)$ all the input trajectories u generated by the GDESC in (7) satisfy the following:

$$|u(\tau)|_{\mathcal{A}_\phi} \le \beta(|\hat{u}(0)|_{\mathcal{A}_\phi},\tau) + 0.5\nu \tag{11}$$

for all $\tau\ge 0$, provided $|\hat{u}(0)|_{\mathcal{A}_\phi}\le\Delta$ and $|\xi(0)|\le\Delta$, where β is the same $\mathcal{KL}$ function of system (10); see [6], [5], and [19] for similar results under different notation and/or definitions of the gains k_i, $i\in\{1,2,3\}$, which might scale the time argument of (11). This fact highlights an important property of system (7): as $(\varepsilon_0,\varepsilon_1,a,\varepsilon_2)\to 0^+$, the transient performance of the control signal is approximately characterized by the transient performance of the reduced nominal average dynamics. Since for the GD flow (10) we know the form of β, for any $\nu>0$, we can compute an approximate lower bound τ_v^* for the amount of time τ needed in (11) to have $|u(\tau)|_{\mathcal{A}_\phi}\le\nu$ for all $\tau\ge\tau_v^*$. By direct computation, we obtain the following:

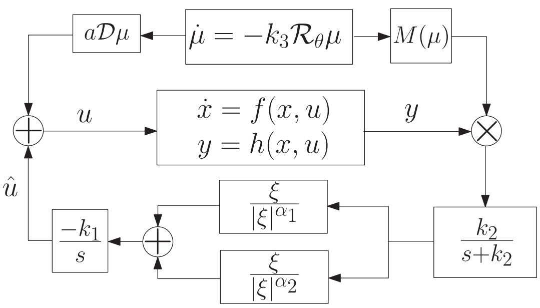

Fig. 1. Example of closed-loop system with a fixed-time gradient-based extremum seeking controller with $\alpha_1\in(0,1)$ and $\alpha_2<0$.

$$k\tau_v^* = \frac{1}{\kappa}\log\left(2\sqrt{\frac{L}{\kappa}}\frac{|\hat{u}(0)|_{\mathcal{A}_\phi}}{\nu}\right) \tag{12}$$

which shows that for constant values of (k,ν) (resp. (τ_v^*,ν)), the value of τ_v^* (resp. k) grows logarithmically with $|\hat{u}(0)|_{\mathcal{A}_\phi}$. Since (11) holds only when $|\hat{u}(0)|_{\mathcal{A}_\phi}\le\Delta$, the maximum value of $k\tau_v^*$ over all initial conditions satisfying $|\hat{u}(0)|_{\mathcal{A}_\phi}\le\Delta$ also grows logarithmically with Δ, for any parameters $(\varepsilon_0,\varepsilon_1,a,\varepsilon_2)$ in the GDESC that guarantee the satisfaction of the bound (11).

B. ES Controllers With Fixed-Time Convergence Bounds

In order to improve the convergence properties of the GDESC, we can consider a new class of ESCs that generate bounds of the form (11) with the additional property that $\beta\in\mathcal{KL}_\mathcal{T}$. To achieve this property, let us assume first that ξ and $\hat{u}$ are scalars, and consider the dynamics (7) with the following map:

$$F_u(\hat{u},\xi) = \mathrm{sign}(\xi)|\xi|^{1-\alpha_1} + \mathrm{sign}(\xi)|\xi|^{1-\alpha_2}$$

with $\alpha_1\in(0,1)$ and $\alpha_2<0$. Since $\xi\in\mathbb{R}$, we can write $\mathrm{sign}(\xi)=\xi/|\xi|$, which leads to the function

$$F_u(\hat{u},\xi) = -k\left(\frac{\xi}{|\xi|^{\alpha_1}} + \frac{\xi}{|\xi|^{\alpha_2}}\right) \tag{13}$$

which is defined to be zero whenever $\xi=0$. Fig. 1 shows a scheme of the closed-loop system (7) with an ESC implementing learning dynamics characterized by the function (13). In general, the mapping F_u is not Lipschitz continuous without further conditions on (α_1,α_2). However, even in cases when F_u is discontinuous at the point $\xi=0$, the existence of generalized solutions in the sense of Krasovskii can always be guaranteed [25, Lemma 5.26]. Functions of this form have been extensively studied in the literature of fixed-time stabilization by using the notion of homogeneity (in the bilimit), see for instance [26, Sec. 5.1], [14, Sec. 1], [23, Sec. 4-A], [27, Ex. 1], or [28, Lemma 2.1]. However, in the context of ESC, they remained unexplored.

The closed-loop system of Fig. 1 can be studied by following similar steps as in the previous section. In this case, instead of (10), the reduced nominal average dynamics are given by

$$\frac{\mathrm{d}\hat{u}_r}{\mathrm{d}\tau} = -k\left(\frac{\nabla\phi(\hat{u}_r)}{|\nabla\phi(\hat{u}_r)|^{\alpha_1}} + \frac{\nabla\phi(\hat{u}_r)}{|\nabla\phi(\hat{u}_r)|^{\alpha_2}}\right) \tag{14}$$

which can be analyzed by using the smooth Lyapunov function $V_G(\hat{u}_r)=\frac{1}{2}(\phi(\hat{u}_r)-\phi(u^*))^2$, which, under Assumption 3, is radially unbounded and positive definite with respect to u^*. The time derivative of V_G along the solutions of system (14) satisfies $\frac{\mathrm{d}V_G(\hat{u}_r(\tau))}{\mathrm{d}\tau} \le -k(c_1V_G(\hat{u}_r)^{\gamma_1}+c_2V_G(\hat{u}_r)^{\gamma_2})$ for all $\hat{u}_r\ne u^*$, where $\tilde{\alpha}_1=2-\alpha_1>0$, $\tilde{\alpha}_2=2-\alpha_2>0$, $c_1:=2^{\frac{2+3\tilde{\alpha}_1}{4}}\kappa^{\frac{\tilde{\alpha}_1}{2}}>0$, $c_2:=2^{\frac{2+3\tilde{\alpha}_2}{4}}\kappa^{\frac{\tilde{\alpha}_2}{2}}>0$, $\gamma_1:=\frac{2+\tilde{\alpha}_1}{4}\in(0,1)$, and $\gamma_2:=\frac{2+\tilde{\alpha}_2}{4}>1$. It follows by [26, Th. 5.8] and the smoothness of V_G that the point u^*

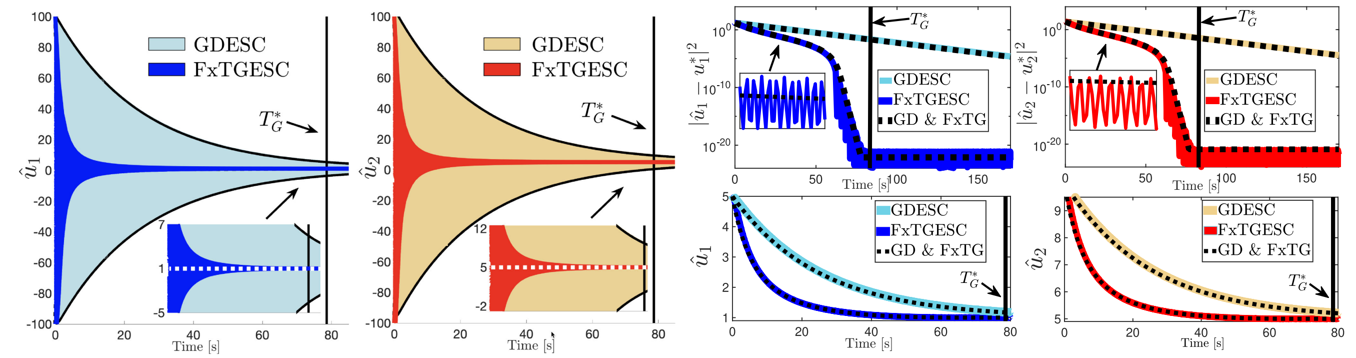

Fig. 2. Left: Approximate (finite-time) reachable set of the GDESC and the FxTESC with $\hat{u}(0) \in K = [-100\,100] \times [-100\,100]$. The insets illustrate the convergence of all solutions of the FxTESC to the optimal point; Right: Time history of *one* solution of the GDESC and the FxTESC with identical initialization, in logarithmic scale (top) and linear scale (bottom). The dashed lines show the trajectories of the GD and FxTG flows.

is UGFxTS for system (14). Moreover, if we set $\alpha_2 = -\alpha_1$, the assumptions of Lemma 1 hold with $\alpha = 2/\alpha_1$ and $\sqrt{ab} = 4k\kappa$, and we obtain the following estimate on the bound for the settling time $T_k(\cdot)$ of (14):

$$T_G^* := \frac{\pi}{2k\alpha_1\kappa}. \tag{15}$$

Finally, note that all the previous computations hold when ξ and $\hat{u}_r$ are vectors in $\mathbb{R}^n$, and an explicit function $\beta_{G,k} \in \mathcal{KL}_{\mathcal{T}}$ for system (14) can be computed by using Lemma 3 in the Appendix. Therefore, for the ESC of Fig. 1, the dynamics (14) can be seen as general fixed-time gradient (FxTG) flows in $\mathbb{R}^n$ with the UGFxTS property.

Remark 1: FxTG flows have been recently studied in the context of model-based optimization and fixed-time stabilization in [28, Lemma 2.1], [15], and [16]. Continuity of the function (13) can be guaranteed as in [1] and [15]. □

The key implication of the fixed-time bound (15), which is *independent* of the initial conditions of system (14), is that for positive values of (k, κ) and $p = (\nu, \alpha_1)$, with $\alpha_1 \in (0, 1)$, the value of τ_ν^* obtained from (12) will be larger than T_G^* whenever $\hat{u}_r(0) \in \Omega_{G,p} := \{u_{r0} \in \mathbb{R}^n : |u_{r0}|_{\mathcal{A}_\phi} > 0.5\nu \exp(\frac{\pi}{2\alpha_1})\}$. Therefore, if for any pair $\Delta > \nu > 0$, the parameters $(\varepsilon_0, \varepsilon_1, a, \varepsilon_2)$ of the FxTESC shown in Fig. 1 can be selected such that the bound (11) holds with the $\mathcal{KL}_{\mathcal{T}}$ function $\beta_{G,k}$ (a result that we will establish in the following section), then the convergence time of the FxTESC will outperform the convergence time of the GDESC for all initial conditions $\hat{u}_r(0)$ in the set $\Omega_{G,p} \cap (\mathcal{A}_\phi + \Delta\mathbb{B})$, which is nonempty when Δ is sufficiently large or ν is sufficiently small.

Remark 2: Since the bound (15) is independent of the initial conditions of the system, a universal gain k can now be used to induce a desired convergence time T_G^* via the class $\mathcal{KL}_{\mathcal{T}}$ function $\beta_{G,k}$. This property is fundamentally different from the asymptotic or exponential (semiglobal practical) convergence properties of the smooth ESCs considered in [3], [5], [18], and [19]. Note, however, that the parameters $(\varepsilon_0, \varepsilon_1, a, \varepsilon_2)$ will still depend on ν and Δ, since their role is to guarantee that the ESC approximates the behavior (on compact sets) of its reduced nominal average dynamics (14). □

Example 1: In order to illustrate the previous discussion, let us consider a simple plant in $\mathbb{R}^2$ with $f(x,u) = 10 \times [-x_1 + u_1, -x_2 + u_2]^\top$ and output $y = (x_1 - 1)^2 + (x_2 - 5)^2$. Since $f(x,u)$ describes a stable linear system that generates bounded states under bounded inputs, Assumption 1 holds. We set $\Delta = 100$, and we simulate the closed-loop system (7) using the following parameters: $a = 0.01, \varepsilon_0 = 4 \times 10^{-5}, \varepsilon_1 = 1 \times 10^{-4}, \varepsilon_2 = 5 \times 10^{-1}$, and $k = 0.02$, which satisfy the relations of (6). For the oscillator (2), we used $2\pi\theta_1 = 3.5, 2\pi\theta_2 = 4$, and $\mu(0) = [0, 1, 0, 1]^\top$. For the FxTESC with learning dynamics (13), we used $\alpha_1 = 0.5 = -\alpha_2$. Since in this case $\phi(u) = (u_1 - 1)^2 + (u_2 - 5)^2$, it follows that Assumption 3 holds with $\kappa = L = 2$. Using (15), we obtain $T_G^* = 78.53$. Fig. 2 compares the behavior of the trajectories of the GDESC and the FxTESC. We emphasize that both algorithms used the same parameters $(a, k, \varepsilon_0, \varepsilon_1, \varepsilon_2)$. In the left figures, we have numerically approximated the reachable set of both algorithms from initial conditions with $\hat{u}(0) \in [-100\,100] \times [-100\,100]$, and $x(0) = \xi(0) = [1, 1]^\top$ by running 1×10^3 simulations with random initializations on this set. The insets show that all solutions generated by the FxTESC converge to a ν-neighborhood (with $\nu = 1 \times 10^{-7}$) of the optimal point $u^* = [1, 5]^\top$ before the time T_G^*. On the other hand, the right plots show the time history of *one* solution of the GDESC and FxTESC, respectively, with identical initialization. As expected, and as shown by the dashed lines, the trajectories generated by both ESCs are almost identical to the trajectories of their reduced nominal average dynamics (10) and (14). In particular, as highlighted in the upper logarithmic plot, the trajectory generated by the FxTESC approximately inherits the "fixed-time convergence property" of system (14). □

Remark 3 (Finite-Time Versus Fixed-Time Stability in ESC): In contrast to the property of UGFxTS, the property of *finite-time* stability is characterized by a class-$\mathcal{KL}$ function that satisfies $\lim_{s \to T(r)} \beta(r, s) = 0$, but where $T(\cdot)$ is not necessarily uniformly bounded, see [29]. The ESC shown in Fig. 1 can induce this weaker property by using $\alpha_1 = \alpha_2 = 1$, which generates reduced nominal average dynamics given by the discontinuous flow $\frac{d\hat{u}}{d\tau} = -2k\frac{\nabla\phi(\hat{u})}{|\nabla\phi(\hat{u})|}$, studied in [30] using generalized solutions in the context of differential inclusions. Under κ-strong convexity of ϕ and L-globally Lipschitz of $\nabla\phi$, the lower bound τ_v^* on the convergence time of this system satisfies $k\tau_v^* \leq (2\kappa)^{-1} L |\hat{u}_r(0)|_{\mathcal{A}_\phi}$, which *grows linearly* with $|\hat{u}_r(0)|_{\mathcal{A}_\phi}$. Thus, in the ESC case, the bound on $k\tau_v^*$ would also grow linearly with Δ, which is a weaker property compared to the constant bound (15). Other optimization flows with finite-time convergence properties are presented in [30] and [17]. Note that when $\alpha_1 = \alpha_2 = 0$ in (13), the FxTESC of Fig. 1 reduces to the standard GDESC. □

Next, we formalize and generalize the previous discussion by characterizing an entire family of FxTESCs.

IV. Gradient-Based Fixed-Time ES Controllers

Consider the closed-loop system (7) with general learning dynamics in (4) now modeled as follows:

$$\dot{\hat{u}} \in -k_1 F_u(\hat{u}, \xi) \tag{16}$$

where $F_u : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map, and $k_1 = \varepsilon_0 k$.

Assumption 4: The set-valued mappings $F_u(\cdot,\cdot)$ and $\hat{F}_G(\cdot) := F_u(\cdot, \nabla\phi(\cdot))$ satisfy the *Basic Conditions*. □

The regularity properties of Assumption 4 are standard in the analysis of nonsmooth systems, and they also hold when F_u and $\hat{F}_G$ are single-valued continuous functions. However, by working with differential inclusions, we will be able to consider ESCs with learning dynamics that are not necessarily continuous. In this case, solutions must be understood in a generalized sense by considering the set-valued map $\hat{F}_G^K(\hat{u}) := \bigcap_{\delta>0} \overline{\text{con}}\hat{F}_G(\hat{u}+\delta\mathbb{B})$, where $\overline{\text{con}}(\cdot)$ stands for the closed convex hull, see [9, Sec. 6.1], for examples, in the context of ESC. When $\hat{F}_G$ is LB, the set-valued map $\hat{F}_G^K$ satisfies the *Basic Conditions* [25, Lemma 5.16].

The following stability assumption characterizes the ESCs considered in this article.

Assumption 5: For each $k>0$, system $\dot{\hat{u}} \in -k\hat{F}_G(\hat{u})$ renders the set $\mathcal{A}_\phi$ UGFxTS with some $\beta_{G,k} \in \mathcal{KL}_\mathcal{T}$, with continuous settling time function satisfying $T_k(r) \leq T_k^*$ for all $r \geq 0$. □

Remark 4: Assumption 5 can be certified via Lyapunov functions (c.f. Lemma 1 or [26, Th. 5.8] for differential inclusions), or by studying the homogeneity properties in the bi-limit of the mapping $\hat{F}_G$, see [14], [26], [31], and [15], [16] for different examples of gradient-based optimization dynamics $\dot{\hat{u}} \in -k\hat{F}_G(\hat{u})$ that satisfy Assumption 5. □

We are now ready to state the first main result of this article.

Theorem 1: Suppose that Assumptions 1, 2, 4, and 5 hold. Then, $\forall$ $k>0$ and $\forall$ $\Delta > \nu > 0$, $\exists$ $\varepsilon_2^* > 0$ such that $\forall$ $\varepsilon_2 \in (0,\varepsilon_2^*)$, $\exists$ $a^*>0$ such that $\forall$ $a \in (0,a^*)$, $\exists$ $\varepsilon_1^*>0$ such that $\forall$ $\varepsilon_1 \in (0,\varepsilon_1^*)$, $\exists$ $\varepsilon_0^*>0$ such that $\forall$ $\varepsilon_0 \in (0,\varepsilon_0^*)$, all solutions of (7) with learning dynamics (16), and $|\hat{u}(0)|_{\mathcal{A}_\phi} \leq \Delta$, $|\xi(0)| \leq \Delta$, induce the bound

$$|u(\tau)|_{\mathcal{A}_\phi} \leq \beta_{G,k}(|\hat{u}(0)|_{\mathcal{A}_\phi},\tau) + \nu, \quad \forall \tau \geq 0$$

and $\beta_{G,k}(|\hat{u}(0)|_{\mathcal{A}_\phi},\tau) = 0$ for all $\tau \geq T_k^*$. □

Proof: Let k, Δ, and ν be given. Without loss of generality we assume $\nu \in (0,1)$. Let Assumption 5 generate the function $\beta_{G,k} \in \mathcal{KL}_\mathcal{T}$. We define the set $\tilde{K} := \left\{u \in \mathbb{R}^n : |u|_{\mathcal{A}_\phi} \leq \beta_{G,k}\left(\max_{y \in \mathcal{A}_\phi + \Delta\mathbb{B}} |y|_{\mathcal{A}_\phi}, 0\right) + 1\right\}$. By construction, this set is compact since without loss of generality $\beta_{G,k}$ can be taken to be continuous or to be upper bounded by a continuous class $\mathcal{KL}$ function [25, pp. 69]. Thus, there exists $M>0$ such that $\tilde{K} \subset M\mathbb{B}$. By continuity of $\nabla\phi$, there exists $\tilde{M} > \Delta$ such that $|\tilde{F}_G(u,a)| + \nu \leq \tilde{M}$ for all $|u| \leq M$ and all $a \in (0,1)$, where $\tilde{F}_G$ is defined in (9). Using this construction, we divide the proof in two main steps.

Step 1: Stability: The closed-loop system (7) is in standard form for the application of singular perturbation theory for nonsmooth systems (see [20] and [22]). The boundary layer dynamics are $\dot{x} = f(x,\hat{u}+a\tilde{\mu})$, $\dot{\hat{u}}=0$, $\dot{\xi}=0$, $\dot{\mu}=0$. By Assumption 1, the plant dynamics (3) have a well-defined quasi-steady-state manifold $x^* = \ell_x(\hat{u}+a\tilde{\mu})$. Therefore, the closed-loop system has a well-defined reduced system, given by the following:

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}\tau} \in -kF_u(\hat{u},\xi), \quad \varepsilon_1\frac{\mathrm{d}\mu}{\mathrm{d}\tau} = -2\pi\mathcal{R}_\theta\mu, \ \mu \in \mathbb{T}^n, \tag{17a}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = -\frac{1}{\varepsilon_2}\left(\xi - F_G(\phi(\hat{u}+a\tilde{\mu}),\tilde{\mu})\right) \tag{17b}$$

where we used $\phi(\hat{u}+a\tilde{\mu}) = h(\ell_x(\hat{u}+a\tilde{\mu}),\hat{u}+a\tilde{\mu})$. Since the dynamics of μ render forward invariant (and UGAS) the set $\mathbb{T}^n$, we focus on the properties of the states $(\hat{u},\xi)$. Indeed, note that since $0<\varepsilon_1 \ll \varepsilon_2$, system (17) is also in the standard form for the application of singular perturbation theory. The fast dynamics correspond to the linear oscillator that generates sinusoidal functions $\tau \mapsto \tilde{\mu}(\tau)$. The reduced dynamics are obtained by using [2, Lemma 6] and by averaging the dynamics of $(\hat{u},\xi)$ along the trajectories $\tilde{\mu}$. The resulting average system has state $\zeta^A = [\hat{u}^{A\top},\xi^{A\top}]^\top$ and dynamics

$$\frac{\mathrm{d}\hat{u}^A}{\mathrm{d}\tau} \in -kF_u(\hat{u}^A,\xi^A), \quad \varepsilon_2\frac{\mathrm{d}\xi^A}{\mathrm{d}\tau} = -\xi^A + \tilde{F}_G(\hat{u}^A,a) \tag{18}$$

with $\tilde{F}_G$ given by (9). When $a=0$, this system is also in the standard form for the application of singular perturbation theory, with ε_2 acting as small parameter. Using the definition of $\tilde{F}_G$ and the exponential stability properties of the low-pass filter in (17), we obtain the reduced nominal average dynamics $\frac{\mathrm{d}\hat{u}_r}{\mathrm{d}\tau} \in -kF_u(\hat{u}_r,\nabla\phi(\hat{u}_r)) = -k\hat{F}_G(\hat{u}_r)$. By Assumption 5, this system renders the set $\mathcal{A}_\phi$ UGFxTS with pair $(\beta_{G,k},T_k^*)$.

Now, for the purpose of analysis, let us restrict the dynamics (18) to evolve in the compact flow set $C = \tilde{K} \times \bar{M}\mathbb{B}$, where $\bar{M} := \tilde{M} + 1$; and the dynamics (17) and (7) to evolve in the compact flow set $C = \tilde{K} \times \bar{M}\mathbb{B} \times \mathbb{T}^n \times \Xi$. Applying [21, Th. 2] and [25, Th. 7.21] to the restricted system (18), we immediately obtain that the compact set $\mathcal{A}_\zeta := \mathcal{A}_\phi \times \bar{M}\mathbb{B}$ is GPAS as $(a,\varepsilon_2) \to 0^+$ with $\beta_{G,k} \in \mathcal{KL}_\mathcal{T}$. Since by the definition of solutions to (1), we have that $|\xi^A(\tau)|_{\bar{M}\mathbb{B}} = 0$ for all $\tau \in \text{dom}(\zeta^A)$, we also have that $|\zeta^A(\tau)|_{\mathcal{A}_\zeta} = |\hat{u}^A(\tau)|_{\mathcal{A}_\phi}$. Thus, for each $\nu' \in (0,0.5\nu)$ there exists $\varepsilon_2^*>0$ such that for all $\varepsilon_2 \in (0,\varepsilon_2^*)$ there exists $a^*>0$ such that for each $a \in (0,a^*)$ every solution of the restricted system (18) satisfies the bound

$$|\zeta^A(\tau)|_{\mathcal{A}_\zeta} \leq \beta_{G,k}(|\zeta^A(0)|_{\mathcal{A}_\zeta},\tau) + 0.5\nu'. \tag{19}$$

Next, since the fast oscillator dynamics of (17) render UGAS the set $\mathbb{T}^n$, and also generate a well-defined average system corresponding to (18), by Lemma 2 in the Appendix, we can directly establish that the restricted system (17) renders the set $\mathcal{A}_\zeta \times \mathbb{T}^n$ GPAS as $(\varepsilon_1,a,\varepsilon_2) \to 0^+$ with class $\mathcal{KL}_\mathcal{T}$ function $\beta_{G,k}$. This implies that for each $\nu' \in (0,0.5\nu)$ there exists $\varepsilon_2^*>0$ such that for each $\varepsilon_2 \in (0,\varepsilon_2^*)$ there exists $a^* \in (0,\nu'/2)$, such that for each $a \in (0,a^*)$ there exists $\varepsilon_1^*>0$, such that for each $\varepsilon_1 \in (0,\varepsilon_1^*)$ each solution of the restricted system (17) satisfies the bound $|\zeta(\tau)|_{\mathcal{A}_\zeta} \leq \beta_{G,k}(|\zeta(0)|_{\mathcal{A}_\zeta},\tau) + \frac{\nu'}{2}$, for all $\tau \in \text{dom}(\zeta,\mu)$, with $\zeta = (\hat{u},\xi)$. Since $|\mu(\tau)|_{\mathbb{T}^n} = 0$, it follows that $|\hat{\zeta}(\tau)|_{\tilde{\mathcal{A}}} = |\zeta(\tau)|_{\mathcal{A}_\zeta}$ for all $\tau \in \text{dom}(\hat{\zeta})$, where $\hat{\zeta} = [\hat{u}^\top,\xi^\top,\mu^\top]^\top$ and $\tilde{\mathcal{A}} = \mathcal{A}_\zeta \times \mathbb{T}^n$. Thus, the overall state $\hat{\zeta}$ of the restricted system (17) satisfies $|\hat{\zeta}(\tau)|_{\mathcal{A}_\phi} \leq \beta_{G,k}(|\hat{\zeta}(0)|_{\mathcal{A}_\phi},\tau) + \frac{\nu'}{2}$, for all $\tau \in \text{dom}(\hat{\zeta})$. Finally, since by Assumption 1, the plant dynamics have a well-defined UGAS quasi-steady-state manifold $\ell_x(\cdot)$, we can apply again Lemma 2 in the Appendix to establish that the restricted system (7) renders the set $\tilde{\mathcal{A}} \times \Xi$ GPAS as $(\varepsilon_0,\varepsilon_1,a,\varepsilon_2) \to 0^+$ with $\beta_{G,k} \in \mathcal{KL}_\mathcal{T}$. Since $|(\hat{u},\xi,\mu)(\tau)|_{\tilde{\mathcal{A}}} = |\hat{u}(\tau)|_{\mathcal{A}_\phi}$ for all $\tau \in \text{dom}(x,\hat{u},\xi,\mu)$, by using the definition of u in (4), and the facts that $a \in (0,\nu'/2)$ and $|\mu|=1$, we obtain the $\mathcal{KL}_\mathcal{T}$ bound of the theorem.

Step 2: Completeness of Solutions: We now show completeness of solutions for the unrestricted system (7) with learning dynamics (16), flow set $C = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}^n \times \Xi$, and initial conditions satisfying $|\hat{u}(0)|_{\mathcal{A}_\phi} \leq \Delta$, and $|\xi(0)| \leq \Delta$. Let the parameters $(\varepsilon_0,\varepsilon_1,a,\varepsilon_2)$ be generated as in Step 1 such that the $\mathcal{KL}_\mathcal{T}$ bound holds for u. Due to this bound, the construction of $\tilde{K}$, and Assumption 1, any solution of system (7) with learning dynamics (16), and length$(\text{dom}(\hat{u},\xi,\mu,x)) < \infty$, must stop due to ξ leaving the set $\bar{M}\mathbb{B}$. To show that this cannot occur, note that by the stability result of Step 1 and the uniform bound on $\hat{u}^A$, every solution of system (18) satisfies $\nabla\phi(\hat{u}^A(\tau)) + \mathcal{O}(a) \in \tilde{M}\mathbb{B}$ for all $\tau \geq 0$. By [32, Lemma 5], the low-pass filter dynamics in (18) render forward invariant the set $\tilde{M}\mathbb{B}$. It follows that every solution ζ^A of system (18) with $\zeta^A(0) \in (\mathcal{A}_\phi + \Delta\mathbb{B}) \times \Delta\mathbb{B}$ is complete. Since $\nabla\phi$ is locally Lipschitz, and $\nabla\phi(\mathcal{A}_\phi) = 0$ due to the optimality of the set $\mathcal{A}_\phi$, it follows that since $|\hat{u}^A(\tau)|_{\mathcal{A}_\phi} \leq 0.5\nu'$ for all $\tau \geq T_k^*$, there exists $\varphi > 0$ such that $|\nabla\phi(u^A(\tau)) + \mathcal{O}(a)| \leq \varphi\nu'$ for all $\tau \geq T_k^*$. Thus, by linearity and exponential stability of the low-pass filter, there exists $T'>0$ such that the trajectories ξ^A of (18) with $|\xi^A(0)| \leq \Delta$ also satisfy $|\xi^A(\tau)| \leq 2\varphi\nu'$, for all $\tau \geq T'$. Since the trajectories of system (18) converge to a $2\varphi\nu'$-neighborhood of the set $\mathcal{A}_\phi \times \{0\}$, by [25, Corollary 7.7] for the restricted system (18) there exists a UGAS set $\Omega_{a,\varepsilon_2} \subset (\mathcal{A}_\phi \times \{0\}) + 2\varphi\nu'\mathbb{B}$. By [21, Th. 1 & 2], there exists $\varepsilon_1^{**}>0$ such that for each $\varepsilon_1' \in (0,\min\{\varepsilon_1,\varepsilon_1^{**}\})$ the trajectories $\zeta = (\hat{u},\xi)$ generated by system (17) restricted to $\tilde{K} \times \bar{M}\mathbb{B} \times \mathbb{T}^n$ satisfy: (a) $\exists T''>0$, such that $\zeta(\tau) \in \Omega_{a,\varepsilon_2} + \varphi\nu'\mathbb{B} \subset (\mathcal{A}_\phi \times \{0\}) + 3\varphi\nu'\mathbb{B}$ for all $\tau \geq T''$, such that $\tau \in \text{dom}(\zeta)$; and (b) the trajectories ζ and ζ^A are (τ,ϵ)-close [25, Definition 5.23] with $\tau = T''+1$ and

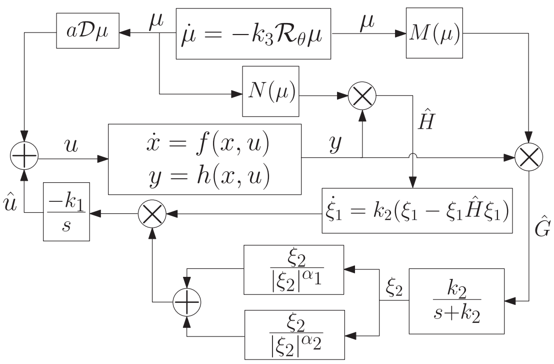

Fig. 3. Example of closed-loop system with a fixed-time Newton-based extremum seeking controller with $\alpha_1 \in (0,1)$ and $\alpha_2 < 0$.

$\epsilon = 0.5\nu'$. Thus, it follows that the trajectories $(\hat{u}, \xi, \mu)$ generated by system (17) with $|\hat{u}(0)|_{\mathcal{A}_\phi} \leq \Delta$ and $|\xi(0)| \leq \Delta$ belong to the set $\tilde{K} \times (\tilde{M} + 0.5\nu')\mathbb{B} \times \mathbb{T}^n$ for all $\tau \geq 0$. Completness of solutions for the closed-loop system (7) follows now by applying again the exact same procedure, using Assumption 1, [25, Corollary 7.7], [21, Th. 1 and 2], and (τ, ϵ)-closeness of solutions between the trajectories $(\hat{u}, \xi)$ of system (7) and (17), and the fact that $\tilde{M} + \nu' < \bar{M}$. ■

Remark 5: The result of Theorem 1 also holds for ESCs that have only finite-time convergence properties (c.f. Remark 3). In this case, Theorem 1 holds with $T_k(\Delta)$ instead of T_k^*. Whereas ESC with finite-time convergence properties has been numerically studied in [9], [11], and [33], to our knowledge, a general convergence result, such as Theorem 1, was absent in the literature. □

V. Newton-Like Fixed-Time ES Controllers

We now extend the previous results to Newton-like ESCs with "fixed-time" convergence properties. In this case, we make the following additional assumption, which is also standard (see [4], [17], and [24]).

Assumption 6: There exists $\tilde{\kappa} > 0$ such that the Hessian of ϕ satisfies $\nabla^2\phi(u) \succeq \tilde{\kappa}I$ for all $u \in \mathbb{R}^n$. □

To further motivate the results of this section, we recall that the reduced average nominal system associated to the standard Newton-like ESC (NESC) of [24] is given by the Newton flow (NF):

$$\frac{\mathrm{d}\hat{u}_r}{\mathrm{d}\tau} = -k\nabla^2\phi(\hat{u}_r)^{-1}\nabla\phi(\hat{u}_r). \tag{20}$$

Using the Lyapunov function $V_N(\hat{u}_r) = \frac{1}{2}|\nabla\phi(\hat{u}_r)|^2$, it follows that $\dot{V}_N = -2kV_N(\hat{u}_r)$, and by using the Comparison Lemma and Assumptions 3 and 6, we obtain that the solutions of (20) satisfy $|\hat{u}_r(\tau)|_{\mathcal{A}_\phi} \leq \frac{L}{\kappa}e^{-k\tau}|\hat{u}(0)|_{\mathcal{A}_\phi}$ for all $\tau \geq 0$. Now, consider the fixed-time Newton-like ESC (FxTNESC) of Fig. 3 that generates the following closed-loop system in the τ-time scale:

$$\begin{pmatrix} \frac{\mathrm{d}\hat{u}}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\xi_1}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\xi_2}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\mu}{\mathrm{d}\tau} \\ \frac{\mathrm{d}x}{\mathrm{d}\tau} \end{pmatrix} = \begin{pmatrix} -kF_u(\hat{u},\xi) \\ -\frac{1}{\varepsilon_2}\left(\xi_1 F_H(y,\tilde{\mu})\xi_1 - \xi_1\right) \\ -\frac{1}{\varepsilon_2}\left(\xi_2 - F_G(y,\tilde{\mu})\right) \\ -\frac{2\pi}{\varepsilon_1}\mathcal{R}_\theta\tilde{\mu} \\ \frac{1}{\varepsilon_0}f(x, \hat{u} + a\tilde{\mu}) \end{pmatrix} \tag{21}$$

where $\mu \in \mathcal{T}^n$, and where the learning dynamics F_u are now defined as follows:

$$F_u(\hat{u}, \xi) := \xi_1\left(\frac{\xi_2}{|\xi_2|^{\alpha_1}} + \frac{\xi_2}{|\xi_2|^{\alpha_2}}\right) \tag{22}$$

with $F_u := 0$ whenever $\xi_2 = 0$. As shown in [2], the parameters (α_1, α_2) can be selected again as in Remark 1 to guarantee continuity of F_u. The input u, the dynamic oscillator, the mappings (F_G, M), and the constants (k_1, k_2, k_3, a) are defined again as in Section III. However, system (21) has an extra state ξ_1 with dynamics depending on the mapping F_H, defined as $F_H(y, \tilde{\mu}) := yN(\tilde{\mu})$, where $N : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is matrix-valued function with entries N_{ij} satisfying $N_{ij} = N_{ji}$, $N_{ij} = \frac{16}{a^2}(\tilde{\mu}_i^2 - \frac{1}{2})\ \forall\ i = j$, and $N_{ij} = \frac{4}{a^2}\tilde{\mu}_i\tilde{\mu}_j\ \forall\ i \neq j$, where $\tilde{\mu}_i$ is the ith entry of the vector $\tilde{\mu}$.

Remark 6: In (21), the state ξ_1 is a matrix of dimension $n \times n$. Therefore, the dynamics of ξ_1 must be understood as a matrix differential equation. This notation, which is used to simplify our presentation, is consistent with the notation used in the Newton-based ESCs of [24]. □

By using [2, Lemma 6], we can analyze system (21) via singular perturbation and averaging theory for nonsmooth systems. In particular, for the reduced dynamics of (21), we can carry out a Taylor expansion of $\phi(\hat{u} + a\tilde{\mu})$ around the point $\hat{u}$ for small values of a, where we now retain the second-order terms: $\phi(\hat{u} + a\tilde{\mu}) = \phi(\hat{u}) + a\tilde{\mu}^\top\nabla\phi(\hat{u}) + \frac{a^2}{2}\tilde{\mu}^\top\nabla^2\phi(\hat{u})\tilde{\mu} + O(a^3)$. Using this expansion, the definitions of the mappings M, N, F_G, and F_H, and [2, Lemma 6], we obtain an average system with state $\zeta^A = (\hat{u}^A, \xi^A)$ and dynamics

$$\frac{\mathrm{d}\hat{u}^A}{\mathrm{d}\tau} = F_u(\hat{u}^A, \xi^A), \quad \varepsilon_2\frac{\mathrm{d}\xi_2^A}{\mathrm{d}\tau} = -\xi_2^A + \tilde{F}_G(\hat{u}^A, a) \tag{23a}$$

$$\varepsilon_2\frac{\mathrm{d}\xi_1^A}{\mathrm{d}\tau} = -\xi_1^A\nabla^2\phi(\hat{u}^A)\xi_1^A + \xi_1^A + O(a) \tag{23b}$$

which is also a singularly perturbed system. When $a = 0$, and for fixed-values of $\hat{u}^A$, the fast dynamics render locally exponentially stable [24, pp. 1761] the quasi-steady-state manifold $\xi^*(\hat{u}^A) = (\nabla^2\phi(\hat{u}^A)^{-1}, \nabla\phi(\hat{u}^A))$. Therefore, the reduced nominal average dynamics of (23) correspond to

$$\dot{\hat{u}}_r = -k\nabla^2\phi(\hat{u}_r)^{-1}\left(\frac{\nabla\phi(\hat{u}_r)}{|\nabla\phi(\hat{u}_r)|^{\alpha_1}} + \frac{\nabla\phi(\hat{u}_r)}{|\nabla\phi(\hat{u}_r)|^{\alpha_2}}\right). \tag{24}$$

Using again V_N, we obtain that $\dot{V}_N(\hat{u}_r) = -k\rho_1 V_N(\hat{u}_r)^{\chi_1} - k\rho_2 V_N(\hat{u}_r)^{\chi_2} < 0$, for all $\hat{u}_r \neq u^*$, where $\rho_1 = 2^{\chi_1} > 0$, $\rho_2 = 2^{\chi_2} > 0$, $\chi_1 = \frac{2-\alpha_1}{2} \in (0.5, 1)$, $\chi_2 = \frac{2-\alpha_2}{2} > 1$. By [26, Th. 5.8] and the smoothness of V_N, system (24) renders the point $\mathcal{A}_\phi = \{u^*\}$ UGFxTS. Moreover, using $\alpha_2 = -\alpha_1$, the assumptions of Lemma 1 hold with $\alpha = 1/\alpha_1$ and $\sqrt{ab} = 2k$, and we obtain the following estimate of the fixed-time convergence bound:

$$T_N^* := \frac{\pi}{2k\alpha_1}. \tag{25}$$

Note that in contrast to T_G^* in (15), the expression for T_N^* is now also independent of the unknown parameters of Assumptions 3 and 6, and an explicit function $\beta_{N,u} \in \mathcal{KL}_\mathcal{T}$ can be obtained for system (24) via Lemma 3 in the Appendix. Therefore, system (24) can be seen as a fixed-time Newton-flow (FxTN), and by averaging and singular perturbation theory we can expect that the FxTNESC will inherit (locally) the same $\mathcal{KL}$ bound of (24).

The following theorem corresponds to the second main result of this article.

Theorem 2: Consider the closed-loop system (21) with learning dynamics (16), and suppose that Assumptions 1, 2, and 6 hold, as well as Assumptions 4 and 5 with $\hat{F}_G$ substituted by $\hat{F}_N(\cdot) := F_u(\cdot, (\nabla^2\phi(\cdot)^{-1}, \nabla\phi(\cdot)))$, and $\beta_{G,k}$ substituted by $\beta_{N,k}$. Then, $\forall\ k > 0\ \exists\ \Delta > 0$, such that $\forall\ \nu \in (0, \Delta)$, $\exists\ \varepsilon_2^* > 0$, such that $\forall\ \varepsilon_2 \in (0, \varepsilon_2^*)$, $\exists\ a^* > 0$, such that $\forall a \in (0, a^*)$, $\exists\ \varepsilon_1^* > 0$, such that $\forall\ \varepsilon_1 \in (0, \varepsilon_1^*)$, $\exists\ \varepsilon_0^* > 0$, such that $\forall\ \varepsilon_0 \in (0, \varepsilon_0^*)$, all solutions of system (21) with $|\hat{u}(0)|_{\mathcal{A}_\phi} \leq \Delta$, $|\xi_1(0)|_{\nabla^2\phi(\mathcal{A}_\phi)^{-1}} \leq \Delta$, and $|\xi_2(0)| \leq \Delta$, induce the bound

$$|u(\tau)|_{\mathcal{A}_\phi} \leq \beta_{N,u}(|\hat{u}(0)|_{\mathcal{A}_\phi}, \tau) + \nu, \quad \forall \tau > 0$$

and $\beta_{N,u}(|\hat{u}(0)|_{\mathcal{A}_\phi}, \tau) = 0$ for all $\tau \geq T_k^*$. □

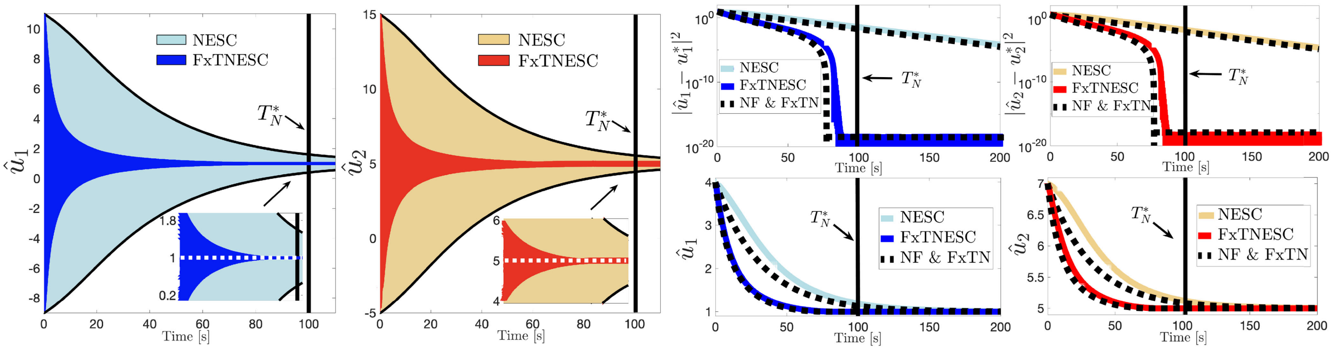

Fig. 4. Left: Approximate (finite-time) reachable set of the NESC and the FxTNESC, with $\hat{u}(0) \in \mathcal{A}_\phi + 10\mathbb{B}$. The insets illustrate the convergence of all solutions of the FxTNESC; Right: Time history of *one* solution of the NESC and the FxTNESC with identical initialization, in logarithmic scale (top) and linear scale (bottom). The dashed lines show the trajectories of the NF and FxTN flows (20) and (24) with identical initialization.

Proof: The proof is almost identical to the proof of Theorem 1, with the difference that the results hold only locally due to the local stability properties of the boundary layer dynamics of system (23). In particular, define $\mathcal{A}_\xi := \{\nabla^2\phi(\mathcal{A}_\phi)^{-1}\} \times \{0\}$, and note that by the stability properties of the boundary layer dynamics of (23), and by the smoothness properties of ϕ, there exists $\delta > 0$ and $a^{**} > 0$ such that for all $\xi(0) \in (\{\nabla^2\phi(\mathcal{A}_\phi)^{-1}\} \times \{0\}) + \delta\mathbb{B}$, $a \in (0, a^{**})$ and all $|\hat{u}^A|_{\mathcal{A}_\phi} \leq \delta$, every solution ξ^A of the boundary layer dynamics of system (23) is complete and satisfies $|\xi^A(\tau)|_{\mathcal{A}_\xi} \leq \delta''\mathbb{B}$ for all time $\tau \geq 0$, and $\delta'' > 0$. In turn, since $\beta_{N,k}$ is a class $\mathcal{KL}_\mathcal{T}$ function, there exists $\Delta > 0$ sufficiently small such that $\beta_{N,k}(\Delta, 0) + 0.5\delta < \delta$, which implies that for Δ sufficiently small the set $\tilde{K}$ used in the proof of Theorem 1 can be constructed such that $\tilde{K} \subset \mathcal{A}_\phi + \delta\mathbb{B}$. From here we can repeat the same Steps 1 and 2 of the proof of Theorem 1. ■

Remark 7: In the above-mentioned discussion, the parameter ε_2 was the same for the dynamic of ξ_1 and ξ_2. However, this was done only to simplify the presentation, and in practice they can be different in order to simplify the tuning of the algorithm. As in Theorem 1, the convergence result of Theorem 2 covers a variety of Newton-based ESCs that go beyond the one presented in Fig. 3, having fixed-time or finite-time (in this case $T_k^* = T_k(\Delta)$) convergence properties, including ESCs with discontinuous vector fields. □

For constant values of $k > 0$, $\alpha \in (0, 1)$, $p = (\nu, \alpha_1)$, and by using the structure of the exponential $\mathcal{KL}$ bound of (20), the value of T_N^* in (25) will be smaller than the convergence time τ_v^* of the NESC whenever the initial conditions $\hat{u}(0)$ of the controllers are in the set $\Omega_{Np} := \{u_{r,0} \in \mathbb{R}^n : \Delta \geq |u_{r,0}|_{\mathcal{A}_\phi} > \frac{\kappa\nu}{2L}\exp(\pi/2\alpha_1)\}$. Since Theorem 2 is a local result, in this case Δ cannot be selected arbitrarily large. However, this does not necessarily imply that the set Ω_{Np} is empty, especially as $\nu \to 0^+$.

Example 2: We consider the same plant and cost function of Example 1, but this time we simulate the closed-loop system using the FxTNESC of Fig. 3. We set $\alpha_1 = 0.5 = -\alpha_2$, and $k = \pi/100$, which *assigns* $T_N^* = 100$ via (25). To guarantee that the ESC behaves (on compact sets) as its average system in the slowest time scale, we use again $a = 0.01$, $\varepsilon_0 = 5 \times 10^{-5}$, $\varepsilon_1 = 1 \times 10^{-4}$, $\varepsilon_{2,\xi_1} = 1 \times 10^1$, $\varepsilon_{2,\xi_2} = 2 \times 10^{-1}$. For the oscillator we used $2\pi\theta_1 \approx 5$, $2\pi\theta_2 \approx 3.5$, and $\mu(0) = [0, 1, 0, 1]^\top$. We further used a low-pass filter to smooth the Hessian estimation. This filter is not necessary for the simulation, but it can simplify the tuning of the Newton-based ESCs (see [24]). We computed again a numerical approximation of the reachable set (for the state $\hat{u}$) of the NESC and the FxTNES from initial conditions satisfying $\hat{u} \in \mathcal{A}_\phi + 10\mathbb{B}$, $\xi_2(0) = 1$, $\xi_1(0) = [0.25, -0.1, -0.1, 0.25]$, via 1×10^3 simulations with random initialization in this set. The result is shown in the left plots of Fig. 4 . The fixed-time convergence property of the proposed FxTNES is further illustrated in the logarithmic scale of the upper right plots, shown in Fig. 4, which also shows the trajectories of the reduced average nominal dynamics (20) and (24) using the same gains k and with identical initialization. As shown in the lower right plot of Fig. 4, the trajectories of the ESCs remain close to the trajectories of their respective reduced average nominal dynamics. □

Remark 8: When the plant (3) is a static map, i.e., $y = h(u)$, one can take $\phi(u) = h(u)$ and $\varepsilon_0 = 1$. In this case, Theorems 1 and 2 recover the results of [1] and [2], which are specialized for the learning dynamics (13) and (22), now with sharper bounds T_G^* and T_N^* given by (15) and (25), respectively. In this case, the convergence time T_N^* can be completely prescribed *a priori* by the user, without the need of retuning the gain k for different initial conditions. A similar observation holds for T_G^* if a lower bound on κ is known *a priori*. When the plant is dynamic, the bounds hold in the t-time scale with T_G^*/ε_0 and T_N^*/ε_0. □

VI. Conclusion

In this article, we introduced a novel class of nonsmooth ESCs with convergence bounds characterized by class-$\mathcal{KL}_\mathcal{T}$ functions that confer suitable transient performance. Our main results can be used for the design and analysis of different averaging-based ESCs that go beyond those considered in this article, and which are not necessarily Lipschitz continuous, or even continuous. In the latter case, the ESCs must be analyzed using the framework of differential inclusions. When the plant is a static map, the convergence time of the algorithms can be prescribed *a priori* by the users without retuning the gain of the learning dynamics for different initial conditions. Two numerical examples were presented to illustrate our theoretical results. Future research directions will focus on FxTESCs for multiagent systems.

Appendix

The following Lemma is a minor extension of [21, Th. 2], for the case when the average dynamics have a compact set that is SGPAS instead of UGAS. The proof follows directly by using [21, Th. 1], and the same steps of the proof of [22, Th. 7], and therefore, it is omitted due to space limitations.

Lemma 2: Consider the singularly perturbed system with state $(x_1, x_2, y) \in \mathbb{R}^n \times \mathbb{X}_2 \times \mathbb{X}_y$ and dynamics

$$\dot{x}_1 \in F_\delta(x), \quad \dot{x}_2 = G_\delta(x, y), \quad \varepsilon\dot{y} = H(x, y) \tag{26}$$

where $\mathbb{X}_2 \subset \mathbb{R}^n$ and $\mathbb{X}_y \subset \mathbb{R}^p$ are compact sets, $x = [x_1^\top, x_2^\top]^\top$, and for each $\delta > 0$ the set-valued mapping $F_\delta : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ satisfies the Basic Conditions, and the mappings $G_\delta : \mathbb{R}^{2n} \times \mathbb{R}^p \to \mathbb{R}^n$ and $H : \mathbb{R}^{2n} \times \mathbb{R}^p \to \mathbb{R}^p$ are Lipschitz continuous. Suppose the following holds.

1) *Existence of Average:* There exists $\delta^* > 0$ such that for each $\delta \in (0, \delta^*)$ there exists a continuous function $G_\delta^A : \mathbb{R}^{2n} \to \mathbb{R}^n$ such that for each compact set $K \subset \mathbb{R}^n$ there exists a class-$\mathcal{L}$ function

σ_K such that for each $L > 0$, and each solution $y_{bl} : [0, L] \to \mathbb{X}_y$ of the system: $(x, y_{bl}) \in K \times \mathbb{X}_2 \times \mathbb{X}_y$, $\dot{x} = 0$, $\dot{y}_{bl} = H(x, y_{bl})$, the following holds $|\frac{1}{L}\int_0^L G_\delta^A(x) - G_\delta(x, y_{bl}(s))ds| \leq \sigma_K(L)$.

2) *SGPAS of Average System:* There exists a compact set $\mathcal{A}_x \subset \mathbb{R}^n$ such that the system $\dot{x}_1^A \in F_\delta(x^A)$, $\dot{x}_2^A = G_\delta^A(x^A)$ renders the set $\mathcal{A}_x \times \mathbb{X}_2$ SGPAS as $\delta \to 0^+$ with $\beta \in \mathcal{KL}$.

Then, system (26) renders the compact set $\mathcal{A}_x \times \mathbb{X}_2 \times \mathbb{X}_y$ SGPAS as $(\varepsilon, \delta) \to 0^+$ with $\beta \in \mathcal{KL}$. □

Lemma 3: Suppose that $V : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies the Assumptions of Lemma 1, and there exist $\lambda_1, \lambda_2, p > 0$ such that $\lambda_1|x|^p \leq V(x) \leq \lambda_2|x|^p$. Then, every solution of (1) satisfies the following:

$$|x(t)| \leq c_1 \tan\left(\max\left\{0, -c_2 t + \arctan\left(c_3|x(0)|^{\frac{p}{2\alpha}}\right)\right\}\right)^{\frac{2\alpha}{p}}$$
$$=: \beta(|x(0)|, t),\ \forall t \geq 0, \quad \text{and} \quad \beta \in \mathcal{KL}_{\mathcal{T}}, \tag{27}$$

where $c_1 := \left(\frac{a}{b}\right)^{\frac{\alpha}{p}}\left(\frac{1}{\lambda_1}\right)^{\frac{1}{p}}$, $c_2 := \frac{\sqrt{ab}}{2\alpha}$, and $c_3 := \sqrt{\frac{b}{a}}\lambda_2^{\frac{1}{2\alpha}}$. □

Proof: : Let $\dot{y} = -ay^{\gamma_1} - by^{\gamma_2}$, with γ_1, γ_2 as in Lemma 1. Using steps as in the proof of [23, Lemma 2], and the fact that $y^* = 0$ is an equilibrium point, it follows that every solution y satisfies $\frac{2\alpha}{\sqrt{ab}}\arctan(\sqrt{\frac{b}{a}}y^{\frac{1}{2\alpha}}(t)) = \max\{0, -t + \frac{2\alpha}{\sqrt{ab}}\arctan(\sqrt{\frac{b}{a}}y(0)^{\frac{1}{2\alpha}})\}$ for all $t \geq 0$. Solving for y and using the generalized Comparison Lemma of [34, Lemma 1], we obtain $V(x(t)) \leq (a/b)^\alpha \tan(\max\{0, -\sqrt{ab}/(2\alpha)t + \arctan(\sqrt{b/a}V(x(0))^{1/2\alpha})\})^{2\alpha}$ for all $t \geq 0$. The result now follows by using the upper and lower bounds of V, and the continuity and monotonicity properties of the functions $\arctan : \mathbb{R}_{\geq 0} \to [0, \pi/2)$ and $\tan : [0, \pi/2) \to \mathbb{R}$. ■

Acknowledgment

The first author would like to thank A. R. Teel for fruitful discussions on singularly perturbed nonsmooth systems.

References

[1] J. I. Poveda and M. Krstic, "Fixed-time gradient-based extremum seeking," in *Proc. Amer. Control Conf.*, 2020, pp. 2838–2843.

[2] J. I. Poveda and M. Krstic, "Fixed-time Newton-based extremum seeking," *in Proc. 21st IFAC World Congr.*, 2020, *arXiv:2012.13015*.

[3] K. B. Ariyur and M. Krstic, *Real-Time Optimization by Extremum-Seeking Control*. Hoboken, NJ, USA: Wiley, 2003.

[4] D. DeHaan and M. Guay, "Extremum-seeking control of state-constrained nonlinear systems," *Automatica*, vol. 41, pp. 1567–1574, 2005.

[5] D. Nesic, Y. Tan, W. H. Moase, and C. Manzie, "A unifying approach to extremum seeking: Adaptive schemes based on estimation of derivatives," in *Proc. IEEE Conf. Decis. Control*, 2010, pp. 4625–4630.

[6] Y. Tan, D. Nesic, and I. M. Mareels, "On non-local stability properties of extremum seeking control," *Automatica*, vol. 42, no. 6, pp. 889–903, 2006.

[7] H. Durr, M. S. Stankovic, C. Ebenbauer, and K. H. Johansson, "Lie bracket approximation of extremum seeking systems,"*Automatica*, vol. 49, pp. 1538–1552, 2013.

[8] V. Grushkovskaya, A. Zuyev, and C. Ebenbauer, "On a class of generating vector fields for the extremum seeking problem: Lie bracket approximation and stability properties," *Automatica*, vol. 94, pp. 151–160, 2018.

[9] J. I. Poveda and A. R. Teel, "A framework for a class of hybrid extremum seeking controllers with dynamic inclusions,"*Automatica*, vol. 76, pp. 113–126, 2017.

[10] R. J. Kutadinata, W. H. Moase, and C. Manzie, "Extremum-seeking in singularly perturbed hybrid systems," *IEEE Trans. Autom. Control*, vol. 62, no. 6, pp. 3014–3020, Jun. 2017.

[11] A. Scheinker and M. Krstic, "Non-C2 lie bracket averaging for nonsmooth extremum seekers," *J. Dyn. Syst., Meas., Control*, vol. 136, no. 1, pp. 1–10, 2014.

[12] H. Rios and A. R. Teel, "A hybrid fixed-time observer for state estimation of linear systems," *Automatica*, vol. 87, pp. 103–112, 2018.

[13] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Trans. Autom. Control*, vol. 57, no. 8, pp. 2106–2110, Aug. 2012.

[14] V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous approximation, recursive observer design, and output feedback," *SIAM J. Control Optim.*, vol. 47, no. 4, pp. 1814–1850, 2008.

[15] K. Garg and D. Panagou, "Fixed-time stable gradient flows: Applications to continuous-time optimization," *IEEE Trans. Autom. Control*, to be published, doi: 10.1109/TAC.2020.3001436.

[16] K. Garg, M. Baranwal, R. Gupta, R. Vasudevan, and D. Panagou, "Fixed-time stable proximal dynamical system for solving mixed variational inequality problems," 2019, *arXiv:1908.03517*.

[17] O. Romero and M. Benosman, "Finite-time convergence of continuous-time optimization algorithms via differential inclusions," *in Proc. Conf. Neural Inf. Process. Syst.*, 2019, pp. 1–11.

[18] A. R. Teel, L. Moreau, and D. Nesic, "A unified framework for input-to-state stability in systems with two time scales," *IEEE Trans. Autom. Control*, vol. 48, no. 9, pp. 1526–1544, Sep. 2003.

[19] D. Nesic, A. Mohammadi, and C. Manzie, "A framework for extremum seeking control of systems with parameter uncertainties," *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 435–448, Feb. 2013.

[20] W. Wang, A. Teel, and D. Nesic, "Analysis for a class of singularly perturbed hybrid systems via averaging," *Automatica*, vol. 48, no. 6, pp. 1057–1068, 2012.

[21] W. Wang, A. R. Teel, and D. Nesic, "Averaging in singularly perturbed hybrid systems with hybrid boundary layer systems," *in Proc. 51st IEEE Conf. Decis. Control*, 2012, pp. 6855–6860.

[22] J. I. Poveda and N. Li, "Robust hybrid zero-order optimization algorithms with acceleration via averaging in continuous time," *Automatica*, vol. 123, 2021, Art. no. 109361.

[23] S. Parsegov, A. Polyakov, and P. Shcherbakov, "Nonlinear fixed-Time control protocol for uniform allocation of agents on a segment," in *Proc. 51st IEEE Conf. Decis. Control*, 2012, pp. 7732–7737.

[24] A. Ghaffari, M. Krstic, and D. Nesic, "Multivariable Newton-based extremum seeking," *Automatica*, vol. 48, pp. 1759–1767, 2012.

[25] R. Goebel, R. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems*. Princeton, NJ, USA: Princeton Univ. Press, 2012.

[26] E. Bernau, D. Efimov, W. Perruquetti, and A. Polyakov, "On homogeneity and its application in sliding mode control," *J. Franklin Inst.*, vol. 351, pp. 1866–1901, 2014.

[27] T. Menard, E. Moulay, and W. Perruqueti, "Fixed-time observer with simple gains for uncertain systems,"*Automatica*, vol. 81, pp. 438–446, 2017.

[28] Z. Zuo, Q. Han, and B. Ning, *Fixed-Time Cooperative Control of Multi-Agent Systems*. Berlin, Germany: Springer, 2019.

[29] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[30] J. Cortes, "Finite-time convergent gradient flows with applications to network consensus," *Automatica*, vol. 42, no. 11, pp. 1993–2000, 2006.

[31] X. Pan, Z. Liu, and Z. Chen, "Distributed optimization with finite-time convergence via discontinuous dynamics," *in Proc. 37th Chin. Control Conf.*, 2018, pp. 6665–6669.

[32] S. Park, N. Martins, and J. Shamma, "Payoff dynamics model and evolutionary dynamics model: Feedback and convergence to equilibria," 2020, *arXiv:1903.02018v4*.

[33] L. Wang, S. Chen, K. Ma, and H. Zhao, "On robust convergence of perturbation based extremum seeking," *in Proc. Chin. Autom. Congr.*, 2019, pp. 4873–4878.

[34] A. R. Teel and L. Praly, "On assigning the derivative of a disturbance attenuation control Lyapunov function," *Math. Control Signals Syst.*, vol. 13, pp. 95–124, 2000.