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Adaptive-gain observer-based stabilization of stochastic strict-feedback systems with sensor uncertainty☆

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ABSTRACT

We study the adaptive output-feedback stabilization problem of stochastic strict-feedback systems with sensor uncertainty. Specifically, we consider the simultaneous presence of sensor uncertainty, unknown growth rate and stochastic disturbance, which has not been treated heretofore. By developing a new stochastic adaptive dual-domination approach, an adaptive observer and an output-feedback controller are designed, in which two gains are suitably selected to dominate the unknown sensor sensitivity and unknown growth rate, respectively. By using the nonnegative semimartingale convergence theorem, it is proved that the closed-loop system has an almost surely unique solution on $[0, +\infty)$ and that regulation to the equilibrium at the origin of the closed-loop system is achieved almost surely. Finally, two simulation examples are given to illustrate the control design.

1. Introduction

Since stochastic noise and nonlinearity exist widely in applications, the research of stochastic nonlinear control has drawn considerable attention and is gaining importance in econometrics, biology, environmental science and other areas. Over the past two decades, significant results have been obtained on the controller design for stochastic nonlinear systems. For the state-feedback control design, Deng and Krstic (1997a, 1997b) and Deng, Krstic, and Williams (2001) develop stochastic backstepping control design with the quartic Lyapunov functions; Pan and Basar (1998, 1999) focus on the optimal and near-optimal controller design under risk-sensitive cost function criterion. When it turns to the output-feedback control design, Deng and Krstic (1999) present the first result on global output-feedback stabilization in probability for stochastic nonlinear continuous-time systems. Furthermore, Deng and Krstic (2000) generalize the results in Deng and Krstic (1999) to systems with noise whose covariance is time varying and bounded but the bound is not known a priori. Since then, significant contributions have been made in studying the output-feedback control of stochastic nonlinear systems with different structures by many researchers. For example, Liu and Zhang (2006) address the output-feedback design for systems in observer canonical form under long-term average tracking risk-sensitive cost criteria; Liu, Zhang, and Jiang (2007) investigate the decentralized adaptive output-feedback stabilization for large-scale stochastic nonlinear systems with uncertainties; Wu, Xie, Shi, and Xia (2009) design output-feedback controllers for stochastic systems with Markovian switching; Li, Xie, and Zhang (2011) solve the output-feedback stabilization problem for stochastic high-order nonlinear systems by completely removing the power order restriction and largely relaxing the nonlinear growth condition; Li and Liu (2017) study the adaptive output-feedback control problem by developing a general stochastic convergence theorem.

It should be emphasized that all the above-mentioned results (Deng & Krstic, 1999, 2000; Li & Liu, 2017; Li et al., 2011; Liu & Zhang, 2006; Liu et al., 2007; Wu et al., 2009), on the output-feedback control of stochastic nonlinear systems, do not consider the sensor uncertainty in their systems' outputs. However, in practice, as shown in circuits and electrical devices (Carr, 1993) and mechanical systems (Kolovsky, 1999), the sensor sensitivity $\theta(t)$ in the system output is not always a constant. Due to manufacturing reasons, there always exists a sensitivity error in $\theta(t)$. For example, as demonstrated in Lantto (1999), the displacement sensor of a magnetic bearing suspension system experiences $\pm 10\%$ sensitivity error. Recently, motivated by Carr (1993), Kolovsky (1999) and Lantto (1999), Chen, Qian, Sun, and Liang (2018) propose a dual-domination approach to solve the output feedback stabilization problem for nonlinear systems with unknown measurement sensitivity in the output. However, the

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results in Chen et al. (2018) contain two limitations: *the growth rate for nonlinear functions should be accurately known and there is no noise in the studied system*.

In many applications, systems do not merely suffer from unknown sensor uncertainty but are often subject to stochastic perturbations and unknown growth rate, simultaneously, in their dynamics (Hinrichsen & Pritchard, 1996; Ugrinovskii, 1998; Wonham, 1970). For example, as demonstrated by Ugrinovskii (1998), the two-mass spring system is a typical stochastic physical system. Once the performance of the spring has deteriorated through wear, the value of the spring coefficient is deemed to belong between suitable bound and is thus unknown. Besides, for the output-feedback control of the spring system, the displacement sensor most commonly used is an inductive type sensor with excellent linearity and signal-to-noise ratio (Lantto, 1999). The sensor sensitivity is practically the same in both X- and Y-directions, but may differ from its assumed value because of manufacturing reasons. In view of the above-mentioned facts, the two-mass spring system can be practically modeled as a stochastic nonlinear system with sensor uncertainty and unknown growth rate. Specifically, the sensor uncertainty comes from the displacement sensor and the unknown growth rate is produced by the unknown nominal value of the spring coefficient. To the best of the authors' knowledge, there is no previously published work which can solve such an output-feedback control problem, which contains sensor uncertainty, unknown growth rate and stochastic noise simultaneously. From a practical point of view, it is imperative to study the output-feedback control of such systems.

Motivated by the above observations, we study the adaptive output-feedback stabilization control of stochastic strict-feedback systems with unknown growth rate and sensor uncertainty. The contributions of this paper are four-fold:

(1) In contrast to the previous work in the literature (Chen et al., 2018; Deng & Krstic, 1999, 2000; Li & Liu, 2017; Li, Liu, & Feng, 2019; Li et al., 2011; Liu & Zhang, 2006; Liu et al., 2007; Wu et al., 2009), the system model studied in this paper is more general since it considers sensor uncertainty, unknown growth rate and stochastic noise simultaneously.

(2) This paper is not an easy generalization from the deterministic system (Chen et al., 2018) to stochastic systems. In fact, even when there is no noise in the studied systems, the results in this paper are new and more general than those in Chen et al. (2018) since this paper considers an unknown growth rate. For the controller design, due to the effect of unknown growth rate, the information of the bounding function for the system uncertainty cannot be used directly, which makes the constant gain approach in Chen et al. (2018) inapplicable since it cannot guarantee stability. In this paper, we introduce a new dynamic gain in the observer and control design. The existence of stochastic noise and unknown growth rate makes the stability analysis in this paper much more involved and difficult than that of Chen et al. (2018); advanced stochastic analysis techniques are needed to prove the stability.

(3) A new adaptive observer is designed in this paper. On the one hand, we design an adaptive observer driven only by the input u, not using the information of output y. In the existing results (Lei & Lin, 2006; Li & Liu, 2017; Yan & Liu, 2011), the adaptive observers are driven by both u and y. Nevertheless, the existence of unknown sensor sensitivity $\theta(t)$ in the output makes the observers in Lei and Lin (2006), Li and Liu (2017) and Yan and Liu (2011) inapplicable since they require $y = x_1$ (i.e. $\theta(t) \equiv 1$) to construct an estimate error in the stability analysis. On the other hand, by using the information of the allowable sensitivity error, we design a new adaptive law for the dynamic gain in the observer, which can effectively dominate the nonlinear terms arising from the controller and unknown state x_1.

(4) A new domination approach is developed in Proposition 3 to prove the boundedness of the closed-loop system. Firstly, a new change of coordinates is introduced with two constants: the first constant is used to dominate the unknown sensor sensitivity and another suitably constructed constant can effectively dominate the nonlinear terms arising from unknown growth rate and the gains of the observer and controller. Then, by using the Itô formula and the nonnegative semimartingale convergence theorem, we prove boundedness of all the states in the closed-loop system.

The remainder of this paper is organized as follows. Section 2 describes the problem to be investigated. Section 3 shows controller design. Section 4 analyzes stability of the closed-loop system. Section 5 gives two examples to illustrate the theoretical results. Section 6 includes concluding remarks. Appendices A–E collect the useful tools, the proofs of Theorem 1 and Propositions 1–3, respectively.

Notations: $\mathbb{R}^+$ and $\mathbb{R}^n$ denote the set of all nonnegative real numbers and the real n-dimensional space, respectively. For a given vector or matrix X, X^T denotes its transpose, $Tr\{X\}$ denotes its trace when X is square, and $|X|$ is the Euclidean norm of a vector X. Defining $|A| = \left(\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2\right)^{1/2}$ for a matrix $A_{n\times m}$. I_n denotes the n-dimensional identity matrix. $\lambda_{max}(M)$ and $\lambda_{min}(M)$ denote the maximum and minimum eigenvalues of a square matrix M. For any $a, b \in R$, let $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. $C(R^n, R)$ denotes the set of all continuous functions mapping from R^n to R and $\chi_A(\cdot)$ denotes the indicator function of A. C^i denotes the set of all functions with continuous ith derivatives. For any given $V \in C^2$ associated with stochastic system $dx = f(t, x)\mathrm{d}t + g(t, x)\mathrm{d}\omega$, the differential operator $\mathscr{L}$ is defined as $\mathscr{L}V(x) \triangleq \frac{\partial V(x)}{\partial x} f(t, x) + \frac{1}{2}\mathrm{Tr}\left\{g^T(t, x)\frac{\partial^2 V(x)}{\partial x^2} g(t, x)\right\}$.

2. Problem formulation

Consider a class of stochastic nonlinear systems described by

$$dx_i = (x_{i+1} + f_i(t, x))dt + g_i(t, x)d\omega, \ \ i = 1, \dots, n-1, \tag{1}$$

$$dx_n = (u + f_n(t, x))dt + g_n(t, x)d\omega, \tag{2}$$

$$y = \theta(t)x_1, \tag{3}$$

where $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, control input and measurement output. The functions $f_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{1\times r}$ are piecewise continuous in t, locally bounded and locally Lipschitz continuous in x uniformly in $t \in \mathbb{R}^+$, $f_i(t, 0) = 0$, $g_i(t, 0) = 0$, $i = 1, \dots, n$. The sensor sensitivity $\theta(t)$ is an unknown continuous function of $t \in \mathbb{R}^+$. ω is an r-dimensional independent standard Wiener process defined on the complete probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ with a filtration $\mathscr{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all P-null sets).

For system (1)–(3), we need the following assumptions.

Assumption 1. There exists an unknown positive constant c (c is called the unknown growth rate) such that

$$|f_i(t, x)| \leq c(|x_1| + \cdots + |x_i|), \tag{4}$$

$$|g_i(t, x)| \leq c(|x_1| + \cdots + |x_i|), \ \ i = 1, \dots, n. \tag{5}$$

Assumption 2. The sensor sensitivity $\theta(t)$ is an unknown continuous function satisfying $\theta(t) \in [1 - \bar{\theta}, 1 + \bar{\theta}]$, where $0 < \bar{\theta} < 1$ is the allowable sensitivity error.

Remark 1. As demonstrated by Deng and Krstic (1999, 2000), Li and Liu (2017), Li et al. (2011), Liu and Zhang (2006), Liu et al. (2007) and Wu et al. (2009), the linear growth condition (4)–(5)

in Assumption 1 is a natural condition frequently used to solve the output-feedback control problems of stochastic nonlinear systems. In addition, as shown in Chen et al. (2018), Assumption 2 is crucial to guarantee the stability of system (1)–(3). However, it is obvious that system (1)–(3) is substantially different from those in Chen et al. (2018), Deng and Krstic (1999, 2000), Li and Liu (2017), Li et al. (2011), Liu and Zhang (2006), Liu et al. (2007) and Wu et al. (2009). Unlike Deng and Krstic (1999, 2000), Li and Liu (2017), Li et al. (2011), Liu and Zhang (2006), Liu et al. (2007) and Wu et al. (2009), there is sensor sensitivity $\theta(t)$ in (3), which is neither known nor differentiable. Different from Chen et al. (2018), stochastic noise exists in (1)–(2) (the noise ω is an r-dimensional independent standard Wiener process), and the growth rate c of the drift terms and diffusion terms is unknown, see (4)–(5).

The objective of this paper is to design an adaptive output-feedback controller for system (1)–(3) such that the closed-loop system has an almost surely unique solution on $[0,+\infty)$ and that regulation to the equilibrium at the origin of the closed-loop system is achieved almost surely.

3. Adaptive output-feedback controller design

In this section, we first design an observer and then design an adaptive output-feedback controller for system (1)–(3).

Firstly, we choose suitable design parameters $a_i > 0$ and $b_i > 0$, $i = 1, \ldots, n$, such that the matrices A and B are Hurwitz, where

$$A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -b_n & -b_{n-1} & \cdots & -b_1 \end{bmatrix}. \tag{6}$$

For any positive constant σ, there exist $P = P^T$ and $Q = Q^T$ satisfying

$$A^T P + PA \le -2I_n, \tag{7}$$

$$DP + PD \ge 0, \tag{8}$$

$$B^T Q + QB \le -2I_n, \tag{9}$$

$$DQ + QD \ge 0, \tag{10}$$

where $D = \text{diag}[\sigma, 1+\sigma, \ldots, n-1+\sigma]$.

For system (1)–(3), we construct the following observer

$$\dot{\hat{x}}_1 = \hat{x}_2 - La_1\hat{x}_1, \tag{11}$$

$$\dot{\hat{x}}_2 = \hat{x}_3 - L^2 a_2\hat{x}_1, \tag{12}$$

$$\vdots$$

$$\dot{\hat{x}}_n = u - L^n a_n\hat{x}_1, \tag{13}$$

where $L(t) \ge 1$ is a dynamic gain to be designed later.

Introducing the following coordinates

$$e_i = \frac{x_i - \hat{x}_i}{L^{i-1+\sigma}}, \quad i = 1, \ldots, n, \tag{14}$$

$$z_1 = \frac{x_1}{L^\sigma}, \; z_i = \frac{\hat{x}_i}{L^{i-1+\sigma}L_0^{i-1}}, \; v = \frac{u}{L^{n+\sigma}L_0^n}, \; i = 2, \ldots, n, \tag{15}$$

and choosing the output-feedback controller as

$$v = -b_n L^{-\sigma} y - b_{n-1}z_2 - \cdots - b_2 z_{n-1} - b_1 z_n, \tag{16}$$

then from (1)–(3) and (11)–(16) we have

$$de = \left(LAe + L^{1-\sigma}ax_1 - \frac{\dot{L}}{L}De\right)dt + F_e dt + G_e d\omega, \tag{17}$$

$$dz = \Big(LL_0Bz + LL_0B_zb_n(1-\theta(t))z_1 + LD_2e_2 + \frac{L}{L_0}D_1(e_1 - z_1) - \frac{\dot{L}}{L}Dz\Big)dt + F_z dt + G_z d\omega, \tag{18}$$

where $L_0 \ge 1$ is a constant gain to be designed, $e = (e_1, \ldots, e_n)^T$, $z = (z_1, \ldots, z_n)^T$, $a = (a_1, \ldots, a_n)^T$, $D_2 = (1, 0, \ldots, 0)^T$ and

$$B_z = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 \\ a_2 \\ \frac{1}{L_0}a_3 \\ \vdots \\ \frac{1}{L_0^{n-2}}a_n \end{bmatrix}, \; F_e = \begin{bmatrix} \frac{1}{L^\sigma}f_1 \\ \frac{1}{L^{1+\sigma}}f_2 \\ \vdots \\ \frac{1}{L^{n-1+\sigma}}f_n \end{bmatrix}, \tag{19}$$

$$F_z = \begin{bmatrix} \frac{1}{L^\sigma}f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; G_e = \begin{bmatrix} \frac{1}{L^\sigma}g_1 \\ \frac{1}{L^{1+\sigma}}g_2 \\ \vdots \\ \frac{1}{L^{n-1+\sigma}}g_n \end{bmatrix}, \; G_z = \begin{bmatrix} \frac{1}{L^\sigma}g_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{20}$$

Remark 2. The observer (11)–(13) and the controller (16) is different from those in Deng and Krstic (1999, 2000), Li and Liu (2017), Li et al. (2011), Liu and Zhang (2006), Liu et al. (2007) and Wu et al. (2009), in the sense that the information of x_1 is not used. Since the sensor sensitivity $\theta(t)$ in (3) is unknown, the information of the state x_1 is unavailable and it cannot be directly used in constructing the observer and controller.

Choose the Lyapunov function

$$V_1(e,z) = (m_1 + 1)e^T Pe + z^T Qz, \tag{21}$$

where $m_1 = |Q|^2|a|^2$.

In the following theorem, a new stochastic adaptive dual-domination approach is developed to solve the stabilization problem of system (1)–(3). Specifically, a dynamic gain $L(t)$ and a constant gain L_0 are suitably chosen to dominate the unknown growth rate c and the unknown sensor sensitivity $\theta(t)$, respectively. The proof of this theorem can be found in Appendix B.

Theorem 1. *For system (17)–(18), if the allowable sensitivity error satisfies $0 < \bar{\theta} < \min\{1, \frac{1}{b_n|Q|}\}$ and Assumptions 1–2 hold, using the gains $L(t)$ and L_0 defined by*

$$\dot{L} = \left(\frac{\hat{x}_1}{L^\sigma}\right)^2 + \left(\frac{y}{(1-\bar{\theta})L^\sigma}\right)^2, \quad L(0) = 1, \tag{22}$$

$$L_0 \ge \max\left\{1, \frac{m_1 + k_1 + 1}{2\rho}, 12|P|^2\right\}, \tag{23}$$

the unknown growth rate c and the unknown sensor sensitivity $\theta(t)$ is dominated in the sense of the Lyapunov inequality

$$\mathscr{L}V_1 \le -\Big(L(m_1+1) - k_2 - k_3\Big)(|z|^2 + |e|^2), \tag{24}$$

where

$$\rho = 1 - b_n\bar{\theta}|Q|, \tag{25}$$

$$k_1 = 4 + 4|Q|^2 + 2|a||Q| + 2(m_1+1)|P|^2|a|^2, \tag{26}$$

$$k_2 = (2c + c^2)|Q| + (c^2(n+1)n^3 + cn)(m_1+1) \cdot |P| \sum_{i=1}^{n} L_0^{2i-2}, \tag{27}$$

$$k_3 = (c^2(n+1)n^3 + c(n^2 + 2n^{\frac{3}{2}}))(m_1+1)|P|. \tag{28}$$

Remark 3. If the growth rate c of the drift terms and diffusion terms is known, Assumption 1 reduces to the norm assumption frequently used in the output-feedback control of stochastic nonlinear systems, such as that in Deng and Krstic (1999, 2000), Li et al. (2011), Liu and Zhang (2006), Liu et al. (2007), Sun, Shao, Chen, and Meng (2018) and Wu et al. (2009). In this case, from (27)–(28), k_2 and k_3 are completely known. Then, a constant gain rather than a dynamic gain, can be chosen as $L > \frac{k_2+k_3}{m_1+1}$ to make $\mathscr{L}V_1$ in (24) negative definite. From Theorem 1.1 in Deng and Krstic (1999), system (1)–(3) is globally asymptotically stable in probability. However, from Assumption 1, the growth rate c is completely unknown in this paper, which means that k_2 and k_3 in (24) are unknown. In this case, the stability analysis of system (1)–(3) becomes challenging since the negative definiteness of $\mathscr{L}V_1$ in (24) cannot be ensured. To deal with this difficult problem, a dynamic gain $L(t)$ is introduced in (22), and an elaborate stability analysis process is developed in the next section.

4. Stability analysis

In this section, we focus on the stability analysis of the closed-loop system (1)–(3), (11)–(13), (15)–(16) and (22)–(23). Before giving the main results, we first present three fundamental propositions. Specifically, Proposition 1 characterizes the existence and uniqueness of a global solution to the closed-loop system. Then Proposition 2 proves the boundedness of $L(t)$ on $[0, +\infty)$, which is crucial in the whole stability analysis process. After that, Proposition 3 focuses on the property analysis of z, e.

Proposition 1. *If Assumptions 1–2 hold, then the system composed of (17)–(18) and $\dot{L}(t)$ in (22) has an almost surely unique solution $(L(t), z(t), e(t))$ on $[0, +\infty)$.*

Proof. The proof is given in Appendix C.

Proposition 2. *If Assumptions 1–2 hold, then $L(t)$ defined in (22) is bounded on $[0, +\infty)$ a.s.*

Proof. Denoting $\Omega_1 = \{\lim_{t\to+\infty} L(t) = +\infty\}$, we suppose $P\{\Omega_1\} > 0$. By using the nonnegative semimartingale convergence theorem in Lemma A.3, we get a contradiction. Thus, $P\{\Omega_1\} = 0$, which means that $L(t)$ is bounded on $[0, +\infty)$ a.s. The detailed proof process is given in Appendix D.

Proposition 3. *If Assumptions 1–2 hold, then z and e are bounded on $[0, +\infty)$ a.s. Meanwhile, $\int_0^{+\infty} |z|^2 dt < +\infty$ a.s. and $\int_0^{+\infty} |e|^2 dt < +\infty$ a.s.*

Proof. The proof of this proposition includes three steps:

Step 1. We first introduce two new coordinate transformations with L_0 and L^* to rescale the $(x_1, \hat{x}_2, \ldots, \hat{x}_n)^T$-system and $(x_1 - \hat{x}_1, \ldots, x_n - \hat{x}_n)^T$-system as ε-system and ξ-system respectively, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$, $\xi = (\xi_1, \ldots, \xi_n)^T$ and L^* is a design parameter.

Step 2. We then prove

$$\mathscr{L}V_3(\varepsilon, \xi) \le -|\varepsilon|^2 - |\xi|^2 + (2m_2 + m_3)\dot{L}(t), \tag{29}$$

where $V_3(\varepsilon, \xi) = \varepsilon^T P \varepsilon + \xi^T P \xi$, m_2 and m_3 are two positive constants.

Step 3. We finally use the nonnegative semimartingale convergence theorem to prove

$$\int_0^{+\infty} (|e|^2 + |z|^2) ds < +\infty \text{ a.s.}; \tag{30}$$

$$z \text{ and } e \text{ are bounded on } [0, +\infty) \text{ a.s.} \tag{31}$$

The detailed proofs of *Steps 1–3* are given in Appendix E.

Remark 4. From (E.37) and (E.38) in Appendix E, we obtain that the new adaptive law $\dot{L}$ in (22) can dominate the terms arising from unknown state x_1, which is crucial for stability analysis.

Remark 5. From (24), we know that the proof of boundedness of z and e is a difficult problem with the effect of the unknown constants k_2 and k_3. In the proof of Proposition 3 (see Appendix E), a new domination approach is developed to prove boundedness of z and e. Two constants L_0 and L^* are introduced in (E.1) and (E.2) to dominate the unknown terms. Specifically, as proved in Theorem 1, L_0 can dominate the unknown sensor sensitivity $\theta(t)$; L^* is elaborately constructed in (E.3) to dominate $L(t)$, L_0 and unknown terms arising from unknown growth rate c. Combining this domination approach with the nonnegative semimartingale convergence theorem, the boundedness of z and e is proved.

Based on Propositions 1–3, we now give the main results on stability analysis for system (1)–(3) in the following theorem.

Theorem 2. *For system (1)–(3), if the allowable sensitivity error satisfies $0 < \bar{\theta} < \min\{1, \frac{1}{b_n|Q|}\}$ and Assumptions 1–2 hold, using the observer (11)–(13) and controller (16) with the gains $L(t)$ and L_0 defined in (22)–(23), the following conclusions hold:*

(1) The closed-loop system composed of (1)–(3), (11)–(13), (15)–(16) and (22)–(23) has an almost surely unique solution on $[0, +\infty)$ for any initial condition $(x_0, \hat{x}_0) \in R^n \times R^n$;

(2) The closed-loop system is almost surely regulated to the equilibrium at the origin. Specifically, $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \hat{x}(t) = 0$, a.s., where $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^T$.

Proof. From Propositions 1–3, we obtain that the closed-loop system composed of (17)–(18) and (22)–(23) has an almost surely unique solution (L, e, z) on $[0, +\infty)$ and (L, e, z) is bounded on $[0, +\infty)$. By (14)–(15), conclusion (1) holds.

From Proposition 2, we know that $L(t)$ is bounded on $[0, +\infty)$ a.s. By (14)–(15) and Proposition 3 we have

$$\int_0^{+\infty} (|x|^2 + |\hat{x}|^2) ds < +\infty \text{ a.s.}; \tag{32}$$

$$x \text{ and } \hat{x} \text{ are bounded on } [0, +\infty) \text{ a.s.} \tag{33}$$

Noting the drift terms and diffusion terms in the closed-loop system composed of (1)–(3), (11)–(13), (15)–(16) and (22)–(23) are local bounded in x and $\hat{x}$, from (32)–(33) and Lemma A.4 we obtain $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \hat{x}(t) = 0$ a.s.

5. Two simulation examples

In this section, we give two simulation examples to show the effectiveness of the control scheme developed in the last two sections.

Example 1. Consider the following system

$$dx_1 = x_2 dt + c_1 x_1 d\omega, \tag{34}$$

$$dx_2 = u dt + \ln(1 + (x_2^2)^{c_2}) dt + c_3 \sin(x_1) d\omega, \tag{35}$$

$$y = \theta(t) x_1, \tag{36}$$

where $\theta(t) = 1 + 0.25|\sin(10t)|$, c_1, $c_2 \ge 1$ and c_3 are unknown constants.

It is obvious that

$$|c_1 x_1| = c_1 |x_1|, \tag{37}$$

$$|\ln(1 + (x_2^2)^{c_2})| \le (2c_2 - 1)|x_2|, \tag{38}$$

$$|c_3 \sin(x_1)| \le c_3 |x_1|. \tag{39}$$

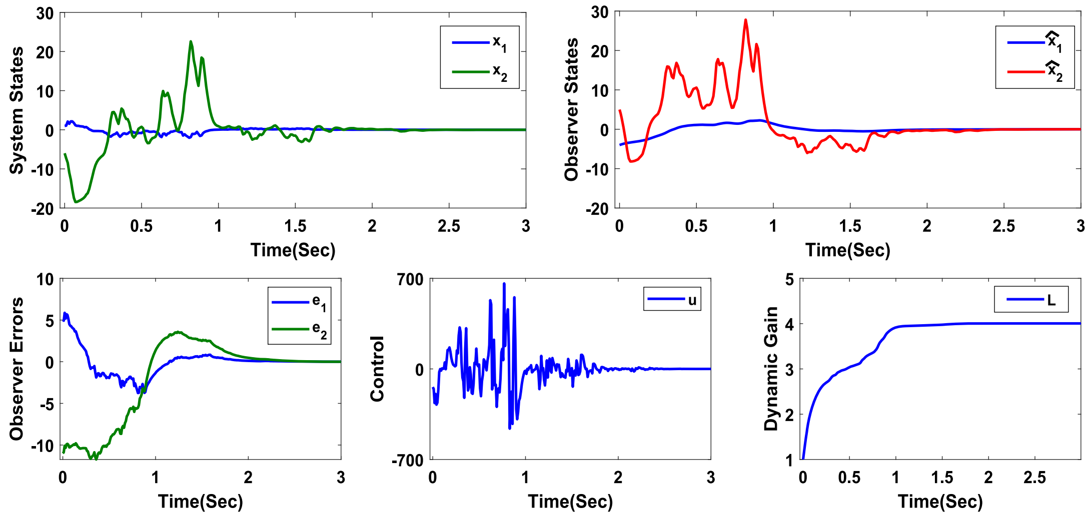

Fig. 1. The response of the closed-loop system (34)–(36) and (40)–(43).

From (37)–(39), Assumption 1 is satisfied with $c = \max\{2c_2 - 1, c_1, c_3\}$.

Choosing the parameters $a_1 = 2, a_2 = 1, b_1 = 1, b_2 = 0.25, \sigma = 0.4$, by (9)–(10) we obtain $\frac{1}{b_2|Q|} = 0.4350$. From the definition of $\bar{\theta}$ in Theorem 1 we have $0 < \bar{\theta} < \min\{1, 0.4350\} = 0.4350$. By Assumption 2 we get $\theta(t) \in (0.5650, 1.4350)$, which means that $\theta(t) = 1 + 0.25|\sin(10t)|$ satisfies Assumption 2.

By following the design procedure developed in Section 3 and choosing $c_1 = 1, c_2 = 2, c_3 = 1.5, L_0 = 15.7$, we can get the observer and controller as

$$\dot{\hat{x}}_1 = \hat{x}_2 - 2L\hat{x}_1, \tag{40}$$

$$\dot{\hat{x}}_2 = u - L^2\hat{x}_1, \tag{41}$$

$$u = -61.6225L^2 y - 15.7L\hat{x}_2, \tag{42}$$

$$\dot{L} = \left(\frac{\hat{x}_1}{L^{0.4}}\right)^2 + \left(\frac{y}{0.75L^{0.4}}\right)^2. \tag{43}$$

In the practical simulation, we randomly set the initial conditions as $x_1(0) = 1$, $x_2(0) = -6$, $\hat{x}_1(0) = -4$, $\hat{x}_2(0) = 5$, $L(0) = 1$. The response of the closed-loop system (34)–(36) and (40)–(43) is given in Fig. 1. From Fig. 1, we observe that $\lim_{t\to+\infty} x_1(t) = \lim_{t\to+\infty} x_2(t) = \lim_{t\to+\infty} \hat{x}_1(t) = \lim_{t\to+\infty} \hat{x}_2(t) = 0$, a.s., from which the effectiveness of the controller is demonstrated.

Example 2. Consider the one-link manipulator which contains motor dynamics and stochastic disturbances; see Fig. 2. The system is described as in Chen, Jiao, Li, and Li (2010) and Xue, Zhang, Zhang, and Xie (2018)

$$D\ddot{q} + B\dot{q} + N\sin(q) = \tau_r + \tau_d, \tag{44}$$

$$M\dot{\tau}_r + H\tau_r = u - K_m\dot{q}, \tag{45}$$

where $q, \dot{q}, \ddot{q}$ denote the link position, velocity and acceleration, respectively. τ_r is the torque produced by the electrical subsystem, and $\tau_d = c_1\sin(q)\dot{\omega}$ is the torque disturbance with unknown constant c_1 and the torque stochastic disturbance ω defined in system (1)–(3). u is the control input used to represent the electromechanical torque. D is the mechanical inertia, B is the coefficient of viscous friction at the joint, N is a positive constant related to the mass of the load and the coefficient of gravity, M is the armature inductance, H is the armature resistance, and K_m is the back electromotive force coefficient.

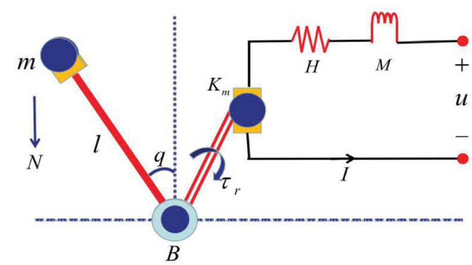

Fig. 2. Model of one-link manipulator.

Introducing $x_1 = MDq, x_2 = MD\dot{q}, x_3 = M\tau_r$, (44)–(45) can be transformed into

$$dx_1 = x_2dt, \tag{46}$$

$$dx_2 = x_3dt - \left(\frac{B}{D}x_2 + MN\sin\left(\frac{x_1}{MD}\right)\right)dt + c_1M\sin\left(\frac{x_1}{MD}\right)d\omega, \tag{47}$$

$$dx_3 = udt - \left(\frac{K_m}{MD}x_2 + \frac{H}{M}x_3\right)dt, \tag{48}$$

$$y = \theta(t)x_1, \tag{49}$$

where $\theta(t) = 1 + 0.09|\sin(t)|$, $c_1 > 0$, $B > 0$ and $H > 0$ are unknown constants. We choose $D = 1$ kg m^2, $N = 4$, $M = 0.5H$, $K_m = 1$ N m/A. It can be verified that Assumption 1 is satisfied with $c = \max\{\frac{B}{D}, \frac{N}{D}, \frac{C_1}{D}, \frac{K_m}{MD}, \frac{H}{M}\}$.

Choosing the parameters $a_1 = 1.5, a_2 = 0.75, a_3 = 0.125, b_1 = 3, b_2 = 3, b_3 = 1, \sigma = 0.3$, by solving the matrix inequalities (9)–(10) we get $\frac{1}{b_3|Q|} = 0.0992$. From the definition of $\bar{\theta}$ in Theorem 1 we have $0 < \bar{\theta} < \min\{1, 0.0992\} = 0.0992$. By Assumption 2 we get $\theta(t) \in (0.9008, 1.0992)$. It is obviously that $\theta(t) = 1 + 0.09|\sin(t)|$ satisfies Assumption 2.

By following the design procedure developed in Section 3 and choosing $c_1 = 2$, $B = 1$ N ms/rad, $H = 0.5\ \Omega$, $L_0 = 17.2$, we can get the observer and controller as

$$\dot{\hat{x}}_1 = \hat{x}_2 - 1.5L\hat{x}_1, \tag{50}$$

$$\dot{\hat{x}}_2 = \hat{x}_3 - 0.75L^2\hat{x}_1, \tag{51}$$

$$\dot{\hat{x}}_3 = u - 0.125L^3\hat{x}_1, \tag{52}$$

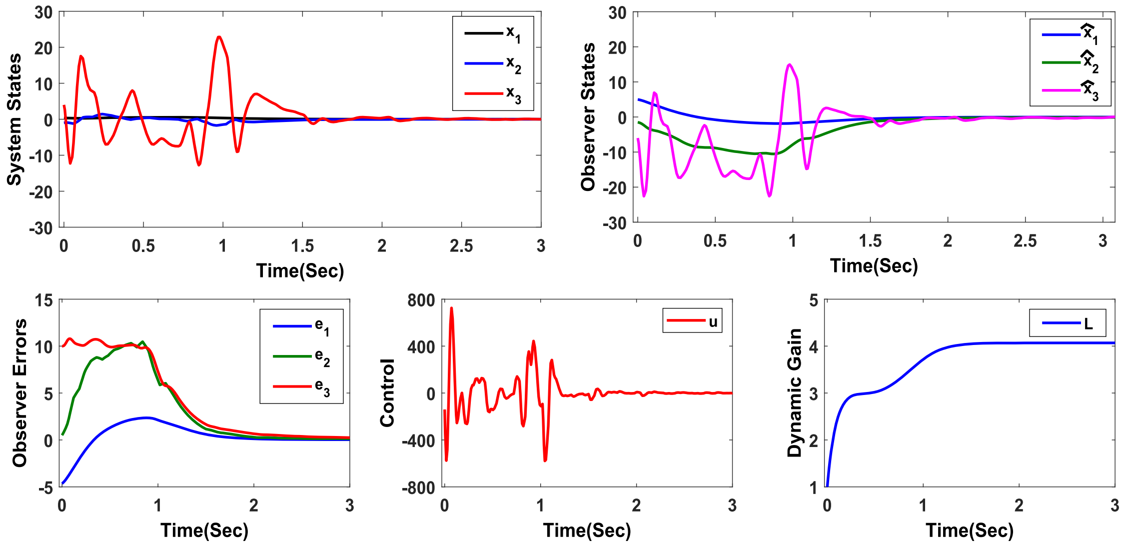

Fig. 3. The response of the closed-loop system (46)–(49) and (50)–(54).

$$u = -5088.4L^3 y - 887.52L^2 \hat{x}_2 - 51.6L\hat{x}_3, \tag{53}$$

$$\dot{L} = \left(\frac{\hat{x}_1}{L^{0.3}}\right)^2 + \left(\frac{y}{0.91L^{0.3}}\right)^2. \tag{54}$$

In the practical simulation, we randomly set the initial conditions as $x_1(0) = 0.35$, $x_2(0) = -1$, $x_3(0) = 4$, $\hat{x}_1(0) = 5$, $\hat{x}_2(0) = -1.5$, $\hat{x}_3(0) = -6$, $L(0) = 1$. The response of the closed-loop system (46)–(49) and (50)–(54) is given in Fig. 3. From Fig. 3, we observe that $\lim_{t\to+\infty} x_i(t) = \lim_{t\to+\infty} \hat{x}_i(t) = 0$, a.s., $i = 1, 2, 3$, from which the effectiveness of the controller is demonstrated.

Remark 6. It can be observed from Examples 1–2 that the system (34)–(36) and (46)–(49) are both with time-varying sensor uncertainty, unknown growth rate and stochastic noise simultaneously. For the output-feedback control of the system (34)–(36) and (46)–(49), the method in Chen et al. (2018) fails since there are unknown growth rate and stochastic noise in the studied system. Although the method in Lei and Lin (2006), Li and Liu (2017) and Yan and Liu (2011) can deal with the unknown growth rate, their adaptive observer designs require $y = x_1$ to construct an estimate error in the stability analysis. In other words, the existence of time-varying sensor uncertainty in system (34)–(36) and (46)–(49) makes the observers in Lei and Lin (2006), Li and Liu (2017) and Yan and Liu (2011) inapplicable since x_1 is unavailable. Our control schemes (40)–(43) and (50)–(54) have the following novel features: (1) To overcome the difficulties caused by the time-varying sensor uncertainty, our adaptive observers are driven only by the input u, not using the information of output y; (2) We design new adaptive laws (43) and (54) for the dynamic gains in the observers and controllers, which dominate the nonlinear terms arising from the controllers and unknown state x_1. By using advanced stochastic analysis techniques developed in Propositions 1–3 and Theorem 2, we prove that the control schemes (40)–(43) and (50)–(54) solve the output-feedback stabilization problems for system (34)–(36) and (46)–(49). It should be emphasized that our control schemes are not routine combinations of the existing methods in Chen et al. (2018), Lei and Lin (2006), Li and Liu (2017) and Yan and Liu (2011). In fact, when the time-varying sensor uncertainty and the unknown growth rate simultaneously appear in stochastic systems, the observer design and the stability analysis are completely different from that in Chen et al. (2018), Lei and Lin (2006), Li and Liu (2017) and Yan and Liu (2011) due to the effect of stochastic noise.

6. Concluding remarks

In this paper we have addressed the adaptive output-feedback stabilization problem of stochastic strict-feedback systems with sensor uncertainty. In contrast to the previous work, the system model studied in this paper is more general since it considers sensor uncertainty, unknown growth rate and stochastic noise simultaneously. In fact, even when there is no noise in the system, the results in this paper are new and more general than the existing results since this paper considers an unknown growth rate. To solve the output-feedback stabilization problem, we first design a new observer, which is driven only by the input u, not using the information of output y. Besides, we construct a new adaptive law for the dynamic gain in the observer, which dominates the nonlinear terms arising from the controller and unknown state x_1. We then develop a stochastic adaptive dual-domination approach to design an adaptive output-feedback controller. For the stability analysis, we propose a new domination approach in Proposition 3 to prove the boundedness of the closed-loop system, and use advanced stochastic analysis techniques to prove that the closed-loop system has an almost surely unique solution on $[0, +\infty)$ and that the closed-loop system is almost surely regulated to the equilibrium at the origin. It is worth emphasizing that the existence of stochastic noise and unknown growth rate makes the stability analysis in this paper much more involved and difficult than that of Chen et al. (2018), Jiang, Xie, and Zhang (2019), Sun et al. (2018) and Xie and Jiang (2019).

For the adaptive output-feedback control of stochastic nonlinear systems, many important issues are still open and worth investigating, such as the adaptive output-feedback controls in the case when there are uncertain parameters in the studied systems, using the control scheme developed in this paper to solve the adaptive output-feedback control problems of stochastic underactuated mechanical system (Li, Liu, & Feng, 2017) and random benchmark systems (Li, Liu, & Feng, 2020), etc.

Acknowledgments

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Appendix A. Useful tools

In this part, four lemmas are collected, which are frequently used in the controller design and stability analysis.

Lemma A.1 (Krstic & Deng, 1998). *For $(x, y) \in R^2$, the following Young's inequality holds:*

$$xy \le \frac{\nu^p}{p}|x|^p + \frac{1}{q\nu^q}|y|^q, \tag{A.1}$$

where $\nu > 0$, the constants $p > 1$ and $q > 1$ satisfy $(p-1)(q-1) = 1$.

Lemma A.2 (Yang & Lin, 2004). *For $p \in [1, +\infty)$ and any $x_i \in R, i = 1, \ldots, n$, the following inequality holds:*

$$(|x_1| + \cdots + |x_n|)^p \le n^{p-1}(|x_1|^p + \cdots + |x_n|^p). \tag{A.2}$$

Lemma A.3 (Mao, 2008). *Let $\mathscr{A}(t), t \ge t_0$ and $\mathscr{U}(t), t \ge t_0$ be two 1-dimensional continuous adapted increasing processes with $\mathscr{A}(t_0) = \mathscr{U}(t_0) = 0$ a.s., $\mathscr{W}(t), t \ge t_0$ be a 1-dimensional continuous local martingale with $\mathscr{W}(t_0) = 0$ a.s., and ξ be a nonnegative $\mathscr{F}_{t_0}$-measurable stochastic variable. If $\mathscr{Y}(t) \triangleq \xi + \mathscr{A}(t) - \mathscr{U}(t) + \mathscr{W}(t), t \ge t_0$ is nonnegative, then*

$$\left\{\lim_{t\to+\infty}\mathscr{A}(t) < +\infty\right\} \subseteq \left\{\lim_{t\to+\infty}\mathscr{Y}(t) \text{ exists and is finite}\right\} \cap \left\{\lim_{t\to+\infty}\mathscr{U}(t) < +\infty\right\} \quad a.s. \tag{A.3}$$

Lemma A.4 (Li & Liu, 2017). *Suppose that system $dx = f(t, x)dt + g(t, x)d\omega$ has a unique solution on $[t_0, +\infty)$, and that $f(t, x)$ and $g(t, x)$ are locally bounded in $x \in R^n$ uniformly in $[t_0, +\infty)$. If the solution $x(t)$ of the above system satisfies*

(i) $x(t)$ is bounded on $[t_0, +\infty)$ a.s.; and

(ii) $\lim_{t\to+\infty}\int_{t_0}^{t}\alpha(x(s))ds$ exists and is finite a.s. with $\alpha \in C(R^n, R)$, then $\lim_{t\to+\infty}\alpha(x(t)) = 0$ a.s.

Appendix B. Proof of Theorem 1

Step 1. *We first prove that V_1 satisfies (24).*

Denoting $V_{11}(e) = e^TPe$, from (7)–(8), (17) and (22) we obtain

$$\begin{aligned}\mathscr{L}V_{11}(e) \le& \Big[\Big(LAe + L^{1-\sigma}ax_1 - \frac{\dot L}{L}De\Big)^TPe + e^TP\Big(LAe + L^{1-\sigma}ax_1 - \frac{\dot L}{L}De\Big)\Big] + 2e^TPF_e + |P||G_e|^2\\ \le& -2L|e|^2 + 2L^{1-\sigma}e^TPax_1 - \frac{\dot L}{L}e^T(DP+PD)e + 2e^TPF_e + |P||G_e|^2\\ \le& -2L|e|^2 + 2L^{1-\sigma}e^TPax_1 + 2e^TPF_e + |P||G_e|^2.\end{aligned} \tag{B.1}$$

Similarly, let $V_{12}(z) = z^TQz$, from (9)–(10), (18) and (22) we get

$$\begin{aligned}\mathscr{L}V_{12}(z) \le& \Big(LL_0Bz + LL_0B_zb_n(1-\theta(t))z_1 + LD_2e_2 + \frac{L}{L_0}D_1(e_1 - z_1) - \frac{\dot L}{L}Dz\Big)^TQz + z^TQ\Big(LL_0Bz\\ &+ LL_0B_zb_n(1-\theta(t))z_1 + LD_2e_2 - \frac{\dot L}{L}Dz + \frac{L}{L_0}D_1(e_1 - z_1)\Big) + 2z^TQF_z + |Q||G_z|^2\\ \le& -2LL_0|z|^2 + 2LL_0z^TQB_zb_n(1-\theta(t))z_1 + 2z^TQ\Big(\frac{L}{L_0}D_1e_1 + LD_2e_2 - \frac{L}{L_0}D_1z_1\Big)\\ &+ 2z^TQF_z + |Q||G_z|^2.\end{aligned} \tag{B.2}$$

By (21) and (B.1)–(B.2) we have

$$\begin{aligned}\mathscr{L}V_1 \le& -2L(m_1+1)|e|^2 + 2L^{1-\sigma}(m_1+1)e^TPax_1 + 2(m_1+1)e^TPF_e + (m_1+1)|P||G_e|^2\\ &- 2LL_0|z|^2 + 2LL_0z^TQB_zb_n(1-\theta(t))z_1 + 2z^TQ\Big(\frac{L}{L_0}D_1e_1 + LD_2e_2 - \frac{L}{L_0}D_1z_1\Big)\\ &+ 2z^TQF_z + |Q||G_z|^2.\end{aligned} \tag{B.3}$$

Noting $x_1 = L^\sigma z_1$, using Lemma A.1 with $\nu = \frac{1}{2}$, $p = q = 2$, $x = 2|e|$ and $y = |P||a||z|$, we have

$$\begin{aligned}&2L^{1-\sigma}(m_1+1)e^TPax_1\\ &\le 2L(m_1+1)|P||a||e||z_1|\\ &\le \frac{1}{2}L(m_1+1)|e|^2 + 2L(m_1+1)|P|^2|a|^2|z|^2.\end{aligned} \tag{B.4}$$

By (4), (14) and (19) we get

$$\begin{aligned}|F_e| \le& c\left(\frac{|x_1|}{L^\sigma} + \frac{|x_1| + |x_2|}{L^{1+\sigma}} + \cdots + \frac{|x_1| + \cdots + |x_n|}{L^{n-1+\sigma}}\right)\\ \le& c\left(\frac{n|x_1|}{L^\sigma} + \frac{(n-1)|x_2|}{L^{1+\sigma}} + \cdots + \frac{|x_n|}{L^{n-1+\sigma}}\right)\\ \le& c\Big(\frac{n|x_1|}{L^\sigma} + \frac{(n-1)(|\hat x_2| + L^{1+\sigma}|e_2|)}{L^{1+\sigma}} + \cdots + \frac{(|\hat x_n| + L^{n-1+\sigma}|e_n|)}{L^{n-1+\sigma}}\Big).\end{aligned} \tag{B.5}$$

From Lemma A.2 we have

$$|e_2| + \cdots + |e_n| \le |e_1| + \cdots + |e_n| \le \sqrt{n}|e|. \tag{B.6}$$

It follows from (15) and (B.5)–(B.6) that

$$\begin{aligned}|F_e| \le& c\Big(n|z_1| + (n-1)L_0|z_2| + \cdots + L_0^{n-1}|z_n| + (n-1)\sqrt{n}|e|\Big)\\ \le& c\left(n\sum_{i=1}^{n}L_0^{i-1}|z_i| + (n-1)\sqrt{n}|e|\right).\end{aligned} \tag{B.7}$$

From (B.7) we obtain

$$\begin{aligned}2(m_1+1)e^TPF_e \le& 2c(m_1+1)|e||P|\Big(n\sum_{i=1}^{n}L_0^{i-1}|z_i| + (n-1)\sqrt{n}|e|\Big)\\ \le& 2cn(m_1+1)|P|\Big(\frac{1}{2}\sum_{i=1}^{n}L_0^{2i-2}|z|^2 + \frac{n+2\sqrt{n}}{2}|e|^2\Big)\\ \le& c(n^2 + 2n^{\frac{3}{2}})(m_1+1)|P||e|^2 + cn(m_1+1)|P|\sum_{i=1}^{n}L_0^{2i-2}|z|^2.\end{aligned} \tag{B.8}$$

From (5), (14)–(15) and (20), taking a similar process as in (B.5)–(B.7) we get

$$|G_e| \le c\left(n\sum_{i=1}^{n}L_0^{i-1}|z_i| + (n-1)\sqrt{n}|e|\right), \tag{B.9}$$

which and Lemma A.2 implies

$$(m_1+1)|P||G_e|^2 \le c^2(n+1)n^3(m_1+1)|P|\left(\sum_{i=1}^{n} L_0^{2i-2}|z|^2+|e|^2\right). \tag{B.10}$$

By (19) and Lemma A.1 we have

$$\begin{aligned}
&2z^TQ\frac{L}{L_0}D_1e_1+2z^TQLD_2e_2\\
&\le 2\frac{L}{L_0}|Q||D_1||e_1||z|+2L|Q||z||e_2|\\
&\le \frac{L}{4}|Q|^2|D_1|^2|e_1|^2+\frac{4L}{L_0^2}|z|^2+\frac{L}{4}|e_2|^2+4L|Q|^2|z|^2\\
&\le \frac{Lm_1}{4}|e|^2+\frac{4L}{L_0^2}|z|^2+\frac{L}{4}|e|^2+4L|Q|^2|z|^2\\
&\le \frac{L}{4}(m_1+1)|e|^2+\frac{4L}{L_0^2}|z|^2+4L|Q|^2|z|^2
\end{aligned} \tag{B.11}$$

and

$$-2z^TQ\frac{L}{L_0}D_1z_1\le 2\frac{L}{L_0}|Q||D_1||z|^2\le\frac{2|a|L}{L_0}|Q||z|^2, \tag{B.12}$$

where $|D_1|\le(\sum_{i=1}^{n}a_i^2)^{\frac{1}{2}}=|a|$, $|D_2|=1$.

From (4)–(5), (15) and (20), using Lemma A.1 we have

$$\begin{aligned}
2z^TQF_z+|Q||G_z|^2&\le 2c|Q||z|^2+c^2|Q||z|^2\\
&=(2c+c^2)|Q||z|^2.
\end{aligned} \tag{B.13}$$

Substituting (B.4), (B.8) and (B.10)–(B.13) into (B.3) we get

$$\begin{aligned}
\mathscr{L}V_1\le&-\frac{5L}{4}(m_1+1)|e|^2+(c^2(n+1)n^3+c(n^2+2n^{\frac{3}{2}}))\\
&\cdot(m_1+1)|P||e|^2+2L(m_1+1)|P|^2|a|^2|z|^2\\
&+(c^2(n+1)n^3+cn)(m_1+1)|P|\sum_{i=1}^{n}L_0^{2i-2}|z|^2\\
&-2LL_0(1-b_n|1-\theta(t)||Q|)|z|^2+\left(\frac{4L}{L_0^2}+4L|Q|^2\right.\\
&\left.+\frac{2|a|L}{L_0}|Q|+(2c+c^2)|Q|\right)|z|^2.
\end{aligned} \tag{B.14}$$

By Assumption 2 and the definition of $\bar\theta$ we have

$$1>1-b_n|1-\theta(t)|\cdot|Q|\ge 1-b_n\bar\theta|Q|=\rho>0. \tag{B.15}$$

By (B.14)–(B.15) we get

$$\begin{aligned}
\mathscr{L}V_1\le&-LL_0\left(2\rho-\frac{4+4|Q|^2+2(m_1+1)|P|^2|a|^2}{L_0}\right.\\
&-\frac{(c^2(n+1)n^3+cn)(m_1+1)|P|\sum_{i=1}^{n}L_0^{2i-2}}{LL_0}\\
&\left.-\frac{2|a||Q|}{L_0}-\frac{(2c+c^2)|Q|}{LL_0}\right)|z|^2-\left(\frac{5L}{4}(m_1+1)\right.\\
&\left.-(c^2(n+1)n^3+c(n^2+2n^{\frac{3}{2}}))(m_1+1)|P|\right)|e|^2\\
\le&-LL_0\left(2\rho-\frac{k_1}{L_0}-\frac{k_2}{LL_0}\right)|z|^2\\
&-\Big(L(m_1+1)-k_3\Big)|e|^2,
\end{aligned} \tag{B.16}$$

where k_1, k_2 and k_3 are defined in (26)–(28). By (23) and (B.16) we get (24).

Step 2. *We then prove that the unknown growth rate c and the unknown sensor sensitivity $\theta(t)$ can be dominated by (22) and (23).*

From (22), (24) and (27)–(28) we know that the dynamic gain $L(t)$ can dominate the unknown k_2 and k_3 arising from the unknown growth rate c. From (23) and (25), we can observe that larger allowable sensitive error $\bar\theta$ yields larger L_0, which means that the constant gain L_0 can dominate the unknown sensor sensitivity $\theta(t)$.

Appendix C. Proof of Proposition 1

For the system composed of (17)–(18) and $\dot L(t)$ in (22), it is obviously that the drift terms and diffusion terms satisfy the local Lipschitz condition. Therefore, by Theorem 3.15 in Mao and Yuan (2006), the considered system has an almost surely unique solution $(L(t),z(t),e(t))$ on $[0,t_f)$, where $t_f=\lim_{\tau\to+\infty}\inf\{t\ge 0:|z(t)|+|e(t)|+|L(t)|\ge\tau\}$. In the following, we prove $t_f=+\infty$.

Substituting (3) and (14)–(15) into (22) yields

$$\begin{aligned}
\dot L&=\left(\frac{\hat x_1}{L^\sigma}\right)^2+\left(\frac{y}{(1-\bar\theta)L^\sigma}\right)^2\\
&=(z_1-e_1)^2+\left(\frac{\theta(t)}{1-\bar\theta}\right)^2z_1^2.
\end{aligned} \tag{C.1}$$

By Assumption 2, Lemma A.2 and (C.1) we have

$$\begin{aligned}
\dot L&\le 2z_1^2+2e_1^2+\left(\frac{1+\bar\theta}{1-\bar\theta}\right)^2z_1^2\\
&\le M_0(|z|^2+|e|^2),
\end{aligned} \tag{C.2}$$

where $M_0=2+(\frac{1+\bar\theta}{1-\bar\theta})^2$.

Choosing $V_2=V_1+\frac{1}{2\lambda}L^2$ with $\lambda=\frac{M_0}{m_1+1}$, from (24) and (C.2) we have

$$\begin{aligned}
\mathscr{L}V_2\le&-(L(m_1+1)-k_2-k_3)(|z|^2+|e|^2)\\
&+\frac{1}{\lambda}M_0L(|z|^2+|e|^2)\\
=&-\left(L\left(m_1+1-\frac{M_0}{\lambda}\right)-k_2-k_3\right)(|z|^2+|e|^2)\\
=&(k_2+k_3)(|z|^2+|e|^2).
\end{aligned} \tag{C.3}$$

From (C.3) and the definitions of V_1 and V_2 we get

$$\mathscr{L}V_2\le\frac{k_2+k_3}{\lambda_{\min}\{Q\}\wedge(m_1+1)\lambda_{\min}\{P\}}V_2. \tag{C.4}$$

By (C.4) and Theorem 3.19 in Mao and Yuan (2006), we obtain $t_f=+\infty$.

Appendix D. Proof of Proposition 2

Denoting $\Omega_1=\{\lim_{t\to+\infty}L(t)=+\infty\}$, we suppose $P\{\Omega_1\}>0$. Defining the stopping time

$$\sigma=\inf\left\{t\ge 0:L(t)\ge\frac{k_2+k_3}{m_1+1}\right\}. \tag{D.1}$$

It is obviously that $\sigma<+\infty$ a.s. on Ω_1. From (17), (18), (21) and (24) we have

$$\begin{aligned}
V_1(t)&=V_1(0)+\int_0^t\mathscr{L}V_1(s)ds+\int_0^t\left(\frac{\partial V_1}{\partial e}G_e+\frac{\partial V_1}{\partial z}G_z\right)d\omega\\
&=V_1(0)+\int_0^{t\wedge\sigma}\mathscr{L}V_1(s)ds+\int_{t\wedge\sigma}^t\mathscr{L}V_1(s)ds\\
&\quad+\int_0^t\left(\frac{\partial V_1}{\partial e}G_e+\frac{\partial V_1}{\partial z}G_z\right)d\omega
\end{aligned}$$

$$
\begin{aligned}
&\le V_1(0)+\int_0^{t\wedge\sigma}(k_2+k_3-L(m_1+1))(|z|^2+|e|^2)ds\\
&\quad-\int_{t\wedge\sigma}^{t}(L(m_1+1)-k_2-k_3)(|z|^2+|e|^2)ds\\
&\quad+\int_0^t\left(\frac{\partial V_1}{\partial e}G_e+\frac{\partial V_1}{\partial z}G_z\right)d\omega\\
&\triangleq V_1(0)+\mathscr{A}_1(t)-\mathscr{U}_1(t)+\mathscr{W}_1(t).
\end{aligned}\tag{D.2}
$$

It can be easily shown that $\mathscr{A}_1(t)$ and $\mathscr{U}_1(t)$ are two continuous adapted increasing processes with $\mathscr{A}_1(0)=\mathscr{U}_1(0)=0$.

For $k\in Z^+$, defining the stopping time

$$
\sigma_k=k\wedge\inf\{t\ge 0:|e|+|z|\ge k\},\tag{D.3}
$$

then from (17)–(18), (21) and (D.2) we have

$$
\begin{aligned}
&\mathscr{W}_1(t\wedge\sigma_k)\\
&=\int_0^t\chi_{\{0\le s\le\sigma_k\}}\left(\frac{\partial V_1}{\partial e}G_e+\frac{\partial V_1}{\partial z}G_z\right)d\omega\\
&=\int_0^t\chi_{\{0\le s\le\sigma_k\}}(2(m_1+1)e^TPG_e+2z^TQG_z)d\omega.
\end{aligned}\tag{D.4}
$$

On the other hand, it follows from the definitions of G_e and G_z in (20) that

$$
E\left\{\int_0^t\chi^2_{\{0\le s\le\sigma_k\}}(2(m_1+1)e^TPG_e+2z^TQG_z)^2dt\right\}<+\infty,\tag{D.5}
$$

by which and Theorem 5.14 in Mao (2008) (P.25), we obtain that $\mathscr{W}_1(t\wedge\sigma_k)$ in (D.4) is a martingale.

Noting that $\{\sigma_k\}$ is nondecreasing and $\lim_{k\to+\infty}\sigma_k=+\infty$ a.s., we know that the continuous adapted process $\mathscr{W}_1(t)$ is a local martingale.

From (D.2) we have

$$
\begin{aligned}
\lim_{t\to+\infty}\mathscr{A}_1(t)&=\int_0^{\sigma}(k_2+k_3-L(m_1+1))(|z|^2+|e|^2)ds\\
&<+\infty\ \text{a.s. on}\ \Omega_1.
\end{aligned}\tag{D.6}
$$

By (D.2) and (D.6), using Lemma A.3 we get

$$
\begin{aligned}
\lim_{t\to+\infty}\mathscr{U}_1(t)&=\int_{\sigma}^{+\infty}(L(m_1+1)-k_2-k_3)(|z|^2+|e|^2)ds\\
&<+\infty\ \text{a.s. on}\ \Omega_1.
\end{aligned}\tag{D.7}
$$

Defining the stopping time

$$
\tilde\sigma=\inf\left\{t\ge 0:L(t)\ge\frac{k_2+k_3+1}{m_1+1}\right\}.\tag{D.8}
$$

We can obtain that $\sigma<\tilde\sigma<+\infty$ a.s. on Ω_1. Then from (D.7) we have

$$
\begin{aligned}
&\int_{\tilde\sigma}^{+\infty}(|z|^2+|e|^2)ds\\
&\le\int_{\tilde\sigma}^{+\infty}(L(m_1+1)-k_2-k_3)(|z|^2+|e|^2)ds\\
&\le\int_{\sigma}^{+\infty}(L(m_1+1)-k_2-k_3)(|z|^2+|e|^2)ds\\
&<+\infty\ \text{a.s. on}\ \Omega_1.
\end{aligned}\tag{D.9}
$$

It follows from (C.2) and (D.9) that

$$
\begin{aligned}
L(+\infty)-L(\tilde\sigma)&=\int_{\tilde\sigma}^{+\infty}\dot Lds\\
&\le M_0\int_{\tilde\sigma}^{+\infty}(|z|^2+|e|^2)\\
&<+\infty\ \text{a.s. on}\ \Omega_1,
\end{aligned}\tag{D.10}
$$

which is a contradiction. Thus, $P\{\Omega_1\}=0$, which means that $L(t)$ is bounded on $[0,+\infty)$ a.s.

Appendix E. Proof of Proposition 3

Step 1. *We first rescale the $(x_1,\hat x_2,\ldots,\hat x_n)^T$-system and $(x_1-\hat x_1,\ldots,x_n-\hat x_n)^T$-system as ε-system and ξ-system respectively, where $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)^T$ and $\xi=(\xi_1,\ldots,\xi_n)^T$.*

Introducing the change of coordinates

$$
\varepsilon_1=\frac{x_1}{L^{*\sigma}},\ \varepsilon_i=\frac{\hat x_i}{L^{*i-1+\sigma}L_0^{i-1}},\quad i=2,\ldots,n\tag{E.1}
$$

and

$$
\xi_i=\frac{x_i-\hat x_i}{(L^*)^{i-1+\sigma}},\quad i=1,\ldots,n,\tag{E.2}
$$

where L^* is a constant satisfying

$$
\begin{aligned}
L^*>\max\Bigg\{&L(+\infty),\ \frac{12L(+\infty)\sqrt{n-1}|P||a|}{L_0},\\
&\frac{1}{L_0}\Big(1+cn|P|\sum_{i=1}^{n}L_0^{2i-2}+c^2(n+1)n^3|P|\sum_{i=1}^{n}L_0^{2i-2}\Big),\\
&3cn^2|P|+c^2(n+1)n^3|P|+1,\ \frac{12c(c+1)|P|}{L_0},\\
&12\sqrt{n}\Big(b_n(1+\bar\theta)+\sum_{i=1}^{n-1}b_i\Big)|P|L(+\infty)\Bigg\}.
\end{aligned}\tag{E.3}
$$

It can be deduced from (1)–(3), (11)–(13) and (E.1)–(E.2) that

$$
\begin{aligned}
d\varepsilon_1&=(L^*L_0\varepsilon_2-L^*L_0a_1\varepsilon_1+L^*\xi_2+L^*L_0a_1\varepsilon_1)dt\\
&\quad+\frac{1}{L^{*\sigma}}f_1dt+\frac{1}{L^{*\sigma}}g_1d\omega,
\end{aligned}\tag{E.4}
$$

$$
\begin{aligned}
d\varepsilon_2&=\Big(L^*L_0\varepsilon_3-L^*L_0a_2\varepsilon_1+\frac{L^2}{L^*L_0}a_2\xi_1+L^*L_0a_2\varepsilon_1\\
&\quad-\frac{L^2}{L^{*1+\sigma}L_0}a_2x_1\Big)dt,
\end{aligned}\tag{E.5}
$$

$$\vdots$$

$$
\begin{aligned}
d\varepsilon_n&=\Big(\frac{u}{L^{*n-1+\sigma}L_0^{n-1}}-L^*L_0a_n\varepsilon_1+\frac{L^n}{(L^*L_0)^{n-1}}a_n\xi_1\\
&\quad+L^*L_0a_n\varepsilon_1-\frac{L^n}{L^{*n-1+\sigma}L_0^{n-1}}a_nx_1\Big)dt
\end{aligned}\tag{E.6}
$$

and that

$$
\begin{aligned}
d\xi_1&=\Big(L^*\xi_2-L^*a_1\xi_1+L^*a_1\xi_1-La_1\xi_1+\frac{La_1}{L^{*\sigma}}x_1\Big)dt\\
&\quad+\frac{f_1}{L^{*\sigma}}dt+\frac{g_1}{L^{*\sigma}}d\omega,
\end{aligned}\tag{E.7}
$$

$$
\begin{aligned}
d\xi_2&=\Big(L^*\xi_3-L^*a_2\xi_1+L^*a_2\xi_1-\frac{L^2}{L^*}a_2\xi_1+\frac{L^2a_2}{L^{*1+\sigma}}x_1\Big)dt\\
&\quad+\frac{f_2}{L^{*1+\sigma}}dt+\frac{g_2}{L^{*1+\sigma}}d\omega,
\end{aligned}\tag{E.8}
$$

$$\vdots$$

$$
\begin{aligned}
d\xi_n&=\Big(-L^*a_n\xi_1-\frac{L^n}{L^{*n-1}}a_n\xi_1+\frac{L^na_n}{L^{*n-1+\sigma}}x_1+L^*a_n\xi_1\Big)dt\\
&\quad+\frac{f_n}{L^{*n-1+\sigma}}dt+\frac{g_n}{L^{*n-1+\sigma}}d\omega.
\end{aligned}\tag{E.9}
$$

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$ and $\xi = (\xi_1, \ldots, \xi_n)^T$, from (E.4)–(E.9) we have

$$d\varepsilon = \Big(L^*L_0A\varepsilon + L\Gamma a\xi_1 + L^*L_0a\varepsilon_1 + L^*D_2\xi_2 - L\Gamma a\varepsilon_1 + B_z\frac{u}{L^{*n-1+\sigma}L_0^{n-1}}\Big)dt + F_\varepsilon^*dt + G_\varepsilon^*d\omega \tag{E.10}$$

and

$$d\xi = (L^*A\xi + L^*a\xi_1 - LMa\xi_1 + LMa\varepsilon_1)dt + F^*dt + G^*d\omega, \tag{E.11}$$

where $\Gamma = \mathrm{diag}[0, \frac{L}{L^*L_0}, \ldots, (\frac{L}{L^*L_0})^{n-1}]$, $M = \mathrm{diag}[1, \frac{L}{L^*}, \cdots, (\frac{L}{L^*})^{n-1}]$ and

$$F_\varepsilon^* = \begin{bmatrix} \frac{1}{L^{*\sigma}}f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad G_\varepsilon^* = \begin{bmatrix} \frac{1}{L^{*\sigma}}g_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{E.12}$$

$$F^* = \begin{bmatrix} \frac{f_1}{L^{*\sigma}} \\ \vdots \\ \frac{f_n}{L^{*n-1+\sigma}} \end{bmatrix}, \quad G^* = \begin{bmatrix} \frac{g_1}{L^{*\sigma}} \\ \vdots \\ \frac{g_n}{L^{*n-1+\sigma}} \end{bmatrix}. \tag{E.13}$$

Step 2. *We then prove*

$$\mathscr{L}V_3(\varepsilon, \xi) \le -|\varepsilon|^2 - |\xi|^2 + (2m_2 + m_3)\dot{L}(t), \tag{E.14}$$

where $V_3(\varepsilon, \xi) = \varepsilon^TP\varepsilon + \xi^TP\xi$, m_2 and m_3 are two positive constants.

Consider the Lyapunov functions $V_{31}(\varepsilon) = \varepsilon^TP\varepsilon$ for ε-system (E.10), using (7) we obtain

$$\mathscr{L}V_{31}(\varepsilon) \le -2L^*L_0|\varepsilon|^2 + 2L\varepsilon^TP\Gamma a\xi_1 + 2L^*L_0\varepsilon^TPa\varepsilon_1 - 2L\varepsilon^TP\Gamma a\varepsilon_1 + 2L^*\varepsilon^TPD_2\xi_2 + 2\varepsilon^TPF_\varepsilon^* + |P||G_\varepsilon^*|^2 + 2\varepsilon^TPB_z\frac{u}{L^{*n-1+\sigma}L_0^{n-1}}. \tag{E.15}$$

Noting $|D_2| = 1$ and $|\Gamma| \le \sqrt{n-1}$, by (E.3) and Lemma A.1 we have

$$2L\varepsilon^TP\Gamma a\xi_1 \le |2L\varepsilon^TP\Gamma a\xi_1| \le 6(n-1)\frac{L}{L_0}|a|^2|P|^2\xi_1^2 + \frac{1}{6}L^*L_0|\varepsilon|^2, \tag{E.16}$$

$$2L^*L_0\varepsilon^TPa\varepsilon_1 \le |2L^*L_0\varepsilon^TPa\varepsilon_1| \le 6L^*L_0|a|^2|P|^2\varepsilon_1^2 + \frac{1}{6}L^*L_0|\varepsilon|^2, \tag{E.17}$$

$$2L^*\varepsilon^TPD_2\xi_2 \le |2L^*\varepsilon^TPD_2\xi_2| \le 6\frac{L^*}{L_0}|P|^2\xi_2^2 + \frac{1}{6}L^*L_0|\varepsilon|^2, \tag{E.18}$$

$$-2L\varepsilon^TP\Gamma a\varepsilon_1 \le 2\sqrt{n-1}L|P||a||\varepsilon|^2 \le \frac{1}{6}L^*L_0|\varepsilon|^2. \tag{E.19}$$

It follows from (4)–(5), (E.1), (E.3) and (E.12) that

$$2\varepsilon^TPF_\varepsilon^* + |P||G_\varepsilon^*|^2 \le 2c(c+1)|P||\varepsilon|^2 \le \frac{1}{6}L^*L_0|\varepsilon|^2. \tag{E.20}$$

By (15), (16), (E.1), $|B_z| = 1$ and Lemma A.2 we get

$$\frac{u}{L^{*n-1+\sigma}L_0^{n-1}} = \frac{L^nL_0}{L^{*n-1}}\Big(-b_n\theta(t)\varepsilon_1 - \sum_{i=1}^{n-1}b_i(\frac{L^*}{L})^{n-i}\varepsilon_{n+1-i}\Big). \tag{E.21}$$

From (E.3), $|B_z| = 1$, Assumption 2 and Lemma A.2 we have

$$\begin{aligned} 2\varepsilon^TPB_z\frac{u}{L^{*n-1+\sigma}L_0^{n-1}} &\le 2LL_0|P||\varepsilon|\Big(b_n(1+\bar\theta)|\varepsilon_1| + \sum_{i=1}^{n-1}b_i|\varepsilon_{n+1-i}|\Big) \\ &\le 2LL_0\Big(b_n(1+\bar\theta) + \sum_{i=1}^{n-1}b_i\Big)|P||\varepsilon|(|\varepsilon_1| + \cdots + |\varepsilon_n|) \\ &\le 2LL_0\sqrt{n}\Big(b_n(1+\bar\theta) + \sum_{i=1}^{n-1}b_i\Big)|P||\varepsilon|^2 \\ &\le \frac{1}{6}L^*L_0|\varepsilon|^2. \end{aligned} \tag{E.22}$$

Substituting (E.16)–(E.20) and (E.22) into (E.15) yields that

$$\mathscr{L}V_{31} \le -L^*L_0|\varepsilon|^2 + 6(n-1)\frac{L}{L_0}|a|^2|P|^2\xi_1^2 + 6L^*L_0|a|^2|P|^2\varepsilon_1^2 + 6\frac{L^*}{L_0}|P|^2\xi_2^2. \tag{E.23}$$

For ξ-system (E.11), by choosing $V_{32}(\xi) = \xi^TP\xi$ and using (7) we get

$$\mathscr{L}V_{32}(\xi) \le -2L^*|\xi|^2 + 2L^*\xi^TPa\xi_1 - 2L\xi^TPMa\xi_1 + 2L\xi^TPMa\varepsilon_1 + 2\xi^TPF^* + |P||G^*|^2. \tag{E.24}$$

Noting $|M| \le \sqrt{n}$, by Lemma A.1 we have

$$2L^*\xi^TPa\xi_1 \le |2L^*\xi^TPa\xi_1| \le 6L^*|P|^2|a|^2\xi_1^2 + \frac{1}{6}L^*|\xi|^2, \tag{E.25}$$

$$-2L\xi^TPMa\xi_1 \le |2L\xi^TPMa\xi_1| \le 6L|PMa|^2\xi_1^2 + \frac{1}{6}L|\xi|^2 \le 6nL|P|^2|a|^2\xi_1^2 + \frac{1}{6}L^*|\xi|^2 \tag{E.26}$$

$$2L\xi^TPMa\varepsilon_1 \le |2L\xi^TPMa\varepsilon_1| \le 6nL|P|^2|a|^2\varepsilon_1^2 + \frac{1}{6}L^*|\xi|^2. \tag{E.27}$$

On the other hand, from (4)–(5), (E.1)–(E.2), (E.13) and Lemma A.1 we get

$$|F^*| \le c\Big(n\sum_{i=1}^{n}L_0^{i-1}|\varepsilon_i| + (n-1)\sqrt{n}|\xi|\Big), \tag{E.28}$$

$$|G^*| \le c\Big(n\sum_{i=1}^{n}L_0^{i-1}|\varepsilon_i| + (n-1)\sqrt{n}|\xi|\Big). \tag{E.29}$$

Hence,

$$2\xi^TPF^* \le 3cn^2|P||\xi|^2 + cn|P|\sum_{i=1}^{n}L_0^{2i-2}|\varepsilon|^2, \tag{E.30}$$

$$|P||G^*|^2 \le c^2(n+1)n^3|P|\Big(\sum_{i=1}^{n}L_0^{2i-2}|\varepsilon|^2 + |\xi|^2\Big). \tag{E.31}$$

Substituting (E.25)–(E.27) and (E.30)–(E.31) into (E.24) we obtain

$$\mathscr{L}V_{32} \le -\Big(\frac{3}{2}L^* - 3cn^2|P| - c^2(n+1)n^3|P|\Big)|\xi|^2 + 6L^*|P|^2|a|^2\xi_1^2 + 6nL|P|^2|a|^2\xi_1^2$$

$$+ 6nL|P|^2|a|^2\varepsilon_1^2 + c^2(n+1)n^3|P|\sum_{i=1}^{n} L_0^{2i-2}|\varepsilon|^2 + cn|P|\sum_{i=1}^{n} L_0^{2i-2}|\varepsilon|^2. \quad \text{(E.32)}$$

Denoting $V_3(\varepsilon,\xi) = V_{31}(\varepsilon)+V_{32}(\xi)$, from (E.23) and (E.32) we get

$$\begin{aligned}\mathscr{L}V_3 &\le -\left(L^*L_0 - c^2(n+1)n^3|P|\sum_{i=1}^{n} L_0^{2i-2} - cn|P|\sum_{i=1}^{n} L_0^{2i-2}\right)|\varepsilon|^2 \\ &- \left(\frac{3}{2}L^* - 3cn^2|P| - c^2(n+1)n^3|P| - 6\frac{L^*}{L_0}|P|^2\right)|\xi|^2 \\ &+ 6(n-1)\frac{L}{L_0}|a|^2|P|^2\xi_1^2 + 6L^*L_0|a|^2|P|^2\varepsilon_1^2 \\ &+ 6L^*|P|^2|a|^2\xi_1^2 + 6nL|P|^2|a|^2\xi_1^2 \\ &+ 6nL|P|^2|a|^2\varepsilon_1^2. \end{aligned} \quad \text{(E.33)}$$

By (23) we obtain $6\frac{L^*}{L_0}|P|^2 \le \frac{1}{2}L^*$. From (E.3) and (E.33) we have

$$\begin{aligned}\mathscr{L}V_3 &\le -|\varepsilon|^2 - |\xi|^2 + \left(\frac{6(n-1)L^*}{L_0}|a|^2|P|^2 + 6nL^*|a|^2|P|^2 + 6L^*|a|^2|P|^2\right)\xi_1^2 + \left(6L^*L_0|a|^2|P|^2 + 6nL^*|a|^2|P|^2\right)\varepsilon_1^2 \\ &= -|\varepsilon|^2 - |\xi|^2 + m_2\xi_1^2 + m_3\varepsilon_1^2, \end{aligned} \quad \text{(E.34)}$$

where

$$m_2 = \left(\frac{6(n-1)L^*}{L_0} + 6nL^* + 6L^*\right)|a|^2|P|^2, \quad \text{(E.35)}$$

$$m_3 = \left(6L^*L_0 + 6nL^*\right)|a|^2|P|^2. \quad \text{(E.36)}$$

From (E.1)–(E.2) we have

$$m_2\xi_1^2 + m_3\varepsilon_1^2 = \frac{m_2(x_1-\hat{x}_1)^2}{L^{*2\sigma}} + \frac{m_3x_1^2}{L^{*2\sigma}} \le \frac{(2m_2+m_3)(x_1^2+\hat{x}_1^2)}{L^{*2\sigma}}. \quad \text{(E.37)}$$

By (3), (22), (E.37) and Assumption 2 we get

$$\begin{aligned} m_2\xi_1^2 + m_3\varepsilon_1^2 &\le (2m_2+m_3)\left[\left(\frac{y}{(1-\bar{\theta})L^{*\sigma}}\right)^2 + \left(\frac{\hat{x}_1}{L^{*\sigma}}\right)^2\right] \\ &\le (2m_2+m_3)\left[\left(\frac{y}{(1-\bar{\theta})L^{\sigma}}\right)^2 + \left(\frac{\hat{x}_1}{L^{\sigma}}\right)^2\right] \\ &\le (2m_2+m_3)\dot{L}(t). \end{aligned} \quad \text{(E.38)}$$

Substituting (E.38) into (E.34) yields (E.14).

Step 3. *We finally prove*

$$\int_0^{+\infty}(|e|^2+|z|^2)ds < +\infty \ a.s.; \quad \text{(E.39)}$$

z and e are bounded on $[0,+\infty)$ *a.s.* (E.40)

From (E.10)–(E.11) and (E.14) we obtain

$$\begin{aligned} V_3(t) &= V_3(0) + \int_0^t \mathscr{L}V_3(s)ds + \int_0^t\left(\frac{\partial V_3}{\partial\varepsilon}G_\varepsilon^* + \frac{\partial V_3}{\partial\xi}G^*\right)d\omega \\ &= V_3(0) + (2m_2+m_3)[L(t)-1] - \int_0^t(|\varepsilon|^2+|\xi|^2)ds \\ &+ \int_0^t\left(\frac{\partial V_3}{\partial\varepsilon}G_\varepsilon^* + \frac{\partial V_3}{\partial\xi}G^*\right)d\omega \\ &\triangleq V_3(0) + \mathscr{A}_2(t) - \mathscr{U}_2(t) + \mathscr{W}_2(t), \end{aligned} \quad \text{(E.41)}$$

where

$$\mathscr{A}_2(t) = (2m_2+m_3)[L(t)-1], \quad \text{(E.42)}$$

$$\mathscr{U}_2(t) = \int_0^t(|\varepsilon|^2+|\xi|^2)ds, \quad \text{(E.43)}$$

$$\mathscr{W}_2(t) = \int_0^t\left(\frac{\partial V_3}{\partial\varepsilon}G_\varepsilon^* + \frac{\partial V_3}{\partial\xi}G^*\right)d\omega. \quad \text{(E.44)}$$

We can find that $\mathscr{A}_2(t)$ and $\mathscr{U}_2(t)$ are two continuous adapted increasing processes with $\mathscr{A}_2(0) = \mathscr{U}_2(0) = 0$. Taking the similar lines as the proof of $\mathscr{W}_1(t)$ to be a local martingale in Proposition 2, we can prove that the continuous adapted process $\mathscr{W}_2(t)$ is a local martingale.

By Proposition 2 we have $\lim_{t\to+\infty} L(t) < +\infty$ a.s. From (E.42) we get

$$\lim_{t\to+\infty}\mathscr{A}_2(t) = (2m_2+m_3)[L(+\infty)-1] < +\infty \text{ a.s.} \quad \text{(E.45)}$$

It can be induced from (E.41), (E.45) and Lemma A.3 that

$$\lim_{t\to+\infty}\mathscr{U}_2(t) = \int_0^{+\infty}(|\varepsilon|^2+|\xi|^2)ds < +\infty \text{ a.s.}, \quad \text{(E.46)}$$

$$\lim_{t\to+\infty} V_3(t) \text{ exists and is finite.} \quad \text{(E.47)}$$

From (14)–(15), (E.1)–(E.2), (E.46)–(E.47) and the definition of $V_3(\varepsilon,\xi)$ we get (E.39) and (E.40).

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