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Stabilization of a 2×2 system of hyperbolic PDEs with recirculation in the unactuated channel[☆]

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ABSTRACT

A result by Guo and Jin (2010) considered boundary control of a string with point damping at the exact midpoint. An interpretation of the model admits a system of first-order hyperbolic (transport) PDEs with actuation at the boundary of one PDE. The unactuated second PDE exhibits recirculation phenomena. The recirculation coupling of these two PDEs gives rise to a stabilization problem with nonlocal terms that prior has not been considered. In this paper, we consider a general class of 2×2 hyperbolic PDEs where both the unactuated PDE and the actuated PDE have strict-feedback recirculation (from the outlet in the "upstream direction"). In addition, the state of the unactuated PDE feeds, in a non-local fashion, into the domain of the actuated PDE. We introduce a novel set of transformations through which we arrive at a simple target system with desirable exponential stability characteristics. A backstepping observer and output feedback controller design are also given for this 2×2 hyperbolic PDE system. We then apply our control design to the string-and-midway-antidamper application found in Guo and Jin and illustrate the result with simulations.

1. Introduction

Systems of coupled partial differential equations have been studied from control-theoretic perspectives due to their wide applicability to flexible structures. In Dáger and Zuazua (2006), the authors study flexible strings. In Pasumarthy (2006), power-preserving interconnections of several port-Hamiltonian systems (possibly infinite-dimensional) are studied. In Lasiecka (2002), a structural-acoustic model of an aircraft cockpit model consisting of coupled wave and plate equations is investigated.

Boundary stabilization of coupled linear first-order hyperbolic systems is widely studied in the literature. Stabilization of two-state hetero-directional systems exhibiting in-domain and boundary coupling is studied in Di Meglio et al. (2011). In Coron et al. (2013), the extension to 2×2 quasi-linear hyperbolic systems are studied. In Di Meglio et al. (2013), an extension to systems of $n+1$ coupled first-order hyperbolic linear PDEs (consisting of n equations convecting in one direction and 1 controlled equation counter-convecting in the opposite direction) are investigated. In Hu et al. (2016), the generalization of Di Meglio et al. (2013) to $n+m$, with m controlled equations, is explored. A result for the output feedback regulation extension to the generalized $n+m$ problem is generated in Deutscher (2017), in which additional disturbances are considered, and an observer designed. A result for the underactuated case of $1+2$ systems (exhibiting an uncontrolled heterodirectional pair) is generated in Chen et al. (2017). An adaptive extension to the output-feedback design for stabilization of $n+1$ first-order hyperbolic systems is found in Anfinsen et al. (2016), allowing the control design to work even for the case of unknown or incorrect constant parameters. This adaptive result is extended to include spatially-varying parameters in the 2×2 case (Anfinsen & Aamo, 2017). In a similar vein, Hasan (2014) contributes the addition of an observer design for $n+1$ first-order hyperbolic systems with additional ODE dynamics. An extension to this problem integrating semilinear dynamics into the case of the 2×2 hyperbolic system with ODE dynamics is generated in Hasan et al. (2016). In Auriol and Di Meglio (2016), a new control design is proposed to achieve stabilization in minimum-time. In Su et al. (2017, 2018), coupled first-order hyperbolic systems with nonlocal terms are studied. However, to our best knowledge, no boundary controller has been developed to compensate for recirculation in the unactuated part of first-order hyperbolic systems.

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Recirculation models the phenomenon of pointwise damping, found in many engineering applications such as power transmission lines, aerial cable systems, and suspension bridges, all of which are represented by string equations with midway point damping. The stabilization of these systems has drawn much attention in the past few decades. In Chen et al. (1987), the decay rate of damping for certain strings is given by solving a simple matrix eigenvalue problem. In Guo and Jin (2010), output feedback control is designed by using an infinite-dimensional observer to achieve an arbitrary decay rate for the string with midway damping. Other studies on strings with point damping can be found in Khapalov (1997), Liu (1988), Ammari et al. (2000), Ho (1993) and the references therein.

It is commonly known that there is some relationship between the coupled first order hyperbolic equations and the wave equations through a Riemann transformation (Su et al., 2017, 2018). However, these cases are primarily restricted to a single wave equation. Upon application of the Riemann transformation on coupled wave equations, recirculation terms arise in the corresponding coupled first-order hyperbolic equations. In this paper, we give the relationship between the coupled first-order hyperbolic equations with recirculation and string equations with a midway point damping (equivalent to two coupled wave equations). As the backstepping approach (Krstic, 2009) is not easily applied directly to the string with a midway point damping, using a decomposition of the string system into coupled first-order hyperbolic system is advantageous for control design. With an additional novel decoupling transformation, the design of a backstepping controller becomes feasible.

In this paper, we examine a more general result in unstable coupled first-order hyperbolic systems with the recirculation phenomenon. Specifically, we study first-order hyperbolic systems in a strict-feedback configuration, where the "downstream" subsystem not only exhibits recirculation, but also is coupled to the "upstream" (controlled) subsystem. In this configuration, the challenge lies in designing a controller to compensate for recirculation through a nontrivial actuation path. We design a backstepping transformation along with a novel decoupling transformation which admits a simple target system. We also give the corresponding backstepping observer and output feedback controller designs. The main contributions hence are:

(1) Stabilization of systems of first-order hyperbolic PDEs with novel recirculation (non-local) coupling
(2) Design of a novel decoupling transformation
(3) Relaxing the restrictions on the damped point found in Guo and Jin (2010) and Liu (1988).

The rest of this paper is organized as follows: In Section 2, we present our main problem. In Section 3, we design the state feedback controller for the coupled first-order hyperbolic system via the backstepping method, and establish exponential stability for the closed-loop system. We design the backstepping observer and output feedback controller in Section 4. In Section 5, we apply our control design for both the midway and non-midway damped string models. Finally, Section 6 gives some concluding remarks.

2. Problem statement

In this paper, we consider the following coupled hyperbolic equations (see Fig. 1)

$$\eta_t(x,t) = g_1(x)\eta(1,t) - \lambda\eta_x(x,t) + \int_x^1 f(x,y)\eta(y,t)dy \tag{1}$$

$$\begin{aligned}\xi_t(x,t) = \xi_x(x,t) + h(x)\eta(x,t) + \int_0^1 f_{21}(x,y)\eta(y,t)dy \\ + \int_0^x f_{22}(x,y)\xi(y,t)dy \\ + g_2(x)\eta(1,t) + g_3(x)\xi(0,t)\end{aligned} \tag{2}$$

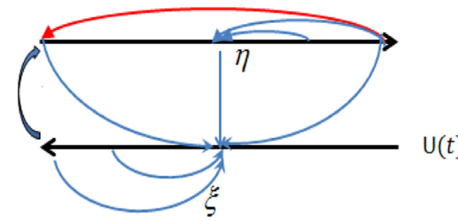

Fig. 1. Flow diagram of the open-loop system (1)–(4).

$$\eta(0,t) = c_1\xi(0,t) + c_2\eta(1,t) + \int_0^1 d(y)\eta(y,t)dy \tag{3}$$

$$\xi(1,t) = U(t) \tag{4}$$

where $U(t)$ is the control law and $0 < \lambda \le 1, c_1, c_2 \in \mathbb{R}$, $f(x,y) \in C(D_1)$, $f_{21}(x,y) \in C([0,1]\times[0,1])$, $f_{22}(x,y) \in C(D)$, where $D_1 = \{(x,y)|0 \le x \le y \le 1\}$, $D = \{(x,y)|0 \le y \le x \le 1\}$, $g_1(x), g_2(x), h(x), d(x) \in C([0,1])$. (3) is a recirculation boundary condition, which is novel, which introduce instability. The other coupling terms on the right side of Eqs. (1) and (2) (see Su et al., 2017, 2018) may introduce instability, which is studied in Su et al. (2017, 2018) .

3. Stabilization of system (1)–(4)

We will design four transformations in succession to stabilize system (1)–(4). These transformations can be composed together to admit a single transformation, but we have chosen to leave them separate during the analysis as to not obfuscate their respective purposes.

3.1. Control design

The first backstepping transformation is as follows:

$$\alpha(x,t) = \eta(x,t) - \int_x^1 q(x,y)\eta(y,t)dy \tag{5}$$

with its associated inverse transformation:

$$\eta(x,t) = \alpha(x,t) + \int_x^1 p(x,y)\alpha(y,t)dy \tag{6}$$

The objective of (5) is to shift the unstable non-local couplings of (1) into the $x = 0$ boundary, and (5) will admit the following intermediate target system:

$$\alpha_t(x,t) = -\lambda\alpha_x(x,t) \tag{7}$$

$$\begin{aligned}\xi_t(x,t) = \xi_x(x,t) + \int_0^1 H(x,y)\alpha(y,t)dy + g_2(x)\alpha(1,t) \\ + \int_0^x f_{22}(x,y)\xi(y,t)dy + h(x)\alpha(x,t) \\ + g_3(x)\xi(0,t) + \int_x^1 h(x)p(x,y)\alpha(y,t)dy\end{aligned} \tag{8}$$

$$\alpha(0,t) = c_1\xi(0,t) + c_2\alpha(1,t) + \int_0^1 G(y)\alpha(y,t)dy \tag{9}$$

$$\xi(1,t) = U(t) \tag{10}$$

where

$$H(x,y) = f_{21}(x,y) + f_{21}(x,y)\int_0^y q(z,y)dz$$

$$G(y) = (d(y) - q(0,y))\left[1 + \int_0^y p(z,y)dz\right]$$

The kernels $q(x,y)$ and $p(x,y)$ are given by the following first-order hyperbolic PDEs defined on the triangle $\{(x,y)|0 \le x \le y \le 1\}$:

$$\lambda q_x(x,y) + \lambda q_y(x,y) = f(x,y) - \int_x^y q(x,z)f(z,y)dz$$
$$\lambda q(x,1) = \int_x^1 q(x,y)g_1(y)dy - g_1(x)$$
$$\lambda p_x(x,y) + \lambda p_y(x,y) = f(x,y) + \int_x^y p(x,z)f(z,y)dz$$
$$\lambda p(x,1) = -g_1(x).$$

Next, the following transformation

$$\beta(x,t) = \xi(x,t) - \int_0^x k(x,y)\xi(y,t)dy - \int_0^x l(x,y)\alpha(y,t)dy - \int_x^1 \rho(x,y)\alpha(y,t)dy \tag{11}$$

and its associated inverse transformation

$$\xi(x,t) = \beta(x,t) + \int_0^x m(x,y)\beta(y,t)dy + \int_0^x n(x,y)\alpha(y,t)dy + \int_x^1 \sigma(x,y)\alpha(y,t)dy \tag{12}$$

are utilized to shift the unstable terms found in (2) to the $x = 1$ boundary, where they can be neutralized via the boundary controller. The transformation (11) will admit the system

$$\alpha_t(x,t) = -\lambda\alpha_x(x,t) \tag{13}$$
$$\beta_t(x,t) = \beta_x(x,t) \tag{14}$$
$$\alpha(0,t) = c_1\beta(0,t) + c_2\alpha(1,t) + \int_0^1 G(y)\alpha(y,t)dy \tag{15}$$
$$\beta(1,t) = W_1(t) \tag{16}$$

where

$$W_1(t) = U(t) - \int_0^1 [k(1,y)\xi(y,t) + l(1,y)\alpha(y,t)]\,dy. \tag{17}$$

We treat W_1 as a pseudo-controller to be designed. After W_1 is determined, the controller U can be expressed.

The kernels $k(x,y)$ and $l(x,y)$ are defined on the triangle $\{(x,y)|0 \le y \le x \le 1\}$, and are given by

$$k_y(x,y) + k_x(x,y) = \int_y^x k(x,z)f_{22}(z,y)dz - f_{22}(x,y)$$
$$l_x(x,y) - \lambda l_y(x,y) = -H(x,y) + k(x,y)c(y) + \lambda l(x,0)G(y) + \int_0^x k(x,z)H(z,y)dz$$
$$k(x,0) = \lambda l(x,0)c_1$$
$$\lambda l(x,x) + l(x,x) = \rho(x,x) + \lambda\rho(x,x) - c(x)$$

while the kernel $\rho(x,y)$ is defined on the triangle $\{(x,y)|0 \le x \le y \le 1\}$ and obeys the following PDE:

$$\rho_x(x,y) - \lambda\rho_y(x,y) = -H(x,y) + \int_0^x k(x,z)H(z,y)dz + \lambda l(x,0)G(y), \tag{18}$$
$$\rho(0,y) = 0 \tag{19}$$
$$\lambda\rho(x,1) = -g_2(x) + \int_0^x k(x,y)g_2(y)dy + \lambda l(x,0)c_2, \tag{20}$$

Then, the following proposed decoupling transformation

$$w(x,t) = c_1\beta(x,t) + c_2\alpha(1-\lambda x,t) + \int_{\lambda x}^1 G(y)\alpha(y-\lambda x,t)dy \tag{21}$$

and its inverse

$$\beta(x,t) = \frac{1}{c_1}w(x,t) - \frac{c_2}{c_1}\alpha(1-\lambda x,t) - \frac{1}{c_1}\int_{\lambda x}^1 G(y)\alpha(y-\lambda x,t)dy \tag{22}$$

will admit the system

$$\alpha_t(x,t) = -\lambda\alpha_x(x,t), \tag{23}$$
$$w_t(x,t) = w_x(x,t) + \lambda G(\lambda x)w(0,t), \tag{24}$$
$$\alpha(0,t) = w(0,t), \tag{25}$$
$$w(1,t) = W(t), \tag{26}$$

where

$$W(t) = c_1W_1(t) + c_2\alpha(1-\lambda,t) + \int_\lambda^1 G(y)\alpha(y-\lambda,t)dy$$

Again, W acts as a pseudo-controller to be designed. Once W is determined, W_1 can be expressed. The decoupling transformation (21) takes the instability in the $x = 0$ boundary of α and shifts it into the interior of w.

Finally, we use the transformation

$$z(x,t) = w(x,t) - \int_0^x \theta(x,y)w(y,t)dy \tag{27}$$

and its inverse transformation

$$w(x,t) = z(x,t) + \int_0^x \zeta(x,y)z(y,t)dy \tag{28}$$

to shift the instability in w to the boundary $x = 1$, where it can be neutralized by the boundary controller. The kernels $\theta(x,y)$, $\zeta(x,y)$ satisfy the following PDEs:

$$\theta_x(x,y) + \theta_y(x,y) = 0 \tag{29}$$
$$\theta(x,0) = \int_0^x \theta(x,y)\lambda G(\lambda y)dy - \lambda G(\lambda x) \tag{30}$$
$$\zeta_x(x,y) + \zeta_y(x,y) = 0 \tag{31}$$
$$\zeta(x,0) = -\lambda G(\lambda x) \tag{32}$$

The resulting (terminal) target system is:

$$\alpha_t(x,t) = -\lambda\alpha_x(x,t) \tag{33}$$
$$z_t(x,t) = z_x(x,t) \tag{34}$$
$$\alpha(0,t) = z(0,t) \tag{35}$$
$$z(1,t) = 0. \tag{36}$$

As one may notice upon inspection, the terminal target system is a simple cascade of two first-order hyperbolic PDEs. It is quite trivial to see that this cascade is exponentially (in fact, finite-time) stable.

3.2. Summary of the transformations and controllers

From composing transformations (5), (11), (21) and (28) which transform system (1)–(4) into system (33)–(36), the aggregate transformation from $(\eta,\xi) \to (\alpha,z)$ can be found to be:

$$\alpha(x,t) = \eta(x,t) - \int_x^1 q(x,y)\eta(y,t)dy \tag{37}$$

$$
\begin{aligned}
&z(x,t) = w(x,t) - \int_0^x \theta(x,y)w(y,t)dy \\
&= c_1\xi(x,t) - c_1\int_0^x [k(x,y)\xi(y,t) + l(x,y)\eta(y,t)]\,dy \\
&\quad + c_1\int_0^x l(x,y)\int_y^1 q(y,z)\eta(z,t)dzdy \\
&\quad - c_1\int_x^1 \rho(x,y)\left[\eta(y,t) - \int_y^1 q(y,z)\eta(z,t)dz\right]dy \\
&\quad + c_2\eta(1-\lambda x,t) - c_2\int_{1-\lambda x}^1 q(1-\lambda x,z)\eta(z,t)dz \\
&\quad + \int_{\lambda x}^1 G(z)\left[\eta(z-\lambda x,t) - \int_{z-\lambda y}^1 q(z-\lambda x,s)\eta(s,t)ds\right]dz \\
&\quad - c_1\int_0^x \theta(x,y)\xi(y,t)dy \\
&\quad + \int_0^x \theta(x,y)\int_0^y [k(y,z)\xi(z,t) + l(y,z)\eta(z,t)]\,dzdy \\
&\quad - \int_0^x \theta(x,y)\int_0^y l(y,z)\int_z^1 q(z,s)\eta(s,t)dsdzdy \\
&\quad + \int_0^x \theta(x,y)\int_y^1 \rho(y,z)\left[\eta(z,t) - \int_z^1 q(z,s)\eta(s,t)ds\right]dzdy \\
&\quad - c_2\int_0^x \theta(x,y)\left[\eta(1-\lambda y,t) - \int_{1-\lambda y}^1 q(1-\lambda y,z)\eta(z,t)dz\right]dy \\
&\quad - \int_0^x \theta(x,y)\int_{\lambda y}^1 G(z)\eta(z-\lambda y,t)dzdy \\
&\quad + \int_0^x \theta(x,y)\int_{\lambda y}^1 G(z)\int_{z-\lambda y}^1 q(z-\lambda y,s)\eta(s,t)dsdzdy
\end{aligned}
\tag{38}
$$

From the boundary condition, we get the control law

$$
\begin{aligned}
U(t) = &-\frac{c_2}{c_1}\eta(1-\lambda,t) + \int_0^1 \mathscr{K}(x,y)\xi(y,t)dy \\
&+ \int_0^1 \mathscr{L}(x,y)\eta(y,t)dy
\end{aligned}
\tag{39}
$$

where the kernels $\mathscr{K}, \mathscr{L}$ are given by

$$
\mathscr{K}(x,y) = k(1,y) + \theta(1,y) + \int_y^1 \theta(1,z)k(z,y)dz \tag{40}
$$

$$
\begin{aligned}
\mathscr{L}(x,y) = l(1,y) &+ \int_0^y \Bigg(l(1,z) - \frac{1}{c_1}\theta(1,z)\rho(z,y) \\
&\quad + \frac{1}{c_1}\int_0^z \theta(1,z)l(z,s)q(s,y)ds \\
&\quad + \frac{1}{c_1}\int_z^y \theta(1,z)\rho(z,s)q(s,y)ds\Bigg)dz \\
&+ \int_y^1 \Bigg(-\frac{1}{c_1}\theta(1,z)l(z,y) \\
&\quad + \frac{1}{c_1}\int_0^y \theta(1,z)l(z,s)q(s,y)ds\Bigg)dz \\
&+ \mathbf{1}_{[1-\lambda,1]}(y)\Bigg[\frac{c_2}{c_1}q(1-\lambda,y) - \frac{c_2}{c_1}\theta(1,\frac{1-y}{\lambda}) \\
&\quad - \frac{c_2}{c_1}\int_{\frac{1-y}{\lambda}}^1 \theta(1,z)q(1-\lambda z,y)dz \\
&\quad + \frac{1}{c_1}\int_0^{\frac{1-y}{\lambda}} \theta(1,z)G(y+\lambda z)dz \\
&\quad - \frac{1}{c_1}\int_{\frac{1-y}{\lambda}}^1\int_{\lambda z}^1 \theta(1,z)G(s)q(s-\lambda z,y)dsdz \\
&\quad - \frac{1}{c_1}\int_0^{\frac{1-y}{\lambda}}\int_{\lambda z}^{y+\lambda z} \theta(1,z)G(s)q(s-\lambda z,y)dsdz\Bigg] \\
&+ \mathbf{1}_{[0,1-\lambda]}(y)\Bigg[-\frac{1}{c_1}G(y+\lambda) \\
&\quad + \frac{1}{c_1}\int_\lambda^{y+\lambda} G(z)q(z-\lambda,y)dz \\
&\quad + \frac{1}{c_1}\int_0^1 \theta(1,z)G(y+\lambda z)dz \\
&\quad - \frac{1}{c_1}\int_0^1\int_{\lambda z}^{y+\lambda z} \theta(1,z)G(s)q(s-\lambda z,y)dsdz\Bigg].
\end{aligned}
\tag{41}
$$

3.3. Closed-loop system stability

Lemma 1. *The target system (33)–(36) is exponentially stable in the L^2- sense.*

The stability of system (33)–(36) can be easily obtained using the following Lyapunov–Krasovskii functional:

$$
V(t) = \int_0^1 \left[e^{-\delta_1 x}\alpha(x,t)^2 + c_2 e^{\delta_2 x}z(x,t)^2\right]dx \tag{42}
$$

with $\delta_1, \delta_2 > 0$ and $c_2 > \lambda$. The details of the proof are omitted.

Since the transformations (5), (11), (21) and (28) are all invertible, we can infer the exponential stability property for (1)–(4). One can show the stability of the original system by consecutively applying inverse transformations and applying the boundedness of the forward and inverse kernels. We summarize our result in Theorem 2.

Theorem 2. *The closed-loop system (1)–(4) with the control law $U(t)$ given by (39) is exponentially stable in the L^2-sense.*

3.4. Kernel PDEs and well-posedness

The kernels $q(x,y)$ and $p(x,y)$ obey the following PDEs:

$$
\begin{aligned}
\lambda q_x(x,y) + \lambda q_y(x,y) &= f(x,y) - \int_x^y q(x,z)f(z,y)dz \\
\lambda q(x,1) &= \int_x^1 q(x,y)g_1(y)dy - g_1(x) \\
\lambda p_x(x,y) + \lambda p_y(x,y) &= f(x,y) + \int_x^y p(x,z)f(z,y)dz \\
\lambda p(x,1) &= -g_1(x).
\end{aligned}
$$

These PDEs can be readily solved by applying the Laplace transformation. The kernels $\rho(x,y)$, $l(x,y)$, $k(x,y)$ satisfy the following system of PDEs

$$
\begin{aligned}
\rho_x(x,y) - \lambda\rho_y(x,y) &= -H(x,y) + \int_0^x k(x,z)H(z,y)dz \\
&\quad + \lambda l(x,0)G(y) - h(x)p(x,y)
\end{aligned}
\tag{43}
$$

$$
k_y(x,y) + k_x(x,y) = \int_y^x k(x,z)f_{22}(z,y)dz - f_{22}(x,y) \tag{44}
$$

$$
\begin{aligned}
l_x(x,y) - \lambda l_y(x,y) &= -H(x,y) + k(x,y)h(y) + \lambda l(x,0)G(y) \\
&\quad + \int_0^x k(x,z)H(z,y)dz \\
&\quad - \int_y^x k(x,z)h(z)p(z,y)dz
\end{aligned}
\tag{45}
$$

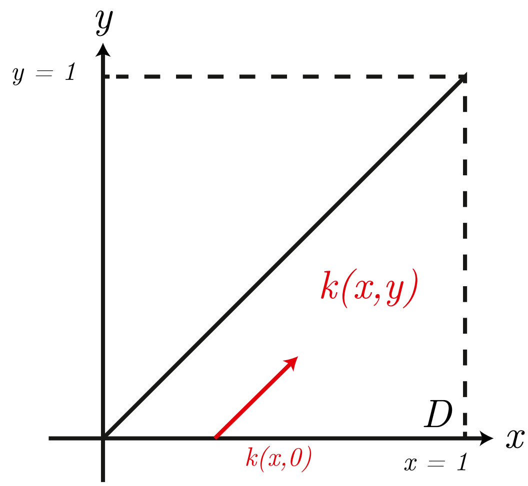

Fig. 2. Characteristic curves for $k(x, y)$.

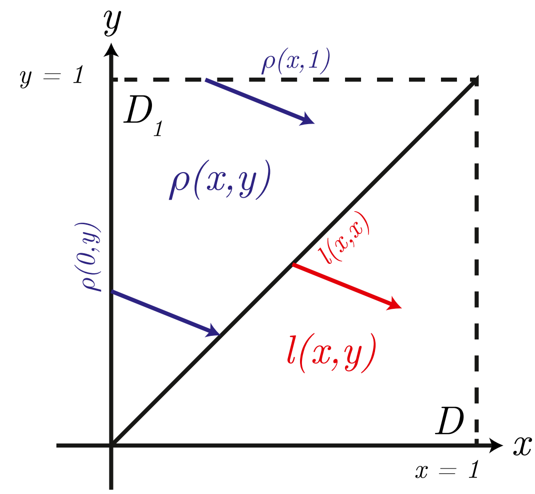

Fig. 3. Characteristic curves for $\rho(x, y)$, $l(x, y)$.

$$\lambda\left[\rho(x,1)-c_2 l(x,0)\right] = -g_2(x)+\int_0^x k(x,y)g_2(y)dy \tag{46}$$

$$k(x,0)-c_1\lambda l(x,0) = -g_3(x)+\int_0^x k(x,y)g_3(y)dy \tag{47}$$

$$\lambda l(x,x)+l(x,x) = \rho(x,x)+\lambda\rho(x,x)-h(x), \tag{48}$$

$$\rho(0,y) = 0 \tag{49}$$

For the details of the derivation of the kennels, see Appendix. The following lemma establishes well-posedness of the PDEs (43)–(49).

Lemma 3. *The kernel PDEs (43)–(49) have a unique solution in $C^1(D_1)\times C^1(D)\times C^1(D)$, where $D_1 = \{(x,y)|0\le x\le y\le 1\}$ and $D = \{(x,y)|0\le y\le x\le 1\}$. Moreover,*

$$|\rho(x,y)| \le Me^{M(x-y)},\ |k(x,y)| \le Me^{M(x-y)},$$
$$|l(x,y)| \le Me^{M(x-y)},$$

where $M = \max\{c_1\bar{c}+c_1\bar{h}+c_1\bar{g}+2\bar{f}, \bar{c}+\bar{h}+\lambda\bar{g}, \bar{g}+\bar{f}\}$, and $\bar{c} = \max_{x\in[0,1]}\{h(x)\}$, $\bar{h} = \max_{x\in[0,1]\times y\in[0,1]}\{H(x,y)\}$, $\bar{f} = \max_{x\in[0,1]\times y\in[0,1]}\{f_{22}(x,y)\}$, $\bar{g} = \max_{x\in[0,1]}\{G(x), g_3(x)\}$.

Proof. We use the method of characteristics for $k(x,y)$, $\rho(x,y)$ and $l(x,y)$. This will admit integral equation forms, to which we apply the method of successive approximations.

To solve $\rho(x,y)$, we first use (49) as the characteristic initial condition when $\lambda x+y\le 1$, and (46) as the characteristic initial condition when $\lambda x+y > 1$. To solve $k(x,y)$, we use (47) as the characteristic initial condition. To solve $l(x,y)$, we use (48) as the initial condition for the characteristics. Then, one can manipulate the resulting integral equations into an iterative form, and directly apply the method of successive approximations.

Precisely, $\rho(x,y)$ is a piecewise function which can be rewritten as: for $\lambda x+y\le 1$ (see Fig. 2)

$$\begin{aligned}\rho(x,y) &= \int_0^x\int_0^s k(s,z)H(z,-\lambda s+y+\lambda x)dzds\\ &-\int_0^x \left[H(s,\lambda x-\lambda s+y)+h(s)p(s,\lambda x-\lambda s+y)\right]ds\\ &+\lambda\int_0^x l(s,0)G(-\lambda s+y+\lambda x)ds\end{aligned}\tag{50}$$

and for $\lambda x+y>1$ (see Fig. 3),

$$\begin{aligned}\rho(x,y) &= \frac{1}{\lambda}g_2\left(\frac{y+\lambda x-1}{\lambda}\right)+c_2 l\left(\frac{y+\lambda x-1}{\lambda},0\right)\\ &+\int_{\frac{y-1+\lambda x}{\lambda}}^x\int_0^s k(s,z)H(z,-\lambda s+y+\lambda x)dzds\\ &-\int_{\frac{y-1+\lambda x}{\lambda}}^x\left[H(s,\lambda x-\lambda s+y)+h(s)p(s,\lambda x-\lambda s+y)\right]ds\\ &+\lambda\int_{\frac{y-1+\lambda x}{\lambda}}^x l(s,0)G(-\lambda s+y+\lambda x)ds\\ &+\frac{1}{\lambda}\int_0^{\frac{y+\lambda x-1}{\lambda}} k\left(\frac{y+\lambda x-1}{\lambda},y\right)g_2(y)dy.\end{aligned}\tag{51}$$

Note that along the characteristic $\lambda x+y = 1$, the two solutions coincide. This, coupled with the PDE (43), guarantees a C^1 solution.

Similarly, we can solve $k(x,y)$ in the integral form as

$$k(x,y) = K_0(x,y)+K[k,l](x,y) \tag{52}$$

where

$$\begin{aligned}K_0(x,y) &= -\int_{x-y}^x f_{22}(s,y)ds-g_3(x-y),\\ K[k,l](x,y) &= \lambda c_1 l(x-y,0)+\int_0^{x-y}k(x-y,z)g_3(z)dz\\ &\quad+\int_{x-y}^x\int_{s+y-x}^s k(s,z)f_{22}(z,s+y-x)dzds.\end{aligned}$$

Likewise, $l(x,y)$ can be solved as: for $\lambda x+y\le 1$,

$$\begin{aligned}l(x,y) &= \int_{\frac{y+\lambda x}{1+\lambda}}^x k(s,-\lambda s+y+\lambda x)c(-\lambda s+y+\lambda x)ds\\ &+\int_{\frac{y+\lambda x}{1+\lambda}}^x\int_0^x k(s,z)H(z,-\lambda s+y+\lambda x)dzds\\ &+\int_{\frac{y+\lambda x}{1+\lambda}}^x\left[\lambda l(s,0)G(\lambda x-\lambda s+y)-H(s,\lambda x-\lambda s+y)\right]ds\\ &+\int_{\frac{\lambda x}{1+\lambda}}^x\int_{\lambda x+y-\lambda s}^s k(s,z)h(z)p(z,\lambda x+y-\lambda s)dzds\\ &+\int_0^{\frac{y+\lambda x}{1+\lambda}}\int_0^s k(s,z)H(z,-\lambda s+y+\lambda x)dzds\end{aligned}$$

$$+\int_0^{\frac{y+\lambda x}{1+\lambda}}[\lambda l(s,0)G(\lambda x-\lambda s+y)-H(s,\lambda x-\lambda s+y)]\,ds$$
$$-\int_0^{\frac{y+\lambda x}{1+\lambda}}h(s)p(s,-\lambda s+y+\lambda x)ds-\frac{h(\frac{y+\lambda x}{1+\lambda})}{1+\lambda} \tag{53}$$

and for $\lambda x+y>1$,

$$l(x,y)=\int_{\frac{y+\lambda x}{1+\lambda}}^{x}k(s,-\lambda s+y+\lambda x)c(-\lambda s+y+\lambda x)ds$$
$$+\int_{\frac{y+\lambda x}{1+\lambda}}^{x}\int_0^{x}k(s,z)H(z,-\lambda s+y+\lambda x)dzds$$
$$+\int_{\frac{y+\lambda x}{1+\lambda}}^{x}[\lambda l(s,0)G(\lambda x-\lambda s+y)-H(s,\lambda x-\lambda s+y)]\,ds$$
$$+\int_{\frac{\lambda x}{1+\lambda}}^{x}\int_{\lambda x+y-\lambda s}^{s}k(s,z)h(z)p(z,\lambda x+y-\lambda s)dzds$$
$$\int_{\frac{y-1+\lambda x}{\lambda}}^{\frac{y+\lambda x}{1+\lambda}}\int_0^{s}k(s,z)H(z,-\lambda s+y+\lambda x)dzds$$
$$+\int_{\frac{y-1+\lambda x}{\lambda}}^{\frac{y+\lambda x}{1+\lambda}}[\lambda l(s,0)G(\lambda x-\lambda s+y)-H(s,\lambda x-\lambda s+y)]\,ds$$
$$+\frac{1}{\lambda}g_2\left(\frac{y+\lambda x-1}{\lambda}\right)+c_2 l\left(\frac{y+\lambda x-1}{\lambda},0\right)$$
$$+\frac{1}{\lambda}\int_0^{\frac{y+\lambda x-1}{\lambda}}k\left(\frac{y+\lambda x-1}{\lambda},x\right)g_2(x)dx$$
$$-\int_{\frac{y-1+\lambda x}{\lambda}}^{\frac{y+\lambda x}{1+\lambda}}h(s)p(s,-\lambda s+y+\lambda x)ds-\frac{h(\frac{y+\lambda x}{1+\lambda})}{1+\lambda}. \tag{54}$$

Since $0<\lambda\le 1$, when $y=0$, $\lambda x\le 1$, so $l(x,0)$ can be written in the integral form as

$$l(x,0)=L_0(x)+L[k,l](x) \tag{55}$$

where

$$\begin{aligned}L_0[k,l](x)=&-\int_0^{x}H(s,-\lambda s+\lambda x)ds\\&-\int_0^{\frac{\lambda x}{1+\lambda}}h(s)p(s,-\lambda s+\lambda x)ds-\frac{h(\frac{\lambda x}{1+\lambda})}{1+\lambda},\\L[k,l](x)=&\int_{\frac{\lambda x}{1+\lambda}}^{x}k(s,-\lambda s+\lambda x)c(-\lambda s+\lambda x)ds\\&+\int_0^{x}\int_0^{s}k(s,z)H(z,-\lambda s+\lambda x)dzds\\&+\int_0^{x}\lambda l(s,0)G(-\lambda s+\lambda x)ds\\&+\int_{\frac{\lambda x}{1+\lambda}}^{x}\int_{\lambda x-\lambda s}^{s}k(s,z)h(z)p(z,\lambda x-\lambda s)dzds.\end{aligned}$$

Since $\rho(x,y)$ and $l(x,y)$ are two continuous functions in x and y depending on $k(x,y)$, $l(x,0)$, we will only need to show that $k(x,y)$ and $l(x,0)$ exist. Let us solve $k(x,y)$ and $l(x,0)$ by using the method of successive approximations. Set

$$\begin{aligned}k^0(x,y)&=K_0(x,y),\quad l^0(x,0)=0,\\k^{n+1}(x,y)&=K_0(x,y)+K[k^n,l^n](x,y),\\l^{n+1}(x,0)&=L[k^n,l^n](x)\end{aligned}$$

for $n=0,1,\ldots$ and consider the differences

$$\Delta k^{n+1}=k^{n+1}-k^n,\quad \Delta l^{n+1}=l^{n+1}-l^n$$

with $\Delta k^0=K_0$, $\Delta l^0=0$. It is easy to see that Δk^n, Δl^n satisfy the integral relationships

$$\Delta k^{n+1}=K[\Delta k^n,\Delta l^n](x,y),\quad \Delta l^{n+1}=L[\Delta k^n,\Delta l^n](x). \tag{56}$$

Let us assume that

$$|\Delta k^n|\le M^n\frac{(x-y)^n}{n!},\quad |\Delta l^n|\le M^n\frac{x^n}{n!}. \tag{57}$$

Denoting

$$\begin{aligned}\bar c&=\max_{x\in[0,1]}\{h(x)\},\\\bar h&=\max_{x\in[0,1]\times y\in[0,1]}\{H(x,y)+h(x)p(x,y)\},\\\bar f&=\max_{x\in[0,1]\times y\in[0,1]}\{f_{22}(x,y)\},\\\bar g&=\max_{x\in[0,1]}\{G(x)\},\\M&=\max\{c_1\bar c+c_1\bar h+c_1\bar g+2\bar f,\bar c+\bar h+\lambda\bar g,\bar g+\bar f\}.\end{aligned}$$

then

$$\begin{aligned}&|\Delta k^{n+1}|=K[\Delta k^n,\Delta l^n](x,y)\\=&\ \lambda c_1\int_{\frac{\lambda(x-y)}{1+\lambda}}^{x-y}\Delta k^n(s,-\lambda s+\lambda(x-y))c(-\lambda s+\lambda(x-y))ds\\&+\lambda c_1\int_{\frac{\lambda(x-y)}{1+\lambda}}^{x-y}\int_0^{x-y}\Delta k^n(s,z)H(z,-\lambda s+\lambda(x-y))dzds\\&+\lambda c_1\int_{\frac{\lambda(x-y)}{1+\lambda}}^{x-y}\lambda\Delta l^n(s,0)G(-\lambda s+\lambda(x-y))ds\\&+\int_{x-y}^{x}\int_{s+y-x}^{s}\Delta k^n(s,z)f_{22}(z,s+y-x)dzds\\&+\bar f\int_0^{x-y}\Delta k^n(x-y,z)dz\\\le&\ \lambda c_1\int_{\frac{\lambda(x-y)}{1+\lambda}}^{x-y}M^n\frac{(s+\lambda s-\lambda(x-y))^n}{n!}c(-\lambda s+\lambda(x-y))ds\\&+\lambda c_1\int_{\frac{\lambda(x-y)}{1+\lambda}}^{x-y}\int_0^{x-y}M^n\frac{(s-z)^n}{n!}H(z,-\lambda s+\lambda(x-y))dzds\\&+\lambda c_1\int_{\frac{\lambda(x-y)}{1+\lambda}}^{x-y}\lambda M^n\frac{s^n}{n!}G(-\lambda s+\lambda(x-y))ds\\&+\int_{x-y}^{x}\int_{s+y-x}^{s}M^n\frac{(s-z)^n}{n!}f_{22}(z,s+y-x)dzds\\&+\bar f\int_0^{x-y}M^n\frac{(x-y-z)^n}{n!}dz+\bar fM^n\frac{(x-y)^{n+1}}{(n+1)!}\\\le&\ \frac{\lambda}{1+\lambda}c_1\bar cM^n\frac{(x-y)^{n+1}}{(n+1)!}+c_1\bar hM^n\frac{(x-y)^{n+2}}{(n+2)!}\\&+\frac{\lambda^2}{1+\lambda}c_1\bar gM^n\frac{(x-y)^{n+1}}{(n+1)!}+\bar fM^n\frac{(x-y)^{n+1}}{(n+1)!}\\\le&\ M^{n+1}\frac{(x-y)^{n+1}}{(n+1)!}\end{aligned}$$

and

$$\begin{aligned}&|\Delta l^{n+1}(x,0)|=L[\Delta k^n,\Delta l^n](x)\\=&\int_{\frac{\lambda x}{1+\lambda}}^{x}\Delta k^n(s,-\lambda s+\lambda x)c(-\lambda s+\lambda x)ds\\&+\int_0^{x}\int_0^{s}\Delta k^n(s,z)H(z,-\lambda s+\lambda x)dzds\\&+\int_0^{x}\lambda\Delta l^n(s,0)G(-\lambda s+\lambda x)ds\end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{\lambda x}{1+\lambda}}^{x}\int_{\lambda x-\lambda s}^{s}\Delta k^{n}(s,z)h(z)p(z,\lambda x-\lambda s)dzds\\
&\le \int_{\frac{\lambda x}{1+\lambda}}^{x} M^{n}\frac{(s+\lambda s-\lambda x)^{n}}{n!}c(-\lambda s+\lambda x)ds\\
&+\bar{h}\int_{0}^{x}\int_{0}^{s}M^{n}\frac{(s-z)^{n}}{n!}dzds+\bar{g}\int_{0}^{x}\lambda M^{n}\frac{s^{n}}{n!}ds\\
&+\bar{h}\int_{\frac{\lambda x}{1+\lambda}}^{x}\int_{\lambda x-\lambda s}^{s}\Delta k^{n}(s,z)dzds\\
&\le \lambda M^{n+1}\frac{x^{n+1}}{(n+1)!}.
\end{aligned}
$$

By induction (57) is proved. It is easy to verify that $F[k,l](x,y)$ and $G[k,l](x,y)$ are continuous operators (see Di Meglio et al., 2013), therefore, the series

$$
k(x,y)=\sum_{n=0}^{\infty}\Delta k^{n}(x,y),\qquad l(x,0)=\sum_{n=0}^{\infty}\Delta l^{n}(x,0)
$$

uniformly converges to the solution of (52), (55) with $n\to\infty$. Also, we show that

$$
|k(x,y)|\le Me^{M(x-y)},\qquad |l(x,0)|\le Me^{Mx}. \tag{58}
$$

Similar to the proof of $k(x,y)$, by using (51), (53), (54) and (58), we have

$$
\begin{aligned}
|\rho(x,y)|&\le Me^{M(x-y)},\\
|l(x,y)|&\le Me^{M(x-y)}.
\end{aligned}
$$

The proof of the uniqueness of this solution is very similar with (Su et al., 2018), so we omit the details. ■

In order to solve for $m(x,y)$, $n(x,y)$, $\sigma(x,y)$, we rewrite transformation (11) as

$$
\begin{aligned}
&\xi(x,t)-\int_{0}^{x}k(x,y)\xi(y,t)dy\\
&=\beta(x,t)+\int_{0}^{x}l(x,y)\alpha(y,t)dy+\int_{x}^{1}\rho(x,y)\alpha(y,t)dy.
\end{aligned} \tag{59}
$$

From Lemma 3, $k(x,y)$ is continuous, and therefore exists a unique continuous inverse kernel $m(x,y)$ defined on D such that (e.g. Vazquez, 2006)

$$
\begin{aligned}
\xi(x,t)=\ &\beta(x,t)+\int_{0}^{x}l(x,y)\alpha(y,t)dy+\int_{x}^{1}\rho(x,y)\alpha(y,t)dy\\
&+\int_{0}^{x}m(x,y)\Big(\beta(y,t)+\int_{0}^{y}l(y,z)\alpha(z,t)dz\\
&+\int_{y}^{1}\rho(y,z)\alpha(z,t)dz\Big)dy
\end{aligned}
$$

which yields the following inverse transformation

$$
\begin{aligned}
\xi(x,t)=\ &\beta(x,t)+\int_{0}^{x}m(x,y)\beta(y,t)dy\\
&+\int_{0}^{x}n(x,y)\alpha(y,t)dy+\int_{x}^{1}\sigma(x,y)\alpha(y,t)dy
\end{aligned} \tag{60}
$$

where

$$
\begin{aligned}
n(x,y)&=l(x,y)+\int_{y}^{x}m(x,z)l(z,y)dz+\int_{0}^{y}m(x,z)\rho(z,y)dz\\
\sigma(x,y)&=\rho(x,y)+\int_{0}^{x}m(x,z)\rho(z,y)dz.
\end{aligned}
$$

Together with Lemma 3, the well-posedness of $m(x,y)$, $n(x,y)$ and $\sigma(x,y)$ is established.

We derive kernel $\theta(x,y)$ next. By taking the time and space derivative of $z(x,t)$, we obtain the following PDE for θ:

$$
\theta_x(x,y)+\theta_y(x,y)=0, \tag{61}
$$

$$
\theta(x,0)=\int_{0}^{x}\theta(x,y)\lambda G(\lambda y)dy-\lambda G(\lambda x) \tag{62}
$$

By repeating the similar computations for $\theta(x,y)$, we can also show that kernel $\zeta(x,y)$ satisfy

$$
\zeta_x(x,y)+\zeta_y(x,y)=0, \tag{63}
$$

$$
\zeta(x,0)=-\lambda G(\lambda x) \tag{64}
$$

The well-posedness of $\theta(x,y)$, $\zeta(x,y)$ are also given in Krstic and Smyshlyaev (2008).

4. Output feedback controller for (1)–(4)

We suppose the only available measurement of our system is the signal $\eta(1,t)$. We propose the following output injection observer for system (1)–(4):

$$
\begin{aligned}
\hat{\eta}_t(x,t)=\ &-\lambda\hat{\eta}_x(x,t)+g_1(x)\eta(1,t)\\
&+\int_{x}^{1}f(x,y)\hat{\eta}(y,t)dy+r(x)[\eta(1,t)-\hat{\eta}(1,t)]
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
\hat{\xi}_t(x,t)=\ &\hat{\xi}_x(x,t)+\int_{0}^{1}f_{21}(x,y)\hat{\eta}(y,t)dy\\
&+\int_{0}^{x}f_{22}(x,y)\hat{\xi}(y,t)dy+s(x)[\eta(1,t)-\hat{\eta}(1,t)]\\
&+g_2(x)\eta(1,t)+h(x)\hat{\eta}(x,t)+g_3(x)\hat{\xi}(0,t)
\end{aligned} \tag{66}
$$

$$
\hat{\eta}(0,t)=c_1\hat{\xi}(0,t)+c_2\eta(1,t)+\int_{0}^{1}d(y)\hat{\eta}(y,t)dy \tag{67}
$$

$$
\hat{\xi}(1,t)=U(t). \tag{68}
$$

One can establish that the observer error

$$
\left(\tilde{\eta}(x,t),\ \tilde{\xi}(x,t)\right)=\left(\eta(x,t)-\hat{\eta}(x,t),\ \xi(x,t)-\hat{\xi}(x,t)\right)
$$

will satisfy the following PDE system:

$$
\tilde{\eta}_t(x,t)=r(x)\tilde{\eta}(1,t)-\lambda\tilde{\eta}_x(x,t)+\int_{x}^{1}f(x,y)\tilde{\eta}(y,t)dy \tag{69}
$$

$$
\begin{aligned}
\tilde{\xi}_t(x,t)=\ &\tilde{\xi}_x(x,t)+\int_{0}^{1}f_{21}(x,y)\tilde{\eta}(y,t)dy\\
&+g_3(x)\tilde{\xi}(0,t)+\int_{0}^{x}f_{22}(x,y)\tilde{\xi}(y,t)dy\\
&+h(x)\tilde{\eta}(x,t)+s(x)\tilde{\eta}(1,t)
\end{aligned} \tag{70}
$$

$$
\tilde{\eta}(0,t)=c_1\tilde{\xi}(0,t)+\int_{0}^{1}d(y)\tilde{\eta}(y,t)dy \tag{71}
$$

$$
\tilde{\xi}(1,t)=0. \tag{72}
$$

The observer gains $r(x)$, $s(x)$ should be chosen in a manner such that the error system (69)–(72) exhibits exponential stability properties. We will solve the stabilization problem for (69)–(72) by using a two-step integral transformation approach. Much like with the control design, we seek invertible transformations from system (69)–(72) to the exponentially stable target system

$$
\tilde{\alpha}_t(x,t)=-\lambda\tilde{\alpha}_x(x,t), \tag{73}
$$

$$
\tilde{\beta}_t(x,t)=\tilde{\beta}_x(x,t), \tag{74}
$$

$$
\tilde{\alpha}(0,t)=c_1\tilde{\beta}(0,t), \tag{75}
$$

$$
\tilde{\beta}(1,t)=0. \tag{76}
$$

We begin by defining the following transformation

$$\tilde{\beta}(x,t) = \tilde{\xi}(x,t) - \int_0^x \bar{k}(x,y)\tilde{\xi}(y,t)dy - \int_0^x \bar{l}(x,y)\tilde{\eta}(y,t)dy - \int_x^1 \bar{\rho}(x,y)\tilde{\eta}(y,t)dy \tag{77}$$

which admits the following intermediate target system:

$$\tilde{\eta}_t(x,t) = r(x)\tilde{\eta}(1,t) - \lambda\tilde{\eta}_x(x,t) + \int_x^1 f(x,y)\tilde{\eta}(y,t)dy \tag{78}$$

$$\tilde{\beta}_t(x,t) = \tilde{\beta}_x(x,t) \tag{79}$$

$$\tilde{\eta}(0,t) = c_1\tilde{\beta}(0,t) + \int_0^1 \bar{G}(y)\tilde{\eta}(y,t)dy \tag{80}$$

$$\tilde{\beta}(1,t) = 0 \tag{81}$$

where $\bar{G}(y) = c_1\bar{\rho}(0,y) + d(y)$. The gain $s(x)$ can then be determined to be

$$s(x) = \int_0^x \bar{k}(x,y)s(y)dy + \int_0^x \bar{l}(x,y)r(y)dy - \lambda\rho(x,1) + \int_x^1 \bar{\rho}(x,y)r(y)dy \tag{82}$$

where $\bar{k}(x,y)$, $\bar{l}(x,y)$, $\bar{\rho}(x,y)$ obey the following PDEs:

$$\bar{k}_x(x,y) + \bar{k}_y(x,y) = \int_y^x \bar{k}(x,z)f_{22}(z,y)dz - f_{22}(x,y) \tag{83}$$

$$\bar{l}_x(x,y) - \lambda\bar{l}_y(x,y) = \int_0^x \bar{k}(x,z)f_{21}(z,y)dz + \bar{k}(x,y)h(y) + \int_y^x \bar{l}(x,z)f(z,y)dz - f_{21}(x,y) + \lambda\bar{l}(x,0)d(y) \tag{84}$$

$$\bar{\rho}_x(x,y) - \lambda\bar{\rho}_y(x,y) = \int_0^x \left[\bar{k}(x,z)f_{21}(z,y) + \bar{l}(x,z)f(z,y)\right]dz - \int_x^y \bar{\rho}(x,z)f(z,y)dz - f_{21}(x,y) + \lambda\bar{l}(x,0)d(y) \tag{85}$$

$$\bar{\rho}(x,x) + \lambda\bar{\rho}(x,x) = h(x) + \lambda\bar{l}(x,x) + \bar{l}(x,x) \tag{86}$$

$$c_1\lambda\bar{l}(x,0) = g_3(x) + \bar{k}(x,0) - \int_0^x \bar{k}(x,y)g_3(y)dy \tag{87}$$

$$\bar{k}(1,y) = 0, \quad \bar{l}(1,y) = 0. \tag{88}$$

We now define the second transformation as

$$\tilde{\alpha}(x,t) = \tilde{\eta}(x,t) - \int_x^1 \bar{q}(x,y)\tilde{\eta}(y,t)dy \tag{89}$$

with the associated inverse transformation

$$\tilde{\eta}(x,t) = \tilde{\alpha}(x,t) + \int_x^1 \bar{p}(x,y)\tilde{\alpha}(y,t)dy$$

mapping the intermediate target system (78)–(81) into the cascade of two first-order hyperbolic PDEs:

$$\tilde{\alpha}_t(x,t) = -\lambda\tilde{\alpha}_x(x,t), \quad \tilde{\beta}_t(x,t) = \tilde{\beta}_x(x,t),$$
$$\tilde{\alpha}(0,t) = c_1\tilde{\beta}(0,t), \quad \tilde{\beta}(1,t) = 0.$$

The gain $r(x)$ is determined to be:

$$r(x) = \int_x^1 \bar{q}(x,y)r(y)dy - \lambda\bar{q}(x,1) \tag{90}$$

with the kernel $\bar{q}(x,y)$ obeying the PDE

$$\lambda\bar{q}_x(x,y) + \lambda\bar{q}_y(x,y) = f(x,y) - \int_x^y \bar{q}(x,z)f(z,y)dz \tag{91}$$

$$\bar{q}(0,y) = \bar{G}(y). \tag{92}$$

The well-posedness of kernels $\bar{q}(x,y)$ and $\bar{p}(x,y)$ is given in Krstic and Smyshlyaev (2008).

The problem then reduces to showing that the PDE system (83)–(88) is well-posed. Once a solution of $\bar{k}(x,y)$, $\bar{l}(x,y)$, $\bar{\rho}(x,y)$ and $\bar{q}(x,y)$ is found, the observer gain $r(x)$ can be obtained from (90), and $s(x)$ is obtained from (82) and $r(x)$. Intuitively, one can expect this cascaded structure of the gains $r(x)$ and $s(x)$ due to the configuration of the model.

The following lemma establishes the well-posedness of the PDE (83)–(88).

Lemma 4. *The kernel PDE system (83)–(88) has a unique solution in $C^1(D_1)\times C^1(D)\times C^1(D)$, where $D_1 = \{(x,y)|0 \le x \le y \le 1\}$, $D = \{(x,y)|0 \le y \le x \le 1\}$. The kernels $\bar{m}(x,y)$, $\bar{n}(x,y)$, $\bar{\sigma}(x,y)$ of inverse transformation*

$$\tilde{\xi}(x,t) = \tilde{\beta}(x,t) + \int_0^x \bar{m}(x,y)\tilde{\beta}(y,t)dy + \int_0^x \bar{n}(x,y)\tilde{\alpha}(y,t)dy + \int_x^1 \bar{\sigma}(x,y)\tilde{\alpha}(y,t)dy \tag{93}$$

likewise admits a unique $C^1(D_1)\times C^1(D)\times C^1(D)$ function.

This lemma can be proved by the method of characteristics and successive approximations approach. The proof directly follows from the proof of Lemma 3, and thus we omit the details here.

The exponential stability (in the L^2-sense) of the target system (73)–(76) and invertibility of transformation (77) and (89) imply the exponential stability of error system (69)–(72) (in L^2-sense). The result is formulated in the following theorem.

Theorem 5. *Let $\bar{\rho}(x,y)$, $\bar{k}(x,y)$, $\bar{l}(x,y)$ be the solution to (83)–(88) and let $\bar{q}(x,y)$ be the solution to (91)–(92). The observer gains $r(x)$ and $s(x)$ are given by (90)* and *(82). Then the error system (69)–(72) is exponentially stable in the L^2-sense.*

Due to the separation principle for linear systems, the exponentially convergent observer is independent of the control input. Thus, the observer can be combined with the backstepping control law derived earlier in the paper to solve the output-feedback problem.

Theorem 6. *Let $\bar{\rho}(x,y)$, $\bar{k}(x,y)$, $\bar{l}(x,y)$ be the solution to (83)–(88) and $\bar{q}(x,y)$ be the solution to (91)–(92). The observer gains $r(x)$ and $s(x)$ are given by (90)* and *(82). The system consisting of (1)–(3) with the control law*

$$U(t) = -\frac{c_2}{c_1}\hat{\eta}(1-\lambda,t) + \int_0^1 \mathscr{K}(x,y)\hat{\xi}(y,t)dy + \int_0^1 \mathscr{L}(x,y)\hat{\eta}(y,t)dy \tag{94}$$

coupled with the observer system (65)–(68) is exponentially stable in the L^2-sense, where $\mathscr{K}(x,y)$ and $\mathscr{L}(x,y)$ are given by (40) and *(41) respectively.*

Proof. The transformation

$$\hat{\alpha}(x,t) = \hat{\eta}(x,t) - \int_x^1 q(x,y)\hat{\eta}(y,t)dy \tag{95}$$

and

$$\hat{z}(x,t) = w(x,t) - \int_0^x \theta(x,y)w(y,t)dy$$
$$= c_1\hat{\xi}(x,t) - c_1\int_0^x \left[k(x,y)\hat{\xi}(y,t) + l(x,y)\hat{\eta}(y,t)\right]dy$$

$$
\begin{aligned}
&+c_1\int_0^x l(x,y)\int_y^1 q(y,\hat z)\hat\eta(z,t)dzdy\\
&-c_1\int_x^1 \rho(x,y)\left[\hat\eta(y,t)-\int_y^1 q(y,\hat z)\hat\eta(z,t)dz\right]dy\\
&+c_2\hat\eta(1-\lambda x,t)-c_2\int_{1-\lambda x}^1 q(1-\lambda x,\hat z)\hat\eta(z,t)dz\\
&+\int_{\lambda x}^1 G(z)\left[\hat\eta(z-\lambda x,t)-\int_{\hat z-\lambda y}^1 q(z-\lambda x,s)\hat\eta(s,t)ds\right]dz\\
&-\int_0^x \theta(x,y)\left[c_1\hat\xi(y,t)+c_2\hat\eta(1-\lambda y,t)\right]dy\\
&+\int_0^x \theta(x,y)\int_0^y\left[k(y,\hat z)\hat\xi(z,t)+l(y,\hat z)\hat\eta(z,t)\right]dzdy\\
&-\int_0^x \theta(x,y)\int_0^y l(y,\hat z)\int_z^1 q(z,s)\hat\eta(s,t)dsdzdy\\
&+\int_0^x \theta(x,y)\int_y^1 \rho(y,\hat z)\hat\eta(z,t)dzdy\\
&-\int_0^x \theta(x,y)\int_y^1 \rho(y,\hat z)\int_z^1 q(z,s)\hat\eta(s,t)dsdzdy\\
&+c_2\int_0^x \theta(x,y)\int_{1-\lambda y}^1 q(1-\lambda y,\hat z)\hat\eta(z,t)dzdy\\
&+\int_0^x \theta(x,y)\int_{\lambda y}^1 G(z)\int_{\hat z-\lambda y}^1 q(z-\lambda y,s)\hat\eta(s,t)dsdzdy\\
&-\int_0^x \theta(x,y)\int_{\lambda y}^1 G(z)\hat\eta(z-\lambda y,t)dady
\end{aligned}
\tag{96}
$$

maps (65)–(68) into

$$
\begin{aligned}
\hat\alpha_t(x,t)&=-\lambda\hat\alpha_x(x,t)+\left[r(x)+\int_x^1 q(x,y)r(y)dy\right]\tilde\eta(1,t)\\
\hat z_t(x,t)&=\hat z_x(x,t)+\left[E_2(x)-\int_0^x \theta(x,y)E_2(y)dy\right]\tilde\eta(1,t)\\
\hat\alpha(0,t)&=\hat z(0,t),\qquad \hat z(1,t)=0,
\end{aligned}
$$

where

$$
\begin{aligned}
E_1(x)=\ &r(x)-\int_0^x k(x,y)s(y)dy\\
&-\int_0^x l(x,y)\left(r(y)+\int_y^1 q(y,s)r(s)ds\right)dy\\
&-\int_x^1 \rho(x,y)\left(r(y)+\int_y^1 q(y,s)r(s)ds\right)dy
\end{aligned}
$$

and

$$
\begin{aligned}
E_2(x)=\ &c_1E_1(x)\\
&+c_2\left(r(1-\lambda x)+\int_{1-\lambda x}^1 q(1-\lambda x,s)r(s)ds\right)\\
&-\int_{\lambda x}^1 G(y)\left(r(y-\lambda x)+\int_{y-\lambda x}^1 q(y-\lambda x,s)r(s)ds\right)dy.
\end{aligned}
$$

From Theorem 5, we can note that the $(\tilde\eta,\tilde\xi)$ system (69)–(72) is exponentially stable. The $(\tilde\eta,\tilde\xi)$ system acts as the input to the exponentially stable $(\hat\alpha,\hat z)$ system. Thus, the cascade interconnection of these two exponentially stable systems $(\hat\alpha,\hat z,\tilde\eta,\tilde\xi)$ will likewise be exponentially stable in L^2-sense. By applying the invertible coordinate transformation (95)–(96), one can show exponential stability of $(\hat\eta,\hat\xi,\tilde\eta,\tilde\xi)$. It directly follows that $(\eta,\xi,\hat\eta,\hat\xi)$ is exponentially stable. ■

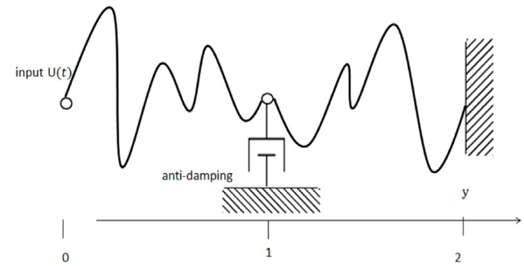

Fig. 4. String with anti-damping in the middle of domain.

5. Application to the point-damped string

In this section, we show that our result on our model can be applied to solve the point-damped string problem. The damped point can be any arbitrary interior point, which is an important extension to the existing literature where the point-damper can only apply to a specific subset of rational points in Guo and Jin (2010) and Liu (1988) (see Fig. 4).

5.1. Stabilization of the point damped string

The string model with a point damper is governed by the following PDE

$$u_{tt}(y,t)=u_{yy}(y,t),\tag{97}$$
$$u(l^-,t)=u(l^+,t),\tag{98}$$
$$u_y(l^-,t)-u_y(l^+,t)=qu_t(l,t),\tag{99}$$
$$u(2,t)=0,\tag{100}$$
$$u_y(0,t)=U(t)\tag{101}$$

where $y\in[0,2]$, $l\in[0,1]$, $t>0$, $u(y,t)$ is the displacement of the string, $U(t)$ is the control, $q>0$, $q\neq l^2+(2-l)^2$ is a constant number.

We introduce a new variable $v(x,t)=[v_1(x,t),v_2(x,t)]^T$ for $x\in[0,1]$ and $t\geq 0$, where

$$v_1(x,t)=u(lx,t),\quad v_2(x,t)=u(2-(2-l)x,t),\quad 0\leq x\leq 1.$$

Then the system (97)–(101) is transformed into an equivalent system:

$$v_{tt}(x,t)=\frac{1}{\rho_i^2}v_{xx}(x,t),\tag{102}$$
$$v_1(1,t)=v_2(1,t),\tag{103}$$
$$\rho_1v_{1x}(1,t)+\rho_2v_{2x}(1,t)=qv_{1,t}(1,t),\tag{104}$$
$$v_{1,x}(0,t)=U(t),\tag{105}$$
$$v_2(0,t)=0\tag{106}$$

where $x\in[0,1]$, $t>0$, $\rho_1=l$, $\rho_2=2-l$, and $v_1(x,t)$, $v_2(x,t)$ propagate with different wave speeds. Moreover, the boundary (104) makes the system (102)–(106) anti-stable.

We proceed to transform system (102)–(106) into a 4×4 system of first-order transport equations which convect in opposing directions, to which our method can be applied (with some slight manipulations). To achieve this, we define the following transformation:

$$\bar\varphi_i(x,t)=v_{i,x}(x,t)+\rho_iv_{i,t}(x,t),\quad i=1,2,\tag{107}$$
$$\bar\psi_i(x,t)=v_{i,x}(x,t)-\rho_iv_{i,t}(x,t),\quad i=1,2\tag{108}$$

together with the inverse given by

$$v_{i,x}=\frac{\bar\varphi_i(x,t)+\bar\psi_i(x,t)}{2},\qquad v_{i,t}=\frac{\bar\varphi_i(x,t)-\bar\psi_i(x,t)}{2\rho_i}.\tag{109}$$

Let $W(t) = U(t) - \rho_1 v_{1,t}(0,t)$, where W is treated as a pseudo-control to be designed. Eqs. (102)–(106) are transformed into

$$\bar{\varphi}_{i,t}(x,t) = \frac{1}{\rho_i}\bar{\varphi}_{i,x}(x,t), \quad i = 1,2, \tag{110}$$

$$\bar{\psi}_{i,t}(x,t) = -\frac{1}{\rho_i}\bar{\psi}_{i,x}(x,t), \quad i = 1,2, \tag{111}$$

$$\bar{\varphi}_1(1,t) = \frac{\rho_2^2-\rho_1^2-q}{\rho_1^2+\rho_2^2-q}\bar{\psi}_1(1,t) - \frac{2\rho_1\rho_2}{\rho_1^2+\rho_2^2-q}\bar{\psi}_2(1,t) \tag{112}$$

$$\bar{\varphi}_2(1,t) = \frac{-2\rho_1\rho_2}{\rho_1^2+\rho_2^2-q}\bar{\psi}_1(1,t) + \frac{\rho_1^2-\rho_2^2-q}{\rho_1^2+\rho_2^2-q}\bar{\psi}_2(1,t) \tag{113}$$

$$\bar{\psi}_2(0,t) = \bar{\varphi}_2(0,t) \tag{114}$$

$$\bar{\psi}_1(0,t) = W(t). \tag{115}$$

By defining $\varphi_i(x,t) = \bar{\varphi}_i(1-x,t)$ and $\psi_i(x,t) = \bar{\psi}_i(1-x,t)$, (110)–(115) can also be written as

$$\varphi_{i,t}(x,t) = -\frac{1}{\rho_i}\varphi_{i,x}(x,t), \quad i = 1,2,$$

$$\psi_{i,t}(x,t) = \frac{1}{\rho_i}\psi_{i,x}(x,t), \quad i = 1,2,$$

$$\varphi_1(0,t) = \frac{\rho_2^2-\rho_1^2-q}{\rho_1^2+\rho_2^2-q}\psi_1(0,t) - \frac{2\rho_1\rho_2}{\rho_1^2+\rho_2^2-q}\psi_2(0,t)$$

$$\varphi_2(0,t) = -\frac{2\rho_1\rho_2}{\rho_1^2+\rho_2^2-q}\psi_1(0,t) + \frac{\rho_1^2-\rho_2^2-q}{\rho_1^2+\rho_2^2-q}\psi_2(0,t)$$

$$\psi_2(1,t) = \varphi_2(1,t)$$

$$\psi_1(1,t) = W(t).$$

We combine $\varphi_2(x,t)$ and $\psi_2(x,t)$ into one state using the following definition

$$\eta(x,t) = \begin{cases} \varphi_2(2x,t), & x \in [0,\frac{1}{2}], \\ \psi_2(2-2x,t), & x \in [\frac{1}{2},1]. \end{cases}$$

Denoting $\gamma(x,t) = \varphi_1(x,t)$ and $\xi(x,t) = \psi_1(x,t)$, we arrive at the system

$$\gamma_t(x,t) = -\frac{1}{\rho_1}\gamma_x(x,t) \tag{116}$$

$$\eta_t(x,t) = -\frac{1}{2\rho_2}\eta_x(x,t) \tag{117}$$

$$\xi_t(x,t) = \frac{1}{\rho_1}\xi_x(x,t) \tag{118}$$

$$\gamma(0,t) = \frac{\rho_2^2-\rho_1^2-q}{\rho_1^2+\rho_2^2-q}\xi(0,t) - \frac{2\rho_1\rho_2}{\rho_1^2+\rho_2^2-q}\eta(1,t) \tag{119}$$

$$\eta(0,t) = -\frac{2\rho_1\rho_2}{\rho_1^2+\rho_2^2-q}\xi(0,t) + \frac{\rho_1^2-\rho_2^2-q}{\rho_1^2+\rho_2^2-q}\eta(1,t) \tag{120}$$

$$\xi(1,t) = W(t). \tag{121}$$

Thus, the joint string with point-damping model can be transformed into a $(2+1)\times(2+1)$ system, to which our controller can be directly applied.

One can immediately see that the (γ,η,ξ) system can be broken into a cascade, where (η,ξ) act as an input to the first-order hyperbolic PDE γ via the boundary condition $\gamma(0,t)$. Thus, if we can stabilize (η,ξ), then the composite system (γ,η,ξ) will be stabilized as well, from the cascading structure. We restate sub-system (η,ξ) below

$$\eta_t(x,t) = -\frac{1}{2\rho_2}\eta_x(x,t) \tag{122}$$

$$\xi_t(x,t) = \frac{1}{\rho_1}\xi_x(x,t) \tag{123}$$

$$\eta(0,t) = \frac{-2\rho_1\rho_2}{\rho_1^2+\rho_2^2-q}\xi(0,t) + \frac{\rho_1^2-\rho_2^2-q}{\rho_1^2+\rho_2^2-q}\eta(1,t) \tag{124}$$

$$\xi(1,t) = W(t) \tag{125}$$

where $x \in [0,1]$, $t > 0$, and $W(t)$ is the controller.

By applying the result from Section 3, the transformation $(\gamma,\eta,\xi) \to (\gamma,\alpha,z)$ is determined to be:

$$\alpha(x,t) = \eta(x,t),$$

$$z(x,t) = \frac{-2\rho_1\rho_2}{\rho_1^2+\rho_2^2-q}\xi(x,t) + \frac{\rho_1^2-\rho_2^2-q}{\rho_1^2+\rho_2^2-q}\eta\left(1-\frac{\rho_1 x}{2\rho_2},t\right).$$

We can also write $u \to (\gamma,\alpha,z)$ to be

$$\gamma(x,t) = u_x(l-lx,t) + lu_t(l-lx,t), \tag{126}$$

$$\alpha(x,t) = \begin{cases} u_x(4x-2lx+l,t) \\ \quad +\rho_2 u_t(4x-2lx+l,t), & x \in [0,\frac{1}{2}], \\ u_x(4-4x+2lx-l,t) \\ \quad -\rho_2 u_t(4-4x+2lx-l,t), & x \in [\frac{1}{2},1], \end{cases} \tag{127}$$

and

$$z(x,t) = \begin{cases} -\frac{2\rho_1\rho_2}{\rho_1^2+\rho_2^2-q}\Big(u_x(l-lx,t) - lu_t(l-lx,t)\Big) \\ +\frac{\rho_1^2-\rho_2^2-q}{\rho_1^2+\rho_2^2-q}\Big(u_x(4-l-lx,t) \\ +\rho_2 u_t(4-l-lx,t)\Big), x \in [\frac{2-l}{l},1], \\ -\frac{2\rho_1\rho_2}{\rho_1^2+\rho_2^2-q}\left(u_x(l-lx,t) - lu_t(l-lx,t)\right) \\ +\frac{\rho_1^2-\rho_2^2-q}{\rho_1^2+\rho_2^2-q}\Big(u_x(l+lx,t) \\ -\rho_2 u_t(l+lx,t)\Big), x \in [0,\frac{2-l}{l}] \end{cases} \tag{128}$$

The pseudo-controller $W(t)$ can be determined to be

$$\begin{aligned} W(t) &= -\frac{\rho_1^2+\rho_2^2-q}{2\rho_1\rho_2}\eta\left(1-\frac{1}{4-2l},t\right) \\ &= -\frac{\rho_1^2+\rho_2^2-q}{2\rho_1\rho_2}v_{2x}\left(\frac{2-2l}{2-l},t\right), \end{aligned} \tag{129}$$

which will admit the controller

$$\begin{aligned} U(t) &= -\frac{\rho_1^2+\rho_2^2-q}{2\rho_1\rho_2}v_{2x}(0,t) + v_{1t}(0,t) \\ &= \frac{\rho_1^2+\rho_2^2-q}{2\rho_1\rho_2}u_x(2l,t) + u_t(0,t), \end{aligned} \tag{130}$$

It has been previously noted that if the control law $W(t)$ makes the sub-system (η,ξ) exponentially stable, then the same control law stabilizes (γ,η,ξ) exponentially. The following corollary directly follows from the application of Theorem 2.

Corollary 7. *Suppose that $q > 0$ and $q \neq l^2 + (2-l)^2$. Then the closed-loop system (97)–(101) with the control law $U(t)$ given by (130) is exponentially stable in $\mathcal{H} = H^1(0,2)\times L^2(0,2)$.*

Note that the due to the transformation (107)–(108), the stability in L^2 in the spatial derivative will admit H^1 stability in the original wave coordinates.

5.2. Numerical simulation of the string with midway point-damping

In this section, we apply our theory to the point-damped string modeled by (97)–(101) where the point-damper is located

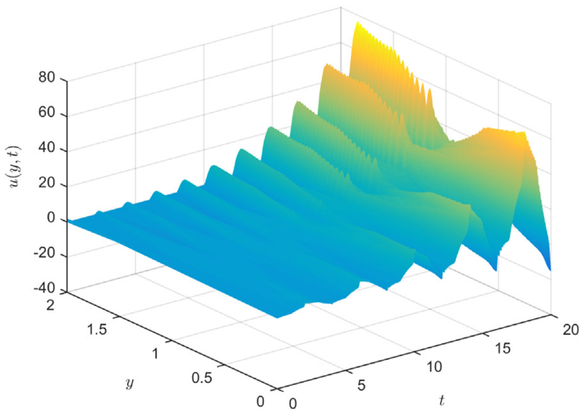

Fig. 5. $u(x, t)$ of the open-loop system (97)–(101) with $U = 0$.

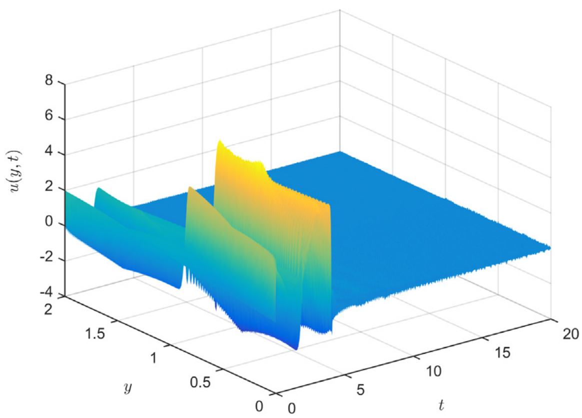

Fig. 6. $u(x, t)$ of the closed-loop system (97)–(101) with the control law U given by (131).

midway, a special case when $l = 1$. The controller, as per (130), is found to be

$$U(t) = \frac{2-q}{2} u_x(2, t) + u_t(0, t). \tag{131}$$

The finite difference method is used for the PDEs (97)–(101) to numerically compute the state in the open-loop case ($U = 0$) and in the closed-loop case with the control law U given by (131). The parameters are chosen to be $q = 6$ and $u(x, 0) = x$. Fig. 5 shows that the displacement $u(x, t)$ for the open-loop system is unstable. Fig. 6 shows that the displacement $u(x, t)$ in the closed-loop system converges to zero, exhibiting the effectiveness of applying closed-loop control.

5.3. Comparison with Guo and Jin's methodology

We present the differences between our methodology and Guo and Jin's methodology in Guo and Jin (2010) applied to the same midway point-damped string (97)–(101). We have taken the liberty of transforming Guo and Jin's target system (in wave variables) into a $(2 + 1) \times (2 + 1)$ first-order hyperbolic system for a more direct comparison:

$$\gamma_t(x, t) = -\gamma_x(x, t) \tag{132}$$

$$\alpha_t(x, t) = -\frac{1}{2}\alpha_x(x, t) \tag{133}$$

$$z_t(x, t) = z_x(x, t) \tag{134}$$

$$\gamma(0, t) = \frac{c}{2+c} z(0, t) - \frac{2}{2+c}\alpha(1, t) \tag{135}$$

Table 1
Comparison of controller $U(t)$.

Variable	Our controller	Guo and Jin's controller
$u_t(0, t)$	1	$\frac{2(q+c)}{4+qc}$
$u_x(2, t)$	$\frac{2-q}{2}$	$-\frac{q(q+c)}{4+qc}$

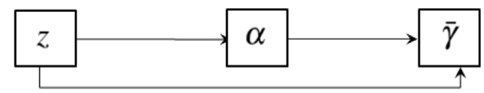

Fig. 7. Flow diagram of our target system.

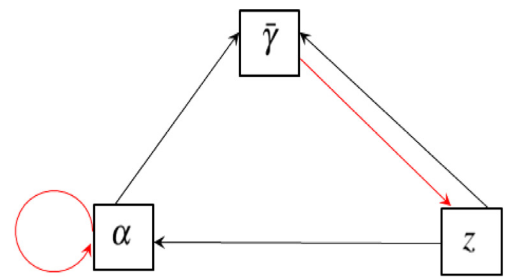

Fig. 8. Flow diagram of Guo and Jin's target system (132)–(137).

$$\alpha(0, t) = -\frac{2}{2+c} z(0, t) + \frac{c}{2+c}\alpha(1, t) \tag{136}$$

$$z(1, t) = -\gamma(1, t) \tag{137}$$

with the corresponding transformation $u \to (\gamma, \alpha, z)$

$$\begin{aligned} \gamma(x, t) &= \frac{2-c}{q+2}\left(u_x(1-x, t) + u_t(1-x, t)\right) \\ &\quad - \frac{q+c}{q+2}\left(u_x(x+1, t) + u_t(x+1, t)\right), \\ \alpha(x, t) &= \begin{cases} u_x(2x+1, t) + u_t(2x+1, t), & x \in [0, \frac{1}{2}] \\ u_x(3-2x, t) - u_x(3-2x, t), & x \in [\frac{1}{2}, 1] \end{cases} \\ z(x, t) &= \frac{2-c}{q+2} u_x(1-x, t) - \frac{2q+c+2}{q+2} u_t(1-x, t) \\ &\quad + \frac{q+c}{q+2}\left(u_x(x+1, t) - u_t(x+1, t)\right) \end{aligned}$$

and controller

$$U(t) = \frac{2(q+c)}{4+qc}\left(\frac{q}{2} u_x(2, t) + u_t(0, t)\right) \tag{138}$$

where $c > 0$, $c \neq 2$ is the designed parameter outlined in Guo and Jin (2010).

Next, we give a comparison between ours and Guo and Jin's methodologies in Guo and Jin (2010) from three different aspects.

From Table 1, we see the control law derived in our paper maintains the same structure with Guo and Jin's control law, but have different control gains. When $q > 2$, $c > 2$ or $q < 2$, $c < 2$, Guo and Jin's control law is more effective, and when $q > 2$, $c < 2$ or $q > 2$, $c < 2$, our control law is more effective.

From Figs. 7 and 8, it is easily seen that Guo and Jin's target system (132)–(137) still exhibits recirculation/feedback behavior, while our target system is a simple cascaded system whose exponential stability property can easily be shown. This is an important distinction to make, as one can synthesize a composite Lyapunov function from each subsystem property. In the case of Guo and Jin's target system, one must attempt to use a small-gain type approach, due to the feedback structure of the system. Often times, the stability of these feedback structured systems is not immediately clear. In contrast, in our result (as a cascade), one can simply analyze the Lyapunov functions of each subsystem independently, and compose them as a linear combination to synthesize the composite Lyapunov function. Such an approach for cascaded systems can be systematically applied, leading to a far more obvious and less complex result for stability analysis. Moreover, as a cascade of first-order *hyperbolic* systems, finite-time convergence can be achieved.

6. Concluding remarks

This paper deals with coupled first-order hyperbolic equations with recirculation terms, which are motivated by the midway point-damped string model. A new decoupling transformation together with the well-studied backstepping transformation is presented, allowing us to design a controller to make the closed-loop system exponentially stable. The result is then applied to both the midway and non-midway point damped string models, and comparisons made to similar work found in Guo and Jin (2010). This paper thus presents novel work on the coupled PDEs with new coupling structures.

Appendix

First, we derive the kernel function of $q(x, y)$. Taking the time and space derivative of $\alpha(x, t)$ respectively,

$$\begin{aligned}\alpha_t(x,t) &= -\lambda\eta_x(x,t) + g_1(x)\eta(1,t) + \int_x^1 f(x,y)\eta(y,t)dy \\ &\quad +\lambda q(x,1)\eta(1,t) - \lambda\int_x^1 q_y(x,y)\eta(y,t)dy \\ &\quad -\lambda q(x,x)\eta(x,t) - \int_x^1 q(x,y)g_1(y)dy\eta(1,t) \\ &\quad -\int_x^1 q(x,y)\int_y^1 f(y,z)\eta(z,t)dzdy \\ \alpha_x(x,t) &= \eta_x(x,t) + q(x,x)\eta(x,t) - \int_x^1 q_x(x,y)\eta(y,t)dy,\end{aligned}$$

we have

$$\begin{aligned}0 &= \alpha_t(x,t) + \lambda\alpha_x(x,t) \\ &= g_1(x)\eta(1,t) + \int_x^1 f(x,y)\eta(y,t)dy \\ &\quad +\lambda q(x,1)\eta(1,t) - \lambda\int_x^1 q_y(x,y)\eta(y,t)dy \\ &\quad -\int_x^1 q(x,y)g_1(y)dy\eta(1,t) - \lambda\int_x^1 q_x(x,y)\eta(y,t)dy \\ &\quad -\int_x^1\int_x^y q(x,z)f(z,y)dz\eta(y,t)dy.\end{aligned}$$

We find the kernel function of $q(x, y)$ is dictated by the following PDE:

$$\begin{aligned}\lambda q_x(x,y) + \lambda q_y(x,y) &= f(x,y) - \int_x^y q(x,z)f(z,y)dz, \\ \lambda q(x,1) &= \int_x^1 q(x,y)g_1(y)dy - g_1(x).\end{aligned}$$

Similar computation shows that $p(x, y)$ is governed by a similar PDE:

$$\begin{aligned}\lambda p_x(x,y) + \lambda p_y(x,y) &= f(x,y) + \int_x^y p(x,z)f(z,y)dz, \\ \lambda p(x,1) &= -g_1(x).\end{aligned}$$

The well-posedness of the kernel PDEs governing $q(x, y)$ and $p(x, y)$ is given in Krstic and Smyshlyaev (2008). Next, we solve for the kernels of $k(x, y)$, $l(x, y)$ and $\rho(x, y)$. By taking the time and space derivative of $\beta(x, t)$ respectively,

$$\begin{aligned}\beta_t(x,t) &= \xi_t(x,t) - \int_0^x k(x,y)\xi_t(y,t)dy \\ &\quad -\int_0^x l(x,y)\alpha_t(y,t)dy - \int_x^1 \rho(x,y)\alpha_t(y,t)dy \\ &\quad -\int_0^1 l(x,y)\alpha_t(y,t)dy \\ &= \xi_x(x,t) + \int_0^1 H(x,y)\alpha(y,t)dy + g_2(x)\alpha(1,t) \\ &\quad +\int_0^x f_{22}(x,y)\xi(y,t)dy + h(x)\alpha(x,t) \\ &\quad -\int_0^x k(x,y)h(y)\alpha(y,t)dy \\ &\quad -\int_0^x k(x,y)g_2(y)dy\alpha(1,t) \\ &\quad -k(x,x)\xi(x,t)dy + k(x,0)\xi(0,t)dy \\ &\quad +\int_0^x k_y(x,y)\xi(y,t)dy \\ &\quad -\int_0^x\int_0^x k(x,z)H(z,y)dz\alpha(y,t)dy \\ &\quad -\int_x^1\int_0^x k(x,z)H(z,y)dz\alpha(y,t)dy \\ &\quad -\int_0^x\int_y^x k(x,z)f_{22}(z,y)dz\xi(y,t)dy \\ &\quad -\int_0^x k(x,y)g_3(y)dy\xi(0,t) + g_3(x)\xi(0,t) \\ &\quad +\lambda l(x,x)\alpha(x,t) - \lambda l(x,0)c_1\xi(0,t) \\ &\quad -\lambda l(x,0)c_2\alpha(1,t) - \lambda\int_0^x l_y(x,y)\alpha(y,t)dy \\ &\quad -\lambda l(x,0)\int_0^1 G(y)\alpha(y,t)dy \\ &\quad +\lambda\rho(x,1)\alpha(1,t) - \lambda\rho(x,x)\alpha(x,t) \\ &\quad -\lambda\int_x^1 \rho_y(x,y)\alpha(y,t)dy \\ &\quad +\int_x^1 h(x)p(x,y)\alpha(y,t)dy \\ &\quad +\int_0^x\int_y^x k(x,z)h(z)p(z,y)dz\alpha(y,t)dy \\ \beta_x(x,t) &= \xi_x(x,t) - \int_0^x k_x(x,y)\xi(y,t)dy - k(x,x)\xi(x,t) \\ &\quad -l(x,x)\alpha(x,t) - \int_0^x l_x(x,y)\alpha(y,t)dy \\ &\quad +\rho(x,x)\alpha(x,t) - \int_x^1 \rho_x(x,y)\alpha(y,t)dy,\end{aligned}$$

we have

$$\begin{aligned}0 &= \beta_t(x,t) - \beta_x(x,t) \\ &= \int_0^1 H(x,y)\alpha(y,t)dy + \int_0^x f_{22}(x,y)\xi(y,t)dy\end{aligned}$$

$$
\begin{aligned}
&+ g_2(x)\alpha(1,t) - \int_0^x k(x,y)h(y)\alpha(y,t)dy \\
&+ h(x)\alpha(x,t) - \int_0^x k(x,y)g_2(y)dy\alpha(1,t) \\
&+ k(x,0)\xi(0,t) + \int_0^x k_y(x,y)\xi(y,t)dy \\
&- \int_0^x \int_0^x k(x,z)H(z,y)dz\alpha(y,t)dy \\
&- \int_x^1 \int_0^x k(x,z)H(z,y)dz\alpha(y,t)dy \\
&- \int_0^x \int_y^x k(x,z)f_{22}(z,y)dz\xi(y,t)dy \\
&+ \lambda l(x,x)\alpha(x,t) - \lambda l(x,0)c_1\xi(0,t) \\
&- \lambda l(x,0)c_2\alpha(1,t) - \int_0^x k(x,y)g_3(y)dy\xi(0,t) \\
&+ g_3(x)\xi(0,t) - \lambda l(x,0)\int_0^1 G(y)\alpha(y,t)dy \\
&- \lambda \int_0^x l_y(x,y)\alpha(y,t)dy + \lambda\rho(x,1)\alpha(1,t) \\
&- \lambda\rho(x,x)\alpha(x,t) - \lambda\int_x^1 \rho_y(x,y)\alpha(y,t)dy \\
&+ \int_0^x k_x(x,y)\xi(y,t)dy + l(x,x)\alpha(x,t) \\
&+ \int_0^x l_x(x,y)\alpha(y,t)dy + \int_x^1 \rho_x(x,y)\alpha(y,t)dy \\
&- \rho(x,x)\alpha(x,t) + \int_x^1 h(x)p(x,y)\alpha(y,t)dy \\
&+ \int_0^x \int_y^x k(x,z)h(z)p(z,y)dz\alpha(y,t)dy.
\end{aligned}
$$

Therefore, the kernels $\rho(x,y)$, $l(x,y)$, $k(x,y)$ must satisfy

$$
\begin{aligned}
\rho_x(x,y) - \lambda\rho_y(x,y) &= -H(x,y) + \int_0^x k(x,z)H(z,y)dz \\
&\quad + \lambda l(x,0)G(y) - h(x)p(x,y), \\
k_y(x,y) + k_x(x,y) &= \int_y^x k(x,z)f_{22}(z,y)dz - f_{22}(x,y), \\
l_x(x,y) - \lambda l_y(x,y) &= -H(x,y) + k(x,y)h(y) + \lambda l(x,0)G(y) \\
&\quad + \int_0^x k(x,z)H(z,y)dz \\
&\quad - \int_y^x k(x,z)h(z)p(z,y)dz, \\
\lambda\rho(x,1) &= -g_2(x) + \int_0^x k(x,y)g_2(y)dy \\
&\quad + \lambda l(x,0)c_2, \\
k(x,0) &= \lambda l(x,0)c_1 + \int_0^x k(x,y)g_3(y)dy \\
&\quad - g_3(x), \\
\lambda l(x,x) + l(x,x) &= \rho(x,x) + \lambda\rho(x,x) - h(x), \\
\rho(0,y) &= 0
\end{aligned}
$$

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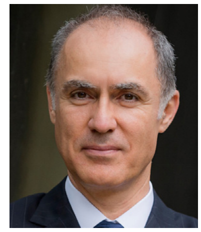

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