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Brief paper

Delay-adaptive control for linear systems with distributed input delays ☆

Yang Zhu [a], Miroslav Krstic [b], Hongye Su [c,*]

[a] School of Electrical Engineering, Tel-Aviv University, Israel
[b] Department of Mechanical and Aerospace Engineering, University of California, San Diego, USA
[c] State Key Laboratory of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, China

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ABSTRACT

A majority of existing literature on adaptive method of time-delay systems concentrate on uncertainty in plant parameters or discrete input delays. This paper proposes a systematic adaptive control approach to solve stabilization problems of linear systems with unknown distributed input delays. Under the rescaled unity-interval notation, the uncertain delay leads the input vector to consist of unknown functions and unknown parameters as well. To resolve the coexistent uncertainties in delay and input vector, a reduction-based change of variable and a backstepping–forwarding transformation of the finite-dimensional plant state and the infinite-dimensional actuator state are introduced. Making use of these conversions, the certainty-equivalence-based control law and the Lyapunov-based update law are developed for adaptive stabilization.

1. Introduction

1.1. Literature review

This year is the 60th anniversary of Otto J. Smith's 1959 publication (Smith, 1959) of a control design idea for the compensation of input delays, which is commonly referred to as the Smith predictor. Over the past 60 years, the predictor feedback has been demonstrated to be quite effective in compensating large delays, and major breakthroughs have been successively reported, such as the framework of the "reduction" approach in Artstein (1982). The standard predictor-based compensation for input delay has the premise that the delay value of every actuator channel is known. However, the fact is that such a prior knowledge of delay length may be hard to acquire in practice. Early work on adaptive control of time-delay systems mainly focused on the uncertainty of plant parameters rather than actuator delays (Bekiaris-Liberis, Jankovic, & Krstic, 2013; Niculescu & Annaswamy, 2003; Ortega & Lozano, 1988; Yildiz, Annaswamy, Kolmanovsky, & Yanakiev, 2010; Zhou, Wen, & Wang, 2009). One could argue that the situation with a highly uncertain delay, even if the plant parameters are known, is a more difficult problem than all of delay-free adaptive control of the 1970–1990s.

Due to the infinite-dimensional nature of time-delay systems, conventional adaptive control methods for finite-dimensional ordinary differential equation (ODE) systems (Goodwin & Sin, 2014) cannot be directly applied to address uncertain delays. Instead, borrowing ideas from adaptive control approaches for infinite-dimensional partial differential equation (PDE) systems (Anfinsen & Aamo, 2019; Smyshlyaev & Krstic, 2010), a few of progresses on adaptive control to handle unknown discrete input delays were recently reported in Bresch-Pietri, Chauvin, and Petit (2012), Bresch-Pietri and Krstic (2009), Zhu, Krstic, and Su (2017, 2018a, 2018b, 2019) and Zhu, Su, and Krstic (2015).

Unfortunately, the results for unknown discrete input delays (Bresch-Pietri et al., 2012; Bresch-Pietri & Krstic, 2009; Zhu et al., 2017, 2018a, 2018b, 2019, 2015) are not applicable to the case of unknown distributed input delays, as the finite-dimensional state of the plant and the infinite-dimensional actuator state are not in the strict-feedback form. References (Bekiaris-Liberis & Krstic, 2011) provided a novel backstepping–forwarding transformation for stability analysis of linear systems with distributed input delays. However, it assumes the delay to be known.

1.2. Contribution and organization

In this paper, we proposes an adaptive control scheme to deal with stabilization problems of linear systems with unknown

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* Corresponding author.
E-mail addresses: yangzhu@mail.tau.ac.il (Y. Zhu), krstic@ucsd.edu (M. Krstic), hysu@iipc.zju.edu.cn (H. Su).

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distributed input delays. To lay a foundation for adaptive stabilization, a predictor-based feedback scheme in the rescaled unity-interval notation under the known delay is provided in Section 2, in which the control scheme is parameterizable in delay and the boundary of spatial variable of PDE is fixed. Section 3 handles the unknown delay. When the delay is uncertain, a key challenge is that the input vector may also contain unknown functions and unknown parameters. Thus two different cases of the input vector are taken into account: (1) the input vector is a constant vector independent of delay (Section 3.1), (2) the input vector is a continuous vector-valued function of delay (Section 3.2). To solve the coexistent uncertainties in delay and input vector, the reduction-based change of variable and the backstepping–forwarding transformation, of certainty-equivalence type, are introduced to convert the ODE–PDE cascade consisting of the finite-dimensional plant state and the infinite-dimensional actuator state into a 'target' system. Making use of these conversions, the certainty-equivalence-based control law and the Lyapunov-based update law are developed for adaptive stabilization.

Notation:

- For a finite-dimensional ODE vector $X(t)$, its Euclid norm is denoted by $|X(t)|$.
- For an infinite-dimensional PDE scalar function $u(x,t)$, $u:[0,1]\times\mathbb{R}^+\to\mathbb{R}$,

$$\|u(x,t)\| = \left(\int_0^1 u(x,t)^2dx\right)^{1/2} \qquad \blacksquare$$

2. Predictor feedback under known delay

Consider linear systems with distributed input delays as follows:

$$\dot{X}(t) = AX(t) + \int_0^D B(D-\sigma)U(t-\sigma)d\sigma \tag{1}$$

where $X(t)\in\mathbb{R}^n$ is the plant state, $U(\theta)\in\mathbb{R}$ for $\theta\in[t-D,t]$ is the actuator state, $U(t)\in\mathbb{R}$ is the control input, $D>0$ is the constant delay, and A, B are the system matrix and input vector of appropriate dimensions, respectively. The input vector $B(\cdot)$ is a continuous real-valued vector function defined on $[0,D]$. For notational simplicity, the system is assumed to be single input. The results of this paper can be straightforwardly extended to the multi-input case, when the delays are the same in each individual input channel.

In this section, we consider the simplest stabilization problem of (1) by assuming the delay D is known.

Through a multi-variable function in *rescaled unity interval* notation

$$u(x,t) = U(t+D(x-1)) = U(t-\sigma), \quad x\in[0,1], \sigma\in[0,D], \tag{2}$$

the system (1) is converted into the ODE–PDE cascade

$$\dot{X}(t) = AX(t) + D\int_0^1 B(Dx)u(x,t)dx \tag{3}$$

$$Du_t(x,t) = u_x(x,t), \quad x\in[0,1] \tag{4}$$

$$u(1,t) = U(t) \tag{5}$$

where $u_t(x,t)$ and $u_x(x,t)$ denote the partial derivatives of $u(x,t)$ with respect to t and x, respectively. The finite-dimensional ODE state $X(t)$ and infinite-dimensional PDE state $u(x,t)$ for $x\in[0,1]$ are assumed to be measurable. It is evident (2) is a solution of the transport PDE (4)–(5). The control objective is to stabilize (3)–(5).

The reduction-based change of variable is introduced as

$$Z(t) = X(t) + D^2\int_0^1\int_0^x e^{-AD(x-y)}B(Dy)dyu(x,t)dx \tag{6}$$

Taking the time-derivative of (6) along (3)–(5) and using the integration by parts in x, we get

$$\begin{aligned}
\dot{Z}(t) &= AX(t) + D\int_0^1 B(Dx)u(x,t)dx \\
&\quad + D\int_0^1\int_0^x e^{-AD(x-y)}B(Dy)dyu_x(x,t)dx \\
&= AX(t) + D\int_0^1 B(Dx)u(x,t)dx \\
&\quad + D\int_0^1 e^{-AD(1-y)}B(Dy)dyu(1,t) \\
&\quad - D\int_0^1 B(Dx)u(x,t)dx \\
&\quad + D^2\int_0^1\int_0^x Ae^{-AD(x-y)}B(Dy)dyu(x,t)dx \\
&= AZ(t) + D\int_0^1 e^{-AD(1-x)}B(Dx)dxU(t)
\end{aligned} \tag{7}$$

Assumption 1. For the system (3) and (7), the pair $\left(A, D\int_0^1 e^{-AD(1-x)}B(Dx)dx\right)$ is stabilizable. There exist a vector K to make $A + D\int_0^1 e^{-AD(1-x)}B(Dx)dxK$ Hurwitz. Namely, there exist matrices $P=P^T>0$ and $Q=Q^T>0$ such that

$$\begin{aligned}
&\left(A + D\int_0^1 e^{-AD(1-x)}B(Dx)dxK\right)^T P \\
&\quad + P\left(A + D\int_0^1 e^{-AD(1-x)}B(Dx)dxK\right) = -Q \qquad \blacksquare
\end{aligned} \tag{8}$$

The controller is designed as

$$U(t) = u(1,t) = KZ(t) \tag{9}$$

Thus the system (7) becomes

$$\dot{Z}(t) = A_{cl}Z(t) \tag{10}$$

where

$$A_{cl} = A + D\int_0^1 e^{-AD(1-x)}B(Dx)dxK \tag{11}$$

For stability analysis, the invertible backstepping–forwarding transformation is brought in as

$$w(x,t) = u(x,t) - Ke^{A_{cl}D(x-1)}Z(t) \tag{12}$$

Take the partial derivatives of (12) with respect to x and t, respectively,

$$w_x(x,t) = u_x(x,t) - Ke^{A_{cl}D(x-1)}A_{cl}DZ(t) \tag{13}$$

$$w_t(x,t) = u_t(x,t) - Ke^{A_{cl}D(x-1)}A_{cl}Z(t) \tag{14}$$

Multiplying (14) with D and minus (13), we have

$$Dw_t(x,t) = w_x(x,t) \tag{15}$$

Combining (10) with (15), substituting $x=1$ into (12) and utilizing (9), the target system for analysis is obtained as follows:

$$\dot{Z}(t) = A_{cl}Z(t) \tag{16}$$

$$Dw_t(x,t) = w_x(x,t), \quad x\in[0,1] \tag{17}$$

$$w(1,t) = 0 \tag{18}$$

Remark 1. As illustrated in Fig. 1, a few of conversions are employed in above control scheme. Firstly, through the multi-variable function (2), the original system with distributed input

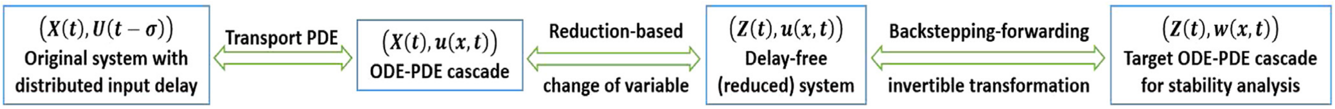

Fig. 1. Conversion between the original system (1) and target system (16)–(18).

delay (1) is represented by the ODE–PDE cascade (3)–(5). Secondly, by the reduction-based change of variable (6), the stabilization problem of ODE plant (3) is reduced to the stabilization problem of delay-free system (7). Finally, under the backstepping–forwarding transformation (12) and the control law (9), the ODE (7) and PDE (4)–(5) are transformed into the target system (16)–(18), which is convenient for stability analysis. ■

Remark 2. An alternative representation of the control scheme (2)–(18) is listed as follows:

$$u(x,t) = U(t+x-D) = U(t-\sigma), x \in [0,D], \sigma \in [0,D], \tag{19}$$

$$\dot{X}(t) = AX(t) + \int_0^D B(x)u(x,t)dx \tag{20}$$

$$u_t(x,t) = u_x(x,t), \quad x \in [0,D] \tag{21}$$

$$u(D,t) = U(t) = KZ(t) \tag{22}$$

$$Z(t) = X(t) + \int_0^D \int_0^x e^{-A(x-y)}B(y)dyu(x,t)dx \tag{23}$$

$$w(x,t) = u(x,t) - Ke^{A_{cl}(x-D)}Z(t) \tag{24}$$

$$\dot{Z}(t) = A_{cl}Z(t), \quad A_{cl} = A + \int_0^D e^{-A(D-x)}B(x)dxK \tag{25}$$

$$w_t(x,t) = w_x(x,t), \quad x \in [0,D] \tag{26}$$

$$w(D,t) = 0 \tag{27}$$

It is evident that (19)–(27) is equivalent to (2)–(18). The main difference is that (2)–(18) is parameterized in D, whereas (19)–(27) is non-parameterized in D. When the delay D is unknown and a time-varying signal $\hat{D}(t)$ is employed to estimate D, it is inconvenient for the adaptive control to be applied to (19)–(27). For example, a moving boundary $u(\hat{D}(t), t)$ appears in (24) which renders the boundary condition (27) non-homogeneous such that $w(\hat{D}(t),t) = u(\hat{D}(t),t) - u(D,t) \neq 0$. The estimate appears in the limit of integration of (23) such that $Z(t) = X(t) + \int_0^{\hat{D}(t)} \int_0^x e^{-A(x-y)}B(y)dyu(x,t)dx$, which makes it difficult to get the error $\tilde{D}(t) = D - \hat{D}(t)$ for the estimator design. That is the reason why the feedback scheme in *rescaled unity-interval* notation (2)–(18) is introduced to lay a foundation for delay-adaptive control, rather than applying adaptive control directly to (19)–(27). ■

Theorem 1. *The closed-loop system consisting of the plant (3)–(5) and the controller (9) is exponentially stable in the sense of the norm*

$$|X(t)|^2 + \int_0^1 u^2(x,t)dx \quad \blacksquare \tag{28}$$

Proof. Consider the Lyapunov candidate

$$V(t) = Z^T(t)PZ(t) + D\int_0^1 (1+x)w(x,t)^2dx \tag{29}$$

Bearing (8) and (18) in mind, taking the time derivative of (29) along the target system (16)–(18) and using the integration by parts in x, we get

$$\begin{aligned}
\dot{V}(t) &= Z^T(t)\left(A_{cl}^TP + PA_{cl}\right)Z(t) \\
&\quad + D\int_0^1 2(1+x)w(x,t)w_t(x,t)dx \\
&= -Z^T(t)QZ(t) + \int_0^1 2(1+x)w(x,t)w_x(x,t)dx \\
&= -Z^T(t)QZ(t) + 4w(1,t)^2 - 2w(0,t)^2 \\
&\quad - \int_0^1 2w(x,t)^2dx - \int_0^1 2(1+x)w_x(x,t)w(x,t)dx \\
&= -Z^T(t)QZ(t) - w(0,t)^2 - \int_0^1 w(x,t)^2dx \\
&\le -\lambda_{\min}(Q)|Z(t)|^2 - \int_0^1 w(x,t)^2dx \\
&\le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\lambda_{\max}(P)|Z(t)|^2 - \frac{1}{2D}D\int_0^1 (1+x)w(x,t)^2dx \\
&\le -\min\left\{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}, \frac{1}{2D}\right\}V(t)
\end{aligned} \tag{30}$$

It is evident that (30) implies the exponential stability in the sense of the norm $|X(t)|^2 + \int_0^1 w(x,t)^2dx$ by the comparison principle. Then making use of the inverse transformations of (6) and (12), the exponential stability in the sense of the norm $|X(t)|^2 + \int_0^1 u(x,t)^2dx$ is derived. ■

3. Predictor feedback under unknown delay

In this section, we consider that the delay D is unknown. Since D is uncertain, the vector $B(Dx)$ for $x \in [0,1]$ in (3) may contain unknown variables so that it cannot be employed directly for control design. Two different cases are taken into account as follows:

- $B(Dx)$ is a constant vector independent of Dx,
- $B(Dx)$ is a continuous vector-valued function of Dx.

3.1. $B(Dx)$ is a constant vector independent of Dx

This section addresses the relatively simple case where the vector $B(Dx)$ for $x \in [0,1]$ in (3) (i.e., $B(D-\sigma)$ for $\sigma \in [0,D]$ in (1)) is a known constant vector independent of Dx such that

$$B(Dx) = B(D-\sigma) = B \tag{31}$$

Then the system (1) is reduced to

$$\dot{X}(t) = AX(t) + B\int_0^D U(t-\sigma)d\sigma \tag{32}$$

and the ODE–PDE cascade (3)–(5) is accordingly reduced to

$$\dot{X}(t) = AX(t) + DB\int_0^1 u(x,t)dx \tag{33}$$

$$Du_t(x,t) = u_x(x,t) \tag{34}$$

$$u(1,t) = U(t) \tag{35}$$

The control objective is to stabilize (33)–(35) when the delay D is unknown.

Remark 3. On the basis of the framework (2)–(18), the key idea of certainty-equivalence-based adaptive control is to use an estimate to replace the unknown delay in (6), (9) and (12). And the Lyapunov-based update law is employed to cancel the estimation error term in the time-derivative of Lyapunov function.

A non-trivial problem is to deal with the unknown D^2 in (6). One intuitive method is to estimate D^2 by $\hat{D}(t)^2$ where $\hat{D}(t)$ is an estimate of D. The other possible method is to treat D^2 as a whole body such that $\theta = D^2$ and estimate θ by $\hat{\theta}(t)$. However, both of methods will produce the mismatch term for update law design. The available method is to regard D^2 as a product of two parameters such that $D^2 = D \cdot D_1$ where $D_1 = D$ and estimate D and D_1 with different update laws, respectively. ■

For using projector operators in the update law later, we make the following assumption.

Assumption 2. There exist known constants $\underline{D}$ and $\overline{D}$ such that

$$0 < \underline{D} \le D = D_1 \le \overline{D} \quad ■ \tag{36}$$

In order to stabilize (33)–(35), the following assumption is required.

Assumption 3. For the system (33), the pair (A, β) is stabilizable where

$$\beta = D\int_0^1 e^{-AD(1-x)}Bdx = D_1\int_0^1 e^{-AD(1-x)}Bdx \tag{37}$$

There exists a vector $K(\beta)$ to make $A + \beta K(\beta)$ Hurwitz. Namely, there exist matrices $P(\beta) = P(\beta)^T > 0$ and $Q(\beta) = Q(\beta)^T > 0$ such that

$$(A+\beta K(\beta))^T P(\beta) + P(\beta)(A+\beta K(\beta)) = -Q(\beta) \quad ■ \tag{38}$$

Denote $\hat{D}(t)$ and $\hat{D}_1(t)$ as the estimates of D and D_1 with estimation errors satisfying

$$\tilde{D}(t) = D - \hat{D}(t) \tag{39}$$

$$\tilde{D}_1(t) = D_1 - \hat{D}_1(t) = D - \hat{D}_1(t) \tag{40}$$

The delay-adaptive control scheme is listed as follows:

The control law is

$$U(t) = u(1,t) = K(\hat{\beta}(t))Z(t) \tag{41}$$

where $K(\hat{\beta}(t))$ is chosen to let

$$A_{cl}(\hat{\beta}(t)) = A + \underbrace{\hat{D}_1(t)\int_0^1 e^{-A\hat{D}(t)(1-x)}Bdx}_{\hat{\beta}(t)}K(\hat{\beta}(t)) \tag{42}$$

be Hurwitz and

$$Z(t) = X(t) + \hat{D}(t)\hat{D}_1(t)\int_0^1\int_0^x e^{-A\hat{D}(t)(x-y)}Bdyu(x,t)dx \tag{43}$$

The update laws are

$$\dot{\hat{D}}_1(t) = \gamma_{D_1}\mathrm{Proj}_{[\underline{D},\overline{D}]}\{\tau_{D_1}(t)\}, \quad \gamma_{D_1} > 0 \tag{44}$$

$$\tau_{D_1}(t) = \frac{1/gZ^T(t)P(\hat{\beta}(t))f_{D_1}(t) - \int_0^1(1+x)w(x,t)h_{D_1}(x,t)dx}{1+\Xi(t)} \tag{45}$$

$$\dot{\hat{D}}(t) = \gamma_D\mathrm{Proj}_{[\underline{D},\overline{D}]}\{\tau_D(t)\}, \quad \gamma_D > 0 \tag{46}$$

$$\tau_D(t) = \frac{1/gZ^T(t)P(\hat{\beta}(t))f_D(t) - \int_0^1(1+x)w(x,t)h_D(x,t)dx}{1+\Xi(t)} \tag{47}$$

where $P(\hat{\beta}(t))$ satisfies (38) and $g > 0$ is a designing coefficient,

$$w(x,t) = u(x,t) - K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)}Z(t) \tag{48}$$

$$\Xi(t) = Z^T(t)P(\hat{\beta}(t))Z(t) + g\int_0^1(1+x)w(x,t)^2dx \tag{49}$$

$$f_{D_1}(t) = \int_0^1 Bu(x,t)dx \tag{50}$$

$$h_{D_1}(x,t) = K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)}f_{D_1}(t) \tag{51}$$

$$\begin{aligned} f_D(t) &= \hat{D}_1(t)\int_0^1 Bu(x,t)dx \\ &\quad - \hat{D}_1(t)\int_0^1 e^{-A\hat{D}(t)(1-x)}Bdxu(1,t) \\ &\quad - \hat{D}(t)\hat{D}_1(t)\int_0^1\int_0^x Ae^{-A\hat{D}(t)(x-y)}Bdyu(x,t)dx \end{aligned} \tag{52}$$

$$h_D(x,t) = K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)}\left(f_D(t) + A_{cl}(\hat{\beta}(t))Z(t)\right) \tag{53}$$

and the projector operator is defined as

$$\mathrm{Proj}_{[\underline{D},\overline{D}]}\{\tau\} = \begin{cases} 0, & \hat{D}_1(t) = \underline{D} \ \ and \ \ \tau < 0 \\ 0, & \hat{D}_1(t) = \overline{D} \ \ and \ \ \tau > 0 \\ \tau, & else \end{cases} \tag{54}$$

$$\mathrm{Proj}_{[\underline{D},\overline{D}]}\{\tau\} = \begin{cases} 0, & \hat{D}(t) = \underline{D} \ \ and \ \ \tau < 0 \\ 0, & \hat{D}(t) = \overline{D} \ \ and \ \ \tau > 0 \\ \tau, & else \end{cases} \tag{55}$$

Theorem 2. *Consider the closed-loop system consisting of the plant (33)–(35) and the adaptive controller (41)–(55). All the states ($X(t)$, $u(x,t)$, $\hat{D}(t)$, $\hat{D}_1(t)$) of the closed-loop system are globally bounded and the regulation of $X(t)$ and $U(t)$ such that $\lim_{t\to\infty} X(t) = \lim_{t\to\infty} U(t) = 0$ is achieved.* ■

Proof. Taking the time-derivative of (43) along (33)–(35), and using the integration by parts in x, we obtain

$$\begin{aligned} \dot{Z}(t) &= AX(t) + D\int_0^1 Bu(x,t)dx + \phi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) \\ &\quad + \frac{\hat{D}(t)\hat{D}_1(t)}{D}\int_0^1\int_0^x e^{-A\hat{D}(t)(x-y)}Bdyu_x(x,t)dx \\ &= AX(t) + D\int_0^1 Bu(x,t)dx + \phi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) \\ &\quad + \frac{\hat{D}(t)\hat{D}_1(t)}{D}\int_0^1 e^{-A\hat{D}(t)(1-y)}Bdyu(1,t) \\ &\quad - \frac{\hat{D}(t)\hat{D}_1(t)}{D}\int_0^1 Bu(x,t)dx \\ &\quad + \frac{\hat{D}(t)\hat{D}_1(t)}{D}\int_0^1\int_0^x A\hat{D}(t)e^{-A\hat{D}(t)(x-y)}Bdyu(x,t)dx \\ &= AX(t) + D\int_0^1 Bu(x,t)dx + \phi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) \\ &\quad + \left(1 - \frac{\tilde{D}(t)}{D}\right)\hat{D}_1(t)\int_0^1 e^{-A\hat{D}(t)(1-x)}Bdxu(1,t) \\ &\quad - \left(1 - \frac{\tilde{D}(t)}{D}\right)\hat{D}_1(t)\int_0^1 Bu(x,t)dx \\ &\quad + \left(1 - \frac{\tilde{D}(t)}{D}\right)\hat{D}_1(t)\hat{D}(t) \\ &\quad \times \int_0^1\int_0^x Ae^{-A\hat{D}(t)(x-y)}Bdyu(x,t)dx \end{aligned} \tag{56}$$

where

$$\begin{aligned} &\phi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) \\ &= \int_0^1\int_0^x \left[\left(\dot{\hat{D}}(t)\hat{D}_1(t) + \hat{D}(t)\dot{\hat{D}}_1(t)\right)I - \hat{D}(t)\hat{D}_1(t)A\dot{\hat{D}}(t)(x-y)\right] \\ &\quad \times e^{-A\hat{D}(t)(x-y)}Bdyu(x,t)dx \end{aligned} \tag{57}$$

Under the control law (41), the formula (56) becomes

$$\dot{Z}(t) = A_{cl}(\hat{\beta}(t))Z(t) + \tilde{D}_1(t)f_{D_1}(t) + \frac{\tilde{D}(t)}{D}f_D(t) + \phi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) \tag{58}$$

where $A_{cl}(\hat{\beta}(t))$ is given in (42), $f_{D_1}(t)$ and $f_D(t)$ have been defined in (50) and (52).

Taking partial derivatives of (48) with respect to x and t, respectively, we get

$$w_x(x,t) = u_x(x,t) - K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)} \times A_{cl}(\hat{\beta}(t))\hat{D}(t)Z(t) \tag{59}$$

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - \psi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) \\ &\quad - K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)} \\ &\quad \times \left(A_{cl}(\hat{\beta}(t))Z(t) + \tilde{D}_1(t)f_{D_1}(t) + \frac{\tilde{D}(t)}{D}f_D(t)\right) \end{aligned} \tag{60}$$

where

$$\begin{aligned} &\psi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) \\ &= \left(\frac{\partial K(\hat{\beta}(t))}{\partial \hat{D}(t)}\dot{\hat{D}}(t) + \frac{\partial K(\hat{\beta}(t))}{\partial \hat{D}_1(t)}\dot{\hat{D}}_1(t)\right)e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)}Z(t) \\ &\quad + K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)} \\ &\quad \times \left[\left(\frac{\partial A_{cl}(\hat{\beta}(t))}{\partial \hat{D}(t)}\dot{\hat{D}}(t) + \frac{\partial A_{cl}(\hat{\beta}(t))}{\partial \hat{D}_1(t)}\dot{\hat{D}}_1(t)\right)\hat{D}(t) \right. \\ &\quad \left. + A_{cl}(\hat{\beta}(t))\dot{\hat{D}}(t)\right](x-1)Z(t) \\ &\quad + K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)}\phi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) \end{aligned} \tag{61}$$

Multiplying (60) by D and minus (59), and employing the control law (41), we get

$$Dw_t(x,t) = w_x(x,t) - D\psi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) - D\tilde{D}_1(t)h_{D_1}(x,t) - \tilde{D}(t)h_D(x,t) \tag{62}$$

$$w(1,t) = 0, \quad x \in [0,1] \tag{63}$$

where $h_{D_1}(x,t)$ and $h_D(x,t)$ have been defined in (51) and (53). Build the Lyapunov candidate such that

$$V(t) = D\log\left(1+\Xi(t)\right) + \frac{gD}{\gamma_{D_1}}\tilde{D}_1(t)^2 + \frac{g}{\gamma_D}\tilde{D}(t)^2 \tag{64}$$

where $\Xi(t)$ has been defined in (49).

Taking the time derivative of (64) along the target closed-loop system (58) and (62)–(63), we have

$$\begin{aligned} \dot{V}(t) &= \frac{1}{1+\Xi(t)}\left[DZ^T(t)\big(P(\hat{\beta}(t))A_{cl}(\hat{\beta}(t)) \right. \\ &\quad + A_{cl}^T(\hat{\beta}(t))P(\hat{\beta}(t))\big)Z(t) \\ &\quad + DZ^T(t)\varphi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right)Z(t) \\ &\quad + 2Z^T(t)P(\hat{\beta}(t))\left(D\phi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) \right. \\ &\quad \left. + D\tilde{D}_1(t)f_{D_1}(t) + \tilde{D}(t)f_D(t)\right) \\ &\quad + 2g\int_0^1 (1+x)w(x,t)w_x(x,t)dx \\ &\quad - 2Dg\int_0^1 (1+x)w(x,t)\psi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right)dx \\ &\quad - 2g\int_0^1 (1+x)w(x,t)\left(D\tilde{D}_1(t)h_{D_1}(x,t) \right. \\ &\quad \left.\left. + \tilde{D}(t)h_D(x,t)\right)dx\right] \\ &\quad - \frac{2gD}{\gamma_{D_1}}\tilde{D}_1(t)\dot{\hat{D}}_1(t) - \frac{2g}{\gamma_D}\tilde{D}(t)\dot{\hat{D}}(t) \\ &= \frac{1}{1+\Xi(t)}\left[-DZ^T(t)Q(\hat{\beta}(t))Z(t) \right. \\ &\quad - gw(0,t)^2 - g\|w(x,t)\|^2 \\ &\quad + DZ^T(t)\varphi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right)Z(t) \\ &\quad + 2DZ^T(t)P(\hat{\beta}(t))\phi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) \\ &\quad \left. - 2Dg\int_0^1 (1+x)w(x,t)\psi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right)dx\right] \\ &\quad - \frac{2gD}{\gamma_{D_1}}\tilde{D}_1(t)\left(\dot{\hat{D}}_1(t) - \gamma_{D_1}\tau_{D_1}(t)\right) \\ &\quad - \frac{2g}{\gamma_D}\tilde{D}(t)\left(\dot{\hat{D}}(t) - \gamma_D\tau_D(t)\right) \end{aligned} \tag{65}$$

where $\tau_{D_1}(t)$ and $\tau_D(t)$ have been defined in (45) and (47), and $\varphi\left(\dot{\hat{D}}(t), \dot{\hat{D}}_1(t)\right) = \frac{\partial P(\hat{\beta}(t))}{\partial \hat{D}(t)}\dot{\hat{D}}(t) + \frac{\partial P(\hat{\beta}(t))}{\partial \hat{D}_1(t)}\dot{\hat{D}}_1(t)$.

Please note that $\hat{D}_1(t)$ and $\hat{D}(t)$ are bounded as the projector operators ensure them to stay in the interval (36). Making use of Young's and Cauchy–Schwarz inequalities, it is evident that the inverse transformation of (48) implies

$$\|u(x,t)\|^2 \le M_u\left(|Z(t)|^2 + \|w(x,t)\|^2\right) \tag{66}$$

where $M_u > 0$ is a constant.

Utilizing (66) and inequalities $0 < \frac{|\varepsilon|}{1+\varepsilon^2} < 1$ and $0 < \frac{\varepsilon^2}{1+\varepsilon^2} < 1$, it is easy to show that

$$\left|\dot{\hat{D}}_1(t)\right| \le \gamma_{D_1}M_{D_1}\frac{|Z(t)|^2 + \|w(x,t)\|^2}{1+\Xi(t)} \le \gamma_{D_1}\bar{M}_{D_1} \tag{67}$$

$$\left|\dot{\hat{D}}(t)\right| \le \gamma_D M_D\frac{|Z(t)|^2 + \|w(x,t)\|^2}{1+\Xi(t)} \le \gamma_D\bar{M}_D \tag{68}$$

where M_{D_1}, M_D, $\bar{M}_{D_1}$ and $\bar{M}_D$ are positive constants.

Thus we have

$$\begin{aligned} \dot{V}(t) &\le \frac{1}{1+\Xi(t)}\left[-\underline{D}\lambda_{\min}(Q)|Z(t)|^2 - gw(0,t)^2 - g\|w(x,t)\|^2 \right. \\ &\quad \left. + (\gamma_{D_1}+\gamma_D)M\left(|Z(t)|^2 + \|w(x,t)\|^2\right)\right] \end{aligned} \tag{69}$$

where $M > 0$ is a constant. By carefully selecting design coefficients $\lambda_{\min}(Q)$, g, γ_{D_1} and γ_D, it is easy to get

$$\dot{V}(t) \le \frac{N}{1+\Xi(t)}\left[-|Z(t)|^2 - w(0,t)^2 - \|w(x,t)\|^2\right] \tag{70}$$

where $N > 0$ is a constant. Thus $Z(t)$ and $w(x,t)$ are bounded and converge to zero. By inverse conversions of (43) and (48), the original states $X(t)$ and $u(x,t)$ are bounded and converge to zero. Then the proof of Theorem 2 is completed. ■

3.2. *B(Dx) is a continuous vector-valued function of Dx*

This section is concerned with the more challenging case where the n-dimensional input vector $B(Dx)$ for $x \in [0,1]$ in (3) is a continuous function of Dx such that

$$B(Dx) = \begin{bmatrix} \rho_1(Dx) \\ \rho_2(Dx) \\ \vdots \\ \rho_n(Dx) \end{bmatrix} \tag{71}$$

where $\rho_i(Dx)$ for $i = 1, \dots, n$ are *unknown* components of the vector-valued function $B(Dx)$.

On the basis of (71), we further denote

$$\mathscr{B}(x) = DB(Dx) = \sum_{i=1}^{n} D\rho_i(Dx)B_i = \sum_{i=1}^{n} b_i(x)B_i \tag{72}$$

where $b_i(x) = D\rho_i(Dx)$ for $i = 1, \dots, n$ are *unknown* scalar continuous functions of x, and $B_i \in \mathbb{R}^n$ for $i = 1, \dots, n$ are the unit vectors accordingly. A three-dimensional example of (72) is given below.

Example 1.

$$\begin{aligned} DB(Dx) &= D\begin{bmatrix} \frac{1}{Dx+1} \\ \sin Dx \\ e^{Dx} \end{bmatrix} = \frac{D}{Dx+1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + D\sin Dx \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &\quad + De^{Dx}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= b_1(x)B_1 + b_2(x)B_2 + b_3(x)B_3 \quad \blacksquare \end{aligned} \tag{73}$$

As a result, with (72), the system (3)–(5) is rewritten as

$$\dot{X}(t) = AX(t) + \int_0^1 \mathscr{B}(x)u(x,t)dx \tag{74}$$

$$Du_t(x,t) = u_x(x,t), \quad x \in [0,1] \tag{75}$$

$$u(1,t) = U(t) \tag{76}$$

For using projector operators later, the following assumption is assumed.

Assumption 4. There exist known constants $\underline{D}$, $\overline{D}$, $\bar{b}_i$, and known continuous functions $b_i^*(x)$ such that

$$0 < \underline{D} \le D \le \overline{D}, \quad 0 < \int_0^1 \left(b_i(x) - b_i^*(x)\right)^2 dx \le \bar{b}_i \tag{77}$$

for $i = 1, \dots, n$.

To stabilize (74)–(76), the following assumption is required.

Assumption 5. For the system (74), the pair (A, β) is stabilizable where

$$\beta = \int_0^1 e^{-AD(1-x)}\mathscr{B}(x)dx \tag{78}$$

There exists a vector $K(\beta)$ to make $A + \beta K(\beta)$ Hurwitz. Namely, there exist matrices $P(\beta) = P(\beta)^T > 0$ and $Q(\beta) = Q(\beta)^T > 0$ such that

$$(A + \beta K(\beta))^T P(\beta) + P(\beta)(A + \beta K(\beta)) = -Q(\beta) \quad \blacksquare \tag{79}$$

Denote $\hat{D}(t)$ and $\hat{b}_i(x,t)$ as the estimates of D and $b_i(x)$ (for $i = 1, \dots, n$) with estimation errors satisfying

$$\tilde{D}(t) = D - \hat{D}(t) \tag{80}$$

$$\tilde{b}_i(x,t) = b_i(x) - \hat{b}_i(x,t) \tag{81}$$

and

$$\hat{\mathscr{B}}(x,t) = \sum_{i=1}^{n} \hat{b}_i(x,t)B_i \tag{82}$$

The delay-adaptive control scheme is designed as follows:

The control law is

$$U(t) = u(1,t) = K(\hat{\beta}(t))Z(t) \tag{83}$$

where $K(\hat{\beta}(t))$ is chosen to let

$$\begin{aligned} A_{cl}(\hat{\beta}(t)) &= A + \int_0^1 e^{-A\hat{D}(t)(1-x)}\hat{\mathscr{B}}(x,t)dxK(\hat{\beta}(t)) \\ &= A + \underbrace{\int_0^1 e^{-A\hat{D}(t)(1-x)}\sum_{i=1}^{n}\hat{b}_i(x,t)B_i dx}_{\hat{\beta}(t)} K(\hat{\beta}(t)) \end{aligned} \tag{84}$$

be Hurwitz and

$$Z(t) = X(t) + \hat{D}(t)\int_0^1\int_0^x e^{-A\hat{D}(t)(x-y)}\hat{\mathscr{B}}(y,t)dyu(x,t)dx \tag{85}$$

The update laws are

$$\dot{\hat{D}}(t) = \gamma_D \mathrm{Proj}_{[\underline{D},\overline{D}]}\{\tau_D(t)\}, \quad \gamma_D > 0 \tag{86}$$

$$\tau_D(t) = \frac{1/gZ^T(t)P(\hat{\beta}(t))f_D(t) - \int_0^1(1+x)w(x,t)h_D(x,t)dx}{1+\Xi(t)} \tag{87}$$

$$\dot{\hat{b}}_i(x,t) = \gamma_b \mathrm{Proj}\{\tau_{b_i}(x,t)\}, \quad \gamma_b > 0 \tag{88}$$

$$\begin{aligned} \tau_{b_i}(x,t) &= \frac{1/gZ^T(t)P(\hat{\beta}(t))f_{b_i}(x,t)}{1+\Xi(t)} \\ &\quad - \frac{\int_0^1(1+y)w(y,t)h_{b_i}(y,t)dyf_{b_i}(x,t)}{1+\Xi(t)} \end{aligned} \tag{89}$$

where $P(\hat{\beta}(t))$ satisfies (79) and $g > 0$ is a designing coefficient,

$$w(x,t) = u(x,t) - K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)}Z(t) \tag{90}$$

$$\Xi(t) = Z^T(t)P(\hat{\beta}(t))Z(t) + g\int_0^1(1+x)w(x,t)^2dx \tag{91}$$

$$\begin{aligned} f_D(t) &= \int_0^1 \hat{\mathscr{B}}(x,t)u(x,t)dx \\ &\quad - \int_0^1 e^{-A\hat{D}(t)(1-x)}\hat{\mathscr{B}}(x,t)dxu(1,t) \\ &\quad - \hat{D}(t)\int_0^1\int_0^x Ae^{-A\hat{D}(t)(x-y)}\hat{\mathscr{B}}(y,t)dyu(x,t)dx \end{aligned} \tag{92}$$

$$h_D(x,t) = K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)}\left(f_D(t) + A_{cl}(\hat{\beta}(t))Z(t)\right) \tag{93}$$

$$f_{b_i}(x,t) = B_i u(x,t) \tag{94}$$

$$h_{b_i}(x,t) = K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)} \tag{95}$$

for $i = 1, \dots, n$, and the projector operator are defined as

$$\mathrm{Proj}_{[\underline{D},\overline{D}]}\{\tau\} = \begin{cases} 0, & \hat{D}(t) = \underline{D} \ \ \textit{and} \ \ \tau < 0 \\ 0, & \hat{D}(t) = \overline{D} \ \ \textit{and} \ \ \tau > 0 \\ \tau, & \textit{else} \end{cases} \tag{96}$$

$$\mathrm{Proj}\{\tau(x)\} = \begin{cases} \tau(x) - \left(\hat{b}_i(x) - b_i^*(x)\right)\dfrac{\int_0^1\left(\hat{b}_i(x)-b_i^*(x)\right)\tau(x)dx}{\int_0^1\left(\hat{b}_i(x)-b_i^*(x)\right)^2dx}, \\ \textit{if} \ \ \int_0^1\left(\hat{b}_i(x) - b_i^*(x)\right)^2 dx = \bar{b}_i \\ \textit{and} \ \ \int_0^1\left(\hat{b}_i(x) - b_i^*(x)\right)\tau(x)dx > 0, \\ \tau(x), \quad \textit{else} \end{cases} \tag{97}$$

Theorem 3. *Consider the closed-loop system consisting of the plant (74)–(76) and the adaptive controller (83)–(97). All the states $(X(t)$, $u(x,t)$, $\hat{D}(t)$, $\hat{b}_i(x,t))$ of the closed-loop system are globally bounded* *and the regulation of $X(t)$ and $U(t)$ such that $\lim_{t\to\infty} X(t) = \lim_{t\to\infty} U(t) = 0$ is achieved.* ■

Proof. Taking the time-derivative of (85) along (74)–(76), and using the integration by parts in x, we obtain

$$\begin{aligned}
\dot{Z}(t) &= AX(t) + \int_0^1 \mathscr{B}(x)u(x,t)dx + \phi\left(\dot{\hat{D}}(t), \dot{\hat{\mathscr{B}}}(x,t)\right) \\
&\quad + \frac{\hat{D}(t)}{D}\int_0^1\int_0^x e^{-A\hat{D}(t)(x-y)}\hat{\mathscr{B}}(y,t)dy u_x(x,t)dx \\
&= AX(t) + \int_0^1 \mathscr{B}(x,t)u(x,t)dx + \phi\left(\dot{\hat{D}}(t), \dot{\hat{\mathscr{B}}}(x,t)\right) \\
&\quad + \frac{\hat{D}(t)}{D}\int_0^1 e^{-A\hat{D}(t)(1-y)}\hat{\mathscr{B}}(y,t)dy u(1,t) \\
&\quad - \frac{\hat{D}(t)}{D}\int_0^1 \hat{\mathscr{B}}(x,t)u(x,t)dx \\
&\quad + \frac{\hat{D}(t)}{D}\int_0^1\int_0^x A\hat{D}(t)e^{-A\hat{D}(t)(x-y)}\hat{\mathscr{B}}(y,t)dy u(x,t)dx \\
&= AX(t) + \int_0^1 \mathscr{B}(x,t)u(x,t)dx + \phi\left(\dot{\hat{D}}(t), \dot{\hat{\mathscr{B}}}(x,t)\right) \\
&\quad + \left(1 - \frac{\tilde{D}(t)}{D}\right)\int_0^1 e^{-A\hat{D}(t)(1-x)}\hat{\mathscr{B}}(x,t)dx u(1,t) \\
&\quad - \left(1 - \frac{\tilde{D}(t)}{D}\right)\int_0^1 \hat{\mathscr{B}}(x,t)u(x,t)dx \\
&\quad + \left(1 - \frac{\tilde{D}(t)}{D}\right)\hat{D}(t)\int_0^1\int_0^x Ae^{-A\hat{D}(t)(x-y)} \\
&\quad \times \hat{\mathscr{B}}(y,t)dy u(x,t)dx
\end{aligned} \tag{98}$$

where

$$\begin{aligned}
&\phi\left(\dot{\hat{D}}(t), \dot{\hat{\mathscr{B}}}(x,t)\right) \\
&= \int_0^1\int_0^x e^{-A\hat{D}(t)(x-y)}\left[\left(\dot{\hat{D}}(t)I - \hat{D}(t)A\dot{\hat{D}}(t)(x-y)\right)\right. \\
&\quad \left.\times \hat{\mathscr{B}}(y,t) + \hat{D}(t)\sum_{i=1}^n \dot{\hat{b}}_i(y,t)B_i\right]dy u(x,t)dx
\end{aligned} \tag{99}$$

Under the control law (83), the formula (98) becomes

$$\begin{aligned}
\dot{Z}(t) &= A_{cl}(\hat{\beta}(t))Z(t) + \phi\left(\dot{\hat{D}}(t), \dot{\hat{\mathscr{B}}}(x,t)\right) + \frac{\tilde{D}(t)}{D}f_D(t) \\
&\quad + \int_0^1\sum_{i=1}^n \tilde{b}_i(x,t)f_{b_i}(x,t)dx
\end{aligned} \tag{100}$$

where $A_{cl}(\hat{\beta}(t))$ is given in (84), $f_D(t)$ and $f_{b_i}(x,t)$ have been defined in (92) and (94).

Taking partial derivatives of (90) with respect to x and t, respectively, we get

$$\begin{aligned}
w_x(x,t) &= u_x(x,t) - K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)} \\
&\quad \times A_{cl}(\hat{\beta}(t))\hat{D}(t)Z(t)
\end{aligned} \tag{101}$$

$$\begin{aligned}
w_t(x,t) &= u_t(x,t) - \psi\left(\dot{\hat{D}}(t), \dot{\hat{\mathscr{B}}}(x,t)\right) \\
&\quad - K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)}\left(A_{cl}(\hat{\beta}(t))Z(t)\right. \\
&\quad \left. + \frac{\tilde{D}(t)}{D}f_D(t) + \int_0^1\sum_{i=1}^n \tilde{b}_i(y,t)f_{b_i}(y,t)dy\right)
\end{aligned} \tag{102}$$

where

$$\begin{aligned}
&\psi\left(\dot{\hat{D}}(t), \dot{\hat{\mathscr{B}}}(x,t)\right) \\
&= \left(\frac{\partial K(\hat{\beta}(t))}{\partial \hat{D}(t)}\dot{\hat{D}}(t) + \sum_{i=1}^n \frac{\partial K(\hat{\beta}(t))}{\partial \hat{b}_i(x,t)}\dot{\hat{b}}_i(x,t)\right)e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)} \\
&\quad \times Z(t) + K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)}\left[\left(\frac{\partial A_{cl}(\hat{\beta}(t))}{\partial \hat{D}(t)}\dot{\hat{D}}(t)\right.\right. \\
&\quad \left.\left. + \sum_{i=1}^n \frac{\partial A_{cl}(\hat{\beta}(t))}{\partial \hat{b}_i(x,t)}\dot{\hat{b}}_i(x,t)\right)\hat{D}(t) + A_{cl}(\hat{\beta}(t))\dot{\hat{D}}(t)\right](x-1)Z(t) \\
&\quad + K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)}\phi\left(\dot{\hat{D}}(t), \dot{\hat{\mathscr{B}}}(x,t)\right)
\end{aligned} \tag{103}$$

Multiplying (102) by D and minus (101), employing the control law (83), we get

$$\begin{aligned}
Dw_t(x,t) &= w_x(x,t) - D\psi\left(\dot{\hat{D}}(t), \dot{\hat{\mathscr{B}}}(x,t)\right) - \tilde{D}(t)h_D(x,t) \\
&\quad - Dh_{b_i}(x,t)\int_0^1\sum_{i=1}^n \tilde{b}_i(y,t)f_{b_i}(y,t)dy
\end{aligned} \tag{104}$$

$$w(1,t) = 0, \quad x\in[0,1] \tag{105}$$

where $h_D(x,t)$ and $h_{b_i}(x,t)$ have been defined in (93) and (95).

Please note that $\hat{D}(t)$ and $\int_0^1 \hat{b}_i(x,t)dx$ are bounded as the projector operators ensure them to stay in the interval (77). Making use of Young's and Cauchy–Schwarz inequalities, it is evident that the inverse transformation of (90) implies

$$\|u(x,t)\|^2 \le M_u\left(|Z(t)|^2 + \|w(x,t)\|^2\right) \tag{106}$$

where $M_u > 0$ is a constant.

Utilizing (106) and inequalities $0 < \frac{|\varepsilon|}{1+\varepsilon^2} < 1$ and $0 < \frac{\varepsilon^2}{1+\varepsilon^2} < 1$, it is easy to show that

$$\left|\dot{\hat{D}}(t)\right| \le \gamma_D M_D\frac{|Z(t)|^2 + \|w(x,t)\|^2}{1+\varXi(t)} \le \gamma_D\bar{M}_D \tag{107}$$

$$\left|\int_0^1 \dot{\hat{b}}_i(x,t)dx\right| \le \gamma_b M_{b_i}\frac{|Z(t)|^2 + \|w(x,t)\|^2}{1+\varXi(t)} \le \gamma_b\bar{M}_{b_i} \tag{108}$$

where M_D, M_{b_i}, $\bar{M}_D$, $\bar{M}_{b_i}$ are positive constants.

Build the Lyapunov candidate such that

$$V(t) = D\log\left(1+\varXi(t)\right) + +\frac{g}{\gamma_D}\tilde{D}(t)^2 + \sum_{i=1}^n\int_0^1\frac{gD}{\gamma_b}\tilde{b}_i(x,t)^2dx \tag{109}$$

where $\varXi(t)$ has been defined in (91).

Taking the time-derivative of (109) along the target closed-loop system (100), (104)–(105), employing (106)–(108), following a similar procedure of proof of Theorem 1, we get

$$\begin{aligned}
\dot{V}(t) &\le \frac{N}{1+\varXi(t)}\left[-|Z(t)|^2 - w(0,t)^2 - \|w(x,t)\|^2\right] \\
&\quad - \frac{2g}{\gamma_D}\tilde{D}(t)\left(\dot{\hat{D}}(t) - \gamma_D\tau_D(t)\right) \\
&\quad - \sum_{i=1}^n\int_0^1\frac{2gD}{\gamma_b}\tilde{b}_i(x,t)\left(\dot{\hat{b}}_i(x,t) - \gamma_b\tau_{b_i}(x,t)\right)dx \\
&\le \frac{N}{1+\varXi(t)}\left[-|Z(t)|^2 - w(0,t)^2 - \|w(x,t)\|^2\right]
\end{aligned} \tag{110}$$

where $N > 0$ is a constant, and $\tau_D(t)$ and $\tau_{b_i}(x,t)$ are given in (87) and (89). Thus $Z(t)$ and $w(x,t)$ are bounded and converge to zero. By inverse conversions of (85) and (90), the original states $X(t)$ and $u(x,t)$ are bounded and converge to zero. Thus the proof of Theorem 3 is proved.

Remark 4. A special case of (71) is that the input vector $B(Dx)$ for $x \in [0, 1]$ is a continuous function of Dx and is parameterizable in the delay D such that

$$B(Dx) = B_0 + \sum_{i=1}^{p} \rho_i(D) B_i(x) \tag{111}$$

where $B_0 \in \mathbb{R}^n$ is a known constant vector, $B_i(x) : [0, 1] \to \mathbb{R}^n$ for $i = 1, \ldots, p$ are known continuous vector functions of x, and $\rho_i(D)$ for $i = 1, \ldots, p$ are *unknown* constant parameters dependent upon D. A three-dimensional example of (111) is given below.

Example 2.

$$\begin{aligned} B(Dx) &= \begin{bmatrix} Dx + \frac{1}{Dx+D} \\ \sqrt{Dx} + 1 \\ D^2x^2 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + D \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \frac{1}{D} \begin{bmatrix} \frac{1}{x+1} \\ 0 \\ 0 \end{bmatrix} \\ &\quad + \sqrt{D} \begin{bmatrix} 0 \\ \sqrt{x} \\ 0 \end{bmatrix} + D^2 \begin{bmatrix} 0 \\ 0 \\ x^2 \end{bmatrix} \\ &= B_0 + \rho_1(D)B_1(x) + \rho_2(D)B_2(x) \\ &\quad + \rho_3(D)B_3(x) + \rho_4(D)B_4(x) \end{aligned} \tag{112}$$

On the basis of (111), we further denote

$$\mathscr{B}(x) = DB(Dx) = DB_0 + \sum_{i=1}^{p} D\rho_i(D)B_i(x) = \sum_{i=0}^{p} b_i B_i(x) \tag{113}$$

where $b_0 = D$ and $b_i = D\rho_i(D)$ for $i = 1, \ldots, p$ are *unknown* constant parameters with respect to D.

We assume there exist known constants $\underline{b}_i$, $\overline{b}_i$ such that $0 < \underline{b}_i \le b_i \le \overline{b}_i$. Instead of functional estimators in (81), we use scalar estimators $\hat{b}_i(t)$ for $i = 0, \ldots, p$ as the estimates of b_i with estimation errors being

$$\tilde{b}_i(t) = b_i - \hat{b}_i(t) \tag{114}$$

and

$$\hat{\mathscr{B}}(x, t) = \sum_{i=0}^{p} \hat{b}_i(t) B_i(x) \tag{115}$$

The update law of functional adaptation (88)–(89) is reduced to the scalar adaptation below

$$\dot{\hat{b}}_i(t) = \gamma_b \mathrm{Proj}_{[\underline{b}_i, \overline{b}_i]}\{\tau_{b_i}(t)\}, \qquad \gamma_b > 0 \tag{116}$$

$$\tau_{b_i}(t) = \frac{1/gZ^T(t)P(\hat{\beta}(t))f_{b_i}(t) - \int_0^1 (1+x)w(x,t)h_{b_i}(x,t)dx}{1 + \Xi(t)} \tag{117}$$

where everything else are defined the same as those in (89) except for

$$f_{b_i}(t) = \int_0^1 B_i(x)u(x,t)dx \tag{118}$$

$$h_{b_i}(x,t) = K(\hat{\beta}(t))e^{A_{cl}(\hat{\beta}(t))\hat{D}(t)(x-1)} f_{b_i}(t) \tag{119}$$

and

$$\mathrm{Proj}_{[\underline{b}_i, \overline{b}_i]}\{\tau\} = \begin{cases} 0, & \hat{b}_i(t) = \underline{b}_i \quad \textit{and} \quad \tau < 0 \\ 0, & \hat{b}_i(t) = \overline{b}_i \quad \textit{and} \quad \tau > 0 \\ \tau, & \textit{else} \end{cases} \qquad \blacksquare \tag{120}$$

4. Conclusion

This paper presents an adaptive approach for stabilizing linear systems with unknown distributed input delays. The control law is based on the certainty-equivalence principle and the update laws are on the basis of the construction of a Lyapunov function with normalization. Two different cases of input dynamics are taken into account: (1) the input vector is a constant vector independent of delay, (2) the input vector is a continuous vector-valued function dependent upon delay.

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Yang Zhu received his Bachelor degree from Hangzhou Dianzi University, China, in Jul. 2010, and Ph.D. degree at Zhejiang University, China, in Jun. 2015. Supported by China Scholarship Council, he was a Visiting Graduate Student at University of California, San Diego, USA from Sep. 2013 to Oct. 2014. Between Aug. 2015 and Sep. 2016, he was a Post-doc at University of Texas at Dallas, USA. From Feb. 2017 to Oct. 2018, he was a Research Fellow at Nanyang Technological University, Singapore. Since Oct. 2018, he has been a Post-doc at Tel Aviv University, Israel. His research involves control theory with its application to transportation engineering.

Miroslav Krstic serves as Sr. Assoc. Vice Chancellor for Research at UC San Diego, overseeing Organized Research Units, the research scientist and postdoctoral scholar affairs, proposal development services, and research initiatives. He is Distinguished Professor of Mechanical and Aerospace Engineering, holds the Alspach endowed chair, and is the founding director of the Cymer Center for Control Systems and Dynamics. Krstic has been elected Fellow of seven scientific societies - IEEE, IFAC, ASME, SIAM, AAAS, IET (UK), and AIAA (Assoc. Fellow) - and as a foreign member of the

Serbian Academy of Sciences and Arts and of the Academy of Engineering of Serbia. He has received the SIAM Reid Prize, ASME Oldenburger Medal, Nyquist Lecture Prize, Paynter Outstanding Investigator Award, Ragazzini Education Award, IFAC Nonlinear Control Systems Award, Chestnut textbook prize, Control Systems Society Distinguished Member Award, the PECASE, NSF Career, and ONR Young Investigator awards, the Schuck ('96 and'19) and Axelby paper prizes, and the first UCSD Research Award given to an engineer. Krstic has also been awarded the Springer Visiting Professorship at UC Berkeley, the Distinguished Visiting Fellowship of the Royal Academy of Engineering, and the Invitation Fellowship of the Japan Society for the Promotion of Science. He serves as Editor-in-Chief of Systems & Control Letters and has been serving as Senior Editor in Automatica and IEEE Transactions on Automatic Control, as editor of two Springer book series, and has served as Vice President for Technical Activities of the IEEE Control Systems Society and as chair of the IEEE CSS Fellow Committee. Krstic has coauthored thirteen books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.

Hongye Su received his Ph.D. degree from Zhejiang University, China, in 1995. He has been promoted to the full Professor at the same university since 2000. Since 2014, he has held the positions of Director of the Institute of Cyber-Systems and Control, as well as Director of the State Key Laboratory of Industrial Control Technology at Zhejiang University. He was the Dean of the College of Control Science and Engineering at Zhejiang University between 2011 and 2013. He received the National Science Fund for Distinguished Young Scholars in 2000. Then he has been promoted as Chang Jiang Chair Professor since 2013. He has taken charge of a large number of significant research projects including National Basic Research Program of China (973 Program) and National High-technology Research and Development Program of China (863 Program). His main academic activities are in the areas of robust control, nonlinear adaptive control and chemical industrial control.