Prescribed-Time Observers for Linear Systems in Observer Canonical Form

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***Abstract*—For linear systems in the observer canonical form, we introduce a state observer with time-varying gains that tend to infinity as time approaches a prescribed convergence time. The observer is shown to exhibit fixed-time stability with an arbitrary convergence time, which is prescribed by the user irrespective of initial conditions. The output estimation error injection terms are also shown to remain uniformly bounded and converge to zero at the prescribed time.**

***Index Terms*—Linear systems, linear system observers, prescribed time, stability of linear systems.**

I. Introduction

Tactical missile guidance applications typically require a small miss distance to nullify a target, and within a short time-to-go that can usually be estimated using on-board or off-board sensors [1]. In these kinds of applications, wherein initial conditions are uncertain, but excellent state estimates are needed within a known finite time to meet the control objectives, observers that allow for easy prescription of the convergence time irrespective of initial conditions offer a clear advantage over those that do not.

Existing approaches to finite-time state estimation achieve convergence of estimation errors to zero within some finite time, which typically depends on initial conditions, and that time is often unknown [2]–[19]. When the convergence time is bounded for all initial conditions, the approach is said to achieve *fixed-time* stabilization [20]. Depending on the approach employed, and how the observer parameters are coupled with the convergence time, fixed-time convergence in a *prescribed* time, whereby the user can prescribe the convergence time *a priori* and irrespective of initial conditions, is in some cases possible, but in general is not guaranteed, and is typically difficult to implement in practice. Here, we explore an alternative approach to prescribing the observer convergence time, and it is easier to implement in practice.

Recently, Song *et al.*, [21] solved the problem of robust prescribed-time stabilization of nonlinear systems in normal form using an alternative approach to finite-time control that employs feedback with time-varying gains that tend to infinity as time approaches the prescribed convergence time. In this note, we explore this alternative approach to finite-time stabilization further in the context of prescribed-time estimation, which employs time-varying observer gains that tend to infinity as time approaches the prescribed convergence time.

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A. Problem Statement

In this note, we study systems whose solutions are only required to exist on a finite-time interval, $t \in [t_0, t_0 + t_f)$, where $t_0 \geq 0$ is the initial time, and $t_f > t_0$ is the final or terminal time by which to meet the estimation and control objectives. In particular, we restrict our analysis to linear single-input single-output systems in the observer canonical form as

$$\dot{x}_1(t) = x_2(t) - a_{n-1}y(t) \tag{1}$$

$$\vdots$$

$$\dot{x}_{r-1}(t) = x_r(t) - a_{\rho+1}y(t) \tag{2}$$

$$\dot{x}_r(t) = x_{r+1}(t) - a_\rho y(t) + b_\rho u(t) \tag{3}$$

$$\vdots$$

$$\dot{x}_{n-1}(t) = x_n(t) - a_1 y(t) + b_1 u(t) \tag{4}$$

$$\dot{x}_n(t) = -a_0 y(t) + b_0 u(t) \tag{5}$$

$$y(t) = x_1(t) \tag{6}$$

which is written more compactly as

$$\dot{x}(t) = Ax(t) - ay(t) + \begin{bmatrix} 0_{(r-1)\times 1} \\ b \end{bmatrix} u(t)$$

$$y(t) = \mathrm{e}_1^T x(t)$$

where $\mathrm{e}_1 := [1, 0, \ldots, 0]^T \in \mathscr{R}^n$ is the first of the n-dimensional unit vectors, and

$$A := \begin{bmatrix} 0 & I_{n-1} \\ \vdots & \\ 0 & \ldots 0 \end{bmatrix}, \quad a := \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix}, \quad b := \begin{bmatrix} b_\rho \\ \vdots \\ b_0 \end{bmatrix}.$$

Here, $x(t) \in \mathscr{R}^n$ is the state, $u(t) \in \mathscr{R}^1$ is a known and bounded control input, $y(t) \in \mathscr{R}^1$ is the measured output, $n > \rho \geq 0, r = n - \rho$ is the relative degree of the system, and the constant coefficients a_is and b_is are known. Our goal is to obtain perfect estimation of the state within a finite time $0 < T \leq t_f$, in a manner in which T is fixed (independent of initial conditions) and easily prescribed *a priori*. We accomplish this with a dynamic state observer of the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) - ay(t) + \begin{bmatrix} 0_{(r-1)\times 1} \\ b \end{bmatrix} u(t) + \begin{bmatrix} g_1(t-t_0,T) \\ \vdots \\ g_n(t-t_0,T) \end{bmatrix} (y(t) - \hat{x}_1(t)) \quad (7)$$

where the time-varying observer gains $\{g_i(t-t_0,T)\}_{i=1}^n$ are functions of the prescribed convergence time T and must be designed. Defining the observer estimation error as $\tilde{x}_i(t) := x_i(t) - \hat{x}_i(t), i = 1,\ldots,n$ yields the error dynamics

$$\dot{\tilde{x}}_i(t) = \tilde{x}_{i+1}(t) - g_i(t-t_0,T)\tilde{x}_1(t), \quad i = 1,\ldots,n-1 \quad (8)$$

$$\dot{\tilde{x}}_n(t) = -g_n(t-t_0,T)\tilde{x}_1(t). \quad (9)$$

B. Design Approach

We show that by using a particular time-varying change of coordinates of the observer error states, and selecting the time-varying observer gains $\{g_i(t-t_0,T)\}_{i=1}^n$ in a particular way, we achieve a form of *fixed-time stability* for (8) and (9) in a convergence time $T > 0$, which we prescribe *a priori* and irrespective of initial conditions. To motivate our approach, consider the change of coordinates

$$\tilde{\zeta}_i(t) := \mu(t-t_0,T)\tilde{x}_i(t), \quad i = 1,\ldots,n \quad (10)$$

where for $t \in [t_0, t_0+T)$, $\mu(t-t_0,T) : [t_0,t_0+T) \mapsto \mathscr{R}^+$ is a monotonically increasing function (to be defined) having the property that $\mu(t-t_0,T)$ *tends to infinity* as $t \to t_0+T$, where T is prescribed by the user. Clearly, (10) also gives the inverse mapping from $\tilde{\zeta}_i(t)$ to $\tilde{x}_i(t)$ as

$$\tilde{x}_i(t) = \mu(t-t_0,T)^{-1}\tilde{\zeta}_i(t), \quad i = 1,\ldots,n. \quad (11)$$

Since $\mu(t-t_0,T)$ increases monotonically to infinity, then $\mu(t-t_0,T)^{-1}$ *decreases monotonically to zero*. So from (11), if the error states $\tilde{\zeta}_i(t)$ remain finite, then clearly $\tilde{x}_i(t) \to 0$ as $t \to t_0+T$. Therefore, if after transforming the $\tilde{x}_i(t)$ system into the $\tilde{\zeta}_i(t)$ coordinates, we can select the observer gains $\{g_i(t-t_0,T)\}_{i=1}^n$ in a way that *stabilizes* the $\tilde{\zeta}_i(t)$ system, then by the nature of the special relationship between $\tilde{\zeta}_i(t)$ and $\tilde{x}_i(t)$ in (11), we simultaneously enforce the condition that $\tilde{x}_i(t) \to 0$ as $t \to t_0+T$.

Note that in general, the change of coordinates (10) may result in a transformed version of (8) and (9), which is much more complicated than the original one, and finding its inverse may not be trivial. However, it offers the advantage of providing a means of achieving fixed-time convergence in the prescribed time T. Using this approach, the convergence is global (with respect to initial conditions), uniform (with respect to t_0), and the convergence time T is fixed (independent of initial conditions). In application, we calculate the state estimate online using (7) and the time-varying observer gains $\{g_i(t-t_0,T)\}_{i=1}^n$. The latter will be shown to have a special structure and are easy to calculate.

In this framework, as the basis for our design, we define the function $\mu_1(t-t_0,T) : [t_0,t_0+T) \mapsto \mathscr{R}^+$ as

$$\mu_1(t-t_0,T) := \frac{T}{T+t_0-t} \quad (12)$$

which starts from 1 at $t = t_0$ and increases monotonically to infinity as $t \to t_0+T$. Using this function, we define the function $\mu(t-t_0,T) : [t_0,t_0+T) \mapsto \mathscr{R}^+$ as

$$\mu(t-t_0,T) := \mu_1(t-t_0,T)^{n+m} = \frac{T^{n+m}}{(T+t_0-t)^{n+m}} \quad (13)$$

which also starts from 1 at $t = t_0$ and increases monotonically to infinity as $t \to t_0+T$, but can be tuned to do so more quickly than $\mu_1(t-t_0,T)$ through the positive integers n (the order of the system) and $m \geq 1$, a design parameter. For notational convenience, we also define the function $\nu(t-t_0,T) : [t_0,t_0+T) \mapsto \mathscr{R}^+$ as

$$\nu(t-t_0,T) := \mu_1(t-t_0,T)^{-1} = \frac{T+t_0-t}{T} \quad (14)$$

which starts from 1 at $t = t_0$ and decreases monotonically to zero as $t \to t_0+T$.

Notation: To save space, we often drop the explicit $(t-t_0)$ and T dependence of the functions (12)–(14) and write them simply as μ_1, μ, and ν, respectively.

Using these functions, our analysis employs the following definition of fixed-time stability.

Definition 1 (FT-GUAS): The system $\dot{x} = f(x,t)$ (of arbitrary dimension of x) is said to be fixed-time, globally uniformly asymptotically stable in time T (FT-GUAS) over the interval $I_T := [t_0, t_0+T)$ if there exists a class $\mathscr{KL}$ function β such that for all $t \in I_T$

$$|x(t)| \leq \beta(|x(t_0)|, \mu_1(t-t_0,T) - 1)$$

where the function $\mu_1(t-t_0,T)$ is defined in (12).

II. Main Results

Using the functions defined in (12)–(14), it is possible to transform the error system (8), (9) in a way that retains the attractive property of (11), but also results in a transformed system that is easy to stabilize asymptotically. This then provides for fixed-time stabilization of (8) and (9) in the sense of Definition 1. Our transformation result is captured in the following lemma, whose proof is provided in the following section.

Lemma 1: Consider the transformation $\tilde{x}_i(t) \mapsto \tilde{\zeta}_i(t)$ defined by

$$\tilde{\zeta}_i(t) = \mu(t-t_0,T)\tilde{x}_i(t), \quad i = 1,\ldots,n \quad (15)$$

and the transformation $\tilde{\zeta}_i(t) \mapsto \tilde{z}_i(t)$ defined by

$$\tilde{z}_i(t) = \sum_{j=1}^{n} p_{i,j}^*(\mu_1)\tilde{\zeta}_j(t), \quad i = 1,\ldots,n \quad (16)$$

where the functions $\{p_{i,j}^*(\mu_1)\}$ are defined by

$$p_{i,j}^*(\mu_1) := \bar{p}_{i,j}\mu_1^{i-j}, \quad 1 \leq j \leq i \leq n \quad (17)$$

and the coefficients $\{\bar{p}_{i,j}\}$ are constants to be determined. By selecting the $\{\bar{p}_{i,j}\}$ according to

$$\bar{p}_{i,i} = 1 \quad (18)$$

$$\bar{p}_{i,j} = 0, \quad j > i \quad (19)$$

for $\{\bar{p}_{i,j}\}$ with $j \geq i$, the recursion relations

$$\bar{p}_{i,j-1} = -\frac{n+m+i-j}{T}\bar{p}_{i,j} + \bar{p}_{i+1,j}, \quad n-1 \geq i \geq j \geq 2 \quad (20)$$

$$\bar{p}_{n,j-1} = -\frac{2n+m-j}{T}\bar{p}_{n,j}, \quad j = n, n-1, \ldots, 2 \quad (21)$$

for $\{\bar{p}_{i,j}\}$ with $j < i$, and the observer gains $\{g_i(t-t_0,T)\}_{i=1}^{n-1}$ according to

$$g_i(t-t_0,T) = l_i + \left(\frac{n+m+i-1}{T}\bar{p}_{i,1} - \bar{p}_{i+1,1}\right)\mu_1^i - \sum_{j=1}^{i-1} g_j(t-t_0,T)\bar{p}_{i,j}\mu_1^{i-j} \tag{22}$$

and $g_n(t-t_0,T)$ according to

$$g_n(t-t_0,T) = l_n + \frac{2n+m-1}{T}\bar{p}_{n,1}\mu_1^n - \sum_{j=1}^{n-1} g_j(t-t_0,T)\bar{p}_{n,j}\mu_1^{n-j} \tag{23}$$

where the $\{l_i\}_{i=1}^n$ are constants to be selected, the observer error system (8), (9) is transformed into

$$\dot{\tilde{z}}_i(t) = \tilde{z}_{i+1}(t) - l_i\tilde{z}_1(t), \quad i = 1,\dots,n-1 \tag{24}$$

$$\dot{\tilde{z}}_n(t) = -l_n\tilde{z}_1(t). \tag{25}$$

The observer gains $\{g_i(t-t_0,T)\}_{i=1}^n$ provided by (22) and (23) are calculated recursively as follows. Observe from (22) and (23) that if we proceed in the order $i = 1,2,..,n$, then explicit expressions for each $g_i(t-t_0,T)$ are obtained in terms of $\{g_j(t-t_0,T)\}_{j=1}^{i-1}$ and the constants $\{l_j\}_{j=1}^i$, regardless of the values of the latter. Thus, after determining the $\{\bar{p}_{i,j}\}$ according to (18)–(21), and selecting $\{l_i\}_{i=1}^n$ as desired, we calculate the $\{g_i(t-t_0,T)\}_{i=1}^n$ using (22) and (23) by proceeding in the order $i = 1,2,..,n$.

The system (24), (25) is stabilized by selecting the constants $\{l_i\}_{i=1}^n$ to make the polynomial $s^n + l_1 s^{n-1} + \cdots + l_n$ and the companion matrix Λ Hurwitz, where Λ is defined from (24) and (25) as

$$\Lambda := \begin{bmatrix} -l_1 & I_{n-1} \\ \vdots & \\ -l_n & 0 \end{bmatrix}. \tag{26}$$

Then by the nature of the transformations used in Lemma 1, we achieve fixed-time stability of the system (8), (9) in the sense of Definition 1, as stated in the following theorem. Its proof is provided in the following section.

Theorem 1: For the dynamic system (1)–(6) defined over the interval $I_T := [t_0, t_0 + t_f)$, consider the observer (7) having error dynamics (8), (9) and observer gains $\{g_i(t-t_0,T)\}_{i=1}^n$ given by (22) and (23) where the $\{l_i\}_{i=1}^n$ are constants to be selected. If the constants $\{l_i\}_{i=1}^n$ are selected such that the polynomial $s^n + l_1 s^{n-1} + \cdots + l_n$ and the companion matrix (26) are both Hurwitz, then the system (8), (9) has a FT-GUAS equilibrium at the origin, with a prescribed convergence time T, and there exist positive constants $\tilde{M}, \tilde{\delta} > 0$ such that for all $t \in I_T$,

$$|\tilde{x}(t)| \le \nu(t-t_0,T)^{m+1}\tilde{M}\mathrm{e}^{-\tilde{\delta}(t-t_0)}|\tilde{x}(t_0)| \tag{27}$$

where $\nu(t-t_0,T)$ is defined in (14), and $m \ge 1$ is an integer and a design parameter. Furthermore, the output estimation error injection terms $\gamma_i(t-t_0,T) := g_i(t-t_0,T)\tilde{x}_1(t)$ for $i = 1,\dots,n$ remain uniformly bounded over the interval I_T, and also converge to zero as $t \to t_0 + T$.

Remark 1: The coefficients $\{\bar{p}_{i,j}\}$ for $j < i$ defined by the recursion relations (20), (21) are easily calculated offline using the following algorithm. Beginning with (21), since $\bar{p}_{n,n} = 1$ is known from (18), we first obtain the set $\{\bar{p}_{n,j-1}\}$ by working backwards for $j = n, n-1, \dots, 2$. Next, we proceed to solve (20), beginning with $i = n-1$ and $j = i = n-1$ for the set $\{\bar{p}_{n-1,j-1}\}$ with $j = n-1, n-2, \dots, 2$. Since $\bar{p}_{n-1,n-1} = 1$ is known from (18), and $\bar{p}_{n,n-1}$ is known from the previous $i = n$ step, we can again work backwards to obtain the remaining $\{\bar{p}_{n-1,j-1}\}$. We continue this process for $i = n-2,\dots,2$ by working backwards from $j = i, i-1, ..., 2$ to obtain the remaining $\{\bar{p}_{i,j}\}$.

In practice, we obtain the constants $\{\bar{p}_{i,j}\}$ for $j < i$ defined by the recursion relations (20), (21) as described in Remark 1. However, it is possible to obtain explicit solutions for them as shown in the following proposition. The proof is provided in the following section.

Proposition 1: For $\{\bar{p}_{i,j}\}$ with $1 \le j < i < n$, the solution to (20) is expressed as

$$\bar{p}_{i,j} = \frac{(-1)^{i-j}(n+m+i-j-1)!}{T^{i-j}(n+m-1)!} + \sum_{l=0}^{i-j-1}\frac{(-1)^{i-j-(l+1)}(n+m+i-j-1)!}{T^{i-j-(l+1)}(n+m+l)!}\bar{p}_{i+1,i-l}$$

and for $\{\bar{p}_{n,j}\}$ with $1 \le j \le n-1$, the solution to (21) is expressed as

$$\bar{p}_{n,j} = \frac{(-1)^{n-j}(2n+m-j-1)!}{T^{n-j}(n+m-1)!}. \tag{28}$$

Some remarks on the observer (7) and how it compares to existing finite-time observers conclude this section.

A. Implementation

The observer (7) is conceptually easy to understand and implement. Indeed, for system (1)–(6) of any given order n, its construction consists of simply selecting the constants $\{l_i\}_{i=1}^n$ to make the companion matrix Λ Hurwitz, and then (independently) selecting the prescribed convergence time T and the design parameter $m \ge 1$. These are the only "tuning parameters" of the observer, since we have explicit formula for the $\{\bar{p}_{i,j}\}$ in (18)–(21), which are set by n, m, and T, and the time-varying gains $\{g_i(t-t_0,T)\}_{i=1}^n$ in (22) and (23), which are functions of the $\{\bar{p}_{i,j}\}$ and the function $\mu_1(t-t_0,T)$, where the latter is defined by (12) based on T and $t-t_0$.

Although the output estimation error injection terms $\{\gamma_i(t-t_0, T)\}_{i=1}^n$ remain finite (see Theorem 1), in practice the observer (7) will exhibit numerical precision limitations as time tends to $t_0 + T$, because the gains $\{g_i(t-t_0,T)\}_{i=1}^n$ go to infinity. This can be mitigated by either: 1) extending the prescribed convergence time T to sometime larger than t_f, or by 2) "turning OFF" the injections $\{\gamma_i(t-t_0,T)\}_{i=1}^n$ (zero them out) at some time $t_{\text{stop}} < t_0 + T$. Doing either will achieve convergence of the estimation error to within some neighborhood of the origin that could be tuned by the user if desired. For the latter option, one can, in principle, determine such a t_{stop} *a priori* as follows. First, set each of the $\{g_i(t-t_0,T)\}_{i=1}^n$ in (22) and (23) to some maximum value selected by the user, say $g_{\max} = 10^{10}$. Next, solve each expression for $t-t_0$. Finally, define t_{stop} as the minimum of those times. Clearly, for high-order systems, this method becomes difficult, and may not be worth the trouble in practice. A simpler solution is to instead monitor the magnitude of the $\{g_i(t-t_0,T)\}_{i=1}^n$ in real time during estimation until any of them approach $g_{\max}$, and then at that time, turn OFF the injections. Once the injections are turned OFF, the estimation errors will grow again. But if desired, the observer could be reset to begin anew from that time, treating the current time t_{stop} as a new t_0, and the current estimation error $\tilde{x}(t_{\text{stop}})$ as a new initial condition. Notice that in this regard, the observer (7) exhibits a high-gain or brute-force nature "asymptotically," whereas the classical time-invariant high-gain observer is of a brute-force nature all of the time.

B. Existing Methods

In theory, higher-order sliding mode differentiators [3]–[7] and observers based on concepts of homogeneity [8]–[14] can achieve convergence in finite time. However, with both of these approaches, simple, constructive procedures for selecting observer parameters that guarantee convergence in a fixed, *prescribed* time appear to be unavailable for higher-order systems, and parameter tuning must usually be done through trial-and-error in numerical simulation. Approaches that use multiple observers [16]–[19] provide for prescription of the convergence time, but by requiring either delays or a hybrid-systems framework, their implementation is relatively complicated. The implicit Lyapunov function approach [15] also provides for prescribed convergence time, by iterating parameters over a grid until a set of LMI are satisfied, and this can be done offline for linear MIMO systems. However, implementation of this method is also complicated, and ultimately still requires trial-and-error to tune the parameters through numerical simulation. The attractive features of (7) are the simplicity of its design, the explicitness of its gains and parameterization, and most importantly, its ability to easily prescribe the convergence time irrespective of initial conditions.

III. Proofs of Main Results

A. Proof of Lemma 1

We begin this section with a proof of Lemma 1.

Proof: From (16), (17), and (19), we obtain

$$\dot{\tilde{z}}_i = \frac{i-1}{T}\bar{p}_{i,1}\mu_1^i\tilde{\zeta}_1 + \sum_{j=2}^{i}\frac{i-j}{T}\bar{p}_{i,j}\mu_1^{i-j+1}\tilde{\zeta}_j + \sum_{j=1}^{i}\bar{p}_{i,j}\mu_1^{i-j}\dot{\tilde{\zeta}}_j. \tag{29}$$

We rewrite the $\dot{\tilde{\zeta}}_j$ terms as follows. By differentiating (15) and substituting in (8) and (9), the transformed error dynamics are shown to be

$$\dot{\tilde{\zeta}}_i = -g_i\tilde{\zeta}_1 + \frac{\dot{\mu}}{\mu}\tilde{\zeta}_i + \tilde{\zeta}_{i+1}, \quad i = 1,\dots,n-1 \tag{30}$$

$$\dot{\tilde{\zeta}}_n = -g_n\tilde{\zeta}_1 + \frac{\dot{\mu}}{\mu}\tilde{\zeta}_n. \tag{31}$$

We next substitute in (30) and (31) for $\dot{\tilde{\zeta}}_j$ in (29), which will yield a different expression for $i = 1,2,\dots,n-1$ than for $i = n$. After lengthy (but straightforward) calculations, we obtain for $i = 1,2,\dots,n-1$

$$\dot{\tilde{z}}_i = -\left(g_i + \sum_{j=1}^{i-1} g_j\bar{p}_{i,j}\mu_1^{i-j} + \bar{p}_{i+1,1}\mu_1^i - \frac{n+m+i-1}{T}\bar{p}_{i,1}\mu_1^i\right)\tilde{z}_1$$
$$+ \tilde{z}_{i+1} + \sum_{j=2}^{i}\left(\frac{n+m+i-j}{T}\bar{p}_{i,j} + \bar{p}_{i,j-1} - \bar{p}_{i+1,j}\right)\mu_1^{i-j+1}\tilde{\zeta}_j.$$

Then by selecting the $\{\bar{p}_{i,j}\}$ for $j < i$ according to (20), and the gains $\{g_i(t-t_0,T)\}_{i=1}^{n-1}$ according to (22), we obtain (24). Similarly, for $i = n$, we obtain

$$\dot{\tilde{z}}_n = -\left(g_n + \sum_{j=1}^{n-1} g_j\bar{p}_{n,j}\mu_1^{n-j} - \frac{2n+m-1}{T}\bar{p}_{n,1}\mu_1^n\right)\tilde{z}_1$$
$$+ \sum_{j=2}^{n}\left(\frac{2n+m-j}{T}\bar{p}_{n,j} + \bar{p}_{n,j-1}\right)\mu_1^{n-j+1}\tilde{\zeta}_j$$

and by selecting the $\{\bar{p}_{n,j}\}$ for $j < n$ according to (21), and the gains $\{g_i(t-t_0,T)\}_{i=1}^{n}$ according to (22) and (23), we obtain (25). ■

B. Three Useful Lemmas

As shown in Lemma 1 and its proof, the change of coordinates of the observer error system (8), (9) consists of two parts: 1) transformation (15) to scale the error dynamics in a way that facilitates fixed-time stabilization through the approach outlined in Section I-B, and 2) transformation (16) to rewrite the scaled error dynamics into a form that is easily stabilized asymptotically. This two-step change of coordinates is easily expressed as a single transformation, and the resulting transformation is invertible. These results are captured in the following lemmas, whose short proofs are in the Appendix.

Lemma 2: The transformation $\tilde{x}_i(t) \mapsto \tilde{z}_i(t)$ is expressed as

$$\tilde{z}(t) = \mu_1^{m+1}P(\mu_1)\tilde{x}(t) \tag{32}$$

where $P(\mu_1)$ is a lower triangular matrix having elements $\{p_{i,j}(\mu_1)\}$ given by

$$p_{i,j}(\mu_1) = \bar{p}_{i,j}\mu_1^{n+i-j-1}, \quad 1 \le j \le i \le n \tag{33}$$

where the constant coefficients $\{\bar{p}_{i,j}\}$ are defined by (18)–(21).

Lemma 3: The transformation $\tilde{z}_i(t) \mapsto \tilde{x}_i(t)$ is expressed as

$$\tilde{x}(t) = \nu^{m+1}Q(\nu)\tilde{z}(t) \tag{34}$$

where $Q(\nu)$ is a lower triangular matrix having elements $\{q_{i,j}(\nu)\}$ given by

$$q_{i,j}(\nu) = \bar{q}_{i,j}\nu^{n+j-i-1}, \quad 1 \le j \le i \le n \tag{35}$$

where the constant coefficients $\{\bar{q}_{i,j}\}$ can be explicitly obtained from the constant coefficients$\{\bar{p}_{i,j}\}$ defined by (18)–(21). Furthermore, $\bar{q} := \sup_{\nu\in(0,1]}|Q(\nu)|$ is finite.

The proof of Theorem 1 leverages Lemmas 1, 2, 3, and the following lemma, whose proof is a straightforward application of induction and is therefore omitted.

Lemma 4: The time-varying observer gains $\{g_i(t-t_0,T)\}_{i=1}^n$ given by (22) and (23) are ith-order polynomials in μ_1, which we denote as

$$g_i(t-t_0,T) = h_i(\mu_1) := \sum_{k=0}^{i}\bar{h}_{i,k}\mu_1^k, \quad i = 1,2,\dots,n \tag{36}$$

where the coefficients $\{\bar{h}_{i,k}\}$ for $i = 1,2,\dots,n$ and $k = 0,1,\dots,i$ are constants.

C. Proof of Theorem 1

We now have everything we need to prove Theorem 1.

Proof: In the proof of Lemma 1, we showed that substituting (22) and (23) into (8) and (9) yield (24) and (25), which can be written as

$$\dot{\tilde{z}}(t) = \Lambda\tilde{z}(t)$$

where Λ was defined in (26). By selecting the constants $\{l_i\}_{i=1}^n$ to make the polynomial $s^n + l_1s^{n-1} + \cdots + l_n$ and companion matrix Λ Hurwitz, there exist positive constants $M_z, \delta_z > 0$ such that for all $t \in [t_0, t_0+T)$

$$|\tilde{z}(t)| \le M_z\mathrm{e}^{-\delta_z(t-t_0)}|\tilde{z}(t_0)|. \tag{37}$$

From (34) and the last statement in Lemma 3, we obtain the estimate

$$|\tilde{x}(t)| \le |\nu^{m+1}||Q(\nu)||\tilde{z}(t)|$$
$$\le \nu^{m+1}\bar{q}|\tilde{z}(t)| \tag{38}$$

and from (32), we obtain the estimate

$$|\tilde{z}(t_0)| \leq |\mu_1(0,T)^{m+1}||P(\mu_1(0,T))||\tilde{x}(t_0)| = |P(1)||\tilde{x}(t_0)|. \tag{39}$$

Substituting (39) into (37), and then the result into (38) then gives

$$|\tilde{x}(t)| \leq \nu^{m+1}\bar{q}|P(1)|M_z e^{-\delta_z(t-t_0)}|\tilde{x}(t_0)|.$$

Thus, we have obtained (27) with $\tilde{M} := \bar{q}|P(1)|M_z$ and $\tilde{\delta} := \delta_z$.

We now consider our claim regarding the output estimation error injection terms $\{\gamma_i(t-t_0,T)\}_{i=1}^n$. For $i = 1, 2, \ldots, n-1$, multiplying $g_i(t-t_0,T)$ by $\tilde{x}_1(t)$ and then using (15) and (13) to replace $\tilde{x}_1(t)$ gives

$$\begin{aligned}\gamma_i(t-t_0,T) &= g_i(t-t_0,T)\tilde{x}_1(t)\\ &= g_i(t-t_0,T)\frac{1}{\mu_1^{n+m}}\tilde{\zeta}_1(t)\\ &= \nu^m g_i(t-t_0,T)\nu^n \tilde{z}_1(t)\end{aligned} \tag{40}$$

since $\nu = \mu_1^{-1}$ and $\tilde{z}_1(t) = \tilde{\zeta}_1(t)$ by (16)–(19). Now combine $g_i(t-t_0,T)$ with ν^n to define a new polynomial using (36) from Lemma 4, such that for $i = 1, 2, \ldots, n-1$

$$\begin{aligned}\eta_i(\nu) &:= g_i(t-t_0,T)\nu^n\\ &= \sum_{k=0}^{i} \bar{h}_{i,k}\mu_1^k\nu^n\\ &= \sum_{k=0}^{i} \bar{h}_{i,k}\nu^{n-k}.\end{aligned} \tag{41}$$

Then, (40) becomes

$$\gamma_i(t-t_0,T) = \nu^m\eta_i(\nu)\tilde{z}_1(t), \quad i = 1, 2, \ldots, n-1.$$

The transformed error $\tilde{z}_1(t)$ is asymptotically stabilized by our choice of the constants $\{l_i\}_{i=1}^n$. Additionally, from (41), it is clear that for $i = 1, 2, \ldots, n-1$, every $\eta_i(\nu)$ is a polynomial in the argument $\nu \in (0,1]$ for $t \in [t_0, t_0+T)$, thus is bounded and goes to zero as $t \to t_0+T$. Therefore, with $m \geq 1$, $\gamma_i(t-t_0,T)$ is uniformly bounded and goes to zero as $t \to t_0+T$ for $i = 1, 2, \ldots, n-1$. The claim for $i = n$ is proven the same way. ■

D. Proof of Proposition 1

The proof of Proposition 1 concludes this section.

Proof: Both (20) and (21) are linear difference equations that can be viewed as discrete-time dynamic systems stepping backward in time, with j playing the role of time. For $i = n$, (21) is autonomous, whereas for each $i = n-1, n-2, \ldots, 2$, each equation defined by (20) is forced by $\{\bar{p}_{i+1,j}\}$, for $j = i, i-1, \ldots, 2$. Constrained by the latter, we must solve (21) first. Define

$$x_n(j) := \bar{p}_{n,j}, \quad a_n(j) := -\frac{2n+m-j}{T}.$$

Then, (21) with initial condition (18) is written as

$$x_n(j-1) = a_n(j)x_n(j), \quad x_n(n) = 1. \tag{42}$$

The solution to (42) is

$$x_n(n-k) = \Gamma(n-k,n)x_n(n) \tag{43}$$

where

$$\Gamma(i-k,i-l) := \begin{cases}\prod_{s=l}^{k-1} a(i-s), & 0 \leq l < k,\\ 1, & k = l\end{cases} \tag{44}$$

for integers i, k, l. Now consider (44) with $l = 0$ for step k, where $0 < k \leq n-1$ as

$$\begin{aligned}\Gamma(n-k,n) &= \prod_{s=0}^{k-1} a_n(n-s)\\ &= \prod_{s=0}^{k-1}\left(-\frac{n+m+s}{T}\right) \qquad (45)\\ &= \left(\frac{-1}{T}\right)^k (n+m+k-1)(n+m+k-2)\ldots\\ &\quad \times (n+m+1)(n+m). \qquad (46)\end{aligned}$$

Multiplying the numerator and denominator of (46) by $(n+m-1)!$ results in

$$\Gamma(n-k,n) = \frac{(-1)^k(n+m+k-1)!}{T^k(n+m-1)!}, \quad 0 < k \leq n-1. \tag{47}$$

Then the solution to (42) is provided by (43) and (47), and the initial condition $x_n(n) = 1$, which gives

$$x_n(n-k) = \frac{(-1)^k(n+m+k-1)!}{T^k(n+m-1)!}, \quad 0 < k \leq n-1. \tag{48}$$

Define $j := n-k$, so that $k = n-j$. Then, since $x_n(j) = \bar{p}_{n,j}$, (48) is rewritten as

$$x_n(j) = \bar{p}_{n,j} = \frac{(-1)^{n-j}(2n+m-j-1)!}{T^{n-j}(n+m-1)!}, \quad 1 \leq j \leq n-1.$$

Now for the systems defined by (20) for $i = n-1, n-2, \ldots, 2$, define

$$x_i(j) := \bar{p}_{i,j}, \quad a_i(j) := -\frac{n+m+i-j}{T}, \quad u_i(j) := \bar{p}_{i+1,j}.$$

Then, (20) can be expressed as

$$x_i(j-1) = a_i(j)x_i(j) + u_i(j), \quad x_i(i) = 1 \tag{49}$$

whose solution is

$$x_i(i-k) = \Gamma(i-k,i)x_i(i) + \sum_{l=0}^{k-1}\Gamma(i-k,i-(l+1))u_i(i-l). \tag{50}$$

As for $i = n$, the transition from the initial condition is

$$\Gamma(i-k,i) = \prod_{s=0}^{k-1} a_i(i-s) = \frac{(-1)^k(n+m+k-1)!}{T^k(n+m-1)!} \tag{51}$$

by using (45)–(47). Then, for the inputs

$$\begin{aligned}\Gamma(i-k,i-(l+1)) &= \prod_{s=l+1}^{k-1} a_i(i-s)\\ &= \prod_{s=l+1}^{k-1}\left(-\frac{n+m+s}{T}\right)\\ &= \frac{(-1)^{k-(l+1)}(n+m+k-1)!}{T^{k-(l+1)}(n+m+l)!}.\end{aligned} \tag{52}$$

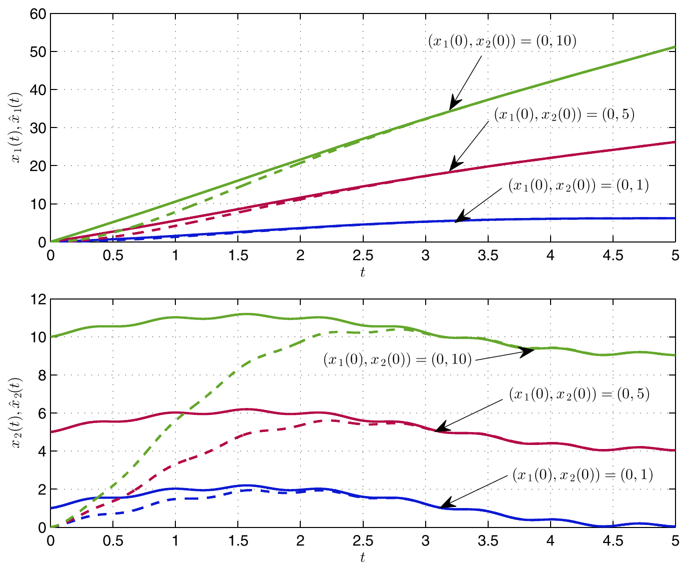

Fig. 1. Time histories of states and state estimates for Example 1, for three sets of initial conditions.

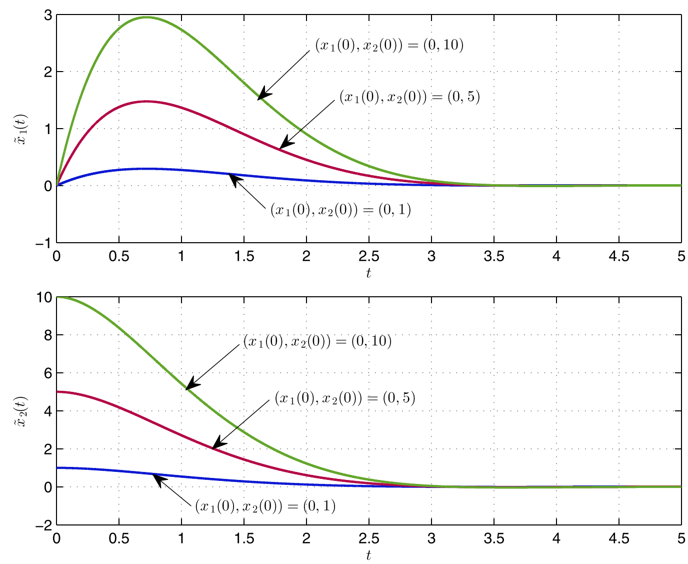

Fig. 2. Time histories of state estimation errors for the simulations shown in Fig. 1.

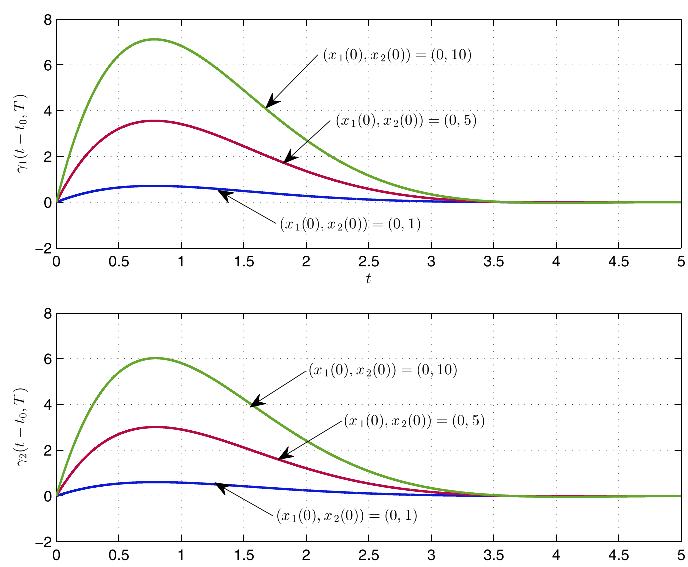

Fig. 3. Time histories of output estimation error injection terms for the simulations shown in Fig. 1.

Then, rewriting (50) with (51) and (52) gives

$$x_i(i-k) = \frac{(-1)^k (n+m+k-1)!}{T^k (n+m-1)!} x_i(i) + \sum_{l=0}^{k-1} \frac{(-1)^{k-(l+1)} (n+m+k-1)!}{T^{k-(l+1)} (n+m+l)!} u_i(i-l). \quad (53)$$

Define $j := i - k$, so that $k = i - j$. Then, (53) is rewritten as

$$x_i(j) = \frac{(-1)^{i-j} (n+m+i-j-1)!}{T^{i-j} (n+m-1)!} + \sum_{l=0}^{i-j-1} \frac{(-1)^{i-j-(l+1)} (n+m+i-j-1)!}{T^{i-j-(l+1)} (n+m+l)!} u_i(i-l) \quad (54)$$

after using $x_i(i) = 1$. Then, since $x_i(j) = \bar{p}_{i,j}$ and $u_i(i-l) = \bar{p}_{i+1,i-l}$, (54) becomes

$$\bar{p}_{i,j} = \frac{(-1)^{i-j} (n+m+i-j-1)!}{T^{i-j} (n+m-1)!} + \sum_{l=0}^{i-j-1} \frac{(-1)^{i-j-(l+1)} (n+m+i-j-1)!}{T^{i-j-(l+1)} (n+m+l)!} \bar{p}_{i+1,i-l}.$$

■

IV. Numerical Simulations

We illustrate the performance of the observer (7) through the following example.

Example 1: Consider the double integrator with single output as

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$
$$y(t) = x_1(t).$$

The observer (7) becomes

$$\dot{\hat{x}}_1(t) = \hat{x}_2(t) + g_1(t - t_0, T)\,(y(t) - \hat{x}_1(t))$$
$$\dot{\hat{x}}_2(t) = u(t) + g_2(t - t_0, T)\,(y(t) - \hat{x}_1(t))$$

which has the error dynamics

$$\dot{\tilde{x}}_1(t) = \tilde{x}_2(t) - g_1(t - t_0, T)\tilde{x}_1(t)$$
$$\dot{\tilde{x}}_2(t) = -g_2(t - t_0, T)\tilde{x}_1(t).$$

Using (18)–(21) and the algorithm provided in Remark 1, we find the constants $\{\bar{p}_{i,j}\}$ to be $\bar{p}_{1,1} = \bar{p}_{2,2} = 1$, $\bar{p}_{1,2} = 0$, and $\bar{p}_{2,1} = -\frac{m+2}{T}$. Then, from (22) and (23), $g_1(t - t_0, T)$ and $g_2(t - t_0, T)$ are

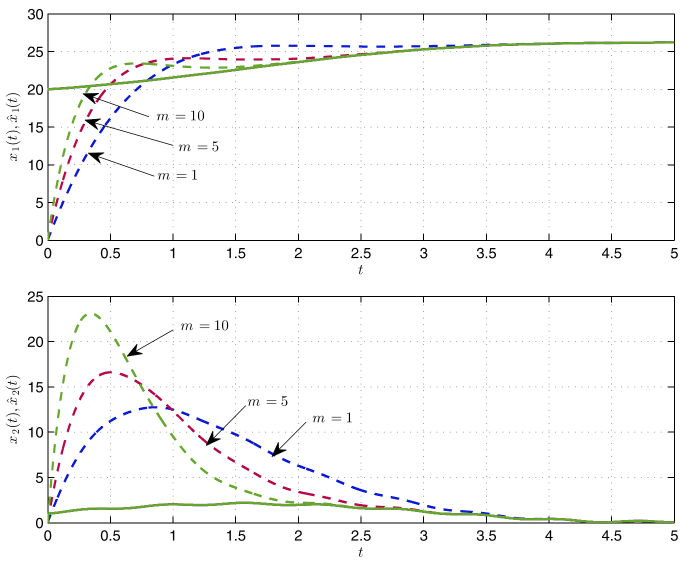

Fig. 4. Time histories of states and state estimates for Example 1, for three values of the observer parameter m.

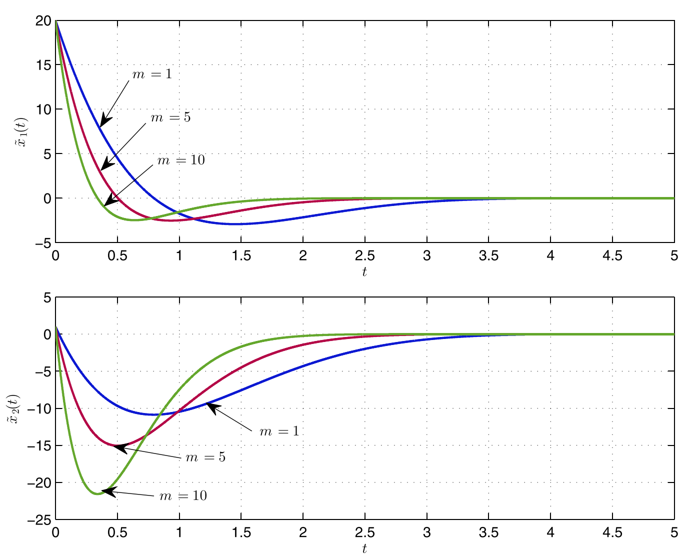

Fig. 5. Time histories of state estimation errors for the simulations shown in Fig. 4.

found to be

$$g_1(t-t_0,T) = l_1 + 2\frac{m+2}{T}\mu_1$$

$$g_2(t-t_0,T) = l_2 + l_1\frac{m+2}{T}\mu_1 + \frac{(m+1)(m+2)}{T^2}\mu_1^2.$$

Fig. 1 illustrates simulation of Example 1 with control input $u(t) = \sin(10t) + \cos(t)$ for three sets of initial conditions: $(x_1(0), x_2(0)) = (0,1)$, $(x_1(0), x_2(0)) = (0,5)$, and $(x_1(0), x_2(0)) = (0,10)$. In all cases, the observer was initialized to $(\hat{x}_1(0), \hat{x}_2(0)) = (0,0)$. The initial time is $t_0 = 0$, and the terminal time is $t_f = 5$. For observer parameters, we selected $l_1 = l_2 = 1$, $m = 1$, and $T = 5$. In Fig. 1, solid lines denote the states, and dashed lines denote the estimates. Fig. 2 shows the corresponding estimation error states versus time. Figs. 1 and 2 show that the convergence of the estimation errors to zero is achieved at the prescribed time $T = 5$, regardless of the initial conditions, which illustrates the fixed-time property of the observer. Fig. 3 shows the corresponding output estimation error injection terms versus time. As proven in Theorem 1, they remain uniformly bounded and converge to zero as $t \to t_0 + T$.

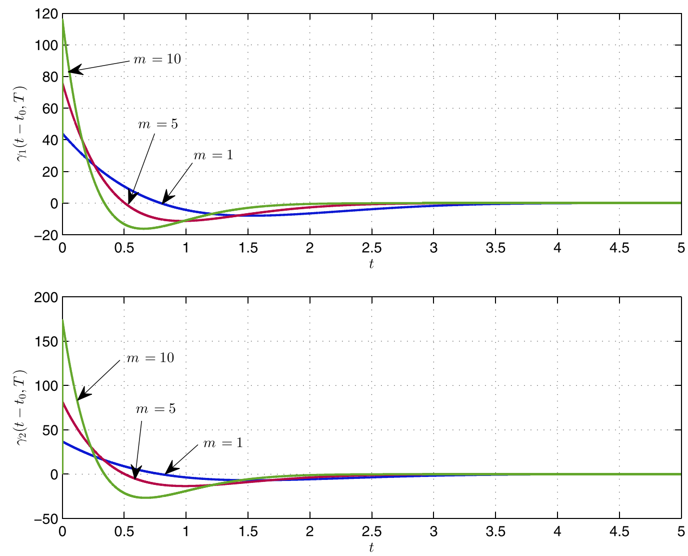

Fig. 6. Time histories of output estimation error injection terms for the simulations shown in Fig. 4.

Figs. 4–6 again illustrate simulation of Example 1 with control input $u(t) = \sin(10t) + \cos(t)$, but with initial conditions $(x_1(0), x_2(0)) = (20,1)$, $(\hat{x}_1(0), \hat{x}_2(0)) = (0,0)$, observer parameters $l_1 = l_2 = 1$ and $T = 5$, and the observer parameter $m \geq 1$ is varied. In Fig. 4, solid lines denote the states, and dashed lines denote the estimates. Figs. 5 and 6 show the corresponding estimation error states and output estimation error injection terms. Figs. 4–6 show that varying the parameter m tunes the transient responses of the estimation error states, but for all m used, the estimation error states converge to zero at the prescribed time $T = 5$.

V. Conclusion

The main benefit of the observer (7) is how easily it allows the user to prescribe the convergence time irrespective of initial conditions. In the presence of significant unmodeled dynamics or measurement disturbances, convergence of the estimation error will degrade to a nonzero neighborhood of the origin. For linear time-invariant systems with piecewise-constant measurement disturbances, the hybrid-system finite-time observer of Li and Sanfelice [19] may be more appropriate than (7).

Appendix

Proof of Lemma 2: The transformation (16) can be written as

$$\tilde{z} = P^*(\mu_1)\tilde{\zeta}$$

where $P^*(\mu_1)$ has elements given by (17)–(21), which show that $P^*(\mu_1)$ is unit lower triangular. Then, substituting in (15) for $\tilde{\zeta}$ and using $\mu = \mu_1^{n+m}$ gives

$$\tilde{z} = P^*(\mu_1)\mu_1^{n+m}\tilde{x}. \tag{55}$$

Now factoring out μ_1^{m+1} to the left and defining the matrix $P(\mu_1) := P^*(\mu_1)\mu_1^{n-1}$ gives (32) and (33). ■

Proof of Lemma 3: From (17)–(21), it is clear that the matrix $P^*(\mu_1)$ is unit lower triangular. Then, its inverse exists and is also unit lower triangular. Define $Q^* := (P^*)^{-1}(\mu_1)$. Since $P^*(\mu_1)$ is invertible, and since $\nu = \mu_1^{-1}$, we obtain from (55)

$$\tilde{x} = \nu^{n+m} Q^* \tilde{z}. \tag{56}$$

Since Q^* is the inverse of $P^*(\mu_1)$, the matrix Q^* obeys

$$P^*(\mu_1) Q^* = I_n \tag{57}$$

where I_n is the $n \times n$ identity matrix. After obtaining the elements of $P^*(\mu_1)$ from (17)–(21), we can determine the elements of Q^* by solving (57) using forward substitution. The result of this process yields the matrix $Q^*(\nu)$ with elements $\{q^*_{i,j}(\nu)\}$ of the form

$$q^*_{i,j}(\nu) = \bar{q}_{i,j} \nu^{j-i}$$

where the $\{\bar{q}_{i,j}\}$ are known functions of the known constants $\{\bar{p}_{i,j}\}$. Having determined the $\{q^*_{i,j}(\nu)\}$ in this way, we return to (56), which now reads

$$\tilde{x} = \nu^{n+m} Q^*(\nu) \tilde{z}. \tag{58}$$

Now factoring out ν^{m+1} to the left and defining the matrix $Q(\nu) := \nu^{n-1} Q^*(\nu)$ gives (34) and (35).

The finiteness of $\bar{q}$ follows from the fact that $|Q(\nu)|$ is a continuous function of a bounded argument $\nu \in (0, 1]$. ■

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