Prescribed-Time Consensus and Containment Control of Networked Multiagent Systems

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***Abstract*—In this paper, we present a new prescribed-time distributed control method for consensus and containment of networked multiple systems. Different from both regular finite-time control (where the finite settling time is not uniform in initial conditions) and the fixed-time control (where the settling time cannot be preassigned arbitrarily), the proposed one is built upon a novel scaling function, resulting in prespecifiable convergence time (the settling time can be preassigned as needed within any physically allowable range). Furthermore, the developed control scheme not only ensures that all the agents reach the average consensus in prescribed finite time under undirected connected topology, but also ensures that all the agents reach a prescribed-time consensus with the root's state being the group decision value under the directed topology containing a spanning tree with the root as the leader. In addition, we extend the result to prescribed-time containment control involving multiple leaders under directed communication topology. Numerical examples are provided to verify the effectiveness and the superiority of the proposed control.**

***Index Terms*—Containment, directed topology, networked multiple systems, prescribed-time consensus.**

I. Introduction

As FINITE time convergence is of special interest to many important applications that involve multiagent systems (MASs) [1], the past few years have witnessed sustained growing interest in finite time control of MAS, leading to fruitful results on finite time consensus and/or containment of MAS in [2]–[19]. It is noted that, however, most existing finite time control methods cannot guarantee the convergence within preset finite time in that the actual converging time period is not uniform nor prespecifiable. This is because the finite time T^* is determined by $T^* \leq [(V(t_0)^{1-\alpha})/(\gamma(1-\alpha))]$ [20], with $\gamma > 0$ and $0 < \alpha < 1$ being design parameters, and $V(t_0)$ being the Lyapunov function of system initial states, from which it is seen that the finite time T^* depends on both the initial conditions and the other design parameters, thus cannot be preset explicitly. In other words, for a given T^*, it is nontrivial to determine the corresponding value for the crucial parameters α and γ etc. The fixed-time consensus has also been investigated in the existing literatures such as in [21]–[23]. It should be emphasized that, although uniform in initial conditions, the settling time in fixed-time control cannot be preassigned arbitrarily (within any physically possible range) because the upper bound of the settling time is subject to certain restrictions. Moreover, the existing finite time control methods do not always lead to smooth control action. For instance, in [2]–[7], signum function is used to achieve finite time consensus of MAS under certain topology conditions, which makes the control action discontinuous. The methods suggested in [8]–[18] involve the fractional power state feedback, rendering control action nonsmooth.

The above analysis indicates that the problem of global and smooth distributed consensus and containment control for networked multiple systems featured with uniform prespecified finite convergence time has not been adequately addressed.

In this paper, we develop a new prescribed-time distributed control method for consensus and containment of MAS in which not only the control action is C^1 smooth everywhere but also the finite convergence time can be explicitly prespecified. For technical tractability and fair comparison, we focus on MAS with single integrator under communication constraints as considered by many other researchers (see [2]–[11]) and we make use of regular (rather than signum function or fractional power) feedback of the relative states of neighboring agents to solve the commonly and extensively studied finite time consensus and containment problem with a completely different approach. The contribution and novelty of the proposed solution is threefold.

1) Different from the traditional finite-time control methods which are based on the signum function or fraction

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power state feedback and thus are discontinuous or nonsmooth, the proposed finite time control is based on regular feedback of local system states, thus it generates smooth (C^1) control action everywhere.

2) In contrast to most existing finite-time control methods where the finite convergence time is determined by system initial condition or a number of design parameters, the proposed one renders the finite convergence time fully independent of initial condition and any other parameter, thus it can be uniformly prespecified.
3) By direct use of regular feedback of local system states, instead of fractional power state feedback, the control scheme is developed based on standard Lyapunov stability theory without involving fractional Lyapunov differential inequality, thus avoiding the technical difficulty in control design and stability analysis arising from fractional order dynamic systems.

Our method, gaining its inspiration from [24], is independent of and differs from [24] in several aspects.

1) We target to networked multiple systems with limited communication connection, whereas [24] is on finite-time control of single system.
2) We do not use the state transformation method as in [24], thus skillfully avoid using direct feedback of the state from each agent (direct state feedback might cause the state of each agent to converge to zero instead of reaching consensus around a nonzero value). Here only the relative states information of neighboring agents is utilized and there is no need for global information of the common origin.
3) Our control scheme allows the system to operate beyond the time interval T, in contrast to [24] that only works for $t \in [t_0, t_0 + T)$.

This paper is also different from [25] because in [25] only one leader is addressed with the aid of observer, while in this paper, three cases are considered, including leaderless consensus, leader–follower consensus with one leader and containment with multiple leaders, without using the observer.

II. Preliminaries

Throughout this paper, we use the following notations: $R^{n\times m}$ represents the set of $n\times m$ real matrices; I_N denotes the identity matrix of dimension N; 1_N (0_N) represents a column vector with each entry being 1 (0); and $\otimes$ denotes the Kronecker product.

A directed graph $\mathcal{G}$ is a pair (ι, ε) [26]–[28], with $\iota = \{\iota_1, \ldots, \iota_N\}$ being the set of nodes and $\varepsilon \subseteq \iota \times \iota$ the set of edges. $J = \{1, \ldots, N\}$ denotes the set of node indexes. The directed edge $\varepsilon_{ij} = (\iota_i, \iota_j)$ denotes that node ι_j can obtain information from ι_i, and $\mathcal{N}_i = \{\iota_j \in \iota | (\iota_j, \iota_i) \in \varepsilon\}$ is the set of in-neighbors of node ι_i. The weighted adjacency matrix is $\mathcal{A} = [a_{ij}] \in R^{N\times N}$, where $\varepsilon_{ji} \in \varepsilon \Leftrightarrow a_{ij} > 0$, otherwise, $a_{ij} = 0$. In addition, $a_{ii} = 0$ for all $i \in J$. The in-degree matrix is $\mathcal{B} = \mathrm{diag}(\mathcal{B}_1, \ldots, \mathcal{B}_N) \in R^{N\times N}$, with $\mathcal{B}_i = \sum_{j\in\mathcal{N}_i} a_{ij}$ being the weighted in-degree of node ι_i. The Laplacian matrix is defined as $L = [l_{ij}] = \mathcal{B} - \mathcal{A}$.

III. Problem Statement and Motivations

In the context of finite time control for MAS, the following single integrator model has been commonly utilized [2]–[11]:

$$\dot{x}_i = u_i, \quad i = 1, \ldots, N \tag{1}$$

where $x_i \in R^m$ and $u_i \in R^m$ are the system state and control input, respectively.

We say a control protocol u_i ($i \in J$) solves a finite-time consensus problem, if it solves a consensus problem, and for any given initial states, there exist a finite time T^* and a real number/vector x^* such that for all $i \in J$, $x_i \to x^*$ as $t \to T^*$ and $x_i = x^*$ when $t \geq T^*$. If $x^* = \sum_{i=1}^{N} x_i(t_0)/N$, it solves the finite-time average-consensus problem [29].

The standard first-order finite-time consensus protocol investigated by several researchers (e.g., [2]–[11]) can be unified as follows:

$$u_i = k \sum_{j\in\mathcal{N}_i} a_{ij}\mathrm{sign}\big(x_j - x_i\big)|x_j - x_i|^{\alpha_{ij}} \tag{2}$$

where $0 \leq \alpha_{ij} < 1$ and $k > 0$ is the control gain. Clearly, protocol (2) covers several different cases: when $\alpha_{ij} = 1$, it reduces to the typical asymptotical consensus protocol studied in [29]; when $\alpha_{ij} = 0$, it reduces to the finite-time protocol established in [2]–[6], which is discontinuous due to the using of the signum function; when $0 < \alpha_{ij} < 1$, it corresponds to the finite-time consensus control scheme employed in [8]–[11], which is continuous but nonsmooth with respect to state variables. It is worth mentioning that with $0 < \alpha_{ij} < 1$, the finite time T^*, within which the consensus is achieved, is determined by $T^* = [(V(t_0)^{1-\alpha})/(\gamma(1-\alpha))]$, where $\gamma > 0$ is some constant related to the control gain k and the fraction index α_{ij} as well as the second smallest eigenvalue $\lambda_2(L)$ of the Laplacian matrix L (which depends on the communication topology structure), and α is related to α_{ij}. We see at least three issues associated with the finite time T^*.

1) T^* depends not only on the control parameters k and α_{ij}, as well as the topology structure but also on the initial condition $V(t_0)$.
2) One can enlarge k or reduce α_{ij} (producing a larger γ or a smaller α) to obtain a smaller T^*, however, the control magnitude becomes larger with a smaller fraction index α_{ij}.
3) If a specific T^* is imposed, one has to literally find the corresponding design parameters γ and α (i.e., k and α_{ij}) based on $V(t_0)$ from $T^* = [(V(t_0)^{1-\alpha})/(\gamma(1-\alpha))]$, which cannot be explicitly predetermined due to the fact that α_{ij} is implicitly involved in the function and the initial condition might not be known *a priori*.

In this paper, we provide a solution to circumvent all the aforementioned shortcomings.

IV. Controller Design and Stability Analysis

Before moving on, we first introduce a time-varying scaling function as

$$\mu(t) = \begin{cases} \frac{T^h}{(T+t_0-t)^h}, & t \in [t_0, t_1) \\ 1, & t \in [t_1, \infty) \end{cases} \tag{3}$$

where $h > 2$ is any user-chosen real number, and $T \geq T_s > 0$ with T_s being the time period needed for signal processing/computing and information transmission/communication. Note that μ^{-q} $(q > 0)$ is monotonically decreasing on $[t_0, t_1)$, and $\mu(t_0)^{-q} = 1$ and $\lim_{t\to t_1^-} \mu(t)^{-q} = 0$. In addition

$$\dot{\mu}(t) = \begin{cases} \frac{h}{T}\mu^{1+\frac{1}{h}}, & t \in [t_0, t_1) \\ 0, & t \in [t_1, \infty) \end{cases} \tag{4}$$

here we use the right-hand derivative of $\mu(t)$ at $t = t_1$ as $\dot{\mu}(t_1)$.

Definition 1 [30]: Consider the system defined by

$$\dot{x}(t) = f(t, x(t)), \quad t \in R_+, \quad x(0) = x_0 \tag{5}$$

where $x \in R^m$ is the state vector, $f : R_+ \times R^m \to R^m$ is a nonlinear vector field locally bounded uniformly in time. The origin of system (5) is said to be globally uniformly *finite-time stable* if it is globally uniformly asymptotically stable and there exists a locally bounded function $T : R^m \to R_+ \bigcup\{0\}$, such that $x(t, x_0) = 0$ for all $t \geq T(x_0)$, where $x(t, x_0)$ is an arbitrary solution of the Cauchy problem (5). The function T is called the *settling-time function*.

Definition 2: The origin of system (5) is said to be globally *prescribed-time stable* if it is globally uniformly finite-time stable and the settling-time T is a user-assignable finite constant, i.e., $\forall 0 < T_p \leq T_{\max} < \infty$ (T_p denotes the physically possible time range), T can be prescribed such that $T_p \leq T \leq T_{\max}$, $\forall x_0 \in \Re^m$.

Lemma 1: Consider system (5). Let $V(x(t), t) : \mathcal{U}\times R_+ \to R$ be a continuously differentiable function and $\mathcal{U} \subset R^m$ be a domain containing the origin. If there exists a real constant $b > 0$ such that

$$V(0, t) = 0 \text{ and } V(x(t), t) > 0 \text{ in } \mathcal{U} - \{0\} \tag{6}$$

$$\dot{V} = -bV - 2\,\frac{\dot{\mu}}{\mu}\,V \text{ in } \mathcal{U} \tag{7}$$

on $[t_0, \infty)$, where $\dot{V} = (\partial V/\partial x)(x)f(t, x)$, then the origin of system (5) is prescribed-time stable with the prescribed time T given in (3). If $\mathcal{U} = R^m$, then the origin of system (5) is globally prescribed-time stable with the prescribed time T. In addition, for $t \in [t_0, t_1)$, it holds that

$$V(t) \leq \mu^{-2} \exp^{-b(t-t_0)} V(t_0) \tag{8}$$

and, for $t \in [t_1, \infty)$, it holds that

$$V(t) \equiv 0. \tag{9}$$

The proof of Lemma 1 is given in the Appendix.

The local neighborhood error is introduced as follows:

$$e_i = \sum_{j\in\mathcal{N}_i} a_{ij}\big(x_i - x_j\big), \quad i = 1, \ldots, N. \tag{10}$$

Denote by $E = [e_1^T, \ldots, e_N^T]^T \in R^{mN}$ and $X = [x_1^T, \ldots, x_N^T]^T \in R^{mN}$ such that $E = (L \otimes I_m)X$.

With the above preparation, we are now in a position to present the finite-time consensus control scheme

$$u_i = -\left(k + c\,\frac{\dot{\mu}}{\mu}\right)e_i, \quad i \in J \tag{11}$$

where $k > 0$ and $c > 0$ are design parameters, which can be represented in the following compact form:

$$U = -\left(k + c\,\frac{\dot{\mu}}{\mu}\right)E \tag{12}$$

where $U = [u_1^T, \ldots, u_N^T]^T \in R^{mN}$.

We investigate the prescribed-time consensus control under two cases: 1) undirected connected topology and 2) directed topology having a spanning tree with the root as the leader.

A. Networked MAS Under Undirected Topology

In this section, we establish the result in which the prescribed-time average consensus is achieved under the proposed control scheme (11) [or (12)], with the communication topology among the N agents satisfying the following assumption.

Assumption 1: The communication topology $\mathcal{G}$ is undirected and connected.

Let $x^* = (1/N)\sum_{i=1}^N x_i(t)$, and $\delta_i(t) = x_i(t) - x^*$ $(i = 1, \ldots, N)$, which denotes the disagreement between the state of the ith agent and the average state. Denote by $\delta = [\delta_1^T, \ldots, \delta_N^T]^T \in R^{mN}$, and then

$$\delta = X - (1_N \otimes I_m) \cdot \frac{1}{N} \cdot \big(1_N^T \otimes I_m\big)X. \tag{13}$$

The following lemma is useful for deriving system stability.

Lemma 2 [26]: If the undirected graph $\mathcal{G}$ is connected and $1_N^T X = 0$, then $X^T LX \geq \lambda_2(L)X^T X$, where $\lambda_2(L) > 0$ denotes the second smallest eigenvalue of the Laplacian matrix L.

Now we are ready to state the following result.

Theorem 1: Under Assumption 1, system (1) with the control law (11) [or (12)], where $c \geq 1/\lambda_2(L)$, is globally prescribed-time stable with the settling time T, and the prescribed-time average consensus is achieved in that

$$\|\delta(t)\| \leq \mu(t)^{-1} \exp^{-k\lambda_2(L)(t-t_0)} \|\delta(t_0)\| \tag{14}$$

for all $t \in [t_0, t_1)$. Further, the consensus is kept and U remains zero over $[t_1, \infty)$, and the control input signal remains C^1 smooth and uniformly bounded over the whole time interval $[t_0, \infty)$.

Proof: We first prove that the average consensus is achieved within the prespecified finite time T and then prove that the consensus will be kept over $[t_1, \infty)$ and the control input U remains zero for $t \in [t_1, \infty)$. Namely, we need to consider the following two cases.

Case 1: The average consensus is achieved within T and U is C^1 smooth and uniformly bounded on $[t_0, t_1)$.

Choosing the Lyapunov function candidate as

$$V = \frac{1}{2}\delta^T\delta. \tag{15}$$

By employing (12), (13), and $L1_N = 0_N$, we obtain that

$$\begin{aligned}\big(1_N^T \otimes I_m\big)\dot{X} &= \big(1_N^T \otimes I_m\big)\left[-\left(k + c\,\frac{\dot{\mu}}{\mu}\right)(L \otimes I_m)X\right] \\ &= -\left(k + c\,\frac{\dot{\mu}}{\mu}\right)\big[\big(1_N^T L\big) \otimes I_m\big]X = 0_m\end{aligned} \tag{16}$$

which yields

$$\dot{\delta} = \dot{X} - (1_N \otimes I_m) \cdot \frac{1}{N} \cdot \left(1_N^T \otimes I_m\right)\dot{X} = \dot{X} \tag{17}$$

and also we have

$$\begin{aligned}(L \otimes I_m)\delta &= (L \otimes I_m)X - (L \otimes I_m)(1_N \otimes I_m) \cdot \frac{1}{N}\left(1_N^T \otimes I_m\right)X \\ &= (L \otimes I_m)X - [(L1_N) \otimes I_m] \cdot \frac{1}{N}\left(1_N^T \otimes I_m\right)X \\ &= (L \otimes I_m)X. \end{aligned} \tag{18}$$

Taking the derivative of V from (17) and (18) gives

$$\begin{aligned}\dot{V} &= \delta^T\dot{\delta} = \delta^T\dot{X} = \delta^T\left[-\left(k + c\ \frac{\dot{\mu}}{\mu}\right)(L \otimes I_m)X\right] \\ &= -k\delta^T(L \otimes I_m)\delta - c\ \frac{\dot{\mu}}{\mu}\delta^T(L \otimes I_m)\delta \\ &= -k\sum_{l=1}^{m}\left(\delta^l\right)^T L\delta^l - c\ \frac{\dot{\mu}}{\mu}\sum_{l=1}^{m}\left(\delta^l\right)^T L\delta^l \end{aligned} \tag{19}$$

where $\delta^l = [\delta_{1l}, \ldots, \delta_{Nl}]^T \in R^N$ $(l = 1, \ldots, m)$. Note that

$$\begin{aligned}\left(1_N^T \otimes I_m\right)\delta &= \left(1_N^T \otimes I_m\right)\left[X - (1_N \otimes I_m) \cdot \frac{1}{N}\left(1_N^T \otimes I_m\right)X\right] \\ &= \left(1_N^T \otimes I_m\right)X - \frac{1}{N}\left[\left(1_N^T \cdot 1_N \cdot 1_N^T\right) \otimes I_m\right]X \\ &= \left(1_N^T \otimes I_m\right)X - \left(1_N^T \otimes I_m\right)X = 0_m \end{aligned} \tag{20}$$

from which we easily obtain that

$$1_N^T\delta^l = 0, \quad l = 1, \ldots, m. \tag{21}$$

According to Lemma 2, we derive from (21) and (19) that

$$\begin{aligned}\dot{V} &\le -k\sum_{l=1}^{m}\lambda_2(L)\left(\delta^l\right)^T\delta^l - c\ \frac{\dot{\mu}}{\mu}\sum_{l=1}^{m}\lambda_2(L)\left(\delta^l\right)^T\delta^l \\ &= -k\lambda_2(L)\delta^T\delta - c\ \frac{\dot{\mu}}{\mu}\lambda_2(L)\delta^T\delta. \end{aligned} \tag{22}$$

Note that $c \ge 1/\lambda_2(L)$, it then follows from (22) that:

$$\dot{V} \le -k\lambda_2(L)\delta^T\delta - \frac{\dot{\mu}}{\mu}\delta^T\delta = -2k\lambda_2(L)V - 2\frac{\dot{\mu}}{\mu}V. \tag{23}$$

According to Lemma 1, we have from (23) that

$$V(t) \le \mu(t)^{-2}\exp^{-2k\lambda_2(L)(t-t_0)} V(t_0) \tag{24}$$

on $[t_0, t_1)$. Thus,

$$\|\delta(t)\|^2 \le \mu(t)^{-2}\exp^{-2k\lambda_2(L)(t-t_0)} \|\delta(t_0)\|^2 \tag{25}$$

on $[t_0, t_1)$, which yields (14), and further

$$\|\delta(t)\| \to 0 \ \text{ as } \ t \to t_1^- \tag{26}$$

by noting that $\mu^{-2} \to 0$ as $t \to t_1^-$. From (26) we see that the average consensus is achieved within the prespecified finite time T.

Note that $L_\infty := \{x(t)|x : R_+ \to R, \sup_{t\in R_+}|x(t)| < \infty\}$. By recalling that $0 < \mu^{-1}, \mu^{-(1-1/h)}, \mu^{-1/h} \le 1$ and $0 < \exp^{-k\lambda_2(L)(t-t_0)} \le 1$, we then have

$$\begin{aligned}\|E\| &= \|(L \otimes I_m)X\| = \|(L \otimes I_m)\delta\| \\ &\le m\mu(t)^{-1}\exp^{-k\lambda_2(L)(t-t_0)}\|L\|\|\delta(t_0)\| \\ &\le m\|L\|\|\delta(t_0)\| \in L_\infty \end{aligned} \tag{27}$$

$$\begin{aligned}\left\|\frac{\dot{\mu}}{\mu}E\right\| &= \frac{h}{T}\mu^{\frac{1}{h}}\|(L \otimes I_m)X\| = \frac{h}{T}\mu^{\frac{1}{h}}\|(L \otimes I_m)\delta\| \\ &\le \frac{h}{T}m\mu^{-\left(1-\frac{1}{h}\right)}\exp^{-k\lambda_2(L)(t-t_0)}\|L\|\|\delta(t_0)\| \\ &\le \frac{h}{T}m\|L\|\|\delta(t_0)\| \in L_\infty \end{aligned} \tag{28}$$

both of which yield

$$\begin{aligned}\|U\| &\le k\|E\| + c\ \frac{h}{T}\left\|\frac{\dot{\mu}}{\mu}E\right\| \\ &\le \left(k\mu^{-\frac{1}{h}} + c\right)m\mu^{-\left(1-\frac{1}{h}\right)}\exp^{-k\lambda_2(L)(t-t_0)}\|L\|\|\delta(t_0)\| \\ &\le \left(k + c\ \frac{h}{T}\right)m\|L\|\|\delta(t_0)\| \in L_\infty \end{aligned} \tag{29}$$

on $[t_0, t_1)$. From (29), it is clear that the control input is uniformly bounded on $[t_0, t_1)$.

By examining U and dU/dt from (12) on $[t_0, t_1)$, we get

$$U = -\left(k + c\ \frac{h}{T}\ \mu^{\frac{1}{h}}\right)(L \otimes I_m)X \tag{30}$$

$$\begin{aligned}dU/dt &= -\left(k + c\ \frac{h}{T}\ \mu^{\frac{1}{h}}\right)\dot{E} - c\ \frac{h}{T}\ \frac{1}{h}\ \mu^{\frac{1}{h}-1}\dot{\mu}\ E \\ &= \left(k + c\ \frac{h}{T}\ \mu^{\frac{1}{h}}\right)(L \otimes I_m)\left(k + c\ \frac{h}{T}\ \mu^{\frac{1}{h}}\right)E \\ &\quad - c\ \frac{h}{T}\ \frac{1}{T}\ \mu^{\frac{2}{h}}E \\ &= \left(k^2 + 2kc\ \frac{h}{T}\ \mu^{\frac{1}{h}} + c^2\ \frac{h^2}{T^2}\ \mu^{\frac{2}{h}}\right)\left(L^2 \otimes I_m\right)X \\ &\quad - c\ \frac{h}{T}\ \frac{1}{T}\ \mu^{\frac{2}{h}}(L \otimes I_m)X \end{aligned} \tag{31}$$

from which we see that both U and dU/dt are continuous with respect to X on $[t_0, t_1)$. Since X is continuous with respect to t according to (1), we then conclude that both U and dU/dt are continuous with respect to t on $[t_0, t_1)$, and therefore U is C^1 smooth with respect to t on $[t_0, t_1)$.

Case 2: The consensus is kept and the control input U remains zero over $[t_1, \infty)$.

By choosing the same Lyapunov function as in (15), $V = (1/2)\delta^T\delta$, on $t \in [t_1, \infty)$, and following the same procedure as in (17)–(23) with the control law (11) [or (12)], we readily obtain:

$$\dot{V} \le -2k\lambda_2(L)V \le 0, \quad t \in [t_1, \infty). \tag{32}$$

By noting that X is continuous at $t = t_1$ from (1), we then have $V(t)$ is continuous at $t = t_1$, and thus,

$$V(t_1) = \lim_{t\to t_1^-}\frac{1}{2}\delta^T\delta = 0. \tag{33}$$

Both (32) and (33) yield

$$0 \le V(t) \le V(t_1) = 0, \quad t \in [t_0, \infty). \tag{34}$$

That is, $V(t) \equiv 0$ on $[t_1, \infty)$. Thus $\delta(t) \equiv 0_{mN}$, and then $E \equiv 0_{mN}$ on $[t_1, \infty)$. From the definition of U in (12), we deduce that $U \equiv 0_{mN}$ on $[t_1, \infty)$. All of these imply that the consensus is kept and the control input U remains zero over $[t_1, \infty)$ with the control law (12).

From the proofs for $t \in [t_0, t_1)$ and $t \in [t_1, \infty)$, we see that the average consensus is achieved within the prespecified finite time T and is kept over $[t_1, \infty)$, and further, the control input U is at least C^1 smooth and uniformly bounded on $[t_0, t_1)$ and remains zero over $[t_1, \infty)$.

Now we show that the control input is uniformly bounded and at least C^1 smooth over the whole time interval $[t_0, \infty)$.

In fact, from (29) and the fact that U remains zero over $[t_1, \infty)$, it is straightforward that U is uniformly bounded over $[t_0, \infty)$.

Also, from the above analysis in the two cases, it is clear that U is C^1 smooth with respect to t except at $t = t_1$. So we just need to verify that U and dU/dt exist and are continuous with respect to t at $t = t_1$. First of all, it is clear that $\lim_{t\to t_1^-} U = 0_{mN}$ from the second inequality in (29) and $\lim_{t\to t_1^+} U = 0_{mN} = U(t_1)$ from the proof in case 2, implying that U exists and is continuous at $t = t_1$. Now we examine each term of dU/dt on the right hand of (31) to prove that dU/dt exists and is continuous at $t = t_1$. Upon using the fact that $\mu^{-1} \to 0$, $\mu^{-(1-1/h)} \to 0$, $\mu^{-(1-2/h)} \to 0$ as $t \to t_1^-$, and $\|\mu\delta\| \in L_\infty$ on $[t_0, t_1)$ guaranteed by (14), it is clear that

$$\begin{aligned} &\|(L\otimes I_m)E\| \\ &\quad = \|(L\otimes I_m)(L\otimes I_m)\delta\| = \mu^{-1}\|\left(L^2\otimes I_m\right)\mu\delta\| \\ &\quad \le \mu^{-1}\|(L^2\otimes I_m)\| \times \|\mu\delta\| \to 0 \end{aligned} \tag{35}$$

$$\begin{aligned} &\|\mu^{\frac{1}{h}}(L\otimes I_m)E\| \\ &\quad \le \mu^{-\left(1-\frac{1}{h}\right)}\|\left(L^2\otimes I_m\right)\| \times \|\mu\delta\| \to 0 \end{aligned} \tag{36}$$

$$\begin{aligned} &\|\mu^{\frac{2}{h}}(L\otimes I_m)E\| \\ &\quad \le \mu^{-\left(1-\frac{2}{h}\right)}\|\left(L^2\otimes I_m\right)\| \times \|\mu\delta\| \to 0 \end{aligned} \tag{37}$$

$$\|\mu^{\frac{2}{h}}E\| \le \mu^{-\left(1-\frac{2}{h}\right)}\|(L\otimes I_m)\| \times \|\mu\delta\| \to 0 \tag{38}$$

as $t \to t_1^-$. By inserting (35)-(38) into (31), we obtain $\|dU/dt\| \to 0$ as $t \to t_1^-$. Then we have

$$\lim_{t\to t_1^-} \|dU/dt\| = 0 = \lim_{t\to t_1^+} \|dU/dt\| \tag{39}$$

and therefore,

$$\lim_{t\to t_1^-} dU/dt = 0_{mN} = \lim_{t\to t_1^+} dU/dt \tag{40}$$

which means that dU/dt exists and is continuous at $t = t_1$. By noting that both U and dU/dt exist and are continuous at $t = t_1$, we then concluded from the definitions of the continuousness and smoothness that U is C^1 smooth with respect to t at $t = t_1$, and thus U is C^1 smooth with respect to t over $[t_0, \infty)$. ■

Remark 1: It is worth noting that the gain $(\dot{\mu}/\mu) = (h/T)\mu^{(1/h)}$ on $[t_0, t_1)$ inside the control law (12) plays a crucial role in achieving the prescribed-time convergence. Although $\mu^{(1/h)}$ grows to infinity when t approaches t_1, the control input U remains bounded, as indicated by (29). This is because the term $\mu^{(1/h)}E$ in the control input U is always bounded even if $\mu^{(1/h)}$ going to infinity. Even though, it may happen in practice that the measurement of the neighborhood error signal E could become noisy, which would result in a product of $\mu^{(1/h)}$ and E (with the former growing unbounded, while the later E not decaying fully to zero), the simple and effective way to address this issue is setting T in μ on $[t_0, t_1)$ to a larger value than the desired finite settling time, that is,

$$\mu = \frac{\bar{T}^h}{\left(\bar{T} + t_0 - t\right)^h}, \quad t \in [t_0, t_1) \tag{41}$$

where $\bar{T} > T$. By doing this, it can prevent the gain from becoming infinite over the desired convergence time T but with some sacrifice on the convergence accuracy. In fact, with such defined μ in (41), by choosing the Lyapunov function $V = \delta^T\delta/2$ and following the same procedure as in the proof of (15)–(25), we can still arrive at (14), i.e., $\|\delta(t)\| \le \mu^{-1}\exp^{-k\lambda_2(L)(t-t_0)}\|\delta(t_0)\|$. In this case, although μ^{-1} does not converge to zero, it converges to a small neighborhood of zero as $t \to t_1^-$. Further, since $\dot{V} \le -2k\lambda_2(L)V$ on $[t_1, \infty)$, we see that $V(t)$ decrease consistently and therefore the average error δ decrease consistently as t increases.

Remark 2: It is interesting to note that, in contrast to most existing finite time control methods, the proposed finite time control scheme, as shown in (11), is built not only upon regular (rather than fractional power) state feedback but also upon time-varying (rather than constant) gain. It is such structural feature that renders the convergence time not only finite but also user preassignable. Also, with the time-varying gain, the proposed control avoids excessively large initial driving force as encountered in many high and constant gain-based control methods, because the initial value of the time-varying gain here can be set as small as desired, rendering the initial control effort small. It should be mentioned that the nature of the time-varying gain as involved in the control scheme, although calling for gain updating according to the given simple analytical algorithm, does not cause any technical difficulty for implementation because the computation involved in updating the gain is even simpler than those involved in updating the parameters in traditional adaptive control.

B. Networked MAS Under Directed Topology Having Spanning Tree With the Root As Leader

We establish a prescribed-time consensus problem under the topology that the graph $\mathcal{G}$ has a directed spanning tree with the root node as the leader, in which the consensus is achieved with the root's state being the group decision value.

Assumption 2: The graph $\mathcal{G}$ has a directed spanning tree with the root node ι_i $(i \in J)$ as the leader.

Without losing generality, we assume that the root vertex $i = 1$. Note that the root ι_1 has no neighbors, the Laplacian L is then partitioned as $\begin{bmatrix} 0 & 0_{1\times(N-1)} \\ L_2 & L_1 \end{bmatrix}$, where $L_2 \in R^{(N-1)\times 1}$ and $L_1 \in R^{(N-1)\times(N-1)}$. Under Assumption 2, it can be verified that L_1 is a nonsingular M-matrix (we call $A = [a_{ij}] \in R^{N\times N}$ a nonsingular M-matrix, if $a_{ij} < 0$, $i \ne j$, and all eigenvalues of A have positive real parts), and is diagonally dominant. The following Lemma, borrowing from [31], gives a useful property of L_1.

Lemma 3 [31]: Under Assumption 2, there exists a positive diagonal matrix $\tilde{P} = \mathrm{diag}\{p_2, \ldots, p_N\}$ such that

$$\tilde{Q} = \tilde{P}L_1 + L_1^T\tilde{P} > 0 \tag{42}$$

in which $p_2, \ldots, p_N$ are determined by $[p_2, \ldots, p_N]^T = (L_1^T)^{-1} 1_{N-1}$.

Let

$$z_i(t) = x_i(t) - x_1, \quad i = 2, \ldots, N. \tag{43}$$

Denote by $\tilde{X} = [x_2^T, \ldots, x_N^T]^T \in R^{m(N-1)}$, $\tilde{Z} = [z_2^T, \ldots, z_N^T]^T \in R^{m(N-1)}$, and $\tilde{E} = [e_2^T, \ldots, e_N^T]^T \in R^{m(N-1)}$ with e_i $(i = 2, \ldots, N)$ being defined the same as in (10). By computation, we readily derive that

$$\tilde{E} = (L_1 \otimes I_m)\big[\tilde{X} - (1_{N-1} \otimes I_m)x_1\big] = (L_1 \otimes I_m)\tilde{Z}. \tag{44}$$

Theorem 2: Under Assumption 2, the control law (11) [or (12)], with $c \geq [(2\lambda_{\max}(\tilde{P}))/(\lambda_1(\tilde{Q}))]$, solves a prescribed-time consensus problem with the group decision value x_1, and the consensus is achieved within the prespecified finite time T in that

$$\begin{aligned}\|\tilde{Z}(t)\| \leq\ & \mu(t)^{-1} \exp^{-\frac{k\lambda_1(\tilde{Q})}{2\lambda_{\max}(\tilde{P})}(t-t_0)} \\ & \times \sqrt{\frac{\lambda_{\max}(\tilde{P})}{\lambda_{\min}(\tilde{P})}} \left\|\left(L_1^{-1} \otimes I_m\right)\right\| \cdot \|\tilde{E}(t_0)\|\end{aligned} \tag{45}$$

for all $t \in [t_0, t_1)$. Furthermore, the consensus is kept over $[t_1, \infty)$ in which U remains zero, and the control input U remains C^1 smooth and uniformly bounded over $[t_0, \infty)$.

Proof: We only need to show two cases.

Case 1: The consensus is achieved within T and the control input U is C^1 smooth and uniformly bounded on $[t_0, t_1)$.

Choosing the Lyapunov function candidate as

$$\tilde{V} = \tilde{E}^T\big(\tilde{P} \otimes I_m\big)\tilde{E}. \tag{46}$$

According to (12), we see that

$$\dot{\tilde{X}} = -\left(k + c\,\frac{\dot{\mu}}{\mu}\right)\tilde{E} \tag{47}$$

and x_1 is time-invariant due to the fact that $a_{11} = \cdots = a_{1N} = 0$. Thus we have

$$\begin{aligned}\dot{\tilde{E}} &= (L_1 \otimes I_m)\left[\dot{\tilde{X}} - (1_{N-1} \otimes I_m)\dot{x}_1\right] = (L_1 \otimes I_m)\dot{\tilde{X}} \\ &= -\left(k + c\,\frac{\dot{\mu}}{\mu}\right)(L_1 \otimes I_m)\tilde{E}.\end{aligned} \tag{48}$$

From Lemma 3 and (48), we derive the derivative of $\tilde{V}$ as

$$\begin{aligned}\dot{\tilde{V}} &= 2\tilde{E}^T\big(\tilde{P} \otimes I_m\big)\dot{\tilde{E}} \\ &= 2\tilde{E}^T\big(\tilde{P} \otimes I_m\big)\left[-\left(k + c\,\frac{\dot{\mu}}{\mu}\right)(L_1 \otimes I_m)\tilde{E}\right] \\ &= -\left(k + c\,\frac{\dot{\mu}}{\mu}\right)\tilde{E}^T\big[\big(\tilde{P}L_1 + L_1^T\tilde{P}\big) \otimes I_m\big]\tilde{E} \\ &= -\left(k + c\,\frac{\dot{\mu}}{\mu}\right)\tilde{E}^T\left(\tilde{Q} \otimes I_m\right)\tilde{E} \\ &\leq -k\lambda_1(\tilde{Q})\tilde{E}^T\tilde{E} - c\,\frac{\dot{\mu}}{\mu}\,\lambda_1(\tilde{Q})\tilde{E}^T\tilde{E} \\ &\leq -k\frac{\lambda_1(\tilde{Q})}{\lambda_{\max}(\tilde{P})}\tilde{V} - c\,\frac{\dot{\mu}}{\mu}\,\frac{\lambda_1(\tilde{Q})}{\lambda_{\max}(\tilde{P})}\tilde{V} \\ &\leq -\frac{k\lambda_1(\tilde{Q})}{\lambda_{\max}(\tilde{P})}\tilde{V} - 2\,\frac{\dot{\mu}}{\mu}\,\tilde{V}\end{aligned} \tag{49}$$

where the fact that $c \geq [(2\lambda_{\max}(\tilde{P}))/(\lambda_1(\tilde{Q}))]$ has been used. It thus follows from Lemma 1 that:

$$\tilde{V}(t) \leq \mu^{-2} \exp^{-\frac{k\lambda_1(\tilde{Q})}{\lambda_{\max}(\tilde{P})}(t-t_0)}\,\tilde{V}(t_0) \tag{50}$$

which further implies that

$$\|\tilde{E}(t)\|^2 \leq \mu^{-2} \exp^{-\frac{k\lambda_1(\tilde{Q})}{\lambda_{\max}(\tilde{P})}(t-t_0)}\,\frac{\lambda_{\max}(\tilde{P})}{\lambda_{\min}(\tilde{P})}\|\tilde{E}(t_0)\|^2 \tag{51}$$

and then that

$$\begin{aligned}\|\tilde{Z}(t)\| &= \|(L_1 \otimes I_m)^{-1}\tilde{E}(t)\| \leq \|(L_1 \otimes I_m)^{-1}\| \cdot \|\tilde{E}(t)\| \\ &\leq \mu^{-2} \exp^{-\frac{k\lambda_1(\tilde{Q})}{2\lambda_{\max}(\tilde{P})}(t-t_0)} \\ &\quad \times \sqrt{\frac{\lambda_{\max}(\tilde{P})}{\lambda_{\min}(\tilde{P})}}\|(L_1 \otimes I_m)^{-1}\|\|\tilde{E}(t_0)\|\end{aligned} \tag{52}$$

which yields (45), and further

$$\|\tilde{Z}(t)\| \to 0 \ \text{ as } \ t \to t_1^-. \tag{53}$$

This means that $X \to (1_N \otimes I_m)x_1$ as $t \to t_1^-$, that is, the consensus with the group decision value x_1 is achieved within the prespecified finite time T.

In the following, we prove that the control input is uniformly bounded on $[t_0, t_1)$. Note that $\|E\| = \|\tilde{E}\|$ due to the fact that $e_1 = \sum_{j\in\mathcal{N}_1} a_{1j}(x_1 - x_j) = 0_m$. Thus we get from (51) that

$$\begin{aligned}\|\mu E(t)\|^2 &= \mu^2\|E(t)\|^2 = \mu^2\|\tilde{E}(t)\|^2 \\ &\leq \exp^{-\frac{k\lambda_1(\tilde{Q})}{\lambda_{\max}(\tilde{P})}(t-t_0)}\,\frac{\lambda_{\max}(\tilde{P})}{\lambda_{\min}(\tilde{P})}\|\tilde{E}(t_0)\|^2 \\ &\leq \frac{\lambda_{\max}(\tilde{P})}{\lambda_{\min}(\tilde{P})}\|\tilde{E}(t_0)\|^2 \in L_\infty\end{aligned} \tag{54}$$

on $[t_0, t_1)$, which further implies

$$\|E\| \in L_\infty \ \text{ and } \ \left\|\frac{\dot{\mu}}{\mu}E\right\| = \frac{h}{T}\mu^{-\left(1-\frac{1}{h}\right)}\|\mu E\| \in L_\infty \tag{55}$$

on $[t_0, t_1)$, and therefore,

$$\|U\| \leq k\|E\| + c\left\|\frac{\dot{\mu}}{\mu}E\right\| \in L_\infty \tag{56}$$

on $[t_0, t_1)$, which implies that U is uniformly bounded on $[t_0, t_1)$.

Similar to the proof in (30) and (31), we can derive that U is C^1 smooth on $[t_0, t_1)$.

Case 2: The consensus is kept and the control input U remains zero over $[t_1, \infty)$.

By following the same line as in the proof of (32)–(34), we establish that the consensus is kept over $[t_1, \infty)$, in which U remains zero.

Furthermore, by following the same line as in (35)–(40) we readily get that the control input U is C^1 smooth and uniformly bounded over the whole time interval $[t_0, \infty)$. ∎

V. Extension to the Case With Multiple Leaders

We extend the above obtained consensus results to the case in which there exist multiple leaders in this section. In the presence of multiple leaders, the prescribed-time containment control problem arises.

Suppose that there are M ($M < N$) leader agents and $N-M$ follower agents in the directed graph $\mathcal{G}$, where a leader is an agent that has no in-neighbor and a follower is an agent that has at least one in-neighbor. Without loss of generality, we let $\mathcal{L} = \{1, 2, \ldots, M\}$ and $\mathcal{F} = \{M+1, M+2, \ldots, N\}$ be the leader set and the follower set, respectively. The Laplacian L in such case is represented as

$$\begin{bmatrix} 0_{M\times M} & 0_{M\times(N-M)} \\ L_2 & L_1 \end{bmatrix} \tag{57}$$

where $L_2 \in R^{(N-M)\times M}$ and $L_1 \in R^{(N-M)\times(N-M)}$.

Assumption 3: Suppose that for each follower, there exists at least one leader that has a directed path to it.

Under Assumption 3, we establish one property of L_1 in the following lemma.

Lemma 4: There exists a positive diagonal matrix $P_{\mathcal{F}} = \mathrm{diag}\{p_{M+1}, \ldots, p_N\} \in R^{(N-M)\times(N-M)}$ such that

$$Q_{\mathcal{F}} = P_{\mathcal{F}} L_1 + L_1^T P_{\mathcal{F}} > 0 \tag{58}$$

in which $p_{M+1}, \ldots, p_N$ are chosen as $[p_{M+1}, \ldots, p_N]^T = (L_1^T)^{-1} 1_{N-M}$. In addition, each entry of $-L_1 L_2$ is non-negative, and each row of $-L_1 L_2$ has a sum equal to one.

Proof: The second assertion is well known (see [19, Lemma 4]). In the following, we show the first assertion. According to [19], we know that L_1 is a nonsingular M-matrix under Assumption 3. From [32, Th. 4.25], we deduce that L_1^{-1} exists and is non-negative. Therefore, it satisfies the same condition as in [31, Lemma 4], and then the first assertion is proved as in the proof of [31, Lemma 4]. ■

Definition 3 [19]: We say the MAS (1) achieves containment in prespecified finite time T if for any initial states, there exist non-negative constants β_j ($j \in \mathcal{L}$) satisfying $\sum_{j=1}^{M} \beta_j = 1$ such that for all $t \geq T$ and $i \in \mathcal{F}$, $x_i - \sum_{j=1}^{M} \beta_j x_j = 0_m$.

Let $E_{\mathcal{L}} = [e_1^T, \ldots, e_M^T]^T$, $E_{\mathcal{F}} = [e_{M+1}^T, \ldots, e_N^T]^T$, $x_{\mathcal{L}} = [x_1^T, \ldots, x_M^T]^T$, and $x_{\mathcal{F}} = [x_{M+1}^T, \ldots, x_N^T]^T$. Then it holds

$$\begin{aligned} E_{\mathcal{F}} &= (L_1 \otimes I_m) x_{\mathcal{F}} + (L_2 \otimes I_m) x_{\mathcal{L}} \\ &= (L_1 \otimes I_m)\Big[x_{\mathcal{F}} - \Big(\Big(-L_1^{-1} L_2\Big) \otimes I_m\Big) x_{\mathcal{L}}\Big]. \end{aligned} \tag{59}$$

According to Lemma 4, the prescribed-time containment objective is achieved if $E_{\mathcal{F}}$ converges to zero in the prespecified finite time T.

We present the containment result in the following theorem.

Theorem 3: Consider system (1) under Assumption 3. The control scheme (11) [or (12)], with $c \geq [(2\lambda_{\max}(P_{\mathcal{F}}))/(\lambda_1(Q_{\mathcal{F}}))]$, solves the prescribed-time containment problem and the containment is achieved within the prespecified finite time T in that

$$\begin{aligned} \|E_{\mathcal{F}}(t)\| &\leq \mu^{-1} \exp^{-\frac{k\lambda_1(Q_{\mathcal{F}})}{2\lambda_{\max}(P_{\mathcal{F}})}(t-t_0)} \\ &\quad \times \sqrt{\frac{\lambda_{\max}(P_{\mathcal{F}})}{\lambda_{\min}(P_{\mathcal{F}})}} \|(L_1 \otimes I_m)^{-1}\| \cdot \|E_{\mathcal{F}}(t_0)\| \end{aligned} \tag{60}$$

for all $t \in [t_0, t_1)$. Furthermore, the containment will be kept over $[t_1, \infty)$ in which U remains zero, and the control input signal remains C^1 smooth and uniformly bounded over $[t_0, \infty)$.

Proof: We need to show two cases.

Case 1: The containment is achieved within T and the control input U is C^1 smooth and uniformly bounded on $[t_0, t_1)$.

Choosing the Lyapunov function candidate as

$$V_{\mathcal{F}} = E_{\mathcal{F}}^T (P_{\mathcal{F}} \otimes I_m) E_{\mathcal{F}}. \tag{61}$$

According to (11), we have

$$\begin{aligned} \dot{E}_{\mathcal{F}} &= (L_1 \otimes I_m)\Big[\dot{x}_{\mathcal{F}} - \Big(\Big(-L_1^{-1} L_2 \otimes I_m\Big)\dot{x}_{\mathcal{L}}\Big)\Big] \\ &= (L_1 \otimes I_m)\dot{x}_{\mathcal{F}} = -\left(k + c\,\frac{\dot{\mu}}{\mu}\right)(L_1 \otimes I_m) E_{\mathcal{F}} \end{aligned} \tag{62}$$

in which we have used the fact that $\dot{x}_{\mathcal{L}} = 0$. With $c \geq [(2\lambda_{\max}(P_{\mathcal{F}}))/(\lambda_1(Q_{\mathcal{F}}))]$, we derive from Lemma 4 and (62) that

$$\begin{aligned} \dot{V}_{\mathcal{F}} &= 2E_{\mathcal{F}}^T (P_{\mathcal{F}} \otimes I_m)\dot{E}_{\mathcal{F}} \\ &= 2E_{\mathcal{F}}^T (P_{\mathcal{F}} \otimes I_m)\left[-\left(k + c\,\frac{\dot{\mu}}{\mu}\right)(L_1 \otimes I_m) E_{\mathcal{F}}\right] \\ &= -\left(k + c\,\frac{\dot{\mu}}{\mu}\right) E_{\mathcal{F}}^T \big[\big(P_{\mathcal{F}} L_1 + L_1^T P_{\mathcal{F}}\big) \otimes I_m\big] E_{\mathcal{F}} \\ &= -\left(k + c\,\frac{\dot{\mu}}{\mu}\right) E_{\mathcal{F}}^T (Q_{\mathcal{F}} \otimes I_m) E_{\mathcal{F}} \\ &\leq -k\lambda_1(Q_{\mathcal{F}}) E_{\mathcal{F}}^T E_{\mathcal{F}} - c\,\frac{\dot{\mu}}{\mu}\,\lambda_1(Q_{\mathcal{F}}) E_{\mathcal{F}}^T E_{\mathcal{F}} \\ &\leq -k\frac{\lambda_1(Q_{\mathcal{F}})}{\lambda_{\max}(P_{\mathcal{F}})} V_{\mathcal{F}} - c\,\frac{\dot{\mu}}{\mu}\,\frac{\lambda_1(Q_{\mathcal{F}})}{\lambda_{\max}(P_{\mathcal{F}})} V_{\mathcal{F}} \\ &\leq -\frac{k\lambda_1(Q_{\mathcal{F}})}{\lambda_{\max}(P_{\mathcal{F}})} V_{\mathcal{F}} - 2\,\frac{\dot{\mu}}{\mu}\,V_{\mathcal{F}}. \end{aligned} \tag{63}$$

According to Lemma 1, we get from (63) that

$$V_{\mathcal{F}}(t) \leq \mu^{-2} \exp^{-\frac{k\lambda_1(Q_{\mathcal{F}})}{\lambda_{\max}(P_{\mathcal{F}})}(t-t_0)} V_{\mathcal{F}}(t_0). \tag{64}$$

This implies

$$\begin{aligned} \|E_{\mathcal{F}}(t)\|^2 &\leq \mu^{-2} \exp^{-\frac{k\lambda_1(Q_{\mathcal{F}})}{\lambda_{\max}(P_{\mathcal{F}})}(t-t_0)} \\ &\quad \times \frac{\lambda_{\max}(P_{\mathcal{F}})}{\lambda_{\min}(P_{\mathcal{F}})} \|E_{\mathcal{F}}(t_0)\|^2 \end{aligned} \tag{65}$$

and then

$$\begin{aligned} \|Z_{\mathcal{F}}(t)\| &= \|(L_1 \otimes I_m)^{-1} E_{\mathcal{F}}(t)\| \leq \|(L_1 \otimes I_m)^{-1}\| \|E_{\mathcal{F}}(t)\| \\ &\leq \mu^{-2} \exp^{-\frac{k\lambda_1(Q_{\mathcal{F}})}{2\lambda_{\max}(P_{\mathcal{F}})}(t-t_0)} \\ &\quad \times \sqrt{\frac{\lambda_{\max}(P_{\mathcal{F}})}{\lambda_{\min}(P_{\mathcal{F}})}} \|(L_1 \otimes I_m)^{-1}\| \|E_{\mathcal{F}}(t_0)\| \end{aligned} \tag{66}$$

which yields (60). From (66) we get

$$\|Z_{\mathcal{F}}(t)\| \to 0 \;\text{ as }\; t \to t_1^- \tag{67}$$

that is, $X_{\mathcal{F}} \to (L_1^{-1} L_2 \otimes I_m) X_{\mathcal{L}}$ as $t \to t_1^-$, meaning that the containment is achieved within the prespecified finite time T.

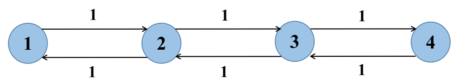
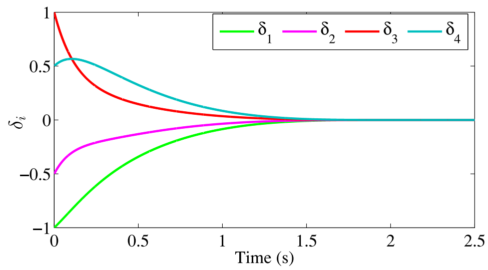
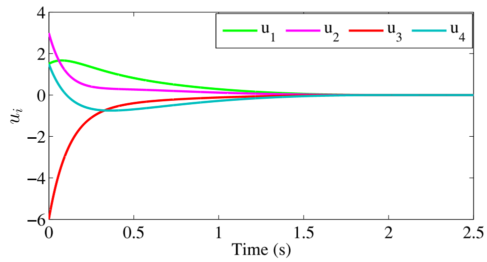


Fig. 1. Communication topology in Example 1.



(a)



(b)

Fig. 2. System response under proposed control (11). (a) δ_i. (b) u_i.

The C^1 smoothness and uniformly boundedness of U on $[t_0, t_1)$ can be shown by following the same line as in the proof of (54)–(56).

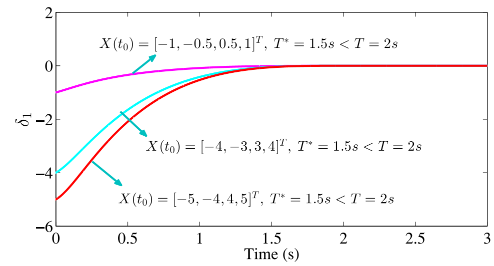
Case 2: The containment is kept and the control input U remains zero over $[t_1, \infty)$.

The prescribed-time containment result and the C^1 smoothness and the uniformly boundedness of U over $[t_0, \infty)$ can be established by following the procedure as used at the end of the proof in Theorem 1. ■

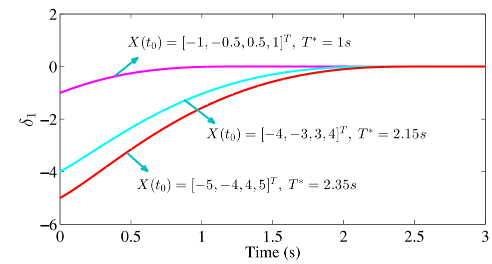
VI. Numerical Simulation

To validate the effectiveness of the proposed prescribed-time consensus and containment control schemes, two examples are simulated in this section.

Example 1 (Consensus): In this example, we compare the performance of the proposed prescribed-time consensus control (11) with the one (2) derived by [9]. The MAS with four members under the communication topology as shown in Fig. 1 is considered. For fair comparison, we need to examine the performance under the same initial control value for both control schemes. This is ensured by setting the control gains for the two control schemes as follows: $k = 0.3$ and $c = 1.8$ for (11) and $k = 3$ for (2). We choose the other design parameters as $h = 3$ and $T = 2$ s in (11) and $\alpha_{ij} = 0.6$ in (2), respectively. Three set of initial states are tested: 1) $X(t_0) = [-1, -0.5, 0.5, 1]^T$; 2) $X(t_0) = [-4, -3, 3, 4]^T$; and 3) $X(t_0) = [-5, -4, 4, 5]^T$, with $t_0 = 0$.

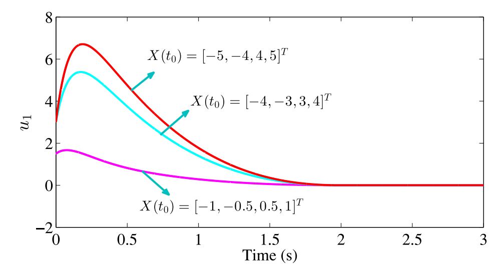


(a)

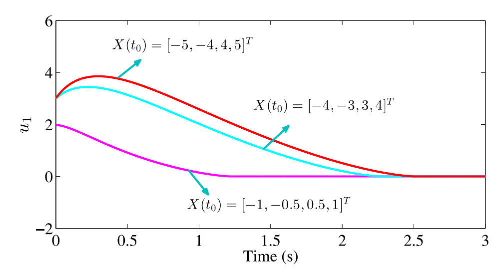


(b)

Fig. 3. Performance comparison between the two control schemes under three different initial conditions. (a) δ_1 under proposed control (11). (b) δ_1 under control (2) in [9].



(a)



(b)

Fig. 4. Comparison of the control inputs under the two control schemes with three different initial states. (a) u_1 under proposed control (11). (b) u_1 under control (2) in [9].

The results are shown in Figs. 2–4, where Fig. 2(a) shows the error convergence and Fig. 2(b) is the control input signal produced by (11), from which we see that the average errors of all the agents converge to zero within the prespecified finite time T and the control input signals are smooth and uniformly bounded. To compare the performance between the prescribed-time control proposed herein and the one developed by [9], three different initial conditions are tested and the

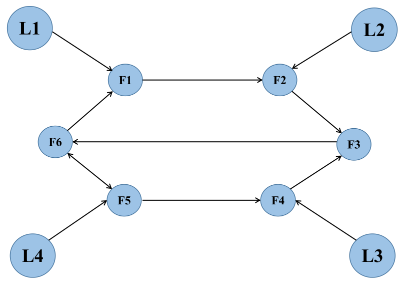


Fig. 5. Communication topology in Example 2.

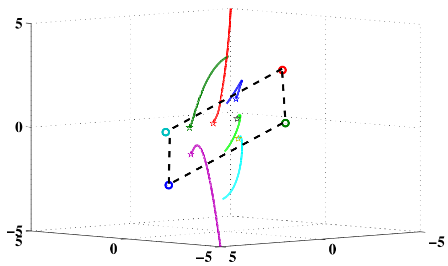
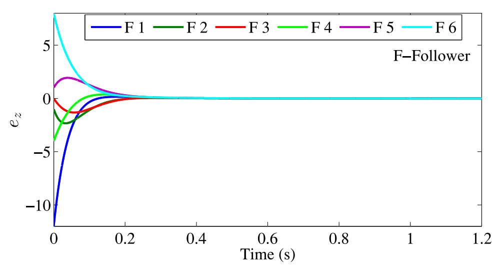
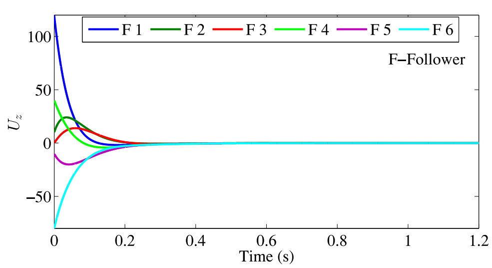


Fig. 6. Trajectory of each agent from initial to final position.



(a)



(b)

Fig. 7. System response at x-axis. (a) e_z. (b) u_z.

results are shown in Figs. 3 and 4. It is seen that the finite time within which the consensus is achieved under (11) does not depend on the initial states $X(t_0)$ nor on other design parameters, and can be uniformly prespecified, whereas the finite convergence time under the traditional finite-time control (2) varies with different initial parameters. Also it is noted from Fig. 5 that the magnitude of the control efforts from both control schemes are comparable, especially our control does not demand excessively large control effort initially or eventually.

Example 2 (Containment): We conduct the simulation on MAS consisting of ten agents with four leaders and six followers to test the effectiveness of the proposed prescribed-time containment control. The communication topology is shown in Fig. 5, with the weight being 1. The initial states of the six followers in x-axis, y-axis, and z-axis are set randomly among $[-5, 5]$. We choose the design parameters as $k = 1$, $c = 3$, $h = 3$, and $T = 1$ s. The containment control results with the proposed control are shown in Figs. 6 and 7. It confirms that, with the proposed prescribed-time containment control method, the convergence time is independent of initial conditions, thus can be uniformly prespecified.

VII. Conclusion

We presented in this paper a new approach for the finite time control of MAS based on time-varying feedback gain. The resultant control is able to achieve consensus within a finite time that can be uniformly prespecified without the need for initial condition and other design parameters. Furthermore, the control is distributed and C^1 smooth everywhere. Extending this method to MAS with more general dynamics represents an interesting topic for future research.

Appendix
Proof of Lemma 1

Proof: We first examine the case of $t \in [t_0, t_1)$ with $t_1 = t_0 + T$. Multiplying μ^2 on both hands of (7) yields

$$\mu^2 \dot{V} \le -b\mu^2 V - 2\mu\dot{\mu}V \tag{68}$$

which further implies

$$\frac{d\left(\mu^2 V\right)}{dt} = \mu^2\dot{V} + 2\mu\dot{\mu}V \le -b\left(\mu^2 V\right). \tag{69}$$

Solving the differential inequality (69) gives

$$\begin{aligned}\mu(t)^2 V(t) &\le \exp^{-b(t-t_0)} \mu(t_0)^2 V(t_0)\\ &= \exp^{-b(t-t_0)} V(t_0)\end{aligned} \tag{70}$$

which then yields (8). Now we consider the case of $t \in [t_1, \infty)$. From (8) we get that $V(t_1) = \lim_{t\to t_1^-} V(t) = 0$, following from the continuity of $V(t)$ as well as $\lim_{t\to t_1^-} \mu_1(t)^{-2h} = 0$. By noting that $b > 0$ and $\dot{\mu}/\mu \ge 0$, we then have from (7) that $\dot{V} \le 0$ on $[t_1, \infty)$, and thus $0 \le V(t) \le V(t_1) = 0$ on $[t_1, \infty)$, which yields (9). On one hand, we see from (6) and (7) that the origin of system (5) is globally asymptotically stable. On the other hand, we have $x(t) \equiv 0$ after $t = t_0 + T$ because $V(x(t), t) \equiv 0$ after $t = t_1 = t_0 + T$. Thus the origin of system (5) is globally prescribed-time stable with the prescribed-time T according to Definition 2. ■

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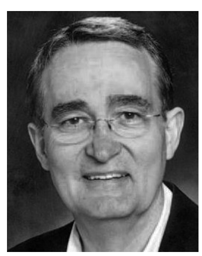

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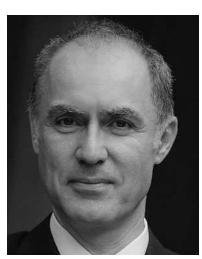

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