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Adaptive error feedback regulation problem for 1D wave equation

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Summary
By using the adaptive control approach, we solve the error feedback regulator problem for the one-dimensional wave equation with a general harmonic disturbance anticollocated with control and with two types of disturbed measurements, ie, one collocated with control and the other anti-collocated with control. Different from the classical error feedback regulator design, which is based on the internal mode principle, we give the adaptive servomechanism design for the system by making use of the measured tracking error (and its time derivative) and the estimation mechanism for the parameters of the disturbance and of the unknown reference. Constructing auxiliary systems and observer and applying the backstepping method for infinite-dimensional system play important roles in the design. The control objective, which is to regulate the tracking error to zero and to keep the states bounded, is achieved.

KEYWORDS
adaptive control, error feedback regulator problem, wave equation

1 | INTRODUCTION

1.1 | Reference review

One of the important problems in control theory is output regulation problem, or alternatively, the servomechanism. This problem addresses designing of a feedback controller to achieve asymptotic tracking of unknown reference signals and asymptotic rejection of undesired disturbances in an uncertain system while maintaining closed-loop stability. Unknown reference signals can alternatively be thought of as measurement disturbances or "noise." The reference and disturbances are usually generated by an exosystem. Generally, there exist two versions of this problem considered. One is the *state feedback regulator problem* where the controller is designed with full information of the state of the plant and exosystem. The other is the more realistic *error feedback regulator problem* (EFRP) where only the components of the tracking error are available for measurement. In this paper, we only focus on the error feedback regulator realization.

In the finite-dimensional system (linear or nonlinear) setting, there are many classical results, which include internal model principle to address this problem (see other works[1-6] and references therein).

Some attempts have been made to extend these classical results to infinite-dimensional systems. In the works of Pohjolainen[7] and Kobayashi,[8] a PI controller was introduced for stable distributed systems with constant disturbance and reference signal. Later, in the works of Byrnes et al[9] and Schumacher,[10] the regulation problem for infinite-dimensional systems with bounded control and observation operator was investigated. Then, the key results in the work of Byrnes et al,[9] which were extended to the regular infinite-dimensional systems with unbounded control and observation operator were reported in the work of Rebarber and Weiss[11] and Natarajan et al.[12] A finite-dimensional output feedback

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regulator for infinite-dimensional systems was developed in the work of Deutscher.[13] Different from the research works just proposed in which the exosystems considered are finite-dimensional, some efforts have been made to focus on the output regulation problem for infinite-dimensional plants driven by infinite-dimensional exosystems. We refer readers to other works.[14-16]

Most of the aforementioned research works about the EFRP focus on the extension of internal model principle theory to infinite-dimensional systems driven by finite-dimensional or infinite-dimensional exosystems. However, there are few results on adaptive servomechanism design for infinite-dimensional systems. Earlier effort on applying adaptive servomechanism to infinite-dimensional systems can be found in the work of Logemann and Ilchmann.[17] In the work of Kobayashi and Oya,[18] an adaptive servomechanism control was designed for a class of distributed parameter system where the input and output operators are collocated and the disturbance is collocated with control. A recent progress has been made in the work of Guo and Guo[19] where an adaptive servomechanism was constructed for one-dimensional (1D) wave equation where the internal stability is needed.

Those generalizations build a theoretical framework, which covers a large class of real systems. However, many real control systems are not included in those abstract frameworks, such as the boundary control partial differential equation (PDE) system, which is anticollocated, or is unstable or even antistable itself. In this situation, the passivity principle cannot be applied. A number of contributions to applying backstepping method for infinite-dimensional system[20] to the stabilization or adaptive stabilization of these PDE systems (please see other works[21-31] and the references therein). Recently, the backstepping-based solution to the output regulation problem for linear 2×2 hyperbolic systems was presented in the work of Deutscher.[32] Adaptive rejection of harmonic disturbance anticollocated with control and the output regulation to zero for 1D wave equation were obtained in the work of Guo et al.[33]

1.2 | Problem formulation and motivation

Let us recall that the EFRP in the finite dimension case. Consider

$$\begin{cases} \dot{x} = Ax + Bu + Pw, \\ \dot{w} = Sw, \\ e = Cx - Qw, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is control input, $e \in \mathbb{R}^m$ is the tracking error, and w models both the reference signal to track and the disturbance to reject. The problem is to find a control law

$$\begin{cases} \dot{\xi} = F\xi + Ge, \\ u = H\xi, \end{cases}$$

such that

1. the origin is an asymptotically stable equilibrium of the closed loop, when the exosystem is disconnected, ie, when $w(t) \equiv 0$; (2)
2. the error $e(t)$ converges to zero, for any initial values $x(0)$, $\xi(0)$, and w_0.

Here, we take the exogenous system in (1) to be a harmonic oscillator

$$\dot{w} = Sw, \, w(0) = w_0 \in \mathbb{R}^n,$$

where $S \in \mathbb{R}^{n\times n}$ whose spectrum only contains simple eigenvalues on the imaginary axis, ie, $i\omega_j$. Then,

$$w(t) = \sum_{j\in\mathcal{J}} e^{i\omega_j t}\langle w_0, \phi_j\rangle \phi_j,$$

where $\phi_j, j \in \mathcal{J}$ is an orthonomal basis $\mathbb{C}^n$.

Thus, the real vectors Pw and Qw contain components, which have the form $a_j \cos\omega_j t + b_j \sin\omega_j t$ with the amplitudes $a_j, b_j, j \in \mathcal{J}$ determined by the initial condition w_0. If we assume the $a_j, b_j, j \in \mathcal{J}$ are unknown, then the EFRP of system (1) has the following statement, which is almost equivalent to (2), ie, the problem is to find a control law such that:

1. the origin is an asymptotically stable equilibrium when the resulting closed loop is disconnected from the disturbance and reference, ie, $a_j = b_j = 0, j \in \mathcal{J}$; (3)
2. the error $e(t)$ converges to zero, for any initial values $x(0)$ and unknown constants $a_j, b_j, j \in \mathcal{J}$,

which inspires us to solve this problem by making use of the adaptive control design method rather than the internal model principle.

In this paper, we consider the error feedback regulation problem for the following wave equation:

$$\begin{cases} y_{tt}(x,t) = y_{xx}(x,t),\ 0<x<1,\ t>0 \\ y_x(0,t) = d(t),\ t \geq 0, \\ y_x(1,t) = u(t),\ t\geq 0 \\ e(t) = y_{\text{out}} - r(t) \to 0, \\ y(x,0) = y_0(x),\quad y_t(x,0) = y_1(x),\ 0 \leq x \leq 1, \end{cases} \tag{4}$$

where $u(t)$ is control input; y_{out} is output to be regulated (in this paper, we consider $y_{\text{out}} = y(1,t)$ and $y_{\text{out}} = y(0,t)$); and $e(t) = y_{\text{out}} - r(t)$ is tracking error, which can be measured. y_0 and y_1 are initial conditions; $d(t)$ represents the general harmonic disturbance, which has the following form:

$$d(t) = \sum_{j=1}^{m} [c_j \cos \omega_j t + d_j \sin \omega_j t].$$

$r(t)$ is the tracking reference signal in the form

$$r(t) = \sum_{j=1}^{n} [a_j \cos \varpi_j t + b_j \sin \varpi_j t].$$

Though our approach applies to this general class of disturbances and tracking reference signal, for simplicity of writing, we take $m = 1$ and $n = 1$, ie,

$$d(t) = c \cos \omega t + d \sin \omega t,$$

and

$$r(t) = a \cos \varpi t + b \sin \varpi t.$$

In this paper, we consider two types of tracking error in this paper. One is

$$e(t) = y(1,t) - r(t),$$

which is collocated with the control, whereas the other is

$$e(t) = y(0,t) - r(t),$$

which is anticollocated with the control. In practice, the reference input to be tracked and the disturbance to be rejected usually are not exactly known signals. The frequencies $\omega, \varpi \neq 0$, and $n\pi + \frac{\pi}{2}$ are assumed to be known (for design purposes), but a, b, c, and d (which determine the amplitudes and the phases) are not known.

Obviously, vector $(\cos \varpi t, \sin \varpi t, \cos \omega t, \sin \omega t)$ is one solution of the following equation with initial value $(w_{10}, w_{20}, w_{30}, w_{40}) = (1,0,1,0)$:

$$\dot{w}(t) = Fw(t),\ w(0) = w_0, \tag{5}$$

where

$$w(t) = (w_1(t), w_2(t), w_3(t), w_4(t))^\top,$$

$$w_0 = (w_{10}, w_{20}, w_{30}, w_{40})^\top,$$

and

$$F = \begin{pmatrix} 0 & -\varpi & 0 & 0 \\ \varpi & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{pmatrix}.$$

The objective of this paper to find the adaptive control law for system (4) such that

1. the resulting closed loop disconnected with disturbance and reference (that is, $a = b = c = d = 0$) will be exponentially stable; and
2. the tracking error $e(t) \to 0, t \to \infty$, for any initial value y_0, y_1 in state space and unknown constants a, b, c, and $d \in \mathbb{R}$.

System (4) is a typical boundary control system with unbounded input and observation operator. In addition, the disturbance and control, in fact, are anticollocated. It is obvious that system (4) without disturbance and control has a zero eigenvalue, which eliminates the assumption in existing literature studies[9,12,13,15,16] that the system operator is always supposed to generate an exponential stable C_0-semigroup. Hence, the EFRP for system (4) cannot be included in the abstract frameworks of the aforementioned literature studies.

This paper is the first to be devoted to solving the adaptive EFRP for PDEs and its contribution may be viewed as an extension of the work of Guo et al[33] but has an essential difference with the work of the aforementioned authors.[33] One of the objective of the work of the aforementioned authors[33] is to regulate the measured output to zero, which means that the output can track the fully known harmonic reference signal. However, this paper is to regulate $e(t) = y(0,t) - r(t)$ or $e(t) = y(1,t) - r(t)$ to zero. The measurement in this paper is the tracking error signal $e(t)$. Since we assume that the harmonic reference signal is unknown, so the outputs $y(0,t)$ and $y(1,t)$ are not known, which bring new difficulty that has to be solved.

The key characteristic of our approach is to construct an auxiliary system in which the control becomes collocated with the disturbance or the measured error becomes system's output, from which we can build the connection between the measured error and the original system. It is a systematic approach, which can be applied to solve the EFRP for other type of PDEs. More precisely, for the case where the output to be regulated is $y_{\text{out}} = y(1,t)$, the method is to use separation of variables to convert the original system into a new system where the boundary output is the measured error and the control becomes collocated with the disturbance; then, based on this new system, an adaptive regulator, which curtains estimators of the parameters of the disturbances and tracking reference, is proposed. It is a finite-dimensional controller. For the case $y_{\text{out}} = y(0,t)$, we have an infinite-dimensional controller, which is more complicated than the previous collocated case. The regulator is found by two auxiliary systems and an adaptive observer.

The traditional output regulation approach is based on the internal model principle and focus on the characterization of the solvability of the EFRP in terms of regulator equations. However, for infinite-dimensional system, the associated regulator equations are usually abstract operator equations, which cannot always be explicitly solved when applying to specific infinite system, eg, PDEs. Our adaptive regulator design is not based on the internal model principle and does not involve the regulator equations, which presents several advantages such as the explicit gain solution and numerical effectiveness. Thus, our adaptive regulator is more implementable for the error feedback regulator realization.

This paper is organized as follows. In the next section, Section 2, we give the collocated adaptive tracking controller design. Section 3 is devoted to the design of the control system with the measured error anticollocated with the control. We present some simulation results illustrating the theory result in Section 4. Conclusions are finally given in Section 5.

2 | COLLOCATE ERROR FEEDBACK REGULATION, $y_{\text{out}} = y(1,t)$

2.1 | Adaptive tracking controller design and main result

This section is devoted to the design of the adaptive tracking controller design for system (4) with the case

$$e(t) = y(1,t) - [a\cos\varpi t + b\sin\varpi t].$$

Inspired by the idea of the motion planning for PDEs in the work of Krstic and Smyshlyaev,[20] we construct an auxiliary system in which the control and the anticollocated disturbance become collocated and the measured error becomes output. To this end, let

$$z(x,t) = y(x,t) - \sec\varpi\cos\varpi x[a\cos\varpi t + b\sin\varpi t] - \frac{\sin\omega(x-1)}{\omega\cos\omega}[c\cos\omega t + d\sin\omega t],\ x \in [0,1],\ t \geq 0. \tag{6}$$

Then, by (4), we obtain the following auxiliary system:

$$\begin{cases} z_{tt}(x,t) &= z_{xx}(x,t), \\ z_x(0,t) &= 0, \\ z_x(1,t) &= u(t) + \varpi\tan\varpi[a\cos\varpi t + b\sin\varpi t] - \sec\omega[c\cos\omega t + d\sin\omega t], \\ z(x,0) &= z_0(x),\quad z_t(x,0) = z_1(x), \end{cases} \tag{7}$$

where

$$z_0(x) = y_0(x) - a \sec \varpi \cos \varpi x - c\frac{\sin \omega(x-1)}{\omega \cos \omega},$$
$$z_1(x) = y_1(x) - b\varpi \sec \varpi \cos \varpi x - d \sec \omega \sin \omega(x-1). \tag{8}$$

Moreover,

$$z(1,t) = y(1,t) - [a \cos \varpi t + b \sin \varpi t] = e(t). \tag{9}$$

Here, and in the rest of this paper, we omit the (obvious) domains for t and x.

The problem becomes how to design controller by using measurement output $z(1,t)$ and its time derivative $z_t(1,t)$ to make (7) stable. We present the adaptive parameters estimator for system (7) as

$$\begin{cases} \dot{\hat{a}}(t) &= r_1 \varpi \tan \varpi z_t(1,t) \cos \varpi t, \\ \dot{\hat{b}}(t) &= r_1 \varpi \tan \varpi z_t(1,t) \sin \varpi t, \\ \dot{\hat{c}}(t) &= -r_2 \sec \omega z_t(1,t) \cos \omega t, \\ \dot{\hat{d}}(t) &= -r_2 \sec \omega z_t(1,t) \cos \omega t, \\ \hat{a}(0) &= \hat{a}_0,\ \hat{b}(0) = \hat{b}_0,\ \hat{c}(0) = \hat{c}_0,\ \hat{d}(0) = \hat{d}_0, \end{cases} \tag{10}$$

and adaptive feedback law

$$u(t) = -k_1 z(1,t) - k_2 z_t(1,t) - \varpi \tan \varpi[\hat{a}(t) \cos \varpi t + \hat{b}(t) \sin \varpi t] + \sec \omega[\hat{c}(t) \cos \omega t + \hat{d}(t) \sin \omega t], \tag{11}$$

where $k_1, k_2, r_j, j = 1, 2$, are positive design parameters. The closed loop of (7) corresponding to (10) and (11) yields

$$\begin{cases} z_{tt}(x,t) = z_{xx}(x,t), \\ z_x(0,t) = 0, \\ z_x(1,t) = -k_1 z(1,t) - k_2 z_t(1,t) + \varpi \tan \varpi[\tilde{a}(t) \cos \varpi t + \tilde{b}(t) \sin \varpi t] \\ \qquad + \sec \omega[\tilde{c}(t) \cos \omega t + \tilde{d}(t) \sin \omega t], \\ \dot{\tilde{a}}(t) = -r_1 \varpi \tan \varpi z_t(1,t) \cos \varpi t, \\ \dot{\tilde{b}}(t) = -r_1 \varpi \tan \varpi z_t(1,t) \sin \varpi t, \\ \dot{\tilde{c}}(t) = -r_2 \sec \omega z_t(1,t) \cos \omega t, \\ \dot{\tilde{d}}(t) = -r_2 \sec \omega z_t(1,t) \sin \omega t, \\ \tilde{a}(0) = \tilde{a}_0,\ \tilde{b}(0) = \tilde{b}_0,\ \tilde{c}(0) = \tilde{c}_0,\ \tilde{d}(0) = \tilde{d}_0. \end{cases} \tag{12}$$

where $\tilde{a}(t) = a - \hat{a}(t), \tilde{b}(t) = b - \hat{b}(t), \tilde{c}(t) = \hat{c}(t) - c$, and $\tilde{d}(t) = \hat{d}(t) - d$ are parameter errors. As explained in the work of Krstic et al,[23] the recommended parameters k_1 and k_2 are chosen so that $\sup\{\lambda : \lambda \in \sigma(A_0)\}$ is small as desired, where the operator $A_0 : D(A_0) \to H^1(0,1) \times L^2(0,1)$ is defined by $A_0(\phi, \psi) = (\psi, \phi'')$ with $D(A_0) = \{(\phi, \psi) \in H^2(0,1) \times H^1(0,1) : \phi'(0) = 0, \phi'(1) = -k_1\phi(1) - k_2\psi(1)\}$. Define the energy function for system (12) as follows:

$$E_z(t) = \frac{1}{2}\int_0^1 \left[z_t^2(x,t) + z_x^2(x,t)\right] \mathrm{d}x + \frac{k_1}{2}[z(1,t)]^2 + \frac{1}{2r_1}\tilde{a}^2(t) + \frac{1}{2r_1}\tilde{b}^2(t) + \frac{1}{2r_2}\tilde{c}^2(t) + \frac{1}{2r_2}\tilde{d}^2(t). \tag{13}$$

A simple computation of the derivative of $E_z(t)$ with respect to t along the solution to (12) shows that

$$\dot{E}_z(t) = -k_2[z_t(1,t)]^2 \leq 0, \tag{14}$$

from which we obtain the feedback law (11) and the update law (10) of $\hat{a}(t), \hat{b}(t), \hat{c}(t)$, and $\hat{d}(t)$.

Let $V = H^3(0,1) \cap D(A)$ with A being defined in $L^2(0,1)$ by

$$\begin{cases} A\phi = -\phi'', \quad \forall \phi \in D(A), \\ D(A) = \left\{\phi \in H^2(0,1) | \phi'(0) = 0, \ \ \phi(1) = 0\right\}. \end{cases} \tag{15}$$

It is seen that A is unbounded self-adjoint positive definite in $L^2(0,1)$ with compact resolvent. A simple computation shows that the eigenpairs $\{(\lambda_n, \phi_n)\}_{n=1}^{\infty}$ of A are

$$\begin{cases} \lambda_n = -\omega_n^2, \ \ \omega_n = i\left(n + \frac{1}{2}\right)\pi, \\ \phi_n(x) = 2\cos \omega_n x = 2\cos\left(n + \frac{1}{2}\right)\pi x. \end{cases} \tag{16}$$

Since $\{\phi_n(x)\}_{n=1}^{\infty}$, defined by (16), forms an orthogonal basis for $L^2(0,1)$, we can then follow the steps as those in the work of Guo and Guo[30] to construct a Galerkin scheme to prove the existence and uniqueness for the classical solution to auxiliary system (12).

Theorem 1. *Suppose that* $(z_0, z_1, \widetilde{a}_0, \widetilde{b}_0, \widetilde{c}_0, \widetilde{d}_0) \in V \times V \times \mathbb{R}^4$, *and they satisfy the following compatible condition:*

$$-k_1 z_0(1) - k_2 z_1(1) + \varpi \tan \varpi \widetilde{a}_0 + \widetilde{c}_0 \sec \omega = 0 \tag{17}$$

and

$$-k_1 z_1(1) - k_2 z_0''(1) + \varpi^2 \tan \varpi[-r_1 z_1(1) + \widetilde{b}_0] + \sec^2 \omega[r_2 z_1(1) + \widetilde{d}_0 \omega \cos \omega] = 0. \tag{18}$$

Then, system (12) admits a unique classical solution z. That is to say, for any time $T > 0$,

$$\begin{cases} z \in L^\infty\left(0, T; H^3(0,1)\right), \quad z_t \in L^\infty\left(0, T; H^2(0,1)\right), \\ z_{tt} \in L^\infty\left(0, T; H^1(0,1)\right), \\ \widetilde{a} \in C^1[0,T], \ \widetilde{b} \in C^1[0,T], \ \widetilde{c} \in C^1[0,T], \ \widetilde{d} \in C^1[0,T] \\ z_{tt}(x,t) = z_{xx}(x,t) \ in \ L^\infty\left(0,T; L^2(0,1)\right), \\ z_x(0,t) = 0, \\ z_x(1,t) = -k_1 z(1,t) - k_2 z_t(1,t) + \varpi \tan \varpi[\widetilde{a}(t) \cos \varpi t + \widetilde{b}(t) \sin \varpi t] \\ \qquad + \sec \omega[\widetilde{c}(t) \cos \omega t + \widetilde{d}(t) \sin \omega t], \\ \quad \dot{\widetilde{a}}(t) = -r_1 \varpi \tan \varpi z_t(1,t) \cos \varpi t, \\ \quad \dot{\widetilde{b}}(t) = -r_1 \varpi \tan \varpi z_t(1,t) \sin \varpi t, \\ \quad \dot{\widetilde{c}}(t = -r_2 \sec \omega z_t(1,t) \cos \omega t, \\ \quad \dot{\widetilde{d}}(t) = -r_2 \sec \omega z_t(1,t) \sin \omega t, \\ \quad \widetilde{a}(0) = \widetilde{a}_0, \ \widetilde{b}(0) = \widetilde{b}_0, \ \widetilde{c}(0) = \widetilde{c}_0, \ \widetilde{d}(0) = \widetilde{d}_0, \\ z(x,0) = z_0(x), \ z_t(x,0) = z_1(x). \end{cases}$$

By the Sobolev embedding theorem, it follows that $z \in C([0,1] \times [0,T])$.

Remark 1. In Theorem 1, condition (17) is the natural compatible condition for the classical solution of (12), and condition (18) is for the existence of the smoother solution that we shall need in the proof of Theorem 2.

Remark 2. Let us remark why the Galerkin method is necessary for the proof of Theorem (1). Actually, we consider (12) and (5) together in the energy state space $\mathcal{H} = \mathcal{V} \times \mathbb{R}^4$

$$\begin{aligned} &\left\langle (u_1, v_1, e, f, g, h, p_1, q_1, p_2, q_2), (u_2, v_2, \hat{e}, \hat{f}, \hat{g}, \hat{h}, \hat{p}_1, \hat{q}_1, \hat{p}_2, \hat{q}_2) \right\rangle_{\mathcal{H}} \\ &= \int_0^1 u_1'(x) u_2'(x) \mathrm{d}x + \int_0^1 v_1(x) v_2(x) \mathrm{d}x + k_1 u_1(1) u_2(1) \\ &+ \left(\frac{e\hat{e}}{r_1} + \frac{f\hat{f}}{r_1} + \frac{g\hat{g}}{r_2} + \frac{h\hat{h}}{r_2} + p_1 \hat{p}_2 + p_2 \hat{p}_2 + p_3 \hat{p}_3 + p_4 \hat{p}_4 \right), \\ &\forall \ (u_1, v_1, e, f, g, h, p_1, q_1, p_2, q_2), (u_2, v_2, \hat{e}, \hat{f}, \hat{g}, \hat{h}, \hat{p}_1, \hat{q}_1, \hat{p}_2, \hat{q}_2) \in \mathcal{H}. \end{aligned}$$

Hence, (12) and (5) can be written as a nonlinear autonomous evolution equation in the state space $\mathcal{H} = \mathcal{V} \times \mathbb{R}^4$

$$\frac{d}{dt}\mathcal{Z}(\cdot, t) = \mathbb{A}\mathcal{Z}(\cdot, t), \mathcal{Z}(\cdot, 0) = \mathcal{Z}_0(\cdot) \in \mathcal{H}, \tag{19}$$

where

$$\begin{cases} \mathcal{Z}(x,t) = \left(z(x,t), z_t(x,t), \widetilde{a}(t), \widetilde{b}(t), \widetilde{c}(t), \widetilde{d}(t), \xi_1(t), \eta_1(t), \xi_2(t), \eta_2(t) \right), \\ \mathcal{Z}_0(x) = \left(z_0(x), z_1(x), \widetilde{a}_0, \widetilde{b}_0, \widetilde{c}_0, \widetilde{d}_0, \xi_{10}, \eta_{10}, \xi_{20}, \eta_{20} \right), \end{cases}$$

and

$$\begin{cases} \mathbb{A}(u, v, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}, \xi_1, \eta_1, \xi_2, \eta_2) = \big(v, u'', -r_1\varpi \tan \varpi v(1)\xi_1, -r_1\varpi \tan \varpi v(1)\eta_1, \\ \qquad -r_2 \sec \omega v(1)\xi_2, -r_2 \sec \omega v(1)\eta_2, -\varpi\eta_1, \varpi\xi_1, -\omega\eta_2, \omega\xi_2\big), \\ D(\mathbb{A}) = \Big\{(u, v, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}, \xi_1, \eta_1, \xi_2, \eta_2) \in H^2(0,1) \times H^1(0,1) \times \mathbb{R}^8 \mid u'(0) = 0, \\ \qquad u'(1) = -k_1 u(1) - k_2 v(1) + \varpi \tan \varpi[\widetilde{a}\xi_1 + \widetilde{b}\eta_1] + \sec \omega[\widetilde{c}\xi_2 + \widetilde{d}\eta_2] \Big\}. \end{cases}$$

Equation (19) is a nonlinear autonomous evolution system. However, same as in the work of Guo and Guo,[30] it seems hard to use nonlinear semigroup to prove its well-posedness due to the lack of dissipativity of $\mathbb{A}$ or any other kind of $\mathbb{A} + \mu I$ for constant $\mu \in \mathbb{R}$. Hence, we invoke the Galerkin method to establish the existence and uniqueness for the solution of Equation (12).

Next, we establish the convergence of auxiliary system (12). To do this, we need the weak solution of (12).

Definition 1. For any initial data $(z_0, z_1, \widetilde{a}_0, \widetilde{b}_0, \widetilde{c}_0, \widetilde{d}_0) \in \mathcal{V} = H^1(0,1) \times L^2(0,1) \times \mathbb{R}^4$, the weak solution $(z, z_t, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d})$ of Equation (12) is defined as the limit of any convergent subsequence of $(z^n, z_t^n, \widetilde{a}^n, \widetilde{b}^n, \widetilde{c}^n, \widetilde{d}^n)$ in the space $L^\infty(0, \infty; \mathcal{V})$, where $(z^n, z_t^n, \widetilde{a}^n, \widetilde{b}^n, \widetilde{c}^n, \widetilde{d}^n)$ is the classical solution (ensured by Theorem 1) with the initial condition (for all $x \in (0,1)$)

$$\left(z^n(x,0), z_t^n(x,0), \widetilde{a}^n(0), \widetilde{b}^n(0), \widetilde{c}^n(0), \widetilde{d}^n(0)\right) = \left(z_0^n(x), z_1^n(x), \widetilde{a}_0^n, \widetilde{b}_0^n, \widetilde{c}_0^n, \widetilde{d}_0^n\right) \in V \times V \times \mathbb{R}^4,$$

which satisfies

$$\lim_{n\to\infty} \left\| \left(z_0^n(x), z_1^n(x), \widetilde{a}_0^n, \widetilde{b}_0^n, \widetilde{c}_0^n, \widetilde{d}_0^n\right) - (z_0, z_1, \widetilde{a}_0, \widetilde{b}_0, \widetilde{c}_0, \widetilde{d}_0) \right\|_{\mathcal{V}} = 0.$$

By (13) and (14), the aforementioned weak solution is well defined, since it does not depend on the choice of initial sequence $(z^n(x,0), z_t^n(x,0), \widetilde{a}^n(0), \widetilde{b}^n(0), \widetilde{c}^n(0), \widetilde{d}^n(0))$. Consequently, $(z, z_t, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}) \in C(0, \infty; \mathcal{V})$. Moreover, by (14), this solution depends continuously on its initial value.

Theorem 2. *Suppose that*

$$\omega, \varpi \neq 0, \ \ n\pi + \frac{\pi}{2}, \ \ n \in \mathbb{Z}. \tag{20}$$

Then, for any initial value $(z_0, z_1, \widetilde{a}_0, \widetilde{b}_0, \widetilde{c}_0, \widetilde{d}_0) \in \mathcal{V}$*, the (weak) solution of system (12) is asymptotically stable in the sense that*

$$\lim_{t\to\infty} \left[\frac{1}{2}\int_0^1 \left[z_t^2(x,t) + z_x^2(x,t)\right] \mathrm{d}x + k_1 z^2(1,t)\right] = 0$$

and

$$\lim_{t\to\infty} \widehat{a}(t) = a, \ \ \lim_{t\to\infty} \widehat{b}(t) = b, \ \ \lim_{t\to\infty} \widehat{c}(t) = c, \qquad \lim_{t\to\infty} \widehat{d}(t) = d.$$

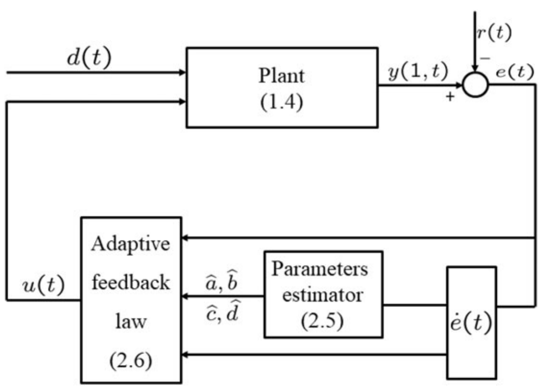

FIGURE 1 Block diagram of the closed-loop system (21)

Now, we are in a position to go back to system (4). Thus, the closed loop of system (4) depicted in Figure 1 corresponding to (9), (10), and (11) is governed by

$$\begin{cases} y_{tt}(x,t) = y_{xx}(x,t), \\ y_x(0,t) = c\cos\omega t + d\sin\omega t, \\ y_x(1,t) = -k_1 e(t) - k_2\dot{e}(t) - \varpi\tan\varpi[\hat{a}(t)\cos\varpi t + \hat{b}(t)\sin\varpi t], \\ \qquad + \sec\omega[\hat{c}(t)\cos\omega t + \hat{d}(t)\sin\omega t], \\ \dot{\hat{a}}(t) = r_1\varpi\tan\varpi\dot{e}(t)\cos\varpi t, \\ \dot{\hat{b}}(t) = r_1\varpi\tan\varpi\dot{e}(t)\sin\varpi t, \\ \dot{\hat{c}}(t) = -r_2\sec\omega\dot{e}(t)\cos\omega t, \\ \dot{\hat{d}}(t) = -r_2\sec\omega\dot{e}(t)\cos\omega t, \\ \hat{a}(0) = \hat{a}_0,\ \hat{b}(0) = \hat{b}_0,\ \hat{c}(0) = \hat{c}_0,\ \hat{d}(0) = \hat{d}_0, \\ e(t) = y(1,t) - [a\cos\varpi t + b\sin\varpi t]. \end{cases} \tag{21}$$

Theorem 3. *Suppose that $\omega \neq 0, n\pi + \frac{\pi}{2}$, $n \in \mathbb{Z}$. For any initial value $(y_0, y_1, \hat{a}_0, \hat{b}_0, \hat{c}_0, \hat{d}_0) \in \mathcal{V}$, there exists a unique (weak) solution to (21) such that $(y(\cdot,t), y_t(\cdot,t), \hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t)) \in C([0,\infty);\mathcal{V})$. Moreover, this closed-loop solution has the following properties.*

i. $\sup_{t\geq 0}\left[\int_0^1 \left[y_t^2(x,t) + y_x^2(x,t)\right]\mathrm{d}x + k_1 y^2(1,t) + \hat{a}^2(t) + \hat{b}^2(t) + \hat{c}^2(t) + \hat{d}^2(t)\right] < \infty.$

ii. $\lim_{t\to\infty}\hat{a}(t) = a,\ \lim_{t\to\infty}\hat{b}(t) = b, \lim_{t\to\infty}\hat{c}(t) = c,\ \lim_{t\to\infty}\hat{d}(t) = d.$

iii. $\lim_{t\to\infty} e(t) = y(1,t) - [a\cos\varpi t + b\sin\varpi t] = 0.$

iv. *When $a = b = c = d = 0$,*

$$\int_0^1 \left[y_t^2(x,t) + y_x^2(x,t)\right]\mathrm{d}x + k_1 y^2(1,t) \leq Me^{-\mu t},$$

for some constants $M, \mu > 0$.

2.2 | Proof of the main result

2.2.1 | Proof of Theorem 2

By density argument, we may assume without loss of generality that the initial value $(z_0, z_1, \tilde{a}_0, \tilde{b}_0, \tilde{c}_0, \tilde{d}_0)$ belongs to $V\times V\times \mathbb{R}^4$ and satisfies compatible conditions (17) and (18). Construct the Lyapunov functional $V_Z(t)$ for system (19) as follows:

$$V_Z(t)(t) = \frac{1}{2}\int_0^1 \left[z_t^2(x,t) + z_x^2(x,t)\right]\mathrm{d}x + \frac{k_2}{2}z^2(1,t) + \left[\frac{\tilde{a}^2(t) + \tilde{b}^2(t)}{2r_1} + \frac{\tilde{c}^2(t) + \tilde{d}^2(t)}{2r_2}\right] + \left[\xi_1^2(t) + \eta_1^2(t)\right] + \left[\xi_2^2(t) + \eta_2^2(t)\right], \tag{22}$$

where $\xi_1(t) = \cos\varpi t, \eta_1(t) = \sin\varpi t, \xi_2(t) = \cos\omega t$, and $\eta_2(t) = \sin\omega t$. A simple computation of the time derivative of $V_Z(t)$ along the solution of system (19) shows

$$\dot{V}_Z(t)(t) = -k_2[z_t(1,t)]^2 \leq 0.$$

This concludes that $V_Z(t) \leq V_Z(0)$; hence,

$$\sup_{t\geq 0}\left[\frac{1}{2}\int_0^1 \left[z_t^2(x,t) + z_x^2(x,t)\right]\mathrm{d}x + k_1 z^2(1,t) + \left|\tilde{a}(t)\right| + |\tilde{b}(t)| + \left|\tilde{c}(t)\right| + |\tilde{d}(t)|\right] < \infty. \tag{23}$$

In particular, one has

$$z_t(1,t) \in L^2(0,\infty). \tag{24}$$

Similarly, define

$$U_z(t) = \frac{1}{2}\int_0^1 \left[z_{xx}^2(x,t) + z_{tx}^2(x,t)\right]\mathrm{d}x + \frac{k_2}{2}z_t^2(1,t).$$

The time derivative of $U_z(t)$ along the solution of (12) can be found as

$$\begin{aligned}\dot{U}_z(t) = & -k_2[z_{tt}(1,t)]^2 - \left[r_1\varpi^2(\tan\varpi)^2 + \frac{r_2}{\cos^2\omega}\right] z_{tt}(1,t)z_t(1,t) \\ & + z_{tt}(1,t)\left[\varpi^2\tan\varpi[\widetilde{b}(t)\cos\varpi t - \widetilde{a}(t)\sin\varpi t] + \frac{\omega}{\cos\omega}[\widetilde{d}(t)\cos\omega t - \widetilde{c}(t)\sin\omega t]\right].\end{aligned} \tag{25}$$

Integrating over $[0,t]$ by part on both sides of (25) gives

$$\begin{aligned}U_z(t) = & U_z(0) - k_2\int_0^t[\ddot{z}(1,s)]^2 ds - \frac{1}{2}\left[r_1\varpi^2(\tan\varpi)^2 + \frac{r_2}{\cos^2\omega}\right][\dot{z}(1,t)]^2 + \frac{1}{2}\left[r_1\varpi^2(\tan\varpi)^2 + \frac{r_2}{\cos^2\omega}\right]z_1^2(1) \\ & + \dot{z}(1,t)\left[\varpi^2\tan\varpi[\widetilde{b}(t)\cos\varpi t - \widetilde{a}(t)\sin\varpi t] + \frac{\omega}{\cos\omega}[\widetilde{d}(t)\cos\omega t - \widetilde{c}(t)\sin\omega t]\right] \\ & - z_1(1)\left[\varpi^2\tan\varpi\widetilde{b}_0 + \frac{\omega}{\cos\omega}\widetilde{d}_0\right] - \varpi^2\tan\varpi\left[\frac{\widetilde{b}^2(t)-\widetilde{b}_0^2}{2r_1} + \frac{\widetilde{a}^2(t)-\widetilde{a}_0^2}{2r_1}\right] - \omega^2\left[\frac{\widetilde{d}^2(t)-\widetilde{d}_0^2}{2r_2} + \frac{\widetilde{c}^2(t)-\widetilde{c}_0^2}{2r_2}\right].\end{aligned} \tag{26}$$

By using Young's inequality in (26), we have the estimation of $U_z(t)$ to be

$$\begin{aligned}U_z(t) \le & \frac{1}{2}z_1^2(1)\left[(\varpi\tan\varpi)^2 r_1 + \frac{r_2}{\cos^2\omega}\right] + \frac{1}{2}\delta k_2 z_t^2(1,t) \\ & + \frac{1}{2\delta k_2}\left[\varpi^2\tan\varpi[\widetilde{b}(t)\cos\varpi t - \widetilde{a}(t)\sin\varpi t] + \frac{\omega}{\cos\omega}[\widetilde{d}(t)\cos\omega t - \widetilde{c}(t)\sin\omega t]\right]^2 \\ & + \left|z_1(1)\left[\varpi^2\tan\varpi\widetilde{b}_0 + \frac{\omega}{\cos\omega}\widetilde{d}_0\right]\right| + |\varpi^2\tan\varpi|\left[\frac{\widetilde{b}^2(t)+\widetilde{b}_0^2}{2r_1} + \frac{\widetilde{a}^2(t)+\widetilde{a}_0^2}{2r_1}\right] \\ & + \omega^2\left[\frac{\widetilde{d}_0^2}{2r_2} + \frac{\widetilde{c}_0^2}{2r_2}\right] - \frac{1}{2}\left[r_1\varpi^2(\tan\varpi)^2 + \frac{r_2}{\cos^2\omega}\right][\dot{z}(1,t)]^2 + U_z(0),\end{aligned} \tag{27}$$

where $\delta > 0$ is a constant that is chosen so that δ satisfies

$$\frac{1}{2}\delta k_2 < \frac{1}{2}\left[r_1\varpi^2(\tan\varpi)^2 + \frac{r_2}{\cos^2\omega}\right]. \tag{28}$$

It is found from (23), (24), (27), and (28) that

$$\sup_{t\ge 0} U_z(t) < \infty,$$

which implies that the trajectory of system (19)

$$\gamma(\mathcal{Z}_0) = \left\{\left(z, z_t, \widetilde{a}(t), \widetilde{b}(t), \widetilde{c}(t), \widetilde{d}(t), \xi_1(t), \eta_1(t), \xi_2(t), \eta_2(t)\right) \mid t \ge 0\right\}$$

is precompact in $\mathcal{H}$. In the light of Lasalle's invariance principle,[34] any solution of system (19) tends to the maximal invariant set of the following:

$$\mathcal{S} = \left\{\left(z, z_t, \widetilde{a}(t), \widetilde{b}(t), \widetilde{c}(t), \widetilde{d}(t), \xi_1(t), \eta_1(t), \xi_2(t), \eta_2(t)\right) \in \mathcal{H} \mid \dot{V}_{\mathcal{Z}}(t) = 0\right\}.$$

Now, by $\dot{V}_{\mathcal{Z}}(t) = 0$, it follows that $z_t(1,t) = 0$, $\widetilde{a} \equiv \widetilde{a}_0$, $\widetilde{b} \equiv \widetilde{b}_0$, $\widetilde{c} \equiv \widetilde{c}_0$, and $\widetilde{d} \equiv \widetilde{d}_0$. Thus, the solution reduces to

$$\begin{cases} z_{tt}(x,t) = z_{xx}(x,t), \\ z_x(0,t) = 0, \\ z_x(1,t) = -k_1 z_0(1) + \widetilde{a}_0\cos\varpi t + \widetilde{b}_0\sin\varpi t + \widetilde{c}_0\cos\omega t + \widetilde{d}_0\sin\omega t, \\ z_t(1,t) = 0. \end{cases} \tag{29}$$

The proof will be accomplished if we can show that (29) admits zero solution only. To this end, we first consider the equation

$$\begin{cases} z_{tt}(x,t) = z_{xx}, \\ z_x(0,t) = 0,\ z_t(1,t) = 0. \end{cases} \tag{30}$$

Introduce a Hilbert space $\mathcal{H} = H^1(0,1) \times L^2(0,1)$ with the inner product

$$\langle(u_1,v_1),(u_2,v_2)\rangle_{\mathcal{H}} = \int_0^1\left[u_1'(x)\overline{u_2'(x)} + v_1(x)\overline{v_2(x)}\right]dx + u_1(1)\overline{u_2(1)}.$$

Define a linear operator $\mathcal{A}$ in $\mathcal{H}$ associated to system (30)

$$\begin{cases} \mathcal{A}(y,z) = (z, y''), \\ D(\mathcal{A}) = \left\{(y,z) \in H^2(0,1) \times H^1(0,1) \mid y'(0) = 0,\ z(1) = 0\right\}. \end{cases} \tag{31}$$

A straightforward calculation and performing of integration by parts shows that $\mathcal{A}$ is skew-symmetric in $\mathcal{H}$. Thus, all eigenvalues of $\mathcal{A}$ are located on the imaginary axis.

Now, we claim that each eigenvalue of $\mathcal{A}$ is geometrical simple, and hence, algebraically simple from general functional analysis theory. To see this, we solve the eigenvalue problem

$$\mathcal{A}(\phi,\psi) = \lambda(\phi,\psi)$$

for any $\lambda \in \sigma_p(\mathcal{A})$. The solution is $\psi = \lambda\phi$ with $\phi \neq 0$ satisfying

$$\begin{cases} \lambda^2\phi(x) - \phi''(x) = 0, \\ \phi'(0) = 0,\ \lambda\phi(1) = 0. \end{cases} \tag{32}$$

Solve (32) in the case where $\lambda = 0$ to give

$$\phi(x) = c \neq 0, \tag{33}$$

where c is a constant. When $\lambda \neq 0$,

$$\phi(x) = e^{\lambda x} + e^{-\lambda x} \tag{34}$$

with $e^{2\lambda} = -1$. Hence, λ is geometrically simple.

Finally, we claim that the spectrum of $\mathcal{A}$ consists of isolated eigenvalues only. In fact, for a given $(f,g) \in \mathcal{H}$ and $\mu \in \rho(\mathcal{A})$, $\mu \neq 0$, solve $(\mu I - \mathcal{A})(\phi,\psi) = (f,g)$, ie,

$$\begin{cases} \phi''(x) = \mu^2\phi(x) - \mu f(x) - g(x), \\ \phi'(0) = 0,\ \phi(1) = \frac{f(1)}{\mu}, \\ \psi(x) = \mu\phi(x) - f(x) \end{cases}$$

to give

$$\begin{cases} \phi(x) = m_1 e^{\mu x} + m_2 e^{-\mu x} - \frac{1}{\mu}\int_0^x \sinh(\mu x - \mu\xi)\,[\mu f(\xi) + g(\xi)]\,\mathrm{d}\xi, \\ \psi(x) = \mu\phi(x) - f(x), \end{cases} \tag{35}$$

where

$$m_1 = \frac{1}{2\mu\cosh\mu}\left[e^{-\mu} f(0) + \int_0^1 \sin h(\mu(1-\xi))\mathrm{d}\xi\right],$$

$$m_2 = \frac{1}{2\mu\cosh\mu}\left[\int_0^1 \sin h(\mu(1-\xi))\mathrm{d}\xi - e^{\mu} f(0)\right].$$

It follows from (35) that

$$(\mu - \mathcal{A})^{-1}(f,g) = (\phi,\psi),\ \forall\,(f,g) \in \mathcal{H},$$

and hence

$$\|(\mu - \mathcal{A})^{-1}(f,g)\|_{H^2(0,1)\times H^1(0,1)} \leq C_1\|(f,g)\|_{\mathcal{H}}$$

for some constant $C_1 > 0$. By the Sobolev embedding theorem, $(\mu I - \mathcal{A})^{-1}$ is compact on $\mathcal{H}$. That is, $\mathcal{A}$ is a skew-adjoint operator with compact resolvent on $\mathcal{H}$. Consequently, the spectrum of $\mathcal{A}$ consists of isolated eigenvalues only.

Furthermore, from (34), we can obtain eigenpairs of $\mathcal{A}$

$$\begin{cases} \lambda_n = \left(n\pi + \frac{\pi}{2}\right) i,\ \ \lambda_{-n} = \overline{\lambda}_n, \\ \Phi_n = \left(\lambda_n^{-1}\phi_n, \phi_n\right),\ \Phi_{-n} = \left(\lambda_{-n}^{-1}\phi_n, \phi_n\right),\ n \in \mathbb{Z}, \end{cases} \tag{36}$$

where $\phi_n(x) = \cos\left(n + \frac{1}{2}\right)\pi x$. By general theory of functional analysis, $\{\Phi_n\}_{n\in\mathbb{Z}}$ forms an orthogonal basis for $\mathcal{H}$. Therefore, the solution of (30) can be represented as

$$(z(\cdot,t), \dot{z}(\cdot,t)) = k_1 a_0(c,0) + \sum_{n=1}^{\infty} a_n e^{\lambda_n t}\Phi_n + \sum_{n=1}^{\infty} a_{-n} e^{\lambda_{-n} t}\Phi_{-n},$$

where the constants $\{a_n\}_{n\in\mathbb{Z}}$ are determined by the initial condition. That is,

$$z_0 = a_0 c + \sum_{n=1}^{\infty}\frac{a_n}{\lambda_n}\phi_n + \sum_{n=1}^{\infty}\frac{a_{-n}}{\lambda_{-n}}\phi_n, \quad z_1 = \sum_{n=1}^{\infty} a_n\phi_n + \sum_{n=1}^{\infty} a_{-n}\phi_n.$$

Hence,

$$
\begin{aligned}
z_x(1,t) &= \sum_{n=1}^{\infty} a_n \frac{\phi_n'(1)}{\lambda_n} e^{\lambda_n t} + \sum_{n=1}^{\infty} a_{-n} \frac{\phi_n'(1)}{\lambda_{-n}} e^{\lambda_{-n} t} \\
&= k_1 a_0 c + \widetilde{a}_0 \cos \varpi t + \widetilde{b}_0 \sin \varpi t + \widetilde{c}_0 \cos \omega t + \widetilde{d}_0 \sin \omega t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&- k_1 a_0 c + \sum_{n=1}^{\infty} a_n \frac{\phi_n'(1)}{\lambda_n} e^{\lambda_n t} + \sum_{n=1}^{\infty} a_{-n} \frac{\phi_n'(1)}{\lambda_{-n}} e^{\lambda_{-n} t} \\
&- \frac{1}{2}[\widetilde{a}_0 - \mathrm{i}\widetilde{b}_0] e^{\mathrm{i}\varpi t} - \frac{1}{2}[\widetilde{a}_0 + \mathrm{i}\widetilde{b}_0] e^{-\mathrm{i}\varpi t} - \frac{1}{2}[\widetilde{c}_0 - \mathrm{i}\widetilde{d}_0] e^{\mathrm{i}\omega t} - \frac{1}{2}[\widetilde{c}_0 + \mathrm{i}\widetilde{d}_0] e^{-\mathrm{i}\omega t} = 0.
\end{aligned} \tag{37}
$$

We now show that $a_{\pm n} = 0$, for all $n \geq 1$. Since otherwise, if there exists $n_0 \geq 1$ such that $|a_{n_0} \frac{\phi_{n_0}'(1)}{\lambda_{n0}}| \neq 0$, then $a_{n_0} \neq 0$ due to the fact $\phi_n'(1) \neq 0$ for all n. Furthermore, the smoothness of the initial value guarantees that $\sum_{n \in \mathbb{Z}, n \neq 0} |a_n \frac{\phi_n'(1)}{\lambda_n}| < \infty$, which implies that there exists an integer $N > n_0$ such that

$$
\sum_{n=N}^{\infty} \left| a_n \frac{\phi_n'(1)}{\lambda_n} \right| < \frac{1}{4} \left| a_{n_0} \frac{\phi_{n_0}'(1)}{\lambda_{n0}} \right|, \quad \sum_{n=N}^{\infty} \left| a_{-n} \frac{\phi_{-n}'(1)}{\lambda_{-n}} \right| < \frac{1}{4} \left| a_{n_0} \frac{\phi_{n_0}'(1)}{\lambda_{n0}} \right|. \tag{38}
$$

Since $\lambda_n \neq \lambda_m$ for any $n, m \in \mathbb{Z}, n \neq m$, and $|\lambda_{n+1} - \lambda_n| = \pi, n \in \mathbb{Z}$, one has, for $t > 0$,

$$
\begin{aligned}
& a_{n_0} \frac{\phi_{n_0}'(1)}{\lambda_{n0}} + \sum_{n=N+1}^{\infty} a_n \frac{\phi_n'(1)}{\lambda_n} e^{(\lambda_n - \lambda_{n_0})t} + \sum_{n=1, n \neq n_0}^{N} a_n \frac{\phi_n'(1)}{\lambda_n} e^{(\lambda_n - \lambda_{n_0})t} + \sum_{n=N+1}^{\infty} a_{-n} \frac{\phi_{-n}'(1)}{\lambda_{-n}} e^{(\lambda_{-n} - \lambda_{n_0})t} + \sum_{n=1}^{N} a_{-n} \frac{\phi_{-n}'(1)}{\lambda_{-n}} e^{(\lambda_{-n} - \lambda_{n_0})t} \\
& -k_1 a_0 c e^{-\lambda_{n_0} t} - \frac{1}{2}[\widetilde{a}_0 - \mathrm{i}\widetilde{b}_0] e^{(\mathrm{i}\varpi - \lambda_{n0})t} - \frac{1}{2}[\widetilde{a}_0 + \mathrm{i}\widetilde{b}_0] e^{-(\mathrm{i}\varpi + \lambda_{n0})t} - \frac{1}{2}[\widetilde{c}_0 - \mathrm{i}\widetilde{d}_0] e^{(\mathrm{i}\omega - \lambda_{n0})t} - \frac{1}{2}[\widetilde{c}_0 + \mathrm{i}\widetilde{d}_0] e^{-(\mathrm{i}\omega + \lambda_{n0})t} = 0.
\end{aligned} \tag{39}
$$

Integrating over $[0, t]$ on both sides of (39) and using (38), and the fact $\mathrm{Re}\lambda_n = 0$, we obtain

$$
\begin{aligned}
\left| a_{n_0} \frac{\phi_{n_0}'(1)}{\lambda_{n0}} \right| t \leq\; & 2 \left| \int_0^t \sum_{n=1, n \neq n_0}^{N} a_n \frac{\phi_n'(1)}{\lambda_n} e^{(\lambda_n - \lambda_{n_0})s} \mathrm{d}s \right| + 2 \left| \int_0^t \sum_{n=1}^{N} a_{-n} \frac{\phi_{-n}'(1)}{\lambda_{-n}} e^{(\lambda_{-n} - \lambda_{n_0})s} \mathrm{d}s \right| \\
& + 2 \left| \int_0^t k_1 a_0 c e^{-\lambda_{n_0} s} \mathrm{d}s \right| + 2 \left| \int_0^t [\widetilde{a}_0 - \mathrm{i}\widetilde{b}_0] e^{(\mathrm{i}\varpi - \lambda_{n0})s} \mathrm{d}s \right| + 2 \left| \int_0^t [\widetilde{a}_0 + \mathrm{i}\widetilde{b}_0] e^{-(\mathrm{i}\varpi + \lambda_{n0})s} \mathrm{d}s \right| \\
& + 2 \left| \int_0^t [\widetilde{c}_0 - \mathrm{i}\widetilde{d}_0] e^{(\mathrm{i}\omega - \lambda_{n0})s} \mathrm{d}s \right| + 2 \left| \int_0^t [\widetilde{c}_0 + \mathrm{i}\widetilde{d}_0] e^{-(\mathrm{i}\omega + \lambda_{n0})s} \mathrm{d}s \right|.
\end{aligned}
$$

Since the right side of the aforementioned equation has an upper bound for all $t \geq 0$, we get that $a_{n_0} = 0$, which is a contradiction. Hence, $a_{\pm n} = 0, n = 1, 2, \cdots$ and by (37), $a_0 = \widetilde{a}_0 = \widetilde{b}_0 = \widetilde{c}_0 = \widetilde{d}_0 = 0$. We have thus proved that $\mathcal{S} = \{(0, 0, 0, 0, 0, 0, 1, 0, 1, 0)\}$, in other words,

$$
\lim_{t \to \infty} \left[\frac{1}{2} \int_0^1 \left[z_t^2(x,t) + z_x^2(x,t) \right] \mathrm{d}x + c_0 z^2(0,t) + \frac{\widetilde{a}^2(t)}{2r_1} + \frac{\widetilde{b}^2(t)}{2r_1} + \frac{\widetilde{c}^2(t)}{2r_2} + \frac{\widetilde{d}^2(t)}{2r_2} \right] = 0.
$$

The proof is complete.

Proof of Theorem 3: For any initial value $(y_0, y_1, \widehat{a}_0, \widehat{b}_0, \widehat{c}_0, \widehat{d}_0) \in \mathcal{V}$, It is obvious from (8) that $(z_0, z_1, \widetilde{a}_0, \widetilde{b}_0, \widetilde{c}_0, \widetilde{d}_0) \in \mathcal{V}$, which implies that there exists a unique solution (weak) $(z, z_t, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}) \in C[0, \infty; \mathcal{V}]$ to (12). This, together with (40), concludes that system (21) admits a unique solution (weak) $(y, y_t, \widehat{a}, \widehat{b}, \widehat{c}, \widehat{d}) \in C([0, \infty; \mathcal{V})$. The first part is proved. Theorem 2 with (6) gives property i, ii, and iii. We say that $a = b = c = d = 0$, which implies there are no disturbance and reference signal, ie, $\widehat{a}(t) = \widehat{b}(t) = \widehat{c}(t) = \widehat{d}(t) \equiv 0$. Thus, iv is valid as a well-known result. The proof is completed.

3 | ANTICOLLOCATED ERROR FEEDBACK REGULATION, $y_{out}=y(0,t)$

3.1 | Adaptive anticollocated tracking controller design and main result

This section is devoted to the design of the adaptive tracking controller for system (4) where $e(t) = y(0,t) - r(t)$. We construct the first auxiliary system in which the measured error becomes output. Let

$$z(x,t) = y(x,t) - \frac{\cos\varpi(1-x)}{\cos\varpi}[a\cos\varpi t + b\sin\varpi t], \quad x\in[0,1], \quad t\geq 0. \tag{40}$$

Then, by (4), we obtain the following auxiliary system:

$$\begin{cases} z_{tt}(x,t) = z_{xx}(x,t), \\ z_x(0,t) = c\cos\omega t + d\sin\omega t + \varpi\tan\varpi[a\cos\varpi t + b\sin\varpi t], \\ z_x(1,t) = u(t), \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x), \end{cases} \tag{41}$$

where

$$z_0(x) = y_0(x) - a\frac{\cos\varpi(1-x)}{\cos\varpi}, \quad z_1(x) = y_1(x) - b\varpi\frac{\cos\varpi(1-x)}{\cos\varpi}. \tag{42}$$

Moreover,

$$z(0,t) = y(0,t) - [a\cos\varpi t + b\sin\varpi t] = e(t). \tag{43}$$

To recover the state of system (41), we design an adaptive observer for system (41) by using the measured output $z(0,t)$ and its time derivative $z_t(0,t)$

$$\begin{cases} \hat{z}_{tt}(x,t) = \hat{z}_{xx}(x,t), \\ \hat{z}_x(0,t) = \hat{c}(t)\cos\omega t + \hat{d}(t)\sin\omega t + \varpi\tan\varpi[\hat{a}(t)\cos\varpi t + \hat{b}(t)\sin\varpi t] \\ \qquad +k_1\left(\hat{z}_t(0,t) - z_t(0,t)\right) + k_2\left(\hat{z}(0,t) - z(0,t)\right), \\ \hat{z}_x(1,t) = u(t), \\ \dot{\hat{c}}(t) = -r_1\left(z_t(0,t) - \hat{z}_t(0,t)\right)\cos\omega t, \\ \dot{\hat{d}}(t) = -r_1\left(z_t(0,t) - \hat{z}_t(0,t)\right)\sin\omega t, \\ \dot{\hat{a}}(t) = -r_2\varpi\tan\varpi\left(z_t(0,t) - \hat{z}_t(0,t)\right)\cos\varpi t, \\ \dot{\hat{b}}(t) = -r_2\varpi\tan\varpi\left(z_t(0,t) - \hat{z}_t(0,t)\right)\sin\varpi t, \\ \hat{c}(0) = \hat{c}_0, \quad \hat{d}(0) = \hat{d}_0, \quad \hat{a}(0) = \hat{a}_0, \quad \hat{b}(0) = \hat{b}_0, \\ \hat{z}(x,0) = \hat{z}_0(x), \quad \hat{z}_t(x,0) = \hat{z}_1(x), \end{cases} \tag{44}$$

where k_1, k_2, r_1, and $r_2 > 0$ are design parameters.

Let $\varepsilon = z - \hat{z}$, $\tilde{c} = c - \hat{c}(t)$, $\tilde{d} = d - \hat{d}(t)$, $\tilde{a} = a - \hat{a}(t)$, and $\tilde{b} = b - \hat{b}(t)$ be parameter estimation error; then, from (44) and (41), ε is governed by

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = k_1\varepsilon_t(0,t) + k_2\varepsilon(0,t) + \tilde{c}(t)\cos\omega t + \tilde{d}(t)\sin\omega t \\ \qquad +\varpi\tan\varpi\left[\tilde{a}(t)\cos\varpi t + \tilde{b}(t)\sin\varpi t\right], \\ \varepsilon_x(1,t) = 0, \\ \dot{\tilde{c}}(t) = r_1\varepsilon_t(0,t)\cos\omega t, \\ \dot{\tilde{d}}(t) = r_1\varepsilon_t(0,t)\sin\omega t, \\ \dot{\tilde{a}}(t) = r_2\varpi\tan\varpi\varepsilon_t(0,t)\cos\varpi t, \\ \dot{\tilde{b}}(t) = r_2\varpi\tan\varpi\varepsilon_t(0,t)\sin\varpi t, \\ \varepsilon(x,0) = \varepsilon_0(x)\,\varepsilon_t(x,0) = \varepsilon_1(x), \\ \tilde{c}(0) = \tilde{c}_0, \quad \tilde{d}(0) = \tilde{d}_0, \quad \tilde{a}(0) = \tilde{a}_0, \quad \tilde{b}(0) = \tilde{b}_0, \end{cases} \tag{45}$$

where

$$\begin{cases} \varepsilon_0(x) = z_0(x) - \hat{z}_0(x), \ \varepsilon_1(x) = z_1(x) - \hat{z}_1(x), \\ \tilde{c}_0 = \hat{c}_0 - c, \ \tilde{d}_0 = \hat{d}_0 - d, \ \tilde{a}_0 = \hat{a}_0 - c, \ \tilde{b}_0 = \hat{b}_0 - b. \end{cases} \tag{46}$$

Let

$$\hat{\varepsilon}(x,t) = \varepsilon(1-x,t).$$

Then, $\widehat{\varepsilon}(x,t)$ satisfies

$$\begin{cases} \widehat{\varepsilon}_{tt}(x,t) = \widehat{\varepsilon}_{xx}(x,t), \\ \widehat{\varepsilon}_{x}(1,t) = -k_1\widehat{\varepsilon}_t(1,t) - k_2\widehat{\varepsilon}(1,t) - \widetilde{c}(t)\cos\omega t - \widetilde{d}(t)\sin\omega t \\ \qquad -\varpi\tan\varpi\left[\widetilde{a}(t)\cos\varpi t + \widetilde{b}(t)\sin\varpi t\right], \\ \widehat{\varepsilon}_x(0,t) = 0, \\ \dot{\widetilde{c}}(t) = r_1\widehat{\varepsilon}_t(1,t)\cos\omega t, \\ \dot{\widetilde{d}}(t) = r_1\widehat{\varepsilon}_t(1,t)\sin\omega t, \\ \dot{\widetilde{a}}(t) = r_2\varpi\tan\varpi\widehat{\varepsilon}_t(1,t)\cos\varpi t, \\ \dot{\widetilde{b}}(t) = r_2\varpi\tan\varpi\widehat{\varepsilon}_t(1,t)\sin\varpi t, \\ \varepsilon(x,0) = \varepsilon_0(x)\ \varepsilon_t(x,0) = \varepsilon_1(x), \\ \widetilde{c}(0) = \widetilde{c}_0, \quad \widetilde{d}(0) = \widetilde{d}_0, \quad \widetilde{a}(0) = \widetilde{a}_0, \quad \widetilde{b}(0) = \widetilde{b}_0. \end{cases} \tag{47}$$

Observe that the structure of (47) is almost same to system (12). We now give the well-posedness and the convergence result directly without proof.

Theorem 4. *Suppose that*

$$\omega \neq 0, n\pi + \frac{\pi}{2},\ n \in \mathbb{Z}. \tag{48}$$

Then, for any initial value $(\varepsilon_0, \varepsilon_1, \widetilde{c}_0, \widetilde{d}_0, \widetilde{a}_0, \widetilde{b}_0) \in \mathcal{V}$, there exists a unique (weak) solution to (45) such that $(\varepsilon, \varepsilon_t, \widetilde{c}, \widetilde{d}, \widetilde{a}, \widetilde{b}) \in C(0,\infty;\mathcal{V})$. Moreover, the solution of system (45) is asymptotically stable in the sense that

$$\lim_{t\to\infty}\left[\frac{1}{2}\int_0^1 \left[\varepsilon_t^2(x,t) + \varepsilon_x^2(x,t)\right]\mathrm{d}x + k_2\varepsilon^2(0,t)\right] = 0$$

and

$$\lim_{t\to\infty}\widehat{c}(t) = c,\ \lim_{t\to\infty}\widehat{d}(t) = d,\ \lim_{t\to\infty}\widehat{a}(t) = a,\ \lim_{t\to\infty}\widehat{b}(t) = b.$$

By the update law of $\widehat{c}(t), \widehat{d}(t), \widehat{a}(t)$, and $\widehat{b}(t)$ in system (45), a formal computation gives

$$\begin{aligned} &\frac{d}{dt}[\widehat{c}(t)\cos\omega t + \widehat{d}(t)\sin\omega t + r_1\varepsilon(0,t)] = \omega[\widehat{d}(t)\cos\omega t - \widehat{c}(t)\sin\omega t], \\ &\frac{d^2}{dt^2}[\widehat{c}(t)\cos\omega t + \widehat{d}(t)\sin\omega t + r_1\varepsilon(0,t)] = -\omega^2[\widehat{c}(t)\cos\omega t + \widehat{d}(t)\sin\omega t], \\ &\frac{d}{dt}[\widehat{a}(t)\cos\varpi t + \widehat{b}(t)\sin\varpi t + r_2\varpi\tan\varpi\varepsilon(0,t)] = \omega[\widehat{b}(t)\cos\varpi t - \widehat{a}(t)\sin\varpi t], \\ &\frac{d^2}{dt^2}[\widehat{a}(t)\cos\varpi t + \widehat{b}(t)\sin\varpi t + r_2\varpi\tan\varpi\varepsilon(0,t)] = -\omega^2[\widehat{a}(t)\cos\varpi t + \widehat{b}(t)\sin\varpi t]. \end{aligned} \tag{49}$$

Now, we construct the second auxiliary system in which the disturbance and reference signal becomes collocated with the control. To do it, let

$$\begin{aligned} p(x,t) = &\widehat{z}(x,t) - \frac{1}{\omega}\sin\omega x[\widehat{c}(t)\cos\omega t + \widehat{d}(t)\sin\omega t + r_1\varepsilon(0,t)] \\ &- \tan\varpi\sin\varpi x[\widehat{a}(t)\cos\varpi t + \widehat{b}(t)\sin\varpi t + r_2\varpi\tan\varpi\varepsilon(0,t)]. \end{aligned} \tag{50}$$

Then, from (44) and (49), we can get the following auxiliary system:

$$\begin{cases} p_{tt}(x,t) = p_{xx}(x,t) - (\omega r_1\sin\omega x + r_2\varpi^3\tan^2\varpi\sin\varpi x)\varepsilon(0,t), \\ p_x(0,t) = -k_1\varepsilon_t(0,t) - (k_2 + r_1 + r_2\varpi^2\tan^2\varpi)\varepsilon(0,t), \\ p_x(1,t) = u(t) - \cos\omega[\widehat{c}(t)\cos\omega t + \widehat{d}(t)\sin\omega t] \\ \qquad -\varpi\sin\varpi[\widehat{a}(t)\cos\varpi t + \widehat{b}(t)\sin\varpi t] - (r_1\cos\omega + r_2\varpi^2\sin\varpi\tan\varpi)\varepsilon(0,t), \\ p(x,0) = p_0(x), \quad p_t(x,0) = p_1(x), \end{cases} \tag{51}$$

where

$$\begin{aligned} p_0(x) &= \widehat{z}_0(x) - \frac{\widehat{c}_0 + r_1\varepsilon_0(0)}{\omega}\sin\omega x - \tan\varpi\sin\varpi x[\widehat{a}_0 + r_2\varpi\tan\varpi\varepsilon_0(0)], \\ p_1(x) &= \widehat{z}_1(x) - \widehat{b}_0\sin\omega x. \end{aligned} \tag{52}$$

Moreover,

$$p(0,t)=\hat{z}(0,t). \tag{53}$$

We present the controller for (51) as follows:

$$\begin{aligned} u(t)= & \cos\omega\left[\hat{c}(t)\cos\omega t+\hat{d}(t)\sin\omega t\right]+\varpi\sin\varpi\left[\hat{a}(t)\cos\varpi t+\hat{b}(t)\sin\varpi t\right]-c_0p(1,t)\\ & -c_1p_t(1,t)-c_0c_1\int_0^1 p_t(\xi,t)\mathrm{d}\xi+(r_1\cos\omega+r_2\varpi^2\sin\varpi\tan\varpi)\varepsilon(0,t), \end{aligned} \tag{54}$$

where c_0 and c_1 are positive design parameters. The closed-loop system of (50) corresponding to controller (54) is

$$\begin{cases} p_{tt}(x,t)=p_{xx}(x,t)-(\omega r_1\sin\omega x+r_2\varpi^3\tan^2\varpi\sin\varpi x)\varepsilon(0,t),\\ p_x(0,t)=-k_1\varepsilon_t(0,t)-(k_2+r_1+r_2\varpi^2\tan^2\varpi)\varepsilon(0,t),\\ p_x(1,t)=-c_0p(1,t)-c_1p_t(1,t)-c_0c_1\int_0^1 p_t(\xi,t)\mathrm{d}\xi,\\ p(x,0)=p_0(x),\quad p_t(x,0)=p_1(x). \end{cases} \tag{55}$$

Define $\mathbf{H}=H^1(0,1)\times L^2(0,1)$, which is a Hilbert space with the two following equivalent norms induced by the inner product:

$$\|(\phi,\psi)\|^2_{(\mathbf{H};\|\cdot\|_1)}=\int_0^1\left[|\phi'(x)|^2dx+|\psi(x)|^2\right]\mathrm{d}x+c_0|\phi(1)|^2,\quad \forall(\phi,\psi)\in\mathbf{H}$$

and

$$\|(\phi,\psi)\|^2_{(\mathbf{H};\|\cdot\|_2)}=\int_0^1\left[|\phi'(x)|^2dx+|\psi(x)|^2\right]\mathrm{d}x+c_0|\phi(0)|^2,\quad \forall\ (\phi,\psi)\in\mathbf{H}.$$

In the rest of this paper, we write norm $\|\cdot\|_{\mathbf{H}}$ without discrimination.

Theorem 5. *For any initial value $(p_0,p_1)\in\mathbf{H}$, there exists a unique (weak) solution to (55) such that $(p,p_t)\in C(0,\infty;\mathbf{H})$. Moreover, the solution of (55) is asymptotically stable in the sense that*

$$\lim_{t\to\infty}E_p(t)=\lim_{t\to\infty}\left[\frac{1}{2}\int_0^1\left[p_x^2(x,t)+p_t^2(x,t)\right]\mathrm{d}x+\frac{1}{2}c_0[p(1,t)]^2\right]=0.$$

Introduce the following transformation (see the works of Krstic et al[23] or Krstic and Smyshlyaev[20 p83]):

$$q(x,t)=[I+P](p)(x,t)=p(x,t)+c_0\int_0^x p(\xi,t)\mathrm{d}\xi, \tag{56}$$

which is invertible. The inverse is given by

$$p(x,t)=q(x,t)-c_0\int_0^x e^{-c_0(x-\xi)}q(\xi,t)\mathrm{d}\xi.$$

It is seen that transformation (56) converts system (55) into the following target system:

$$\begin{cases} q_{tt}(x,t)=q_{xx}(x,t)+[c_0k_2-\omega r_1\sin\omega x+c_0r_1\cos\omega x\\ \qquad -r_2\varpi^3\tan^2\varpi\sin\varpi x+r_2\varpi^2\tan^2\varpi\cos\varpi x]\varepsilon(0,t)+c_0k_1\varepsilon_t(0,t),\\ q_x(0,t)=c_0q(0,t)-k_1\varepsilon_t(0,t)-(k_2+r_1+r_2\varpi^2\tan^2\varpi)\varepsilon(0,t),\\ q_x(1,t)=-c_1q_t(1,t),\\ q(x,0)=q_0(x),\quad q_t(x,0)=q_1(x), \end{cases} \tag{57}$$

where

$$q_0(x)=p_0(x)+c_0\int_0^x p_0(\xi)\mathrm{d}\xi,\ q_1(x)=p_1(x)+c_0\int_0^x q_1(\xi)\mathrm{d}\xi. \tag{58}$$

The target system (57) will be proved to be asymptotically stable later.

Then, controller (54) is obtained in the process of transforming (50) into (57) under the backstepping transformation (56). Notice that controller (54) is expressed by variable p. In order to get the closed loop of system (4), it is necessary to write controller (54) to be expressed by variable $\widehat{z}$. Then, by (50), we rewrite the controller (54) to be

$$
\begin{aligned}
u(t) = & - c_0\widehat{z}(1,t) - c_1\widehat{z}_t(1,t) - c_0c_1\int_0^1 \widehat{z}_t(\xi,t)\mathrm{d}\xi \\
& + \left\{ \left(\cos\omega + \frac{c_0}{\omega}\sin\omega\right)\widehat{c}(t) + \left[c_1\sin\omega + \frac{c_0c_1}{\omega}(1-\cos\omega)\right]\widehat{d}(t) \right\}\cos\omega t \\
& + \left\{ \left(\cos\omega + \frac{c_0}{\omega}\sin\omega\right)\widehat{d}(t) - \left[c_1\sin\omega + \frac{c_0c_1}{\omega}(1-\cos\omega)\right]\widehat{c}(t) \right\}\sin\omega t \\
& + \left\{ (\varpi\sin\varpi + c_0\tan\varpi\sin\varpi)\widehat{a}(t) + [c_1\varpi\tan\varpi\sin\varpi + c_0c_1(\tan\varpi - \sin\varpi)]\,\widehat{b}(t) \right\}\cos\varpi t \\
& + \left\{ (\varpi\sin\varpi + c_0\tan\varpi\sin\varpi)\widehat{b}(t) - [c_1\varpi\tan\varpi\sin\varpi + c_0c_1(\tan\varpi - \sin\varpi)]\,\widehat{a}(t) \right\}\sin\varpi t \\
& + \left[\frac{c_0r_1}{\omega}\sin\omega + r_1\cos\omega + r_2\varpi^2\sin\varpi\tan\varpi + r_2c_0\varpi(\tan\varpi\sin\varpi)^2\right]\left[z(0,t) - \widehat{z}(0,t)\right].
\end{aligned}
\tag{59}
$$

Combined by (4), (44), and (59), the resulting closed loop depicted in Figure 2 is governed by

$$
\begin{cases}
y_{tt}(x,t) = y_{xx}(x,t), \\
y_x(0,t) = c\cos\omega t + d\sin\omega t, \\
y_x(1,t) = \widehat{z}_x(1,t), \\
\widehat{z}_{tt}(x,t) = \widehat{z}_{xx}(x,t), \\
\widehat{z}_x(0,t) = \widehat{c}(t)\cos\omega t + \widehat{d}(t)\sin\omega t \\
\qquad + \varpi\tan\varpi\left[\widehat{a}(t)\cos\varpi t + \widehat{b}(t)\sin\varpi t\right] + k_1\left(\widehat{z}_t(0,t) - \dot{e}(t)\right) + k_2\left(\widehat{z}(0,t) - e(t)\right), \\
\widehat{z}_x(1,t) = -c_0\widehat{z}(1,t) - c_1\widehat{z}_t(1,t) - c_0c_1\int_0^1 \widehat{z}_t(\xi,t)\mathrm{d}\xi \\
\qquad + \left\{ \left(\cos\omega + \frac{c_0}{\omega}\sin\omega\right)\widehat{c}(t) + \left[c_1\sin\omega + \frac{c_0c_1}{\omega}(1-\cos\omega)\right]\widehat{d}(t) \right\}\cos\omega t \\
\qquad + \left\{ \left(\cos\omega + \frac{c_0}{\omega}\sin\omega\right)\widehat{d}(t) - \left[c_1\sin\omega + \frac{c_0c_1}{\omega}(1-\cos\omega)\right]\widehat{c}(t) \right\}\sin\omega t \\
\qquad + \left\{ (\varpi\sin\varpi + c_0\tan\varpi\sin\varpi)\widehat{a}(t) + [c_1\varpi\tan\varpi\sin\varpi + c_0c_1(\tan\varpi - \sin\varpi)]\,\widehat{b}(t) \right\}\cos\varpi t \\
\qquad + \left\{ (\varpi\sin\varpi + c_0\tan\varpi\sin\varpi)\widehat{b}(t) - [c_1\varpi\tan\varpi\sin\varpi + c_0c_1(\tan\varpi - \sin\varpi)]\widehat{a}(t) \right\}\sin\varpi t \\
\qquad + \left[\frac{c_0r_1}{\omega}\sin\omega + r_1\cos\omega + r_2\varpi^2\sin\varpi\tan\varpi + r_2c_0\varpi(\tan\varpi\sin\varpi)^2\right]\left[e(t) - \widehat{z}(0,t)\right], \\
\dot{\widehat{c}}(t) = -r_1\left(\dot{e}(t) - \widehat{z}_t(0,t)\right)\cos\omega t, \\
\dot{\widehat{d}}(t) = -r_1\left(\dot{e}(t) - \widehat{z}_t(0,t)\right)\sin\omega t, \\
\dot{\widehat{a}}(t) = -r_2\varpi\tan\varpi\left(\dot{e}(t) - \widehat{z}_t(0,t)\right)\cos\varpi t, \\
\dot{\widehat{b}}(t) = -r_2\varpi\tan\varpi\left(\dot{e}(t) - \widehat{z}_t(0,t)\right)\sin\varpi t, \\
e(t) = z(0,t) = y(0,t) - [a\cos\omega t + b\sin\omega t] \\
\widehat{a}(0) = \widehat{a}_0, \quad \widehat{b}(0) = \widehat{b}_0, \quad \widehat{c}(0) = \widehat{c}_0, \quad \widehat{d}(0) = \widehat{d}_0, \\
y(x,0) = y_0(x), \; y_t(x,0) = y_1(x).
\end{cases}
\tag{60}
$$

Define $\mathcal{X} = \mathbf{H}\times\mathcal{V}$. Let us consider system (60) in space $\mathcal{X}$.

Theorem 6. *Suppose that $\omega, \varpi \neq 0, n\pi + \frac{\pi}{2}$, $n\in\mathbb{Z}$. For any initial value $(y_0, y_1, \widehat{z}_0, \widehat{z}_1, \widehat{c}_0, \widehat{d}_0, \widehat{a}_0, \widehat{b}_0) \in \mathcal{X}$, there exists a unique (weak) solution to (60) such that $(y(\cdot,t), y_t(\cdot,t), \widehat{z}(\cdot,t), \widehat{z}_t(\cdot,t), \widehat{c}(t), \widehat{d}(t), \widehat{a}(t), \widehat{b}(t)) \in C([0,\infty);\mathcal{X})$. Moreover, this closed-loop solution has the following properties.*

i. $\sup_{t\geq 0}[\int_0^1[y_t^2(x,t) + y_x^2(x,t) + \widehat{z}_t^2(x,t) + \widehat{z}_x^2(x,t)]\mathrm{d}x + \widehat{c}^2(t) + \widehat{d}^2(t) + \widehat{a}^2(t) + \widehat{b}^2(t)] < \infty$.

ii. $\lim_{t\to\infty}\widehat{c}(t) = c$, $\lim_{t\to\infty}\widehat{d}(t) = d$, $\lim_{t\to\infty}\widehat{a}(t) = a$, $\lim_{t\to\infty}\widehat{b}(t) = b$.

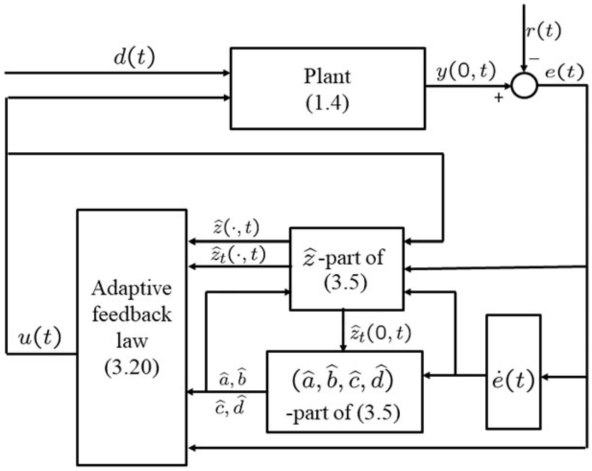

FIGURE 2 Block diagram of the closed-loop system (60)

iii. $\lim_{t\to\infty} e(t) = 0$.
iv. *When* $c = d = a = b = 0$, $\int_0^1 [y_t^2(x,t) + y_x^2(x,t)]\mathrm{d}x + c_0 y^2(0,t) \to 0$ as $t \to \infty$.

3.2 | Proof of the main result

3.2.1 | Proof Theorem 2

For any initial value $(p_0, p_1) \in \mathbf{H}$, it follows from (58) that $(q_0, q_1) \in \mathbf{H}$. By the transformation (56), it is sufficient to prove system (57) has a unique (weak) solution $(q, q_t) \in C(0,\infty;\mathbf{H})$ and asymptotical stabilization of system (57) in the sense that

$$\lim_{t\to\infty} E_q(t) = \lim_{t\to\infty}\left[\frac{1}{2}\int_0^1 \left[q_x^2(x,t) + q_t^2(x,t)\right]\mathrm{d}x + \frac{1}{2}c_0[q(0,t)]^2\right] = 0. \tag{61}$$

Define an operator $\mathbf{A} : \mathbf{D(A)} \to \mathbf{H}$ by

$$\begin{cases} \mathbf{A}(u,v)^\top = (v,u'')^\top, \ \forall (u,v) \in \mathbf{D}(\mathbf{A}) \\ \mathbf{D}(\mathbf{A}) = \left\{(u,v)^\top \in \mathbf{H} |\ \mathbf{A}(u,v)^\top \in \mathbf{H},\ f'(0) = c_0 f(0),\ f'(1) = -c_1 g(1)\right\}. \end{cases} \tag{62}$$

Then, system (57) can be written as

$$\frac{d}{dt}\begin{pmatrix} q(\cdot,t) \\ q_t(\cdot,t)\end{pmatrix} = \mathbf{A}\begin{pmatrix} q(\cdot,t) \\ q_t(\cdot,t)\end{pmatrix} + \begin{pmatrix} 0 \\ f(\cdot,t)\end{pmatrix} + \mathbf{B}\left[-k_1\varepsilon_t(0,t) - (k_2 + r_1 + r_2\varpi^2\tan^2\varpi)\varepsilon(0,t)\right], \tag{63}$$

where $\mathbf{B} = (0, -\delta(x))^\top$ and

$$f(x,t) = [c_0k_2 - \omega r_1 \sin\omega x + c_0 r_1 \cos\omega x - r_2\varpi^3\tan^2\varpi \sin\varpi x + r_2\varpi^2\tan^2\varpi\cos\varpi x]\varepsilon(0,t) + c_0k_1\varepsilon_t(0,t).$$

It is well known that $\mathbf{A}$ generates an exponential stable C_0-semigroup. Then, there exist $K, \mu > 0$ such that

$$\left\|e^{\mathbf{A}t}\right\| \le K e^{-\mu t}. \tag{64}$$

It is a routine exercise that $\mathbf{B}$ and I are admissible for $e^{\mathbf{A}t}$.[35] On the other hand, by Theorem 4, we obtain $\lim_{t\to\infty}\varepsilon(0,t) = 0$. By the proof of Theorem 2, comparing (47) with (12), and noting (24), we have $\varepsilon_t(1,t) \in L^2(0,\infty)$. Therefore, it follows from lemma 2.1 in the work of Zhou and Weiss[36] that system (63) has a unique solution that is asymptotically stable.

3.2.2 | Proof of Theorem 6

Let

$$A_1(x,t) = \frac{\cos\varpi(1-x)}{\cos\varpi}[a\cos\varpi t + b\sin\varpi t]$$

and

$$A_2(x,t) = \frac{1}{\omega}\sin\omega x\left[\hat{c}(t)\cos\omega t + \hat{d}(t)\sin\omega t + r_1\varepsilon(0,t)\right] + \tan\varpi\sin\varpi x\left[\hat{a}(t)\cos\varpi t + \hat{b}(t)\sin\varpi t + r_2\varpi\tan\varpi\varepsilon(0,t)\right].$$

Then, from (41) and (50), together with the fact $\varepsilon = z - \hat{z}$, one has

$$y(x,t) = \varepsilon(x,t) + p(x,t) + A_2(x,t) + A_1(x,t),\ \hat{z}(x,t) = p(x,t) + A_2(x,t). \tag{65}$$

For any initial value $(y_0, y_0, \hat{z}_0, \hat{z}_1, \hat{c}_0, \hat{d}_0, \hat{a}_0, \hat{b}_0) \in \mathcal{V}$, it is obvious from (42),(46), and (52) that $(\varepsilon_0, \varepsilon_1, \tilde{c}_0, \tilde{d}_0, \tilde{a}_0, \tilde{b}_0) \in \mathcal{V}$ and $(p_0, p_1) \in \mathbf{H}$, which implies that there exists a unique solution (weak) $(\varepsilon, \varepsilon_t, \hat{z}, \hat{z}_t, \tilde{c}, \tilde{d}, \tilde{a}, \tilde{b}) \in C([0,\infty); \mathcal{V})$ to (41) and a unique solution $(p, p_t) \in C([0,\infty); \mathbf{H})$ to (55), respectively. It follows from (65) and (50) that system (60) admits a unique solution (weak) $(y, y_t, \hat{a}, \hat{b}, \hat{c}, \hat{d}) \in C([0,\infty); \mathcal{V})$. The first part is proved.

Theorem 4 with (40) gives property i to iii. We say that $a = b = c = d = 0$, which implies there are no disturbance and reference signal, ie, $\hat{a}(t) = \hat{b}(t) = \hat{c}(t) = \hat{d}(t) \equiv 0$. Thus, iv is a well-known result. The proof is completed.

4 | NUMERICAL EXAMPLES

In this section, we present some numerical simulation for illustrating the theory results. In the simulation, the second-order equations in time are firstly converted into a system of two one-order equations, and then the backward Euler method in time and the Chebyshev spectral method in space are used. The grid size is taken as $N = 20$ and time step $dt = 0.001$. We choose $k_1 = 0.9, k_2 = 1.1, \varpi = \frac{\pi}{4}, \omega = \frac{\pi}{3}, r_1 = 1$, and $r_2 = 2$. We set the four unknown parameters to be that $a = 1, b = -1, c = -1.5$, and $d = 1.5$. The initial state for (21) is taken as $y(x,0) = 2x - x^2, y_t(x,0) = -2x + x^2$, $\hat{a}(0) = -1, \hat{b}(0) = 1, \hat{c}(0) = 1.5$, and $\hat{d}(0) = -1.5$. The initial state for (60) is taken as $y(x,0) = 2x - x^2, y_t(x,0) = -2x + x^2$, $\hat{z}(x,0) = -2x + x^2, \hat{z}_t(x,0) = 2x - x^2, \hat{a}(0) = -1, \hat{b}(0) = 1, \hat{c}(0) = 1.5$, and $\hat{d}(0) = -1.5$.

Figures 3 and 4 show that the output signal to regulate tracks asymptotically the references for both collocated and anti-collocated case. Figures 5 and 6 show approximation of the parameters. It is seen that the estimates $\hat{a}(t), \hat{b}(t), \hat{d}(t)$, and $\hat{d}(t)$ approximated, respectively, the system parameters a, b, c, and d. In the collocated error feedback output regulation case, the numerical results for $y(x,t)$ and $z(x,t)$ are presented in Figures 7 and 8. We see that the state of $y(x,t)$ is bounded and "z-part" of system (12) is indeed asymptotically stable. In the anticollocated error feedback output regulation case, the numerical results for $y(x,t)$ and $\varepsilon(x,t)$ are presented in Figures 9 and 10. It is seen that $y(x,t)$ is bounded and "ε-part" of system (45) converges to zero.

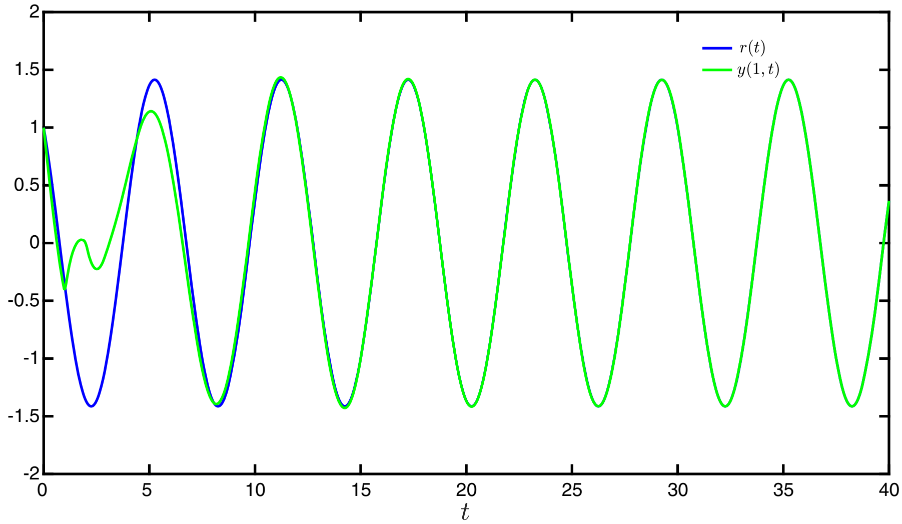

FIGURE 3 The output tracking signal $y(1,t)$ and the reference signal $r(t) = \sin\frac{\pi}{4}t - \cos\frac{\pi}{4}t$ [Colour figure can be viewed at wileyonlinelibrary.com]

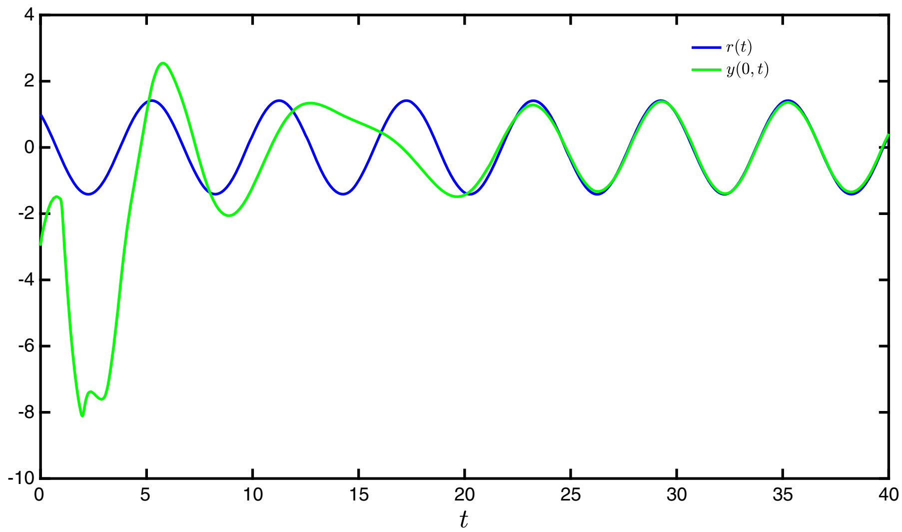

FIGURE 4 The output tracking signal $y(0, t)$ and the reference signal $r(t) = \sin \frac{\pi}{4}t - \cos \frac{\pi}{4}t$ [Colour figure can be viewed at wileyonlinelibrary.com]

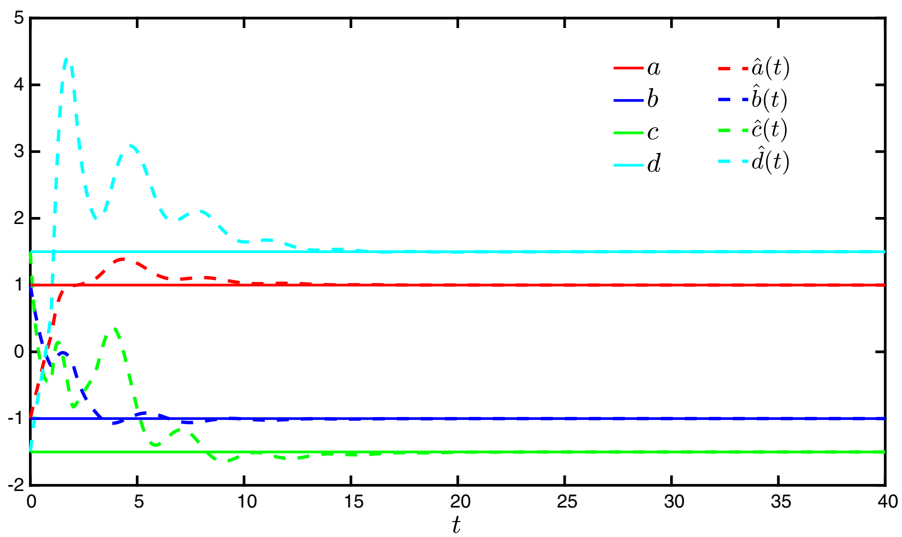

FIGURE 5 Parameters estimations $\hat{a}(t), \hat{b}(t), \hat{c}(t)$, and $\hat{d}(t)$ for system (21) [Colour figure can be viewed at wileyonlinelibrary.com]

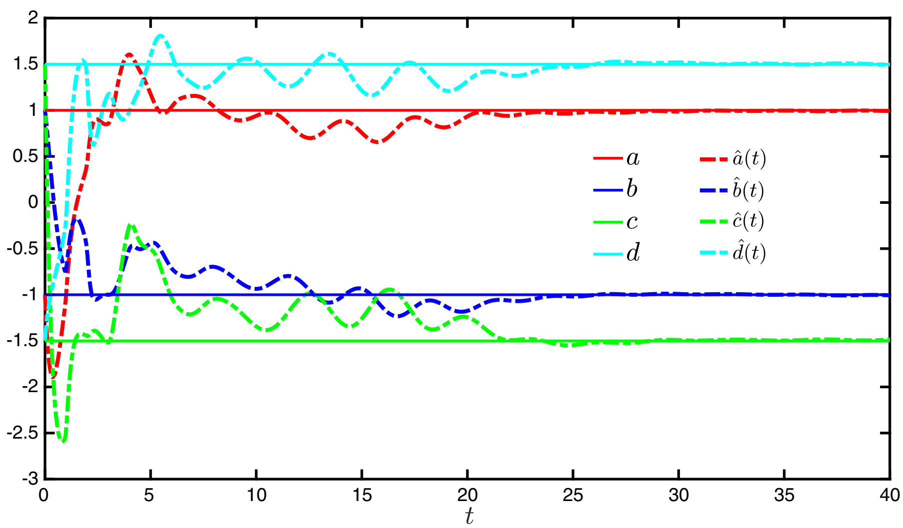

FIGURE 6 Parameters estimations $\hat{a}(t), \hat{b}(t), \hat{c}(t)$, and $\hat{d}(t)$ for system (60) [Colour figure can be viewed at wileyonlinelibrary.com]

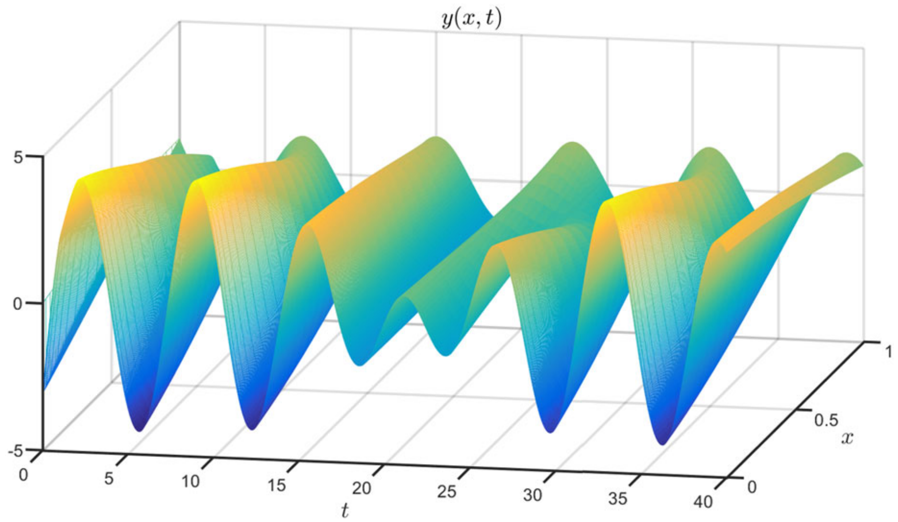

FIGURE 7 The displacement of $y(x, t)$ for system (21) [Colour figure can be viewed at wileyonlinelibrary.com]

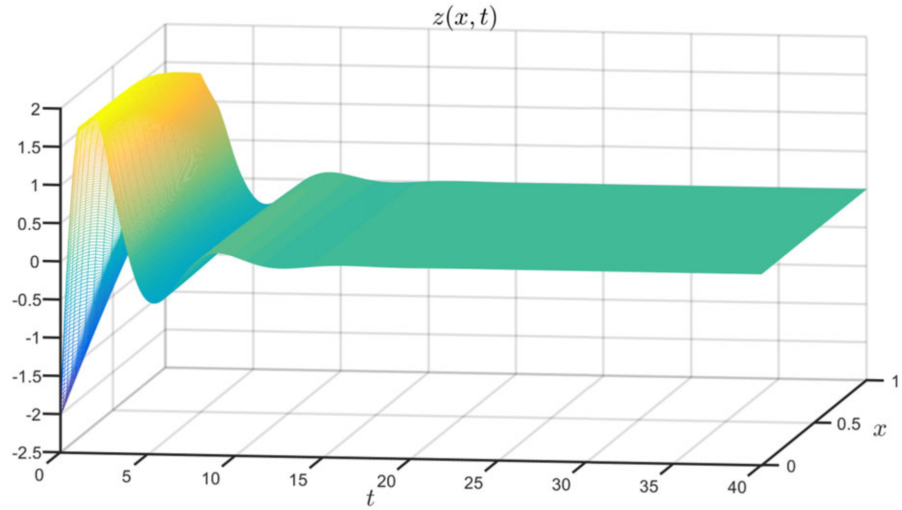

FIGURE 8 The displacement of $z(x, t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

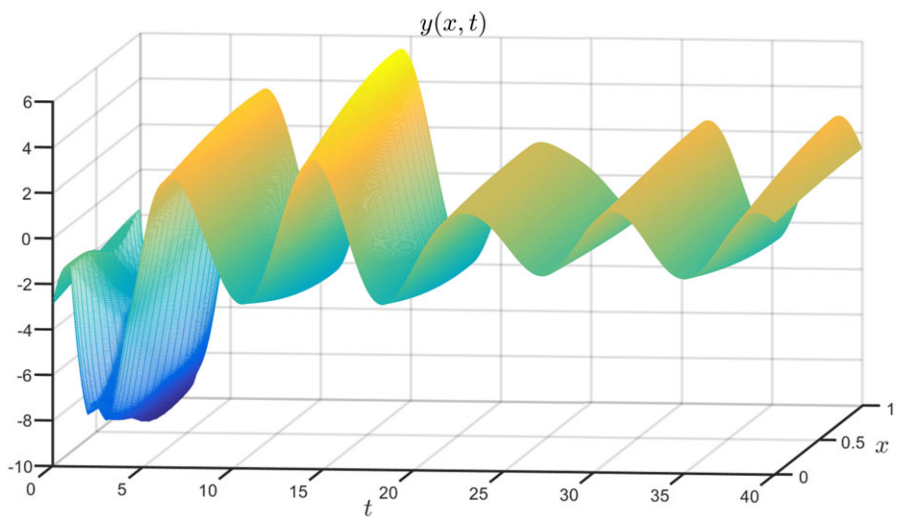

FIGURE 9 The displacement of $y(x, t)$ for system (60) [Colour figure can be viewed at wileyonlinelibrary.com]

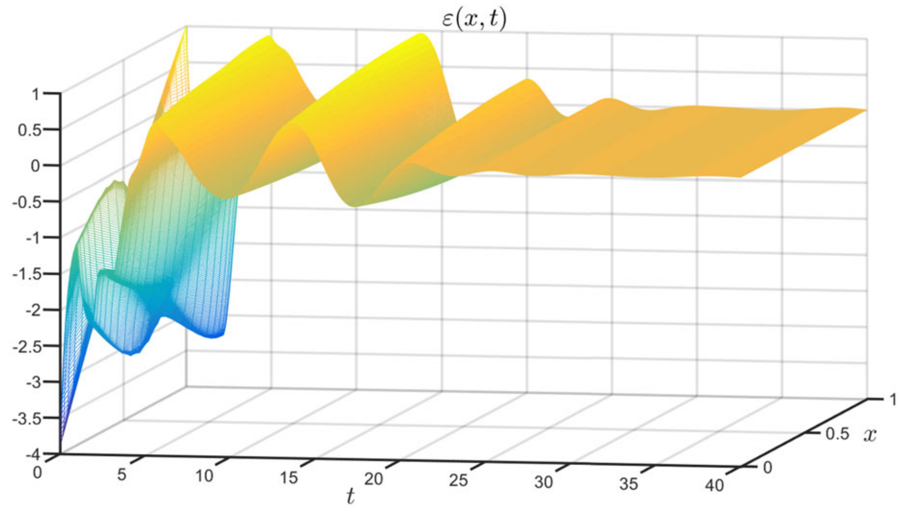

FIGURE 10 The displacement of $\varepsilon(x, t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

5 | CONCLUDING REMARKS

This paper has investigated the adaptive error feedback output regulation problem for 1D wave equation with harmonic disturbance anticollocated with control. We present two different adaptive error feedback output regulator designs for two different types of tracking error. Different from the classical error feedback output regulator design based on the internal mode principle, we first give the adaptive servomechanism design for the system by making use of the measured tracking error(and its time derivative) and the estimation mechanism for the parameters of the disturbances and tracking reference. The key characteristic of our approach is by constructing some auxiliary systems in which the measured error becomes output and the control becomes collocated with the disturbance. The four control objectives are (i) regulate the error output to zero, (ii) keep all the states bounded, (iii) estimate the unknown parameters, and (iv) make the resulting closed loop stable when disconnected with disturbance and reference is obtained. In future works, applying our approach to beam equation seems interesting, and relaxing the harmonic disturbance to general bounded disturbance is also an interesting problem. In addition, a future research direction may be to use adaptive fuzzy control design method in the works of Tong et al[37] and Tong et al[38] to solve output regulation problem for infinite-dimensional systems described by PDEs.

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