Received: 11 July 2017 | Revised: 5 February 2018 | Accepted: 31 March 2018
DOI: 10.1002/rnc.4117

RESEARCH ARTICLE

WILEY

Control of a 2×2 coupled linear hyperbolic system sandwiched between 2 ODEs

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Funding information
National Basic Research Program of China (973 Program), Grant/Award Number: 2014CB049404; Chongqing University Postgraduates Innovation Project, Grant/Award Number: CYD15023

Summary
Motivated by an engineering application in cable mining elevators, we address a new problem on stabilization of 2×2 coupled linear first-order hyperbolic PDEs sandwiched between 2 ODEs. A novel methology combining PDE backstepping and ODE backstepping is proposed to derive a state-feedback controller without high differential terms. The well-posedness and invertibility properties of the PDE backstepping transformation are proved. All states, including coupled linear hyperbolic PDEs and 2 ODEs, are included in the closed-loop exponential stability analysis. Moreover, boundedness and exponential convergence of the designed controller are proved. The performance is investigated via numerical simulation.

KEYWORDS
backstepping, distributed parameter systems, hyperbolic systems, ODE-PDE-ODE

1 | INTRODUCTION

Control of coupled linear hyperbolic PDE systems. The linear 2×2 hyperbolic PDE system can describe a wide range of physical systems, including open channels,[1] gas flow pipelines,[2] and oil wells.[3] Considering the wide range of practical applications, many efforts have been put into stabilizing linear 2 × 2 hyperbolic PDEs in recent years. The boundary stabilization and state estimation for a 2×2 system of first-order hyperbolic linear PDEs with spatially varying coefficients was considered in the work of Vazquez et al.[4] Stabilization of 2 × 2 first-order hyperbolic linear PDEs with uncertain parameters was solved via adaptive control in the work of Anfinsen and Aamo.[5] Disturbance rejection in a 2 × 2 linear coupled hyperbolic system was studied in the works of Aamo[6] and Anfinsen and Aamo.[7] Furthermore, the robust output regulation problem of a coupled 2×2 linear hyperbolic PDE system in the presence of disturbances has been solved in the work of Deutscher.[8] Backstepping design of output-feedback regulators that achieve finite time regulation for boundary controlled linear 2 × 2 hyperbolic systems was presented in the work of Deutscher.[9] Moreover, stabilization of $n + 1$ coupled first-order hyperbolic coupled linear PDEs was considered in the work of Di Meglio et al.[10] Control problem of a first-order hyperbolic linear PDE general system where the number of PDEs in either direction is arbitrary was solved in the work of Hu et al.[11] Disturbance rejection and parameter estimation for this general hyperbolic coupled linear PDE systems were also presented in the work of Anfinsen et al[12,13] A backstepping solution to the output regulation problem for general linear heterodirectional hyperbolic systems with disturbances and spatially varying coefficients was presented in the work of Deutscher.[14]

Control of PDE-ODE systems. Stabilization of PDE-ODE coupled structures has drawn much attention in the last decade when consider the compensation of infinite-dimensional states in the actuating or sensing paths of ODEs, such as time delay,[15,16] vibration string,[17] and diffusion phenomenon.[18] Many results about control of wave PDE-ODE,[17,19] heat PDE-ODE,[20,21] and transport PDE-ODE[22-24] have been obtained so far. The research on coupled linear hyperbolic PDE-ODE coupled system is limited. In a very recent result, the stabilization of a general coupled linear hyperbolic PDE-ODE system was considered in the work of Di Meglio et al,[25] where an ODE is stabilized through compensating linear coupled hyperbolic PDEs in the actuating path. However, except for the input delay compensation of an ODE, with an integration at the input of the transport delay in the work of Krstic,[26] no attempts have been made to address the control problem of a coupled linear hyperbolic system sandwiched between 2 ODEs. In fact, except for the work of Liu and Krstic,[27] when the viscous Burgers' equation with an integration at the input was considered, the present paper is the first when an ODE-PDE cascade is considered, with actuation of the ODE.

Coupled hyperbolic linear ODE-PDE-ODE systems. The ODE-hyperbolic PDE-ODE "sandwich" system can model many physical systems. For example, a mining cable elevator shown in Figure 1A, where the control input drives a drum winding a cable to lift a cargo, the dynamics of the drum could be described by an ODE in the input path of the following coupled linear hyperbolic PDE-ODE, which correspond to the vibration dynamics of the distributed parameter cable and the cargo, respectively. Note that the cable vibrations are described by a wave PDE, which can be converted to 2×2 coupled linear hyperbolic PDEs via introducing the Riemann variables.[23] The vibration control problem of this cable elevator can be regarded as stabilization of a 2×2 coupled linear hyperbolic system sandwiched between 2 ODEs shown in Figure 1B, ie,

$$\dot{X}(t) = AX(t) + Bv(0,t), \tag{1}$$

$$u_t(x,t) = -pu_x(x,t) + c_1 v(x,t), \tag{2}$$

$$v_t(x,t) = pv_x(x,t) + c_2 u(x,t), \tag{3}$$

$$u(0,t) = qv(0,t) + CX(t), \tag{4}$$

$$v(1,t) = z(t), \tag{5}$$

$$\ddot{z}(t) = c_0\dot{z}(t) + ru(1,t) + U(t), \tag{6}$$

$\forall (x,t) \in [0,1]\times[0,\infty)$, where $X(t) \in \mathbb{R}^{n\times 1}$ and $Z^T(t) = [z(t), \dot{z}(t)] = [z_1(t), z_2(t)] \in \mathbb{R}^{2\times 1}$ are ODE states. $u(x,t) \in \mathbb{R}$, $v(x,t) \in \mathbb{R}$ are states of the PDEs. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ satisfy that the pair $[A;B]$ is controllable. $C \in \mathbb{R}^{1\times n}$ and $c_0, c_1, c_2, r, q \in \mathbb{R}$ are arbitrary. p is the arbitrary positive transport velocity. Note that we consider the absolute values of

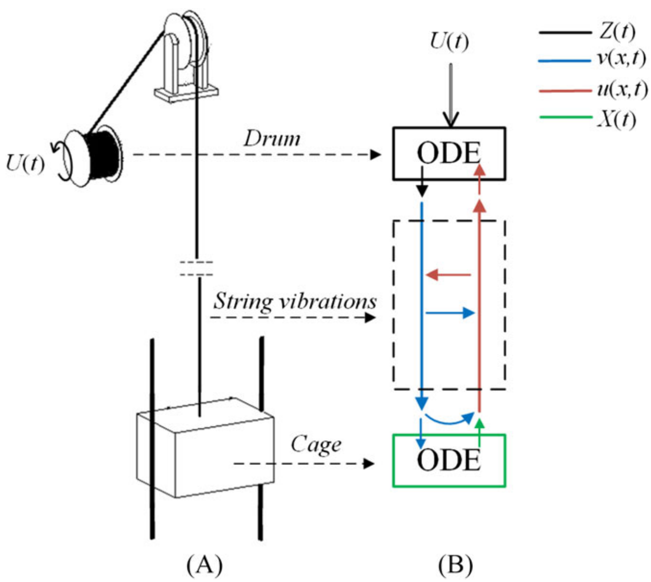

FIGURE 1 A, Mining cable elevator; B, Schematic view of the coupling ODE-PDE-ODE vibration dynamic system [Colour figure can be viewed at wileyonlinelibrary.com]

the transport velocities of (2)-(3) are equal in this paper but with a possible extension to the case where the absolute values of the transport velocities of the 2 transport PDEs are different. $U(t)$ is the control input to be designed. Full relative degree in (5)-(6) is assumed for the design. The control objective here is to exponentially stabilize all ODE states $Z(t), X(t)$ and PDE states $u(x,t), v(x,t)$ by designing a control input $U(t)$. Moreover, the result is extended to a more general system where arbitrary order ODEs sandwiching PDE.

Another physical model of this "sandwich" system is a overhead crane,[28] which consists of a motorized platform (ODE) driving a cable (hyperbolic PDE) connecting a payload (ODE) at the bottom. Besides, oil drilling[23] where a control input acting at the rotary table (ODE) to drive the drilling bit (ODE) through the drilling string (hyperbolic PDE) also can be modeled as hyperbolic PDEs sandwiched by 2 ODEs. In addition to aforementioned applications, a hyperbolic PDE, such as a flow model,[6] with actuator and sensor dynamics can also be described by the "sandwich" system concerned in this paper.

Main contributions:

- We extend the problem in the works of Krstic[26] and Cai et al[29] to a more challenging case where the input dynamics are not first but second order or even m order, and the single transport equation is developed to 2 counterconvecting coupled transport PDEs.
- Additional 2 ODEs are introduced to sandwich the counterconvecting and coupled transport PDEs compared with most exiting results dealing with coupled transport PDEs.[4,11]
- Compared with a very recent result in the work of Di Meglio et al,[25] where an ODE is stabilized through compensating linear coupled hyperbolic PDEs in the actuating path, we need to compensate both the actuated ODE and the linear coupled hyperbolic PDEs in the actuating path to stabilize another ODE.
- It is the first result of stabilizing such an ODE-PDE-ODE "sandwiched" system. Our result is new even if the 2 counterconvecting coupled transport PDEs are replaced by the standard wave PDE.

Organization. The rest of this paper is organized as follows. We seek an infinite-dimensional backstepping transformation that maps the plant into the target system in Section 2. We deal with the input ODE with a number of perturbation terms of PDE states via the ODE backstepping method in Section 3. A controller is proposed and the exponential stability of the closed-loop system is proved by Lyapunov analysis in Section 4. The boundedness and exponential convergence of the controller in the closed-loop system are proved in Section 5. In Section 6, we extend the proposed method and the according proofs to a more general case where the input ODE is arbitrary order. The simulation results are provided in Section 7. The conclusion and future work are presented in Section 8.

2 | BACKSTEPPING FOR PDE-ODE

2.1 | Backstepping transformations and target system

We consider the infinite-dimensional backstepping transformation of the PDE state $u(x,t), v(x,t)$, ie,

$$\alpha(x,t) \equiv u(x,t), \tag{7}$$

$$\beta(x,t) = v(x,t) - \int_0^x \psi(x,y)u(y,t)\mathrm{d}y - \int_0^x \phi(x,y)v(y,t)\mathrm{d}y - \gamma(x)X(t). \tag{8}$$

The kernel equations for $\psi(x,y)$ and $\phi(x,y), \gamma(x)$ are introduced in Section 2.2. The well-posedness of the kernel equations is proved in Section 2.3. Note that the reason why we only apply the backstepping transforation on v is that the source term in (3) is more sensitive to the stability result in the following Lyapunov analysis. Using this partial backstepping transformation (7)-(8) would achieve less calculation of kernels and the simpler structure of the controller.

The inverse of (7)-(8) is considered as

$$u(x,t) \equiv \alpha(x,t), \tag{9}$$

$$v(x,t) = \beta(x,t) - \int_0^x \psi^I(x,y)\alpha(y,t)\mathrm{d}y - \int_0^x \phi^I(x,y)\beta(y,t)\mathrm{d}y - \gamma^I(x)X(t), \tag{10}$$

where $\phi^I(x,y), \psi^I(x,y)$, and $\gamma^I(x)$ are the kernels of the inverse transformation (10), and the well-posedness of them is shown in Section 2.4.

Our aim is to convert the original system (1)-(5) to the following target system:

$$\dot{X}(t) = (A + B\kappa)X(t) + B\beta(0,t), \tag{11}$$

$$\alpha_t(x,t) = -p\alpha_x(x,t) + c_1\beta(x,t) - c_1\int_0^x \psi^I(x,y)\alpha(y,t)\mathrm{d}y, - c_1\int_0^x \phi^I(x,y)\beta(y,t)\mathrm{d}y - c_1\gamma^I(x)X(t), \tag{12}$$

$$\beta_t(x,t) = p\beta_x(x,t), \tag{13}$$

$$\alpha(0,t) = q\beta(0,t) + C_0X(t), \tag{14}$$

where $C_0 = C + q\gamma(0)$. Since the pair $[A; B]$ is controllable, there exists indeed κ such that $A + B\kappa$ is Hurwitz.

Let us now consider the boundary state $\beta(1,t)$. It is easily seen that

$$\begin{aligned}\beta_{tt}(1,t) = {} & v_{tt}(1,t) + p\psi(1,1)u_t(1,t) - p\psi(1,0)u_t(0,t) + p\left(p\psi_y(1,1) + c_2\phi(1,1)\right)u(1,t)\\ & - p\left(p\psi_y(1,0) + c_2\phi(1,0)\right)u(0,t) - p\phi(1,1)v_t(1,t) + (p\phi(1,0) - \gamma(1)B)\,v_t(0,t)\\ & - \gamma(1)A^2X(t) - p\left(c_1\psi(1,1) - p\phi_y(1,1)\right)v(1,t) + \left(pc_1\psi(1,0) - p^2\phi_y(1,0) - \gamma(1)AB\right)v(0,t)\\ & - \int_0^1 \left(p^2\psi_{yy}(1,y) + c_1c_2\psi(1,y)\right)u(y,t)\mathrm{d}y - \int_0^1 \left(p^2\phi_{yy}(1,y) + c_1c_2\phi(1,y)\right)v(y,t)\mathrm{d}y.\end{aligned} \tag{15}$$

Considering (5)-(6), we have

$$v_{tt}(1,t) = c_0v_t(1,t) + ru(1,t) + U(t). \tag{16}$$

Plugging (13) and the inverse transformations (9)-(10) into (15), after a lengthy calculation that involves a change of the order of integration in a double integral, we get

$$\begin{aligned}\beta_{tt}(1,t) = {} & h_1\beta_t(1,t) + h_5\beta(1,t) + U(t) + h_2\alpha_t(1,t) + h_3\beta_t(0,t) + h_4\alpha_t(0,t) + (h_6 + r)\alpha(1,t)\\ & + h_7\beta(0,t) + h_8\alpha(0,t) + \int_0^1 h_9(y)\beta(y,t)\mathrm{d}y + \int_0^1 h_{10}(y)\alpha(y,t)\mathrm{d}y + H_{11}X(t),\end{aligned} \tag{17}$$

which also belongs to the chosen target system. $h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9(y), h_{10}(y)$, and H_{11} are shown in Appendix. Note that (17) is a second-order ODE system $(\beta(1,t), \beta_t(1,t))$ with a number of PDE state perturbation terms.

Remark 1. We can obtain the well-posedness of the closed-loop system through analyzing the well-posedness of the target system (11)-(14), (17). Considering an ODE being well-posed straightly, neglecting that of the input ODE (17), it can be expected that the well-posedness of the target system depends on that of the 2×2 coupled linear hyperbolic PDE-ODE (11)-(14), which can be obtained similarly to the proof of Lemma 1, which will be shown as follows.

2.2 | Kernel equations of the backstepping transformation

Taking the derivative of (8) with respect to x and t, respectively, along the solution of (1)-(4) and substituting the results to (13), we get

$$\begin{aligned}\beta_t(x,t) - p\beta_x(x,t) = {} & v_t(x,t) - \int_0^x \psi(x,y)u_t(y,t)\mathrm{d}y - \int_0^x \phi(x,y)v_t(y,t)\mathrm{d}y - pv_x(x,t) + p\int_0^x \psi_x(x,y)u(y,t)\mathrm{d}y\\ & + p\int_0^x \phi_x(x,y)v(y,t)\mathrm{d}y + p\psi(x,x)u(x,t) + p\phi(x,x)v(x,t) - \gamma(x)\dot{X}(t) + p\gamma'(x)X(t)\\ = {} & c_2u(x,t) + \int_0^x p\psi(x,y)u_x(y,t)\mathrm{d}y - \int_0^x c_1\psi(x,y)v(y,t)\mathrm{d}y - \int_0^x p\phi(x,y)v_x(y,t)\mathrm{d}y\\ & - \int_0^x c_2\phi(x,y)u(y,t)\mathrm{d}y + \int_0^x p\psi_x(x,y)u(y,t)\mathrm{d}y + \int_0^x p\phi_x(x,y)v(y,t)\mathrm{d}y\\ & + p\psi(x,x)u(x,t) + p\phi(x,x)v(x,t) - \gamma(x)\dot{X}(t) + p\gamma'(x)X(t)\\ = {} & (c_2 + 2p\psi(x,x))\,u(x,t) + (p\phi(x,0) - \gamma(x)B - p\psi(x,0)q)\,v(0,t)\\ & + \int_0^x \left(-c_1\psi(x,y) + p\phi_x(x,y) + p\phi_y(x,y)\right)v(y,t)\mathrm{d}y\\ & - \int_0^x \left(c_2\phi(x,y) - p\psi_x(x,y) + p\psi_y(x,y)\right)u(y,t)\mathrm{d}y\\ & + \left(p\gamma'(x) - \gamma(x)A - p\psi(x,0)C\right)X(t) = 0.\end{aligned} \tag{18}$$

For (13) to hold and matching (11), (14) with (1), (4) via the transformations (8), we obtain the following kernel equations:

$$c_2 + 2p\psi(x,x) = 0, \tag{19}$$

$$p\phi(x,0) = \gamma(x)B + p\psi(x,0)q, \tag{20}$$

$$-c_1\psi(x,y) + p\phi_x(x,y) + p\phi_y(x,y) = 0, \tag{21}$$

$$c_2\phi(x,y) - p\psi_x(x,y) + p\psi_y(x,y) = 0, \tag{22}$$

$$p\gamma'(x) - \gamma(x)A - p\psi(x,0)C = 0, \tag{23}$$

$$\gamma(0) = \kappa, \tag{24}$$

where $0 \le y \le x \le 1, t > 0$.

2.3 | Well-posedness of the kernel equations

In Section 2.2, we have obtained the kernel equations (19)-(24) for the transformation (8). In this section, we show the well-posedness of the kernel equations (19)-(24) by using the methods of characteristics and successive approximations.[10]

Lemma 1. *The kernel equations (19)-(24) have a unique solution* $(\psi(x,y), \phi(x,y)) \in C^1(D)\times C^1(D)$ *where* $D = \{(x,y)|0 \le y \le x \le 1\}$.

Proof. The proof of this lemma is presented in Appendix. □

2.4 | Inverse transformation

In order to ensure the invertibility of the transformation (8), in this section, we search for the inverse transformation of (8), which can convert the target system (11)-(14) to the original system (1)-(4).

Recalling the transformation (8) and rewriting it as

$$v(x,t) - \int_0^x \phi(x,y)v(y,t)\mathrm{d}y = \beta(x,t) + \int_0^x \psi(x,y)u(y,t)\mathrm{d}y + \gamma(x)X(t). \tag{25}$$

Considering Lemma 1, $\phi(x,y)$ is continuous, and it can be concluded that a unique continuous $\chi(x,y)$ exists on $D = \{(x,y)|0 \le y \le x \le 1\}$ such that (see, eg, the work of Su et al[30])

$$v(x,t) = \beta(x,t) + \int_0^x \psi(x,y)u(y,t)\mathrm{d}y + \gamma(x)X(t) + \int_0^x \chi(x,y)\left(\beta(y,t) + \int_0^y \psi(y,z)u(z,t)\mathrm{d}z + \gamma(y)X(t)\right)\mathrm{d}y, \tag{26}$$

where the proof can be seen in chapter 9.9 in the work of Vazquez.[31]

Equation (26) can be rewritten in the form of (10) as

$$\begin{aligned} v(x,t) = \beta(x,t) + \int_0^x \chi(x,y)\beta(y,t)\mathrm{d}y + \int_0^x \left(\int_y^x \chi(x,z)\psi(z,y)\mathrm{d}z + \psi(x,y)\right)\alpha(y,t)\mathrm{d}y \\ + \left(\gamma(x) + \int_0^x \chi(x,y)\gamma(y)\mathrm{d}y\right)X(t). \end{aligned} \tag{27}$$

Comparing (27) with (10), we obtain

$$\psi^I(x,y) = -\int_y^x \chi(x,z)\psi(z,y)\mathrm{d}z - \psi(x,y), \tag{28}$$

$$\phi^I(x,y) = -\chi(x,y), \tag{29}$$

$$\gamma^I(x) = -\gamma(x) - \int_0^x \chi(x,y)\gamma(y)\mathrm{d}y. \tag{30}$$

According to the well-posedness of $\psi(x,y)$, $\chi(x,y)$, and $\gamma(y)$ in $D = \{(x,y)|0 \le y \le x \le 1\}$, we can conclude the well-posedness of kernels $\psi^I(x,y), \phi^I(x,y), \gamma^I(x)$ on D in (10), which shows the invertibility between the target system $(\alpha(x,t), \beta(x,t))$ and the original one $(u(x,t), v(x,t))$.

3 | BACKSTEPPING FOR INPUT ODE WITH PDE STATE PERTURBATIONS

The following backstepping transformation for the $(\beta(1,t), \beta_t(1,t))$ system (17) is made

$$y_1(t) = \beta(1,t), \tag{31}$$

$$y_2(t) = \beta_t(1,t) + \tau_1[\beta(1,t)], \tag{32}$$

where τ_1 to be defined in the following steps is the virtual control in the ODE backstepping method.

Step 1. We consider a Lyapunov function candidate as

$$V_{y1} = \frac{1}{2}y_1(t)^2. \tag{33}$$

Taking derivative of (33), we obtain

$$\dot{V}_{y1} = y_1(t)\dot{y}_1(t) = y_1(t)(y_2(t) - \tau_1). \tag{34}$$

Define

$$\tau_1 = \bar{c}_1 y_1(t), \tag{35}$$

where $\bar{c}_1$ is a positive constant to be determined later.

Substituting (35) into (34) yields to

$$\dot{V}_{y1} = -\bar{c}_1 y_1(t)^2 + y_1(t)y_2(t). \tag{36}$$

Step 2. Similarly, a Lyapunov function candidate is considered as

$$V_y = V_{y1} + \frac{1}{2}y_2(t)^2 = \frac{1}{2}y_1(t)^2 + \frac{1}{2}y_2(t)^2. \tag{37}$$

Taking the derivative of (37), we have

$$\dot{V}_y = -\bar{c}_1 y_1(t)^2 + y_1(t)y_2(t) + y_2(t)\left(\beta_{tt}(1,t) + \dot{\tau}_1\right). \tag{38}$$

Recalling (17), we have

$$\begin{aligned}\dot{V}_y = &-\bar{c}_1 y_1(t)^2 + y_1(t)y_2(t) + y_2(t)\Bigg(U(t) + h_1\beta_t(1,t) + h_5\beta(1,t) + h_2\alpha_t(1,t) + h_3\beta_t(0,t) + h_4\alpha_t(0,t)\\ &+ (h_6 + r)\alpha(1,t) + h_7\beta(0,t) + h_8\alpha(0,t) + \int_0^1 h_9(y)\beta(y,t)\mathrm{d}y + \int_0^1 h_{10}(y)\alpha(y,t)\mathrm{d}y + H_{11}X(t) + \dot{\tau}_1\Bigg),\end{aligned} \tag{39}$$

where the gains $h_1, \ldots, h_{10}, H_{11}$ shown in Appendix are related to the kernel functions $(\psi(x,y), \phi(x,y)) \in W^{2,1}(D)$.

Choosing

$$\begin{aligned}U(t) &= -\bar{c}_2 y_2(t) - y_1(t) - \dot{\tau}_1 - h_1\beta_t(1,t) - h_5\beta(1,t) - h_2\alpha_t(1,t) - h_3\beta_t(0,t) - h_4\alpha_t(0,t)\\ &= -(\bar{c}_2 + \bar{c}_1 + h_1)\beta_t(1,t) - (\bar{c}_1\bar{c}_2 + 1 + h_5)\beta(1,t) - h_2\alpha_t(1,t) - h_3\beta_t(0,t) - h_4\alpha_t(0,t),\end{aligned} \tag{40}$$

where $\bar{c}_2$ is a positive constant to be determined later, we have

$$\begin{aligned}\dot{V}_y = &-\bar{c}_1 y_1(t)^2 - \bar{c}_2 y_2(t)^2 + y_2(t)\Bigg((h_6 + r)\alpha(1,t) + h_7\beta(0,t) + h_8\alpha(0,t)\\ &+ \int_0^1 h_9(y)\beta(y,t)\mathrm{d}y + \int_0^1 h_{10}(y)\alpha(y,t)\mathrm{d}y + H_{11}X(t)\Bigg).\end{aligned} \tag{41}$$

4 | CONTROLLER AND STABILITY ANALYSIS

4.1 | Control law

Substituting the PDE transformation (7)-(8) into (40), we get the controller expressed by the original states

$$\begin{aligned}U(t) = &-n_1 v_t(1,t) + n_2 v(1,t) - h_2 u_t(1,t) - n_3 u(1,t) - h_3 v_t(0,t) - n_4 v(0,t) - h_4 u_t(0,t)\\ &+ n_5 u(0,t) + N_8 X(t) + \int_0^1 n_6(y)u(y,t)\mathrm{d}y + \int_0^1 n_7(y)v(y,t)\mathrm{d}y,\end{aligned} \tag{42}$$

where

$$n_1 = \bar{c}_2 + \bar{c}_1 + h_1, \tag{43}$$

$$n_2 = (\bar{c}_2 + \bar{c}_1 + h_1)\phi(1,1) - (\bar{c}_1\bar{c}_2 + 1 + h_5), \tag{44}$$

$$n_3 = (\bar{c}_2 + \bar{c}_1 + h_1)\psi(1,1), \tag{45}$$

$$n_4 = (\bar{c}_2 + \bar{c}_1 + h_1)(\phi(1,0) - \gamma(1)B) - h_3\gamma(0)B, \tag{46}$$

$$n_5 = (\bar{c}_2 + \bar{c}_1 + h_1)\psi(1,0), \tag{47}$$

$$n_6(y) = (\bar{c}_2 + \bar{c}_1 + h_1)(\psi_y(1,y) + c_2\phi(1,y)) + (\bar{c}_1\bar{c}_2 + 1 + h_5)\psi(1,y), \tag{48}$$

$$n_7(y) = (\bar{c}_2 + \bar{c}_1 + h_1)(c_1\psi(1,y) - \phi_y(1,y)) + (\bar{c}_1\bar{c}_2 + 1 + h_5)\phi(1,y), \tag{49}$$

$$N_8 = h_3\gamma(0)A + (\bar{c}_2 + \bar{c}_1 + h_1)\gamma(1)A + (\bar{c}_1\bar{c}_2 + 1 + h_5)\gamma(1). \tag{50}$$

The pending control parameters $\bar{c}_1$ and $\bar{c}_2$ will be determined in the following stability analysis.

By substituting (2)-(3) at $x = 0$ and $x = 1$ into (42), the controller is rewritten as

$$\begin{aligned} U(t) = & -n_1pv_x(1,t) + (n_2 - h_2c_1)v(1,t) + h_2pu_x(1,t) - (n_3 + n_1c_2)u(1,t) - h_3pv_x(0,t) - (n_4 + h_4c_1)v(0,t) \\ & + h_4pu_x(0,t) + (n_5 - h_3c_2)u(0,t) + N_8X(t) + \int_0^1 n_6(y)u(y,t)\mathrm{d}y + \int_0^1 n_7(y)v(y,t)\mathrm{d}y, \end{aligned} \tag{51}$$

which is well defined.

4.2 | Stability analysis of states

Theorem 1. *If initial values* $(u(x,0), v(x,0)) \in W^{2,2}(0,1)$*, for some* $\bar{c}_1$ *and* $\bar{c}_2$*, the closed-loop system consisting of the plant (1)-(6) and the control law (51) is exponentially stable at the origin in the sense of the norm*

$$\left(\int_0^1 u^2(x,t)\mathrm{d}x + \int_0^1 v^2(x,t)\mathrm{d}x + |X(t)|^2 + z_1(t)^2 + z_2(t)^2\right)^{1/2}. \tag{52}$$

Proof. We start from studying the stability of the target system. The equivalent stability property between the target system and the original system is ensured due to the invertibility of the PDE backstepping transformation (7)-(8) and ODE backstepping transformation (31)-(32).

First, we study the stability proof of the target system via Lyapunov analysis of the PDE-ODE system. Second, considering the Lyapunov analysis of the input ODE in Section 3, Lyapunov analysis of the whole ODE-PDE-ODE system is provided, where the control parameters $\bar{c}_1$ and $\bar{c}_2$ in the control law (51) are determined. □

4.2.1 | Lyapunov analysis for the PDE-ODE system

Define the norm

$$\Omega_1(t) = \|\beta(\cdot,t)\|^2 + \|\alpha(\cdot,t)\|^2 + |X(t)|^2, \tag{53}$$

where $\|\beta(\cdot,t)\|^2$ is a compact notation for $\int_0^1 \beta(x,t)^2\mathrm{d}x$.

Consider now a Lyapunov function

$$V_1(t) = X^T(t)P_1X(t) + \frac{a_1}{2}\int_0^1 e^{\delta_1 x}\beta(x,t)^2\mathrm{d}x + \frac{b_1}{2}\int_0^1 e^{-\delta_1 x}\alpha(x,t)^2\mathrm{d}x, \tag{54}$$

where there exists the matrix $P_1 = P_1^T > 0$ being the solution to the Lyapunov equation

$$P_1(A + B\kappa) + (A + B\kappa)^T P_1 = -Q_1, \tag{55}$$

for some $Q_1 = {Q_1}^T > 0$ by recalling $A + B\kappa$ is Hurwitz. The positive parameters a_1, b_1, and δ_1 are to be chosen later.

From (53), we have

$$\theta_{11}\Omega_1(t) \leq V_1(t) \leq \theta_{12}\Omega_1(t), \tag{56}$$

where

$$\theta_{11} = \min\left\{\lambda_{\min}(P_1), \frac{a_1}{2}, \frac{b_1 e^{-\delta_1}}{2}\right\} > 0, \tag{57}$$

$$\theta_{12} = \max\left\{\lambda_{\max}(P_1), \frac{a_1 e^{\delta_1}}{2}, \frac{b_1}{2}\right\} > 0. \tag{58}$$

Time derivative of $V_1(t)$ along (11)-(14) is obtained as

$$\begin{aligned}\dot{V}_1(t) \leq & -\lambda_{\min}(Q_1)|X(t)|^2 + 2X^T P_1 B\beta(0,t) + \frac{p}{2}a_1 e^{\delta_1}\beta(1,t)^2 - \frac{p}{2}a_1\beta(0,t)^2 \\ & - \frac{p}{2}\delta_1 a_1 \int_0^1 e^{\delta_1 x}\beta(x,t)^2 \mathrm{d}x - \frac{p}{2}\delta_1 b_1 \int_0^1 e^{-\delta_1 x}\alpha(x,t)^2 \mathrm{d}x - \frac{p}{2}b_1 e^{-\delta_1}\alpha(1,t)^2 + \frac{p}{2}b_1\alpha(0,t)^2 \\ & + b_1 \int_0^1 e^{-\delta_1 x}\alpha(x,t)c_1\left(\beta(x,t) - \int_0^x \psi^I(x,y)\alpha(y,t)\mathrm{d}y - \int_0^x \phi^I(x,y)\beta(y,t)\mathrm{d}y - \gamma^I(x)X(t)\right)\mathrm{d}x. \end{aligned} \tag{59}$$

Let us consider the final part in (59) first. Using Young's inequality and Cauchy-Schwarz inequality for the final part in (59) yields the existence of $\xi > 0$ such that

$$\int_0^1 e^{-\delta_1 x}\alpha(x,t)c_1\beta(x,t)\mathrm{d}x < \xi\int_0^1 e^{-\delta_1 x}\alpha(x,t)^2\mathrm{d}x + \xi\int_0^1 e^{\delta_1 x}\beta(x,t)^2\mathrm{d}x, \tag{60}$$

$$\int_0^1 e^{-\delta_1 x}\alpha(x,t)c_1\int_0^x \psi^I(x,y)\alpha(y,t)\mathrm{d}y\mathrm{d}x < \frac{\xi}{\delta_1}\int_0^1 e^{-\delta_1 x}\alpha(x,t)^2\mathrm{d}x, \tag{61}$$

$$\int_0^1 e^{-\delta_1 x}\alpha(x,t)c_1\int_0^x \phi^I(x,y)\beta(y,t)\mathrm{d}y\mathrm{d}x < \frac{\xi}{\delta_1}\int_0^1 e^{-\delta_1 x}\alpha(x,t)^2\mathrm{d}x + \frac{\xi}{\delta_1}\int_0^1 e^{\delta_1 x}\beta(x,t)^2\mathrm{d}x, \tag{62}$$

$$\int_0^1 e^{-\delta_1 x}\alpha(x,t)c_1\gamma^I(x)X(t)\mathrm{d}x < \frac{\lambda_{\min}(Q_1)}{4b_1}|X(t)|^2 + \frac{\xi^2 b_1}{\lambda_{\min}(Q_1)}\int_0^1 e^{-\delta_1 x}\alpha(x,t)^2\mathrm{d}x. \tag{63}$$

Recalling (14), applying Young's inequality and substituting (60)-(63) to (59), we obtain

$$\begin{aligned}\dot{V}_1(t) \leq & -\left(\frac{1}{2}\lambda_{\min}(Q_1) - pb_1|C_0|^2\right)|X(t)|^2 - \left(\frac{p}{2}a_1 - pb_1 q^2 - \frac{4|PB|}{\lambda_{\min}(Q_1)}\right)\beta(0,t)^2 \\ & - \left(\frac{p}{2}\delta_1 a_1 - b_1\xi - b_1\frac{\xi}{\delta_1}\right)\int_0^1 \beta(x,t)^2\mathrm{d}x \\ & - \left(\frac{p}{2}\delta_1 b_1 - \frac{2b_1\xi}{\delta_1} - \frac{\xi^2 b_1^2}{\lambda_{\min}(Q_1)} - b_1\xi\right)e^{-\delta_1}\int_0^1 \alpha(x,t)^2\mathrm{d}x \\ & - \frac{p}{2}b_1 e^{-\delta_1}\alpha(1,t)^2 + \frac{p}{2}a_1 e^{\delta_1}\beta(1,t)^2. \end{aligned} \tag{64}$$

Choose parameters b_1, δ_1, and a_1 in sequence to satisfy

$$0 < b_1 < \frac{\lambda_{\min}(Q_1)}{2p|C_0|^2}, \quad \delta_1 > \max\left\{1, \frac{2}{p}\left(3\xi + \frac{\xi^2 b_1}{\lambda_{\min}(Q_1)}\right)\right\}, \tag{65}$$

$$a_1 > \max\left\{\frac{8|PB|}{p\lambda_{\min}(Q_1)} + 2q^2 b_1, \frac{2b_1\xi}{p\delta_1} + \frac{2b_1\xi}{p\delta_1^2}\right\} \tag{66}$$

to make

$$\eta_1 = \frac{1}{2}\lambda_{\min}(Q_1) - pb_1|C_0|^2 > 0, \tag{67}$$

$$\eta_2 = \frac{p}{2}a_1 - pb_1 q^2 - \frac{4|PB|}{\lambda_{\min}(Q_1)} > 0, \tag{68}$$

$$\eta_3 = \frac{p}{2}\delta_1 a_1 - b_1\xi - b_1\frac{\xi}{\delta_1} > 0, \tag{69}$$

$$\eta_4 = \left(\frac{p}{2}\delta_1 b_1 - \frac{2b_1\xi}{\delta_1} - \frac{\xi^2 b_1^2}{\lambda_{\min}(Q_1)} - b_1\xi\right)e^{-\delta_1} > 0. \tag{70}$$

Defining

$$\eta_5 = \frac{p}{2} b_1 e^{-\delta_1} > 0, \quad \eta_6 = \frac{p}{2} a_1 e^{\delta_1} > 0, \tag{71}$$

we arrive at

$$\dot{V}_1(t) \leq -\eta_1 |X(t)|^2 - \eta_2 \beta(0,t)^2 - \eta_3 \int_0^1 \beta(x,t)^2 \mathrm{d}x - \eta_4 \int_0^1 \alpha(x,t)^2 \mathrm{d}x - \eta_5 \alpha(1,t)^2 + \eta_6 \beta(1,t)^2. \tag{72}$$

4.2.2 | Lyapunov analysis for the whole ODE-PDE-ODE system

Recall (37) and define a Lyapunov function

$$V(t) = V_1(t) + V_y(t). \tag{73}$$

Defining the norm

$$\Omega_2(t) = \|\beta(\cdot,t)\|^2 + \|\alpha(\cdot,t)\|^2 + |X(t)|^2 + y_1(t)^2 + y_2(t)^2, \tag{74}$$

we have

$$\theta_{21}\Omega_2(t) \leq V(t) \leq \theta_{22}\Omega_2(t), \tag{75}$$

where

$$\theta_{21} = \min\left\{ \lambda_{\min}(P_1), \frac{a_1}{2}, \frac{b_1 e^{-\delta_1}}{2}, \frac{1}{2} \right\} > 0, \tag{76}$$

$$\theta_{22} = \max\left\{ \lambda_{\max}(P_1), \frac{a_1 e^{\delta_1}}{2}, \frac{b_1}{2}, \frac{1}{2} \right\} > 0. \tag{77}$$

Taking the derivative of (73) and using (72) and (41), we get

$$\begin{aligned}\dot{V} \leq &- \eta_1 |X(t)|^2 - \eta_2 \beta(0,t)^2 - \eta_3 \int_0^1 \beta(x,t)^2 \mathrm{d}x - \eta_4 \int_0^1 \alpha(x,t)^2 \mathrm{d}x - \eta_5 \alpha(1,t)^2 \\ &+ \eta_6 \beta(1,t)^2 - \bar{c}_1 y_1(t)^2 - \bar{c}_2 y_2(t)^2 + y_2(t) \Bigg((h_6 + r)\alpha(1,t) + h_7 \beta(0,t) \\ &+ h_8(q\beta(0,t) + C_0 X(t)) + \int_0^1 h_9(y)\beta(y,t)\mathrm{d}y + \int_0^1 h_{10}(y)\alpha(y,t)\mathrm{d}y + H_{11}X(t) \Bigg),\end{aligned} \tag{78}$$

where (14) is used.

Applying Young's inequality, Cauchy-Schwarz inequality and (31) into (78), we have

$$\begin{aligned}\dot{V} \leq &- \left(\eta_1 - r_1|H_{11}|^2 - r_7 {h_8}^2 |C_0|^2\right)|X(t)|^2 - \left(\eta_2 - h_7^2 r_3 - r_6 {h_8}^2 q^2\right)\beta(0,t)^2 \\ &- \left(\eta_3 - r_5 h_{9\max}^2\right) \int_0^1 \beta(x,t)^2 \mathrm{d}x - \left(\eta_4 - r_2 h_{10\max}^2\right) \int_0^1 \alpha(x,t)^2 \mathrm{d}x \\ &- \left(\eta_5 - (h_6 + r)^2 r_4\right)\alpha(1,t)^2 - (\bar{c}_1 - \eta_6) y_1(t)^2 \\ &- \left(\bar{c}_2 - \left(\frac{1}{4r_1} + \frac{1}{4r_2} + \frac{1}{4r_3} + \frac{1}{4r_4} + \frac{1}{4r_5} + \frac{1}{4r_6} + \frac{1}{4r_7}\right)\right) y_2(t)^2.\end{aligned} \tag{79}$$

We choose positive constants $r_1, r_2, r_3, r_4, r_5, r_6, r_7$

$$\begin{aligned} &r_1 < \frac{\eta_1}{|H_{11}|^2}, r_2 < \frac{\eta_4}{h_{10\max}^2}, r_3 < \frac{\eta_2}{h_7^2}, r_4 < \frac{\eta_5}{(h_6+r)^2}, \\ &r_5 < \frac{\eta_3}{h_{9\max}^2}, r_6 < \frac{\eta_2 - h_7^2 r_3}{h_8^2 q^2}, r_7 < \frac{\eta_1 - r_1|H_{11}|^2}{h_8^2|C_0|^2},\end{aligned} \tag{80}$$

where

$$h_{9\max} = \max_{x\in[0,1]} \{|h_9(x)|\}, \quad h_{10\max} = \max_{x\in[0,1]} \{|h_{10}(x)|\}, \tag{81}$$

and choose the control parameters $\bar{c}_1$ and $\bar{c}_2$

$$\bar{c}_1 > \eta_6, \tag{82}$$

$$\bar{c}_2 > \frac{1}{4}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} + \frac{1}{r_5} + \frac{1}{r_6} + \frac{1}{r_7}\right), \tag{83}$$

such that

$$\begin{aligned}\dot{V} &\leq -\lambda\theta_{22}\Omega_2 - \hat{\eta}_0\beta(0,t)^2 - \hat{\eta}_1\alpha(1,t)^2 \\ &\leq -\lambda V - \hat{\eta}_0\beta(0,t)^2 - \hat{\eta}_1\alpha(1,t)^2, \end{aligned} \tag{84}$$

for some positive λ. Note $\hat{\eta}_0 = \eta_2 - h_7^2 r_3 - r_6 {h_8}^2 q^2 > 0$ and $\hat{\eta}_1 = \eta_5 - (h_6 + r)^2 r_4 > 0$.

From (75) and (84), we can conclude that the target system $(\beta(x,t), \alpha(x,t), X(t), y_1(t), y_2(t))$ is exponentially stable for the norm

$$\left(\int_0^1 \alpha^2(x,t)\mathrm{d}x + \int_0^1 \beta^2(x,t)\mathrm{d}x + |X(t)|^2 + y_1(t)^2 + y_2(t)^2\right)^{1/2}. \tag{85}$$

Using the invertibility between the target system $(\alpha(x,t), \beta(x,t))$ and the original system $(u(x,t), v(x,t))$ via the transformation (8) and its inverse (10), and the invertibility between $(y_1(t), y_2(t))$ and $(\beta(1,t), \beta_t(1,t))$ via the invertible transformations (31), (32), together with (5), we can conclude that the $(v(x,t), u(x,t), X(t), z_1(t), \beta_t(1,t))$ system are exponentially stable for the norm

$$\left(\int_0^1 u^2(x,t)\mathrm{d}x + \int_0^1 v^2(x,t)\mathrm{d}x + |X(t)|^2 + z_1(t)^2 + \beta_t(1,t)^2\right)^{1/2}. \tag{86}$$

Taking the derivative of the inverse transformation (10) and setting $x = 1$, together with (5), we have

$$\begin{aligned}\dot{z}(t) &= v_t(1,t) \\ &= \beta_t(1,t) + p\psi^I(1,1)\alpha(1,t) - p\phi^I(1,1)\beta(1,t) \\ &\quad - p\psi^I(1,0)\alpha(0,t) + \left(p\phi^I(1,0) - \gamma^I(1)B\right)\beta(0,t) \\ &\quad + \int_0^1 \left(\int_y^1 c_1\psi^I(1,\sigma)\psi^I(\sigma,y)\mathrm{d}\sigma - p{\psi_y}^I(1,y)\right)\alpha(y,t)\mathrm{d}y \\ &\quad + \int_0^1 \left(\int_y^1 c_1\psi^I(1,\sigma)\phi^I(\sigma,y)\mathrm{d}\sigma - c_1\psi^I(1,y) + p{\phi_y}^I(1,y)\right)\beta(y,t)\mathrm{d}y \\ &\quad + \left(\int_0^1 c_1\psi^I(1,y)\gamma^I(y)\mathrm{d}y - \gamma^I(1)(A + B\kappa)\right)X(t). \end{aligned} \tag{87}$$

Applying Cauchy-Schwarz inequality into (87), considering the exponential stability results in terms of the norm $\|\alpha(\cdot,t)\|^2 + \|\beta(\cdot,t)\|^2 + |X(t)|^2 + |z(t)|^2 + |\beta_t(1,t)|^2$ shown in (85) and (86), we can obtain the exponential convergence results in terms of $|\dot{z}(t)|^2$, namely, $z_2(t)^2$.

Then, the proof of Theorem 1 is completed.

5 | BOUNDEDNESS AND EXPONENTIAL CONVERGENCE OF THE CONTROLLER $U(T)$

In the last section, we have proposed the controller and proved that all states of PDEs and ODEs are exponentially stable in the closed-loop system including the plant (1)-(6) and the controller (51). Moreover, in this section, we prove the exponential convergence and boundedness of the controller $U(t)$ in the closed-loop system.

Theorem 2. *In the closed-loop system including the plant (1)-(6) and the controller $U(t)$ (51), there exist positive constants λ_2 and Υ_0 making that $|U(t)|$ is bounded and exponentially convergent to zero in the sense of*

$$|U(t)| \leq \Upsilon_0 e^{-\frac{\lambda_2}{2}t}. \tag{88}$$

Considering (51) and Theorem 1, we know that if we want to show the exponential convergence of the bound of the controller (51), exponential convergence of 8 signals $u(1,t), u_x(1,t), v(0,t), v_x(0,t), v(1,t), v_x(1,t), u(0,t)$, and $u_x(0,t)$ in (51) need to be proved, which can be obtained through producing L_2 estimates of $u_x(x,t), v_x(x,t), u_{xx}(x,t), v_{xx}(x,t)$.

Before the proof of Theorem 2, we propose 2 lemmas first. The first one shows the exponential stability estimates in terms of the norm $\|u_x(x,t)\|^2 + \|v_x(x,t)\|^2$. The second one gives the exponential stability estimates in terms of the norm $\|u_{xx}(x,t)\|^2 + \|v_{xx}(x,t)\|^2$.

Lemma 2. *For any initial data* $(u(x,0), v(x,0)) \in H^1(0,1)$*, the closed-loop system* $(u(x,t), v(x,t))$*(1)-(6) with the controller (51) is exponentially stable in the sense of*

$$\|u_x(\cdot,t)\|^2 + \|v_x(\cdot,t)\|^2. \tag{89}$$

Proof. Differentiating (12) and (13) with respect to x, differentiating (14) with respect to t, we have

$$\begin{aligned}\alpha_{xt}(x,t) = &-p\alpha_{xx}(x,t) + c_1\beta_x(x,t) - c_1{\gamma^I}'(x)X(t) - c_1\psi^I(x,x)\alpha(x,t)\\ &- c_1\phi^I(x,x)\beta(x,t) - c_1\int_0^x {\psi_x}^I(x,y)\alpha(y,t)\mathrm{d}y - c_1\int_0^x {\phi_x}^I(x,y)\beta(y,t)\mathrm{d}y,\end{aligned} \tag{90}$$

$$\beta_{xt}(x,t) = p\beta_{xx}(x,t), \tag{91}$$

$$-\alpha_x(0,t) = q\beta_x(0,t) + \frac{1}{p}\left(C_0(A+B\kappa) + c_1\gamma^I(0)\right)X(t) + \frac{1}{p}(C_0B - c_1)\beta(0,t). \tag{92}$$

Considering

$$A_1 = \frac{1}{2}\int_0^1 b_2 e^{-\delta_2 x}\alpha_x(x,t)^2\mathrm{d}x, \tag{93}$$

$$A_2 = \frac{1}{2}\int_0^1 a_2 e^{\delta_2 x}\beta_x(x,t)^2\mathrm{d}x, \tag{94}$$

where b_2 is an arbitrary positive constant, which can adjust the convergence rate, and the positive constants δ_2, a_2 will be chosen later.

Taking the derivative of (93) along (90)-(91), we obtain

$$\begin{aligned}\dot{A}_1 = &-\frac{p}{2}b_2e^{-\delta_2}\alpha_x(1,t)^2 + \frac{p}{2}b_2\alpha_x(0,t)^2 - \frac{p}{2}b_2\delta_2\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)^2\mathrm{d}x\\ &- \int_0^1 b_2e^{-\delta_2 x}\alpha_x(x,t)c_1\psi^I(x,x)\alpha(x,t)\mathrm{d}x - \int_0^1 b_2e^{-\delta_2 x}\alpha_x(x,t)c_1\phi^I(x,x)\beta(x,t)\mathrm{d}x\\ &- \int_0^1 b_2e^{-\delta_2 x}\alpha_x(x,t)c_1\int_0^x {\psi_x}^I(x,y)\alpha(y,t)\mathrm{d}y\mathrm{d}x - \int_0^1 b_2e^{-\delta_2 x}\alpha_x(x,t)c_1\int_0^x {\phi_x}^I(x,y)\beta(y,t)\mathrm{d}y\mathrm{d}x\\ &+ \int_0^1 b_2e^{-\delta_2 x}\alpha_x(x,t)c_1\beta_x(x,t)\mathrm{d}x - \int_0^1 b_2e^{-\delta_2 x}\alpha_x(x,t)c_1{\gamma^I}'(x)X(t)\mathrm{d}x.\end{aligned} \tag{95}$$

Let us consider the last 6 terms in (95) first. Using Young's inequality and Cauchy-Schwarz inequality yields the existence of $\xi_2 > 0$ such that

$$\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)c_1\psi^I(x,x)\alpha(x,t)\mathrm{d}x < \xi_2\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)^2\mathrm{d}x + \xi_2\int_0^1 e^{-\delta_2 x}\alpha(x,t)^2\mathrm{d}x, \tag{96}$$

$$\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)c_1\phi^I(x,x)\beta(x,t)\mathrm{d}x < \xi_2\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)^2\mathrm{d}x + \xi_2\int_0^1 e^{\delta_2 x}\beta(x,t)^2\mathrm{d}x, \tag{97}$$

$$\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)c_1\int_0^x \left({\psi_x}^I(x,y) + \psi^I(x,y)\right)\alpha(y,t)\mathrm{d}y\mathrm{d}x < \frac{\xi_2}{\delta_2}\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)^2\mathrm{d}x + \frac{\xi_2}{\delta_2}\int_0^1 e^{-\delta_2 x}\alpha(x,t)^2\mathrm{d}x, \tag{98}$$

$$\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)c_1\int_0^x \left({\phi_x}^I(x,y) + \phi^I(x,y)\right)\beta(y,t)\mathrm{d}y\mathrm{d}x < \frac{\xi_2}{\delta_2}\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)^2\mathrm{d}x + \frac{\xi_2}{\delta_2}\int_0^1 e^{\delta_2 x}\beta(x,t)^2\mathrm{d}x, \tag{99}$$

$$\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)c_1\beta_x(x,t)\mathrm{d}x < \xi_2\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)^2\mathrm{d}x + \xi_2\int_0^1 e^{\delta_2 x}\beta_x(x,t)^2\mathrm{d}x, \tag{100}$$

$$\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)c_1{\gamma^I}'(x)X(t)\mathrm{d}x, < \xi_2|X(t)|^2 + \xi_2\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)^2\mathrm{d}x. \tag{101}$$

Substituting (96)-(101) into (95) and rewriting it as

$$\begin{aligned}\dot{A}_1 \leq & -\frac{p}{2}b_2e^{-\delta_2}\alpha_x(1,t)^2 + \frac{p}{2}b_2\alpha_x(0,t)^2 \\ & -\left(\frac{p}{2}b_2\delta_2 - 4b_2\xi_2 - \frac{2b_2\xi_2}{\delta_2}\right)\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)^2\mathrm{d}x \\ & +\left(\xi_2 b_2 + \frac{\xi_2 b_2}{\delta_2}\right)\int_0^1 e^{-\delta_2 x}\alpha(x,t)^2\mathrm{d}x + b_2\xi_2|X(t)|^2 \\ & +\left(\xi_2 b_2 + \frac{\xi_2 b_2}{\delta_2}\right)\int_0^1 e^{\delta_2 x}\beta(x,t)^2\mathrm{d}x + \xi_2 b_2\int_0^1 e^{\delta_2 x}\beta_x(x,t)^2\mathrm{d}x. \end{aligned} \tag{102}$$

Taking the derivative of (94), we have

$$\dot{A}_2 = \frac{p}{2}a_2e^{\delta_2}\beta_x(1,t)^2 - \frac{p}{2}a_2\beta_x(0,t)^2 - \frac{p}{2}a_2\delta_2\int_0^1 e^{\delta_2 x}\beta_x(x,t)^2\mathrm{d}x. \tag{103}$$

Defining

$$\bar{A} = A_1 + A_2, \tag{104}$$

taking the derivative of $\bar{A}$, and using (102), (103), we have

$$\begin{aligned}\dot{\bar{A}}(t) \leq & -\frac{p}{2}b_2e^{-\delta_2}\alpha_x(1,t)^2 + \frac{p}{2}b_2\alpha_x(0,t)^2 + \frac{p}{2}a_2e^{\delta_2}\beta_x(1,t)^2 - \frac{p}{2}a_2\beta_x(0,t)^2 \\ & -\left(\frac{p}{2}b_2\delta_2 - 4\xi_2 b_2 - \frac{2\xi_2 b_2}{\delta_2}\right)\int_0^1 e^{-\delta_2 x}\alpha_x(x,t)^2\mathrm{d}x \\ & -\left(\frac{p}{2}a_2\delta_2 - \xi_2 b_2\right)\int_0^1 e^{\delta_2 x}\beta_x(x,t)^2\mathrm{d}x + \left(\xi_2 b_2 + \frac{\xi_2 b_2}{\delta_2}\right)\int_0^1 e^{-\delta_2 x}\alpha(x,t)^2\mathrm{d}x \\ & +\left(\xi_2 b_2 + \frac{\xi_2 b_2}{\delta_2}\right)\int_0^1 e^{\delta_2 x}\beta(x,t)^2\mathrm{d}x + \xi_2 b_2|X(t)|^2. \end{aligned} \tag{105}$$

Considering (104) and recalling (73), we propose a Lyapunov function

$$V_2(t) = \bar{A} + R_1 V. \tag{106}$$

Define the norm

$$\Omega_3(t) = \|\beta_x(\cdot,t)\|^2 + \|\alpha_x(\cdot,t)\|^2 + \|\beta(\cdot,t)\|^2 + \|\alpha(\cdot,t)\|^2 + |X(t)|^2 + \beta(1,t)^2 + y_2(t)^2. \tag{107}$$

We have

$$\theta_{31}\Omega_3(t) \leq V_2(t) \leq \theta_{32}\Omega_3(t), \tag{108}$$

where

$$\theta_{31} = \min\left\{R_1\theta_{21}, \frac{a_2}{2}, \frac{b_2e^{-\delta_2}}{2}\right\} > 0, \tag{109}$$

$$\theta_{32} = \max\left\{R_1\theta_{22}, \frac{a_2e^{\delta_2}}{2}, \frac{b_2}{2}\right\} > 0. \tag{110}$$

Taking the derivative of (106), recalling (105), (84), and applying Cauchy-Schwarz inequality into (92) to rewrite $\alpha_x(0,t)^2$ in (105) as

$$\alpha_x(0,t)^2 \leq 3q^2\beta_x(0,t)^2 + \frac{3}{p^2}\left|C_0(A+B\kappa) + c_1\gamma^I(0)\right|^2|X(t)|^2 + \frac{3}{p^2}(C_0B - c_1)^2\beta(0,t)^2, \tag{111}$$

then we get

$$
\begin{aligned}
\dot{V}_2(t) &= \dot{A} + R_1\dot{V}\\
&\le -\frac{p}{2}b_2e^{-\delta_2}\alpha_x(1,t)^2 - \left(\frac{p}{2}a_2 - \frac{3pb_2q^2}{2}\right)\beta_x(0,t)^2\\
&- \left(\frac{p}{2}b_2\delta_2 - 4\xi_2b_2 - \frac{2\xi_2b_2}{\delta_2}\right)\int_0^1 \alpha_x(x,t)^2\mathrm{d}x - \left(\frac{p}{2}a_2\delta_2 - \xi_2b_2\right)\int_0^1 e^{\delta_2x}\beta_x(x,t)^2\mathrm{d}x\\
&- \left(R_1\hat{\eta}_0 - \frac{3b_2}{2p}(c_1 + C_0B)^2\right)\beta(0,t)^2 - \left(R_1\theta_{22}\lambda - \frac{a_2e^{\delta_2}}{p}\right)y_2(t)^2 - \left(R_1\theta_{22}\lambda - \frac{a_2e^{\delta_2}}{p}\bar{c}_1^2\right)\beta(1,t)^2\\
&- \left(R_1\theta_{22}\lambda - \xi_2b_2 - \frac{\xi_2b_2}{\delta_2}\right)\int_0^1 \alpha(x,t)^2\mathrm{d}x - \left(R_1\theta_{22}\lambda - \xi_2b_2e^{\delta_2} - \frac{\xi_2b_2}{\delta_2}e^{\delta_2}\right)\int_0^1 \beta(x,t)^2\mathrm{d}x\\
&- \left(R_1\theta_{22}\lambda - \xi_2b_2 - \frac{3b_2}{2p}\left|c_1\gamma^I(0) + C_0(A + B\kappa)\right|^2\right)|X(t)|^2 - R_1\hat{\eta}_1\alpha(1,t)^2\\
&\le -\lambda_1\theta_{32}\Omega_3(t) - R_1\hat{\eta}_1\alpha(1,t)^2 - \hat{\eta}_2\beta(0,t)^2 - \hat{\eta}_3\beta_x(0,t)^2\\
&\le -\lambda_1V_2(t) - R_1\hat{\eta}_1\alpha(1,t)^2 - \hat{\eta}_2\beta(0,t)^2 - \hat{\eta}_3\beta_x(0,t)^2,
\end{aligned}
\tag{112}
$$

for some positive λ_1, with the choice

$$
\delta_2 > \max\left\{1, \frac{12\xi_2}{p}\right\},\quad a_2 > \max\left\{\frac{2\xi_2}{\delta_2}, 3b_2q^2\right\}
\tag{113}
$$

and sufficiently large R_1. Note $\hat{\eta}_2 = R_1\hat{\eta}_0 - \frac{3pb_2}{2p^2}(c_1 + C_0B)^2 > 0$ and $\hat{\eta}_3 = \frac{p}{2}a_2 - \frac{3pb_2q^2}{2} > 0$.

Then, recalling (108), (112), we can give the exponential stability estimates of $(\alpha_x(x,t), \beta_x(x,t))$ for the norm

$$
\|\alpha_x(\cdot,t)\|^2 + \|\beta_x(\cdot,t)\|^2.
\tag{114}
$$

Differentiating (9)-(10) with respect to x, we have

$$
u_x(x,t) = \alpha_x(x,t),
\tag{115}
$$

$$
\begin{aligned}
v_x(x,t) = \beta_x(x,t) &- \int_0^x \psi_x^I(x,y)\alpha(y,t)\mathrm{d}y - \int_0^x \phi_x^I(x,y)\beta(y,t)\mathrm{d}y\\
&- \dot{\gamma}^I(x)X(t) - \psi^I(x,x)\alpha(x,t) - \phi^I(x,x)\beta(x,t).
\end{aligned}
\tag{116}
$$

Using the Young and Cauchy-Schwartz inequalities, we get the inequalities

$$
\|v_x(x,t)\|^2 \le 6\left(\|\beta_x(x,t)\|^2 + K_\infty\|\alpha(x,t)\|^2 + L_\infty\|\beta(x,t)\|^2 + \max_{x\in[0,1]}\left\{|\gamma^{I\prime}(x)|^2\right\}|X(t)|^2\right),
\tag{117}
$$

where $K_\infty = \max_{(x,y)\in D}\{|\psi_x(x,y)|^2\} + \max_{x\in[0,1]}\{|\psi^I(x,x)|^2\}$ and $L_\infty = \max_{(x,y)\in D}\{|\phi_x(x,y)|^2\} + \max_{x\in[0,1]}\{|\phi^I(x,x)|^2\}$. Based on the exponential stability estimates in terms of the norms $\|\alpha_x(\cdot,t)\|^2+\|\beta_x(\cdot,t)\|^2$ proved above, together with the exponential stability results in terms of the norm including $\|\alpha(\cdot,t)\|^2 + \|\beta(\cdot,t)\|^2 + |X(t)|^2$ provided in Theorem 1, we obtain the exponential stability estimates in terms of the norm $\|u_x(\cdot,t)\|^2 + \|v_x(\cdot,t)\|^2$.

The proof of Lemma 2 is completed. □

Lemma 3. *For any initial data $(u(x,0), v(x,0)) \in H^2(0,1)$, the closed-loop system $(u(x,t), v(x,t))$(1)-(6) with the controller (51) is exponentially stable in the sense of*

$$
\|v_{xx}(\cdot,t)\|^2 + \|u_{xx}(\cdot,t)\|^2.
\tag{118}
$$

Proof. Twice differentiating (12) and (13) with respect to x, twice differentiating (14) with respect to t, we have

$$\begin{aligned}\alpha_{xxt}(x,t) = & -p\alpha_{xxx}(x,t) + c_1\beta_{xx}(x,t) - c_1\gamma^{I''}(x)X(t) \\ & - c_1\int_0^x \psi_{xx}{}^I(x,y)\alpha(y,t)\mathrm{d}y - c_1\psi^I(x,x)\alpha_x(x,t) \\ & - c_1\int_0^x \phi_{xx}{}^I(x,y)\beta(y,t)\mathrm{d}y - c_1\phi^I(x,x)\beta_x(x,t) \\ & - \left(2c_1\psi_x{}^I(x,x) + c_1\psi_y{}^I(x,x)\right)\alpha(x,t) \\ & - \left(2c_1\phi_x{}^I(x,x) + c_1\phi_y{}^I(x,x)\right)\beta(x,t), \end{aligned} \tag{119}$$

$$\beta_{xxt}(x,t) = p\beta_{xxx}(x,t), \tag{120}$$

and

$$\begin{aligned}\alpha_{xx}(0,t) = {} & q\beta_{xx}(0,t) + \frac{1}{p}C_0B\beta_x(0,t) - \frac{1}{p}c_1\psi^I(0,0)\alpha(0,t) \\ & - \frac{1}{p^2}\left(pc_1\gamma^{I'}(0) - C_0(A+B\kappa)^2 - c_1\gamma(0)(A+B\kappa)\right)X(t) \\ & - \frac{1}{p^2}\left(C_0(A+B\kappa)B + pc_1\phi^I(0,0) - c_1\gamma(0)B\right)\beta(0,t). \end{aligned} \tag{121}$$

Considering

$$B_1 = \frac{1}{2}\int_0^1 b_3e^{-\delta_3x}\alpha_{xx}(x,t)^2\mathrm{d}x, \tag{122}$$

$$B_2 = \frac{1}{2}\int_0^1 a_3e^{\delta_3x}\beta_{xx}(x,t)^2\mathrm{d}x, \tag{123}$$

where the positive constant b_3 can be chosen arbitrarily to adjust the convergence rate, and positive constants δ_3, a_3 will be defined later.

Taking the derivative of (122) along (119)-(120), we have

$$\begin{aligned}\dot{B}_1(t) = & -\frac{p}{2}b_3e^{-\delta_3}\alpha_{xx}(1,t)^2 + \frac{p}{2}b_3\alpha_{xx}(0,t)^2 - \frac{p}{2}b_3\delta_3\int_0^1 e^{-\delta_3x}\alpha_{xx}(x,t)^2\mathrm{d}x \\ & + \int_0^1 b_3e^{-\delta_3x}\alpha_{xx}(x,t)c_1\beta_{xx}(x,t)\mathrm{d}x - \int_0^1 b_3e^{-\delta_3x}\alpha_{xx}(x,t)c_1\gamma^{I''}(x)X(t)\mathrm{d}x \\ & - \int_0^1 b_3e^{-\delta_3}x\alpha_{xx}(x,t)c_1\int_0^x \psi_{xx}{}^I(x,y)\alpha(y,t)\mathrm{d}y\mathrm{d}x \\ & - \int_0^1 b_3e^{-\delta_3x}\alpha_{xx}(x,t)c_1\int_0^x \phi_{xx}{}^I(x,y)\beta(y,t)\mathrm{d}y\mathrm{d}x \\ & - \int_0^1 b_3e^{-\delta_3x}\alpha_{xx}(x,t)(2c_1\psi_x{}^I(x,x) + c_1\psi_y{}^I(x,x))\alpha(x,t)\mathrm{d}x \\ & - \int_0^1 b_3e^{-\delta_3x}\alpha_{xx}(x,t)(2c_1\phi_x{}^I(x,x) + c_1\phi_y{}^I(x,x))\beta(x,t)\mathrm{d}x \\ & - \int_0^1 b_3e^{-\delta_3x}\alpha_{xx}(x,t)c_1\psi^I(x,x)\alpha_x(x,t)\mathrm{d}x \\ & - \int_0^1 b_3e^{-\delta_3x}\alpha_{xx}(x,t)c_1\phi^I(x,x)\beta_x(x,t)\mathrm{d}x. \end{aligned} \tag{124}$$

Now, let us deal with the last 8 terms in (124) by using Young's inequality and Cauchy-Schwarz inequality. Similar to (96)-(101), there exists a $\xi_3 > 0$ such that

$$\int_0^1 e^{-\delta_3x}\alpha_{xx}(x,t)c_1\beta_{xx}(x,t)\mathrm{d}x < \xi_3\int_0^1 e^{-\delta_3x}\alpha_{xx}(x,t)^2\mathrm{d}x + \xi_3\int_0^1 e^{\delta_3x}\beta_{xx}(x,t)^2\mathrm{d}x, \tag{125}$$

$$\int_0^1 e^{-\delta_3x}\alpha_{xx}(x,t)c_1\ddot{\gamma}^I(x)X(t)\mathrm{d}x < \xi_3\int_0^1 e^{-\delta_3x}\alpha_{xx}(x,t)^2\mathrm{d}x + \xi_3|X(t)|^2, \tag{126}$$

$$\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)c_1\int_0^x {\psi_{xx}}^I(x,y)\alpha(y,t)\mathrm{d}y\mathrm{d}x < \frac{\xi_3}{\delta_3}\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)^2\mathrm{d}x + \frac{\xi_3}{\delta_3}\int_0^1 e^{-\delta_3 x}\alpha(x,t)^2\mathrm{d}x, \tag{127}$$

$$\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)c_1\int_0^x {\phi_{xx}}^I(x,y)\beta(y,t)\mathrm{d}y\mathrm{d}x < \frac{\xi_3}{\delta_3}\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)^2\mathrm{d}x + \frac{\xi_3}{\delta_3}\int_0^1 e^{-\delta_3 x}\beta(x,t)^2\mathrm{d}x, \tag{128}$$

$$\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)2c_1\left({\psi_x}^I(x,x)+c_1{\psi_y}^I(x,x)\right)\alpha(x,t)\mathrm{d}x < \xi_3\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)^2\mathrm{d}x + \xi_3\int_0^1 e^{\delta_3 x}\alpha(x,t)^2\mathrm{d}x, \tag{129}$$

$$\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)\left(2c_1{\phi_x}^I(x,x)+c_1{\phi_y}^I(x,x)\right)\beta(x,t)\mathrm{d}x < \xi_3\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)^2\mathrm{d}x + \xi_3\int_0^1 e^{\delta_3 x}\beta(x,t)^2\mathrm{d}x, \tag{130}$$

$$\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)c_1\psi^I(x,x)\alpha_x(x,t)\mathrm{d}x < \xi_3\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)^2\mathrm{d}x + \xi_3\int_0^1 e^{-\delta_3 x}\alpha_x(x,t)^2\mathrm{d}x, \tag{131}$$

$$\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)c_1\phi^I(x,x)\beta_x(x,t)\mathrm{d}x < \xi_3\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)^2\mathrm{d}x + \xi_3\int_0^1 e^{\delta_3 x}\beta_x(x,t)^2\mathrm{d}x. \tag{132}$$

Substituting (125)-(132) into (124), we have

$$\begin{aligned}\dot{B}_1(t) \le & -\frac{p}{2}b_3e^{-\delta_3}\alpha_{xx}(1,t)^2 + \frac{p}{2}b_3\alpha_{xx}(0,t)^2 - \left(\frac{p}{2}b_3\delta_3 - 6\xi_3 b_3 - 2\frac{\xi_3 b_3}{\delta_3}\right)\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)^2\mathrm{d}x \\ & + \xi_3 b_3\int_0^1 e^{\delta_3 x}\beta_{xx}(x,t)^2\mathrm{d}x + \xi_3 b_3\int_0^1 e^{\delta_3 x}\beta_x(x,t)^2\mathrm{d}x + b_3\left(\xi_3+\frac{\xi_3}{\delta_3}\right)\int_0^1 e^{\delta_3 x}\beta(x,t)^2\mathrm{d}x \\ & + b_3\left(\xi_3+\frac{\xi_3}{\delta_3}\right)\int_0^1 e^{-\delta_3 x}\alpha(x,t)^2\mathrm{d}x + \xi_3 b_3\int_0^1 e^{-\delta_3 x}\alpha_x(x,t)^2\mathrm{d}x + \xi_3 b_3|X(t)|^2.\end{aligned} \tag{133}$$

Taking the derivative of (123) along (119)-(120), we have

$$\begin{aligned}\dot{B}_2(t) &= \int_0^1 a_3 e^{\delta_3 x}\beta_{xx}(x,t)\beta_{xxt}(x,t)\mathrm{d}x = p\int_0^1 a_3 e^{\delta_3 x}\beta_{xx}(x,t)\beta_{xxx}(x,t)\mathrm{d}x \\ &= \frac{p}{2}a_3e^{\delta_3}\beta_{xx}(1,t)^2 - \frac{p}{2}a_3\beta_{xx}(0,t)^2 - \frac{p}{2}a_3\delta_3\int_0^1 e^{\delta_3 x}\beta_{xx}(x,t)^2\mathrm{d}x.\end{aligned} \tag{134}$$

Define

$$\bar{B} = B_1 + B_2. \tag{135}$$

Applying Cauchy-Schwarz inequality into (121) as

$$\begin{aligned}\alpha_{xx}(0,t)^2 \le\ & 5q^2\beta_{xx}(0,t)^2 + \frac{5}{p^2}|C_0B|^2\beta_x(0,t)^2 + \frac{5}{p^2}c_1^2\psi^I(0,0)^2\alpha(0,t)^2 \\ & + \frac{5}{p^4}\left(pc_1{\gamma^I}'(0) - C_0(A+B\kappa)^2 - c_1\gamma(0)(A+B\kappa)\right)^2|X(t)|^2 \\ & + \frac{5}{p^4}\left(C_0(A+B\kappa)B + pc_1\phi^I(0,0) - c_1\gamma(0)B\right)^2\beta(0,t)^2,\end{aligned} \tag{136}$$

which is used to replace $\alpha_{xx}(0,t)^2$ in (133), then by recalling (133),(134), the inequality of the derivative of (135) can be obtained as

$$
\begin{aligned}
\dot{\bar{B}} &= \dot{B}_1 + \dot{B}_2 \\
&\leq -\left(\frac{p}{2}b_3\delta_3 - 6\xi_3 b_3 - 2\frac{\xi_3 b_3}{\delta_3}\right)\int_0^1 e^{-\delta_3 x}\alpha_{xx}(x,t)^2\mathrm{d}x \\
&\quad -\left(\frac{p}{2}a_3\delta_3 - \xi_3 b_3\right)\int_0^1 e^{\delta_3} x\beta_{xx}(x,t)^2\mathrm{d}x - \left(\frac{p}{2}a_3 - \frac{5pb_3q^2}{2}\right)\beta_{xx}(0,t)^2 \\
&\quad -\frac{p}{2}b_3 e^{-\delta_3}\alpha_{xx}(1,t)^2 + \xi_3 b_3\int_0^1 e^{\delta_3 x}\beta_x(x,t)^2\mathrm{d}x + b_3\left(\xi_3 + \frac{\xi_3}{\delta_3}\right)\int_0^1 e^{\delta_3 x}\beta(x,t)^2\mathrm{d}x \\
&\quad + b_3\left(\xi_3 + \frac{\xi_3}{\delta_3}\right)\int_0^1 e^{-\delta_3 x}\alpha(x,t)^2\mathrm{d}x + \xi_3 b_3\int_0^1 e^{-\delta_3 x}\alpha_x(x,t)^2\mathrm{d}x + \frac{p}{2}a_3 e^{\delta_3}\beta_{tt}(1,t)^2 \\
&\quad + \left(\frac{5b_3}{2p^3}\left(pc_1\gamma^{I'}(0) - C_0(A+B\kappa)^2 - c_1\gamma(0)(A+B\kappa)\right)^2 + \xi_3 b_3\right)|X(t)|^2 \\
&\quad + \frac{5b_3}{2p^3}\left(C_0(A+B\kappa)B + pc_1\phi^I(0,0) - c_1\gamma(0)B\right)^2\beta(0,t)^2 \\
&\quad + \frac{5b_3}{2p}c_1^2\psi^I(0,0)^2\alpha(0,t)^2 + \frac{5|C_0B|^2 b_3}{2p}\beta_x(0,t)^2.
\end{aligned} \tag{137}
$$

Note that (137) includes a positive term $\beta_{tt}(1,t)$. Substituting (40) into (17) yields

$$
\begin{aligned}
\beta_{tt}(1,t) = &-(\bar{c}_2 + \bar{c}_1)\beta_t(1,t) - (\bar{c}_1\bar{c}_2 + 1)\beta(1,t) + (h_6 + r)\alpha(1,t) + h_7\beta(0,t) + h_8\alpha(0,t) \\
&+ \int_0^1 h_9(y)\beta(y,t)\mathrm{d}y + \int_0^1 h_{10}(y)\alpha(y,t)\mathrm{d}y + H_{11}X(t).
\end{aligned} \tag{138}
$$

Then, applying Cauchy-Schwarz inequality in (138), $\beta_{tt}(1,t)$ can be rewritten as

$$
\begin{aligned}
\beta_{tt}(1,t) \leq\ & 8(\bar{c}_2 + \bar{c}_1)^2\beta_t(1,t)^2 + 8(\bar{c}_1\bar{c}_2 + 1)^2\beta(1,t)^2 + 8(h_6 + r)^2\alpha(1,t)^2 + 8{h_7}^2\beta(0,t)^2 \\
&+ 8{h_8}^2\alpha(0,t)^2 + 8h_{9\,\max}\|\beta(\cdot,t)\|^2 + 8h_{10\,\max}\|\alpha(\cdot,t)\|^2 + 8{H_{11}}^2|X(t)|^2.
\end{aligned} \tag{139}
$$

Replacing $\beta_{tt}(1,t)$ in (137) by (139) will be used in the following Lyapunov analysis.
Define a Lypunov function

$$
V_u = R_2 V_2 + \bar{B}. \tag{140}
$$

Considering the norm

$$
\begin{aligned}
\Omega_4(t) = &\|\beta_{xx}(\cdot,t)\|^2 + \|\alpha_{xx}(\cdot,t)\|^2 + \|\beta_x(\cdot,t)\|^2 + \|\alpha_x(\cdot,t)\|^2 \\
&+ \|\beta(\cdot,t)\|^2 + \|\alpha(\cdot,t)\|^2 + |X(t)|^2 + \beta(1,t)^2 + y_2(t)^2,
\end{aligned} \tag{141}
$$

we have

$$
\theta_{41}\Omega_4(t) \leq V_u(t) \leq \theta_{42}\Omega_4(t), \tag{142}
$$

where

$$
\theta_{41} = \min\left\{R_2\theta_{31}, \frac{a_3}{2}, \frac{b_3 e^{-\delta_3}}{2}\right\} > 0, \tag{143}
$$

$$
\theta_{42} = \max\left\{R_2\theta_{32}, \frac{a_3 e^{\delta_3}}{2}, \frac{b_3}{2}\right\} > 0. \tag{144}
$$

Considering (31), (32), (35), (110), (112), (137), and (139), taking the derivative of V_u, we have

$$
\begin{aligned}
\dot{V}_u &= R_2\dot{V}_2 + \dot{\bar{B}} \\
&\leq -\frac{1}{2}R_2\lambda_1 V_2 - \left(\frac{p}{2}b_3\delta_3 - 6\xi_3 b_3 - 2\frac{\xi_3 b_3}{\delta_3}\right)\int_0^1 \alpha_{xx}(x,t)^2 \mathrm{d}x \\
&\quad - \left(\frac{p}{2}a_3\delta_3 - \xi_3 b_3\right)\int_0^1 e^{\delta_3 x}\beta_{xx}(x,t)^2\mathrm{d}x - \left(\frac{p}{2}a_3 - \frac{5pb_3q^2}{2}\right)\beta_{xx}(0,t)^2 \\
&\quad - \frac{p}{2}b_3 e^{-\delta_3}\alpha_{xx}(1,t)^2 - \left(\frac{1}{2}R_2\lambda_1\theta_{32} - \xi_3 b_3 e^{\delta_3}\right)\int_0^1 \beta_x(x,t)^2\mathrm{d}x \\
&\quad - \left(\frac{1}{2}R_2\lambda_1\theta_{32} - \xi_3 b_3 e^{\delta_3} - \frac{\xi_3 b_3}{\delta_3}e^{\delta_3} - 4pa_3 e^{\delta_3}h_{9\max}\right)\int_0^1 \beta(x,t)^2\mathrm{d}x \\
&\quad - \left(\frac{1}{2}R_2\lambda_1\theta_{32} - \xi_3 b_3 - \frac{\xi_3 b_3}{\delta_3} - 4pa_3 e^{\delta_3}h_{10\max}\right)\int_0^1 \alpha(x,t)^2\mathrm{d}x \\
&\quad - \left(\frac{1}{2}R_2\lambda_1\theta_{32} - \xi_3 b_3\right)\int_0^1 \alpha_x(x,t)^2\mathrm{d}x \\
&\quad - \left[\frac{1}{2}R_2\lambda_1\theta_{32} - \left(\frac{5b_3}{2p^3}\left(pc_1{\gamma^{I}}'(0) - C_0(A+B\kappa)^2 - c_1\gamma(0)(A+B\kappa)\right)^2 + \xi_3 b_3\right)\right. \\
&\qquad \left. - 4pa_3 e^{\delta_3}{H_{11}}^2 - 2\left(\frac{5b_3}{2p}c_1^2\psi(0,0)^2 + 4pa_3 e^{\delta_3}{h_8}^2\right)|C_0|^2\right]|X(t)|^2 \\
&\quad - \left[R_2\hat{\eta}_2 - \left(\frac{5b_3}{2p^3}\left(C_0(A+B\kappa)B + pc_1\phi^I(0,0) - c_1\gamma(0)B\right)^2 + 4pa_3 e^{\delta_3}{h_7}^2\right)\right. \\
&\qquad \left. -2q^2\left(\frac{5b_3}{2p}c_1^2\psi(0,0)^2 + 4pa_3 e^{\delta_3}{h_8}^2\right)\right]\beta(0,t)^2 - \left(R_2\hat{\eta}_3 - \frac{5|C_0B|^2 b_3}{2p}\right)\beta_x(0,t)^2 \\
&\quad - \left(\frac{1}{2}R_2\lambda_1\theta_{32} - 4pa_3 e^{\delta_3}(\bar{c}_2 + \bar{c}_1)^2\right)\beta_t(1,t)^2 \\
&\quad - \left(\frac{1}{2}R_2\lambda_1\theta_{32} - 4pa_3 e^{\delta_3}(\bar{c}_1\bar{c}_2 + 1)^2\right)\beta(1,t)^2 \\
&\quad - \left(R_2R_1\hat{\eta}_1 - 4pa_3 e^{\delta_3}(h_6 + r)^2\right)\alpha(1,t)^2.
\end{aligned} \tag{145}
$$

Choosing

$$
\delta_3 > \max\left\{1, \frac{16\xi_3}{p}\right\}, \quad a_3 > \max\left\{\frac{2\xi_3 b_3}{p\delta_3}, 5b_3q^2\right\}
$$

and large sufficient R_2, we have

$$
\dot{V}_u(t) \leq -\frac{1}{2}R_2\lambda_1 V_2 - \sigma_2\bar{B}, \tag{146}
$$

with the positive constant

$$
\sigma_2 = \frac{2\min\left\{\frac{p}{2}b_3\delta_3 - 6\xi_3 b_3 - 2\frac{\xi_3 b_3}{\delta_3}, \frac{p}{2}a_3\delta_3 - \xi_3 b_3\right\}}{\max\{a_3, b_3\}e^{\delta_3}}. \tag{147}
$$

Then, we arrive at

$$
\dot{V}_u(t) \leq -\lambda_2 V_u(t), \tag{148}
$$

where

$$
\lambda_2 = \min\left\{\frac{1}{2}\lambda_1, \sigma_2\right\}. \tag{149}
$$

Hence,

$$
V_u(t) \leq e^{-\lambda_2 t}V_u(0), \quad \forall t \geq 0. \tag{150}
$$

Then, we get the exponential stability estimates in terms of the norm $\|\alpha_{xx}(\cdot,t)\|^2 + \|\beta_{xx}(\cdot,t))\|^2$.

Twice differentiating (9)-(10) with respect to x, we have

$$\begin{aligned} u_{xx}(x,t) &= \alpha_{xx}(x,t), \\ v_{xx}(x,t) &= \beta_{xx}(x,t) - \gamma^{I''}(x)X(t) \\ &\quad - \left(2\psi^I{}_x(x,x) + \psi^I{}_y(x,x)\right)\alpha(x,t) - \left(2\phi_x{}^I(x,x) + \phi_y{}^I(x,x)\right)\beta(x,t) \\ &\quad - \int_0^x \psi^I{}_{xx}(x,y)\alpha(y,t)\mathrm{d}y - \int_0^x \phi^I_{xx}(x,y)\beta(y,t)\mathrm{d}y \\ &\quad - \psi^I(x,x)\alpha_x(x,t) - \phi^I(x,x)\beta_x(x,t). \end{aligned} \tag{151}$$

Through a similar calculation of (117), considering the exponential stability estimates in terms of the norm $\|\alpha_{xx}(\cdot,t)\|^2 + \|\beta_{xx}(\cdot,t)\|^2$ proved above, and recalling the exponential stability estimates in terms of the norm $\|\alpha_x(\cdot,t)\|^2 + \|\beta_x(\cdot,t)\|^2$ shown in Lemma 2, together with the exponential stability results in terms of the norm including $\|\alpha(\cdot,t)\|^2 + \|\beta(\cdot,t)\|^2 + |X(t)|^2$ provided in Theorem 1, we can conclude the exponential stability estimates in terms of the norm $\|u_{xx}(\cdot,t)\|^2 + \|v_{xx}(\cdot,t)\|^2$.

The proof of Lemma 3 is completed. □

Using Lemma 2 and Lemma 3, we can prove Theorem 2 now.

Proof of Theorem 2. Recalling (51) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |U(t)|^2 \le\ & \big(\bar{\xi}_1 v_x(1,t)^2 + \bar{\xi}_2 v(1,t)^2 + \bar{\xi}_3 u_x(1,t)^2 + \bar{\xi}_4 u(1,t)^2 + \bar{\xi}_5 v_x(0,t)^2 + \bar{\xi}_6 v(0,t)^2 \\ & + \bar{\xi}_7 u_x(0,t)^2 + \bar{\xi}_8 u(0,t)^2 + \bar{\xi}_9 |X(t)|^2 + \bar{\xi}_{10}\|u(\cdot,t)\|^2 + \bar{\xi}_{11}\|v(\cdot,t)\|^2\big), \end{aligned} \tag{152}$$

for some positive constants $\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_4, \bar{\xi}_5, \bar{\xi}_6, \bar{\xi}_7, \bar{\xi}_8, \bar{\xi}_9, \bar{\xi}_{10}, \bar{\xi}_{11}$.

Recalling the exponential estimates in terms of the norms $\|u(\cdot,t)\|_{H_1} + \|v(\cdot,t)\|_{H_1}$ and $\|u(\cdot,t)\|_{H_2} + \|v(\cdot,t)\|_{H_2}$ proved in Lemmas 2 and 3, using Sobolev inequality, we can obtain the exponential estimate in terms of the norm $\|u(\cdot,t)\|_{C^1} + \|v(\cdot,t)\|_{C^1}$, which gives the exponential convergence of $|U(t)|$ by recalling (152) and Theorem 1. The upper boundedness Υ_0 of $|U(t)|$ depends on the initial values of the norms in (152).

The proof of Theorem 2 is completed. □

6 | EXTENSION TO ODES OF ARBITRARY ORDER

In this section, we consider the input ODE is not second but arbitrary m order and provide a sketch of the design and analysis for this general case. Due to the space limitation, we omit some detailed calculations, which can refer to Sections 2 to 5.

Replace (5)-(6) by

$$v(1,t) = C_z Z(t), \tag{153}$$

$$\dot{Z}(t) = A_z Z(t) + B_z U(t) + \phi_z, \tag{154}$$

where $Z(t) \in \mathbb{R}^{m\times 1}, A_z = [0,1,0,\ \dots,0;0,0,1,0,\ \dots,0;\ \dots;0,\ \dots,0,1;a_{z1},\ \dots,a_{zm}] \in \mathbb{R}^{m\times m}$, $a_{z1},\ \dots,a_{zm}$ are arbitrary constants. $B_z = [0,0,\ \dots,1]^T \in \mathbb{R}^{m\times 1}$, $C_z = [1,0,\ \dots,0] \in \mathbb{R}^{1\times m}$, and

$$\phi_z = R_{z1}v(0,t) + R_{z2}u(1,t) + \int_0^1 R_{z3}(x)u(x,t)\mathrm{d}x + \int_0^1 R_{z4}(x)v(x,t)\mathrm{d}x + R_{z5}X(t), \tag{155}$$

with R_{z1}, R_{z2} being arbitrary constants, $R_{z3}(x), R_{z4}(x)$ being arbitrary functions, and R_{z5} being an arbitrary constant matrix. Full relative degree in (153)-(154) is assumed for the design.

6.1 | Control design

Equation (154) where $Z(t) = [z_1(t),\ \dots,z_m(t)]^T$ can be written as the form of a chain of m integrators as

$$\dot{z}_1(t) = z_2(t), \tag{156}$$

$$\dot{z}_2(t) = z_3(t), \tag{157}$$

$$\vdots \tag{158}$$

$$\dot{z}_{m-1}(t) = z_m(t), \tag{159}$$

$$\dot{z}_m(t) = a_{z1}z_1(t) + \cdots + a_{zm}z_m(t) + U(t) + \phi_z. \tag{160}$$

The m order form of (17) is obtained as

$$\begin{aligned}\partial_t^m \beta(1,t) = U(t) &+ \left[q_m\partial_t^{m-1}\beta(1,t) + q_{m-1}\partial_t^{m-2}\beta(1,t) + \cdots + q_2\beta_t(1,t) + q_1\beta(1,t)\right]\\ &+ \left[p_m\partial_t^{m-1}\alpha(1,t) + p_{m-1}\partial_t^{m-2}\alpha(1,t) + \cdots + p_2\alpha_t(1,t) + p_1\alpha(1,t)\right]\\ &+ \left[\bar{q}_m\partial_t^{m-1}\beta(0,t) + \bar{q}_{m-1}\partial_t^{m-2}\beta(0,t) + \cdots + \bar{q}_2\beta_t(0,t) + \bar{q}_1\beta(0,t)\right]\\ &+ \left[\bar{p}_m\partial_t^{m-1}\alpha(0,t) + \bar{p}_{m-1}\partial_t^{m-2}\alpha(0,t) + \cdots + \bar{p}_2\alpha_t(0,t) + \bar{p}_1\alpha(0,t)\right]\\ &+ \int_0^1 \bar{Q}(y)\beta(y,t)\mathrm{d}y + \int_0^1 \bar{P}(y)\alpha(y,t)\mathrm{d}y + H_zX(t),\end{aligned} \tag{161}$$

where $q_m, \ldots, q_1, p_m, \ldots, p_1, \bar{q}_m, \ldots, \bar{q}_1, \bar{p}_m, \ldots, \bar{p}_1, \bar{Q}(y), \bar{P}(y)$, and H_z are coefficients consisting of the kernel functions in the backstepping transformation (8), (10) and the system parameters in (1)-(4), (153)-(154), (155).

The following backstepping transformation for the $(\beta(1,t), \beta_t(1,t), \ldots, \partial_t^{m-1}\beta(1,t))$ system (161) is made

$$y_1(t) = \beta(1,t), \tag{162}$$

$$y_2(t) = \beta_t(1,t) + \tau_1[\beta(1,t)], \tag{163}$$

$$\vdots \tag{164}$$

$$y_m(t) = \partial_t^{m-1}\beta(1,t) + \tau_{m-1}\left[\beta(1,t), \ldots, \partial_t^{m-2}\beta(1,t)\right], \tag{165}$$

where $\tau_1, \ldots, \tau_{m-1}$ defined in the following steps are the virtual controls in the ODE backstepping method.

Step 1. We consider a Lyapunov function candidate as

$$V_{y1} = \frac{1}{2}y_1(t)^2. \tag{166}$$

Taking the derivative of (166), we obtain

$$\dot{V}_{y1} = -\hat{c}_1 y_1(t)^2 + y_1(t)y_2(t), \tag{167}$$

with the choice of

$$\tau_1 = \hat{c}_1 y_1(t), \tag{168}$$

where $\hat{c}_1$ is a positive constant to be determined later.

Step 2. A Lyapunov function candidate is considered as

$$V_{y2} = V_{y1} + \frac{1}{2}y_2(t)^2 = \frac{1}{2}y_1(t)^2 + \frac{1}{2}y_2(t)^2. \tag{169}$$

Taking the derivative of (169), we have

$$\dot{V}_{y2} = -\hat{c}_1 y_1(t)^2 + y_1(t)y_2(t) + y_2(t)(y_3(t) - \tau_2 + \dot{\tau}_1). \tag{170}$$

Choosing $\tau_2 = \dot{\tau}_1 + y_1(t) + \hat{c}_2 y_2(t)$, we have

$$\dot{V}_{y2} = -\hat{c}_1 y_1(t)^2 - \hat{c}_2 y_2(t)^2 + y_2(t)y_3(t). \tag{171}$$

Step 3.

⋮

Step m.-1

Step m. Similarly, a Lyapunov function candidate is considered as

$$V_{ym} = V_{y_{m-1}} + \frac{1}{2}y_m(t)^2 = \frac{1}{2}y_1(t)^2 + \frac{1}{2}y_2(t)^2 + \cdots + \frac{1}{2}y_{m-1}(t)^2 + \frac{1}{2}y_m(t)^2. \tag{172}$$

Taking the derivative of (172), we have

$$\dot{V}_{ym} = -\hat{c}_1 y_1(t)^2 - \hat{c}_2 y_2(t)^2 - \cdots - \hat{c}_{m-1} y_{m-1}(t)^2 + y_{m-1}(t)y_m(t) + y_m(t)\dot{y}_m(t). \tag{173}$$

Considering (161), (173) can be rewritten as

$$\begin{aligned}\dot{V}_{ym} =& -\hat{c}_1 y_1(t)^2 - \hat{c}_2 y_2(t)^2 - \cdots - \hat{c}_{m-1} y_{m-1}(t)^2 + y_{m-1}(t)y_m(t) + y_m(t)\Big[U(t)\\ &+ \left[q_m\partial_t^{m-1}\beta(1,t) + q_{m-1}\partial_t^{m-2}\beta(1,t) + \cdots + q_2\beta_t(1,t) + q_1\beta(1,t)\right]\\ &+ \left[p_m\partial_t^{m-1}\alpha(1,t) + p_{m-1}\partial_t^{m-2}\alpha(1,t) + \cdots + p_2\alpha_t(1,t) + p_1\alpha(1,t)\right]\\ &+ \left[\bar{q}_m\partial_t^{m-1}\beta(0,t) + \bar{q}_{m-1}\partial_t^{m-2}\beta(0,t) + \cdots + \bar{q}_2\beta_t(0,t) + \bar{q}_1\beta(0,t)\right]\\ &+ \left[\bar{p}_m\partial_t^{m-1}\alpha(0,t) + \bar{p}_{m-1}\partial_t^{m-2}\alpha(0,t) + \cdots + \bar{p}_2\alpha_t(0,t) + \bar{p}_1\alpha(0,t)\right]\\ &+\int_0^1 \bar{Q}(y)\beta(y,t)\mathrm{d}y + \int_0^1 \bar{P}(y)\alpha(y,t)\mathrm{d}y + H_zX(t) + \dot{\tau}_{m-1}\Big],\end{aligned} \tag{174}$$

where

$$\tau_{m-1} = \hat{c}_1 y_1^{m-2}(t) + y_1^{m-3}(t) + \hat{c}_2 y_2^{m-3}(t) + y_2^{m-4}(t) + \cdots + \hat{c}_{m-1}y_{m-1}(t), \quad \forall m \geq 4. \tag{175}$$

Note that $y_i^n(t)$ denotes n order derivative of $y_i(t), \forall i = 1, \ldots, m$.
Design the controller as

$$\begin{aligned}U(t) =& -\left[q_m\partial_t^{m-1}\beta(1,t) + q_{m-1}\partial_t^{m-2}\beta(1,t) + \cdots + q_2\beta_t(1,t) + q_1\beta(1,t)\right]\\ &-\left[p_m\partial_t^{m-1}\alpha(1,t) + p_{m-1}\partial_t^{m-2}\alpha(1,t) + \cdots + p_2\alpha_t(1,t)\right]\\ &-\left[\bar{q}_m\partial_t^{m-1}\beta(0,t) + \bar{q}_{m-1}\partial_t^{m-2}\beta(0,t) + \cdots + \bar{q}_2\beta_t(0,t)\right]\\ &-\left[\bar{p}_m\partial_t^{m-1}\alpha(0,t) + \bar{p}_{m-1}\partial_t^{m-2}\alpha(0,t) + \cdots + \bar{p}_2\alpha_t(0,t)\right] - y_{m-1}(t) - \dot{\tau}_{m-1} - \hat{c}_m y_m(t)\\ =& -\left[\left(q_m + \sum_{i=1}^{m-2}\hat{c}_i\right)\partial_t^{m-1}\beta(1,t) + \left(q_{m-1} + m - 2 + \sum_{i=2}^{m-2}\hat{c}_i\hat{c}_{i-1}\right)\partial_t^{m-2}\beta(1,t) + \cdots + q_1\beta(1,t)\right]\\ &-\left[p_m\partial_t^{m-1}\alpha(1,t) + p_{m-1}\partial_t^{m-2}\alpha(1,t) + \cdots + p_1\alpha_t(1,t)\right]\\ &-\left[\bar{q}_m\partial_t^{m-1}\beta(0,t) + \bar{q}_{m-1}\partial_t^{m-2}\beta(0,t) + \cdots + \bar{q}_2\beta_t(0,t)\right]\\ &-\left[\bar{p}_m\partial_t^{m-1}\alpha(0,t) + \bar{p}_{m-1}\partial_t^{m-2}\alpha(0,t) + \cdots + \bar{p}_2\alpha_t(0,t)\right]\\ &- y_{m-1}(t) - \hat{c}_{m-1}\dot{y}_{m-1}(t) - \hat{c}_m y_m(t).\end{aligned} \tag{176}$$

Note that using the transformations (162)-(165), (7)-(8) and the system equations (2)-(3) at $x = 0$ and $x = 1$, the controller (176) can be expressed as a function of the original state $u(x,t), v(x,t)$

$$\begin{aligned}U(t) =& \hat{n}_{m-1}\partial_x^{m-1}v(1,t) + \hat{n}_{m-2}\partial_x^{m-2}v(1,t) + \cdots + \hat{n}_0 v(1,t) + \hat{k}_{m-1}\partial_x^{m-1}u(1,t) + \hat{k}_{m-2}\partial_x^{m-2}u(1,t)\\ &+ \cdots + \hat{k}_0 u(1,t) + \hat{h}_{m-1}\partial_x^{m-1}v(0,t) + \hat{h}_{m-2}\partial_x^{m-2}v(0,t) + \cdots + \hat{h}_0 v(0,t) + \hat{l}_{m-1}\partial_x^{m-1}u(0,t)\\ &+ \hat{l}_{m-2}\partial_x^{m-2}u(0,t) + \cdots + \hat{l}_0 u(0,t) + \hat{D}X(t) + \int_0^1 \hat{N}(y)u(y,t)\mathrm{d}y + \int_0^1 \hat{L}(y)v(y,t)\mathrm{d}y,\end{aligned} \tag{177}$$

which is well defined. $\hat{n}_{m-1}, \ldots, \hat{n}_0, \hat{k}_{m-1}, \ldots, \hat{k}_0, \hat{h}_{m-1}, \ldots, \hat{h}_0, \hat{l}_{m-1}, \ldots, \hat{l}_0$ and $\hat{D}, \hat{N}(y), \hat{L}(y)$ are control gains consisting of the kernels in the backstepping transformations (8), (10), system parameters in (1)-(4), (153)-(154), and control parameters $\hat{c}_1, \ldots, \hat{c}_m, \kappa$.

Now, we get

$$\dot{V}_{ym} = -\hat{c}_1 y_1(t)^2 - \hat{c}_2 y_2(t)^2 - \cdots - \hat{c}_m y_m(t)^2 + y_m(t)\Bigg(\bar{q}_1\beta(0,t) + \bar{p}_1\alpha(0,t) + p_1\alpha(1,t) + \int_0^1 \bar{Q}(y)\beta(y,t)\mathrm{d}y + \int_0^1 \bar{P}(y)\alpha(y,t)\mathrm{d}y + H_z X(t)\Bigg), \tag{178}$$

where $\hat{c}_1, \ldots, \hat{c}_m$ are positive constants to be determined later.

6.2 | Stability analysis of states

Theorem 3. *If initial values* $(u(x,0), v(x,0)) \in W^{m,2}(0,1)$*, the closed-loop system consisting of the plant (1)-(4), (153)-(154), and the control law (177) is exponentially stable at the origin in the sense of the norm*

$$\left(\int_0^1 u^2(x,t)\mathrm{d}x + \int_0^1 v^2(x,t)\mathrm{d}x + |X(t)|^2 + z_1(t)^2 + \cdots + z_m(t)^2\right)^{1/2}. \tag{179}$$

Proof. Recalling (172) and (54), and define a Lyapunov function as

$$V_m(t) = V_1(t) + V_{ym}(t). \tag{180}$$

Defining the norm

$$\Omega_{2m}(t) = \|\beta(\cdot,t)\|^2 + \|\alpha(\cdot,t)\|^2 + |X(t)|^2 + y_1(t)^2 + \cdots + y_m(t)^2, \tag{181}$$

we have

$$\theta_{21}\Omega_{2m}(t) \le V_m(t) \le \theta_{22}\Omega_{2m}(t), \tag{182}$$

where

$$\theta_{21} = \min\left\{\lambda_{\min}(P_1), \frac{a_1}{2}, \frac{b_1 e^{-\delta_1}}{2}, \frac{1}{2}\right\} > 0, \tag{183}$$

$$\theta_{22} = \max\left\{\lambda_{\max}(P_1), \frac{a_1 e^{\delta_1}}{2}, \frac{b_1}{2}, \frac{1}{2}\right\} > 0. \tag{184}$$

Taking the derivative of (180) and using (72) and (178), we get

$$\begin{aligned}\dot{V}_m \le& -\eta_1|X(t)|^2 - \eta_2\beta(0,t)^2 - \eta_3\int_0^1 \beta(x,t)^2\mathrm{d}x - \eta_4\int_0^1 \alpha(x,t)^2\mathrm{d}x - \eta_5\alpha(1,t)^2 \\ &+ \eta_6 y_1(t)^2 - \hat{c}_1 y_1(t)^2 - \hat{c}_2 y_2(t)^2 - \cdots - \hat{c}_{m-1}y_{m-1}(t)^2 - \hat{c}_m y_m(t)^2 + y_m(t)\Bigg(\bar{q}_1\beta(0,t) \\ &+ \bar{p}_1\alpha(0,t) + p_1\alpha(1,t) + \int_0^1 Q(y)\beta(y,t)\mathrm{d}y + \int_0^1 P(y)\alpha(y,t)\mathrm{d}y + H_z X(t)\Bigg).\end{aligned} \tag{185}$$

Recalling (14), then applying Young's inequality, Cauchy-Schwarz inequality, and (162) into (185), we have

$$\begin{aligned}\dot{V}_m \le& -\left(\eta_1 - \hat{r}_1|H_z|^2 - \hat{r}_7\bar{p}_1^2|C_0|^2\right)|X(t)|^2 - \left(\eta_2 - \bar{q}_1^2\hat{r}_3 - \hat{r}_6\bar{p}_1^2 q^2\right)\beta(0,t)^2 \\ &- \left(\eta_3 - \hat{r}_5\bar{Q}_{\max}^2\right)\int_0^1 \beta(x,t)^2\mathrm{d}x - \left(\eta_4 - \hat{r}_2\bar{P}_{\max}^2\right)\int_0^1 \alpha(x,t)^2\mathrm{d}x \\ &- \left(\eta_5 - p_1^2\hat{r}_4\right)\alpha(1,t)^2 - (\hat{c}_1 - \eta_6)y_1(t)^2 - \hat{c}_2 y_2(t)^2 - \cdots - \hat{c}_{m-1}y_{m-1}(t)^2 \\ &- \left(\hat{c}_m - \left(\frac{1}{4\hat{r}_1} + \frac{1}{4\hat{r}_2} + \frac{1}{4\hat{r}_3} + \frac{1}{4\hat{r}_4} + \frac{1}{4\hat{r}_5} + \frac{1}{4\hat{r}_6} + \frac{1}{4\hat{r}_7}\right)\right)y_m(t)^2.\end{aligned} \tag{186}$$

We choose positive constants $\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4, \hat{r}_5, \hat{r}_6, \hat{r}_7$ as

$$\hat{r}_1 < \frac{\eta_1}{|H_z|^2},\ \hat{r}_2 < \frac{\eta_4}{\bar{P}_{\max}^2},\ \hat{r}_3 < \frac{\eta_2}{\bar{q}_1^2},\ \hat{r}_4 < \frac{\eta_5}{p_1^2},$$
$$\hat{r}_5 < \frac{\eta_3}{\bar{Q}_{\max}^2},\ \hat{r}_6 < \frac{\eta_2 - \bar{q}_1^2\hat{r}_3}{\bar{p}_1^2 q^2},\ \hat{r}_7 < \frac{\eta_1 - \hat{r}_1|H_z|^2}{\bar{p}_1^2|C_0|^2}, \tag{187}$$

where

$$\bar{Q}_{\max} = \max_{x\in[0,1]}\{|\bar{Q}(x)|\},\ \ \bar{P}_{\max} = \max_{x\in[0,1]}\{|\bar{P}(x)|\}, \tag{188}$$

and choose the control parameters $\hat{c}_1$ and $\hat{c}_m$ as

$$\hat{c}_1 > \eta_6, \tag{189}$$

$$\hat{c}_m > \frac{1}{4}\left(\frac{1}{\hat{r}_1} + \frac{1}{\hat{r}_2} + \frac{1}{\hat{r}_3} + \frac{1}{\hat{r}_4} + \frac{1}{\hat{r}_5} + \frac{1}{\hat{r}_6} + \frac{1}{\hat{r}_7}\right). \tag{190}$$

Note that positive control parameters $\hat{c}_2, \dots, \hat{c}_{m-1}$ can be chosen arbitrarily to adjust the exponential decay rate of the closed-loop system.

Finally, we arrive at

$$\dot{V}_m(t) \le -\hat{\lambda}V_m(t) - \hat{g}_0\beta(0,t)^2 - \hat{g}_1\alpha(1,t)^2, \tag{191}$$

for some positive $\hat{\lambda}$ and $\hat{g}_0 = \eta_2 - \bar{q}_1^2\hat{r}_3 - \hat{r}_6\bar{p}_1^2q^2 > 0$, $\hat{g}_1 = \eta_5 - p_1^2\hat{r}_4 > 0$.

Through a process similar to (85)-(87), we arrive at Theorem 3. □

6.3 | Boundedness and exponential convergence of the controller $U(t)$(177)

In this section, we prove the exponential convergence and boundedness of the controller $U(t)$(177) in the closed-loop system including the m order input ODE.

Theorem 4. *In the closed-loop system including the plant (1)-(4), (153), (154) and the controller $U(t)$ (177), there exist positive constants λ_m and Υ_{0m} making that $|U(t)|$ is bounded and exponentially convergent to zero in the sense of*

$$|U(t)| \le \Upsilon_{0m}e^{-\frac{\lambda_m}{2}t}. \tag{192}$$

We would produce and analyze L_2 estimates of $u_x(x,t), v_x(x,t), \dots, \partial_x^{m-1}u(x,t), \partial_x^{m-1}v(x,t), \partial_x^m u(x,t), \partial_x^m v(x,t)$ to prove the exponential convergence of the bounds of signals $\partial_x^{m-1}v(1,t), \partial_x^{m-1}u(1,t), \partial_x^{m-1}v(0,t), \partial_x^{m-1}u(0,t), \partial_x^{m-2}u(1,t), \partial_x^{m-2}v(0,t), \partial_x^{m-2}u(0,t), \dots, v(1,t), u(1,t), v(0,t), u(0,t)$ in the controller (177).

Lemma 4. *For any initial data $(u(x,0), v(x,0)) \in H^{m-1}(0,1)$, the closed-loop system $(u(x,t), v(x,t))$(1)-(4), (153), (154) with the controller (177) is exponentially stable in the sense of*

$$\left\|\partial_x^{m-1}u(\cdot,t)\right\|^2 + \left\|\partial_x^{m-1}v(\cdot,t)\right\|^2. \tag{193}$$

Proof. Define a Lyapunov function

$$\bar{B}_{m-1}(t) = \frac{1}{2}\int_0^1 b_{m-1}e^{-\delta_{m-1}x}\partial_x^{m-1}\alpha(x,t)^2\mathrm{d}x + \frac{1}{2}\int_0^1 a_{m-1}e^{\delta_{m-1}x}\partial_x^{m-1}\beta(x,t)^2\mathrm{d}x, \tag{194}$$

where the positive constant b_{m-1} can be chosen arbitrarily to adjust the convergence rate and positive constants a_{m-1}, δ_{m-1} will be defined later.

Taking the derivative of (194) along the system obtained from $m-1$ times differentiating (12)-(13) with respect to x and $m-1$ times differentiating (14) with respect to t, using Cauchy-Schwarz inequality, we can choose positive a_{m-1}

and δ_{m-1} (see Remark 2) such that

$$
\begin{aligned}
\dot{\bar{B}}_{m-1}(t) \leq & -\bar{M}_1\int_0^1 e^{-\delta_{m-1}x}\partial_x^{m-1}\alpha(x,t)^2\mathrm{d}x-\bar{M}_2\int_0^1 e^{\delta_{m-1}x}\partial_x^{m-1}\beta(x,t)^2\mathrm{d}x\\
&-\bar{M}_3\partial_x^{m-1}\beta(0,t)^2-\frac{1}{2}b_{m-1}e^{-\delta_{m-1}}\partial_x^{m-1}\alpha(1,t)^2+\frac{1}{2}a_{m-1}e^{\delta_{m-1}}\partial_x^{m-1}\beta(1,t)^2\\
&+\bar{M}_4\int_0^1 e^{\delta_{m-1}x}\left[\partial_x^{m-2}\beta(x,t)^2+\cdots+\beta_x(x,t)^2+\beta(x,t)^2\right]\mathrm{d}x\\
&+\bar{M}_5\int_0^1 e^{-\delta_{m-1}x}\left[\partial_x^{m-2}\alpha(x,t)^2+\cdots+\alpha_x(x,t)^2+\alpha(x,t)^2\right]\mathrm{d}x\\
&+\bar{M}_6|X(t)|^2+\bar{M}_7\left[\partial_x^{m-2}\beta(0,t)^2+\cdots+\beta_x(0,t)^2+\beta(0,t)^2\right],
\end{aligned}
\tag{195}
$$

where $\bar{M}_1,\bar{M}_2,\bar{M}_3,\bar{M}_4,\bar{M}_5,\bar{M}_6,\bar{M}_7$ are positive constants.

Remark 2. As in (137), we can choose a_{m-1},δ_{m-1} to make sure $m-1$ order terms $\|\partial_x^{m-1}\alpha(\cdot,t)\|^2$, $\|\partial_x^{m-1}\beta(\cdot,t)\|^2$, $\partial_x^{m-1}\beta(0,t)^2$ in $\dot{\bar{B}}_{m-1}$ negative except for $\partial_x^{m-1}\beta(1,t)^2$. The positive term $\partial_x^{m-1}\beta(1,t)^2=\partial_t^{m-1}\beta(1,t)^2$ can be accommodated by the exponential results in the sense of the norms $y_1(t)^2,\ \dots,\ y_m(t)^2$ provided in Theorem 3. Note that as (136), the positive term $\partial_x^{m-1}\alpha(0,t)^2$ can be written as positive terms $\partial_x^{m-1}\beta(0,t)^2,\partial_x^{m-2}\beta(0,t)^2,\ \dots,\beta(0,t)^2$ and $|X(t)|^2$ via using Cauchy-Schwarz inequality into the $m-1$ order time derivative of (14) with (12). As in (137), $\partial_x^{m-1}\beta(0,t)^2$ can be overcame by choosing a_{m-1}, and other positive terms with coefficients $\bar{M}_6,\bar{M}_7$ are kept in (195). These rest positive terms will be overcame in the following steps.

All positive $m-2,\ \dots,1$ order terms can be accommodated by the exponential estimates in the sense of the norm $\|\partial_x^{m-2}\alpha(\cdot,t)\|^2+\|\partial_x^{m-2}\beta(\cdot,t)\|^2,\ \dots,\|\alpha_x(\cdot,t)\|^2+\|\beta_x(\cdot,t)\|^2$, which can be obtained according to Lemma 2 and Lemma 3. Together with the exponential results in the sense of the norm $\|\alpha(\cdot,t)\|^2+\|\beta(\cdot,t)\|^2+|X(t)|^2+y_1(t)^2+\cdots+y_m(t)^2$ provided in Theorem 3, we define a Lyapunov function

$$
V_{u(m-1)}(t)=R_{m-1}\left[\prod_{i=1}^{m-2}R_iV_m(t)+\prod_{i=2}^{m-2}R_i\bar{A}(t)+\prod_{i=3}^{m-2}R_i\bar{B}(t)+\prod_{i=4}^{m-2}R_i\bar{B}_3(t)+\cdots+\bar{B}_{m-2}(t)\right]+\bar{B}_{m-1}(t),
\tag{196}
$$

where $\bar{B}_i(t),\ \ \forall i=3,\ \dots,m-2$ are Lyapunov functions similar to (194), where $m-1$ is replaced by i.

Taking the derivative of (196) and choosing sufficiently large $R_i>0$, we have

$$
\dot{V}_{u(m-1)}(t)=-\lambda_{m-1}V_{u(m-1)}(t)-\hat{g}_2\alpha(1,t)^2-\hat{g}_3\left[\partial_x^{m-1}\beta(0,t)^2+\partial_x^{m-2}\beta(0,t)^2+\cdots+\beta_x(0,t)+\beta(0,t)^2\right],
\tag{197}
$$

for some positive $\lambda_{m-1},\hat{g}_2,\hat{g}_3$.

Now, we obtain the exponential stability estimates in terms of the norms $\|\partial_t^{m-1}\alpha(\cdot,t)\|^2+\|\partial_t^{m-1}\beta(\cdot,t)\|^2$.

Through a similar process with (115)-(117), we can prove Lemma 4. □

Lemma 5. *For any initial data $(u(x,0),v(x,0))\in H^m(0,1)$, the closed-loop system $(u(x,t),v(x,t))$(1)-(4), (153), (154) with the controller (177) is exponentially stable in the sense of*

$$
\|\partial_x^m u(\cdot,t)\|^2+\|\partial_x^m v(\cdot,t)\|^2.
\tag{198}
$$

Proof. Define a Lyapunov function

$$
\bar{B}_m(t)=\frac{1}{2}\int_0^1 b_me^{-\delta_mx}\partial_x^m\alpha(x,t)^2\mathrm{d}x+\frac{1}{2}\int_0^1 a_me^{\delta_mx}\partial_x^m\beta(x,t)^2\mathrm{d}x,
\tag{199}
$$

where the positive constant b_m can be chosen arbitrarily to adjust the convergence rate and positive constants a_m and δ_m will be defined later.

Taking the derivative of (199) along the system obtained from m times differentiating (12)-(13) with respect to x and m times differentiating (14) with respect to t, using Cauchy-Schwarz inequality, we can choose positive a_m and

δ_m (see Remark 3) such that

$$\begin{aligned}\dot{\bar{B}}_m(t) \leq & -M_1\int_0^1 e^{-\delta_m x}\partial_x^m\alpha(x,t)^2\mathrm{d}x - M_2\int_0^1 e^{\delta_3 x}\partial_x^m\beta(x,t)^2\mathrm{d}x \\ & - M_3\partial_x^m\beta(0,t)^2 - \frac{1}{2}b_m e^{-\delta_m}\partial_x^m\alpha(1,t)^2 + \frac{1}{2}a_m e^{\delta_m}\partial_x^m\beta(1,t)^2 \\ & + M_4\int_0^1 e^{\delta_m x}[\partial_x^{m-1}\beta(x,t)^2 + \cdots + \beta_x(x,t)^2 + \beta(x,t)^2]\mathrm{d}x \\ & + M_5\int_0^1 e^{-\delta_3 x}[\partial_x^{m-1}\alpha(x,t)^2 + \cdots + \alpha_x(x,t)^2 + \alpha(x,t)^2]\mathrm{d}x \\ & + M_6|X(t)|^2 + M_7[\partial_x^{m-1}\beta(0,t)^2 + \cdots + \beta_x(0,t)^2 + \beta(0,t)^2],\end{aligned} \tag{200}$$

where $M_1, M_2, M_3, M_4, M_5, M_6, M_7$, and M_8 are positive constants.

Remark 3. We can choose a_m, and δ_m to make sure all m order terms $\|\partial_x^m\alpha(\cdot,t)\|^2$, $\|\partial_x^m\beta(\cdot,t)\|^2$, $\partial_x^m\beta(0,t)^2$ in $\dot{\bar{B}}_m$ negative except for $\partial_x^m\beta(1,t)^2$. Note that as in (136), the positive term $\partial_x^m\alpha(0,t)^2$ can be written as positive terms $\partial_x^m\beta(0,t)^2, \partial_x^{m-1}\beta(0,t)^2, \ldots, \beta(0,t)^2$ and $|X(t)|^2$ via using Cauchy-Schwarz inequality into the m order time derivative of (14) with (12). As in (137), $\partial_x^m\beta(0,t)^2$ can be overcame by choosing a_m, and other positive terms with coefficients M_6, M_7 are kept in (200).

Substituting (176) into (161), using Cauchy-Schwarz inequality, $\partial_x^m\beta(1,t)^2 = \partial_t^m\beta(1,t)^2$ in (200) yields to the positive terms

$$\begin{aligned}M_9\Bigg[&\left[\partial_t^{m-1}\beta(1,t)^2 + \partial_t^{m-2}\beta(1,t)^2 + \cdots + \beta_t(1,t) + \beta(1,t)^2\right] \\ & + \beta(0,t)^2 + \alpha(1,t)^2 + \int_0^1\beta(y,t)^2\mathrm{d}y + \int_0^1\alpha(y,t)^2\mathrm{d}y + |X(t)|^2\Bigg],\end{aligned} \tag{201}$$

where M_9 is a positive constant. Note that $\alpha(0,t)^2$ can be combined into $\beta(0,t)^2$ and $|X(t)|^2$ via applying Cauchy-Schwarz inequality into (14).

According to Lemma 2, Lemma 3, and Lemma 4, we obtain the exponential estimates in sense of $\|\partial_x^{m-1}\alpha(\cdot,t)\|^2 + \|\partial_x^{m-1}\beta(\cdot,t)\|^2 + \cdots + \|\alpha_x(\cdot,t)\|^2 + \|\beta_x(\cdot,t)\|^2$, which accommodate the $m-1, \ldots, 1$ order positive terms in $\dot{\bar{B}}_m$ (200). Together with the exponential results in terms of the norm $\|\alpha(\cdot,t)\|^2 + \|\beta(\cdot,t)\|^2 + |X(t)|^2 + y_1(t)^2 + \cdots + y_m(t)^2$ provided in Theorem 3, define a Lyapunov function

$$V_{um} = R_m V_{u(m-1)} + \bar{B}_m. \tag{202}$$

Taking the derivative of V_{um} and choosing sufficiently large R_m, we arrive at

$$\dot{V}_{um}(t) \leq -\lambda_m V_{um}(t), \tag{203}$$

for some positive λ_m.

Through a similar process with (150)-(151), we arrive at Lemma 5.

Proof of Theorem 4. Using the exponential estimates in terms of the norm $\|u(\cdot,t)\|^2 + \|v(\cdot,t)\|^2 + \|u_x(\cdot,t)\|^2 + \|v_x(\cdot,t)\|^2 + \cdots + \|\partial_x^m u(\cdot,t)\|^2 + \|\partial_x^m v\cdot,t)\|^2 + z_1(t)^2 + z_2(t)^2 + \cdots + z_m(t)^2 + |X(t)|^2$ provided in Lemma 2, Lemma 3, Lemma 4, Lemma 5, and Theorem 3, through a similar process with proof of Theorem 2, we arrive at Theorem 4. ☐

7 | SIMULATION

We use the finite difference method to conduct the simulation with the time interval 0.00025 and spatial interval 0.005. Note that the solutions of the kernel equations (19)-(24), which are coupled linear hyperbolic PDEs in $D = \{(x,y)|0 \leq y \leq x \leq 1\}$, are also solved by the finite difference method. Then, there would be 3 loops corresponding to $t \in [0,\infty)$, $x \in [0,1]$, $y \in [0,x]$ in the programming code of the simulation. We define the system parameters in (1)-(6) as

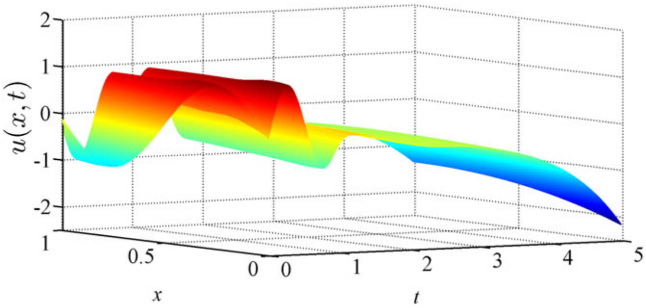

FIGURE 2 Response of $u(x, t)$ in the plant (1)-(6) without control [Colour figure can be viewed at wileyonlinelibrary.com]

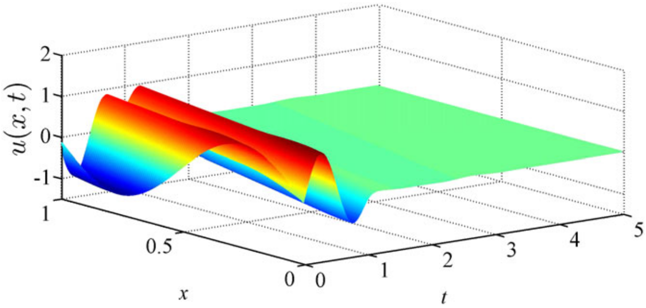

FIGURE 3 Response of $u(x, t)$ in the plant (1)-(6) with the controller (51) [Colour figure can be viewed at wileyonlinelibrary.com]

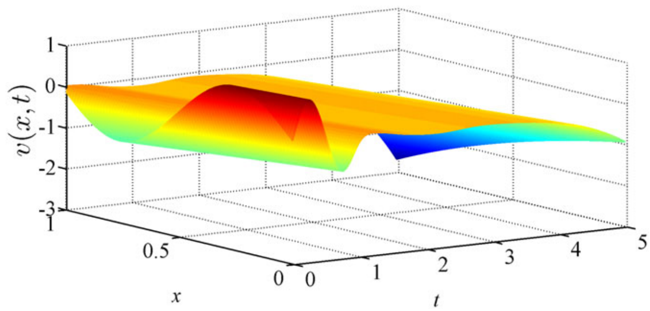

FIGURE 4 Response of $v(x, t)$ in the plant (1)-(6) without control [Colour figure can be viewed at wileyonlinelibrary.com]

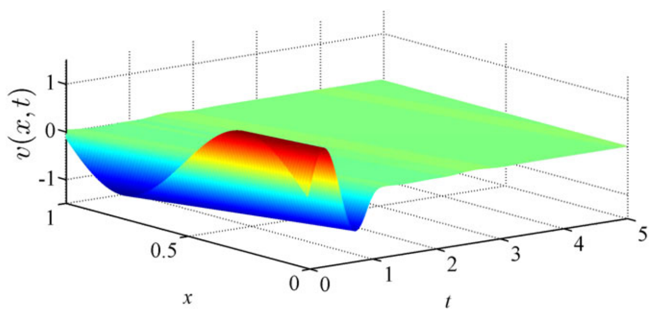

FIGURE 5 Response of $v(x, t)$ in the plant (1)-(6) with the controller (51) [Colour figure can be viewed at wileyonlinelibrary.com]

$[A, B, c_1, c_2, q, p, C, c_0, r] = [0.5, 1, 0.5, 0.5, 1, 1, 1, 1, 1]$ and the control parameters are chosen as $[\kappa, \bar{c}_1, \bar{c}_2] = [-2, 5, 13]$. Both initial conditions of $v(x, t)$ and $u(x, t)$ are defined as $v(x, 0) = u(x, 0) = \sin(2\pi x)$. Then, the initial conditions of $X(t)$ and $z(t)$ are $X(0) = u(0, 0) - v(0, 0) = 0$, $z(0) = v(1, 0) = 0$ according to (4) and (5).

Comparing Figure 2 that shows the open-loop response of $u(x, t)$ and Figure 3 that gives the closed-loop response of $u(x, t)$, as one can observe, in the latter case, convergence to zero is achieved, whereas the states grow unbounded in the former case. Similar results can be obtained via comparing Figure 4 and Figure 5, which show the open-loop and closed-loop responses of $v(x, t)$, respectively.

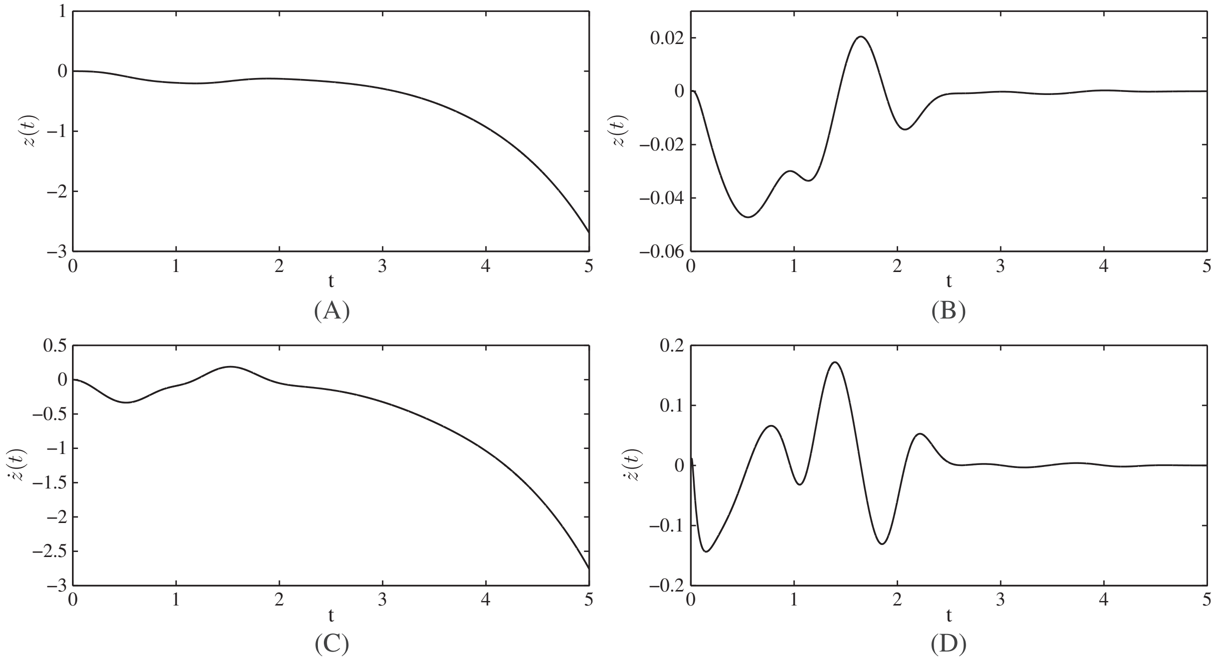

FIGURE 6 Response of input ODE states $z(t), \dot{z}(t)$ in the open-loop and closed-loop cases. A, $z(t)$, open-loop case; B, $z(t)$, closed-loop case; C, $\dot{z}(t)$, open-loop case; D, $\dot{z}(t)$, closed-loop case

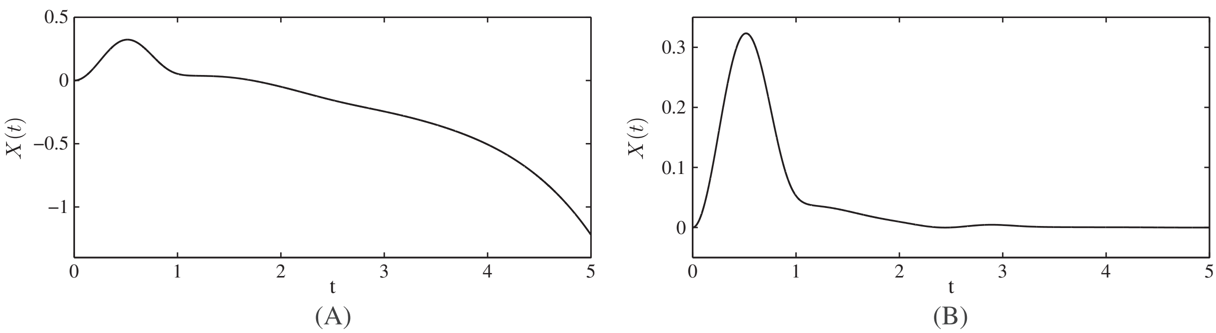

FIGURE 7 Response of ODE state $X(t)$ in the open-loop and closed-loop cases. A, $X(t)$, open-loop case; B, $X(t)$, closed-loop case

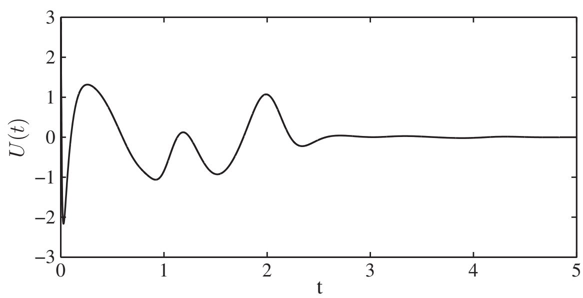

FIGURE 8 Controller $U(t)$(51) in the closed-loop system

In Figure 6, we show the responses of the input ODE states $z(t), \dot{z}(t)$ in both open-loop (left side) and closed-loop (right side) cases. We observe that the states $z(t), \dot{z}(t)$ grow unbounded in the open-loop case and converge to zero under control. Similar results can be observed in Figure 7, which shows both open-loop and closed-loop responses of the ODE state $X(t)$.

In Figure 8, we show the response of the control input $U(t)$(51) in the closed-loop system. As one can observe, $U(t)$ converges to zero.

8 | CONCLUSION

We present a novel methodology combining PDE backstepping and ODE backstepping to stabilize a 2×2 coupled linear hyperbolic system sandwiched between 2 ODEs. All PDE and ODE states are proved exponentially stable in the closed-loop system via Lyapunov analysis. Moreover, the boundedness and exponentially convergence of the designed control input are also proved in this paper. This paper opens a door for stabilization of ODE-PDE-ODE sandwich systems where the 2×2 coupled linear hyperbolic system can be extended to other PDE types.

ACKNOWLEDGEMENT

This work was supported by the National Basic Research Program of China (973 Program) under Grant 2014CB049404 and Chongqing University Postgraduates Innovation Project [CYD15023].

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How to cite this article: Wang J, Krstic M, Pi Y. Control of a 2×2 coupled linear hyperbolic system sandwiched between 2 ODEs. *Int J Robust Nonlinear Control*. 2018;1–30. https://doi.org/10.1002/rnc.4117

APPENDIX

$h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9(y), h_{10}(y), H_{11}$ are shown as

$$
\begin{aligned}
h_1 &= c_0 - p\phi(1,1),\ h_2 = p\psi(1,1),\ h_3 = p\phi(1,0) - \gamma(1)B,\ h_4 = -p\psi(1,0),\\
h_5 &= -pc_1\psi(1,1) + p^2\phi_y(1,1) + p^2\phi(1,1)\phi^I(1,1) - c_0\phi^I(1,1)p,\\
h_6 &= p^2\psi_y(1,1) + pc_2\phi(1,1) + c_0\psi^I(1,1)p - p^2\phi(1,1)\psi^I(1,1),\\
h_7 &= pc_1\psi(1,0) - p^2\phi_y(1,0) - \gamma(1)AB - p^2\phi(1,1)\phi^I(1,0) - c_0\gamma^I(1)B\\
&\quad + p\phi(1,1)\gamma^I(1)B + c_0\phi^I(1,0)p - (p\phi(1,0) - \gamma(1)B)\gamma^I(0)B,\\
h_8 &= p^2\phi(1,1)\psi^I(1,0) - c_0\psi^I(1,0)p - p(p\psi_y(1,0) + c_2\phi(1,0)),
\end{aligned}
$$

$$
\begin{aligned}
h_9(y) &= p\phi(1,1)\psi^I(1,y)c_1 - \int_y^1 p\phi(1,1)\psi^I(1,\delta)c_1\phi^I(\delta,y)\mathrm{d}\delta\\
&\quad + \int_y^1 \left(p^2\phi_{yy}(1,\delta) + c_1c_2\phi(1,\delta)\right)\phi^I(\delta,y)\mathrm{d}\delta - \left(p^2\phi_{yy}(1,y) + c_1c_2\phi(1,y)\right)\\
&\quad + p(c_1\psi(1,1) - p\phi_y(1,1))\phi^I(1,y) + c_0\phi^I_y(1,y)p + \int_y^1 c_0\psi^I(1,\delta)c_1\phi^I(\delta,y)\mathrm{d}\delta\\
&\quad - c_0\psi^I(1,y)c_1 - p^2\phi(1,1)\phi^I{}_y(1,y),\\
h_{10}(y) &= \int_y^1 c_0\psi^I(1,\delta)c_1\psi^I(\delta,y)\mathrm{d}\delta - c_0p\psi^I{}_y(1,y)\\
&\quad - \left(p^2\psi_{yy}(1,y) + c_2c_1\psi(1,y)\right) + p(c_1\psi(1,1) - p\phi_y(1,1))\psi^I(1,y)\\
&\quad + \int_y^1 \left(p^2\phi_{yy}(1,\delta) + c_1c_2\phi(1,\delta)\right)\psi^I(\delta,y)\mathrm{d}\delta\\
&\quad + p^2\phi(1,1)\psi_y{}^I(1,y) - \int_y^1 p\phi(1,1)\psi^I(1,\delta)c_1\psi^I(\delta,y)\mathrm{d}\delta,
\end{aligned}
$$

$$
\begin{aligned}
H_{11} = {} & p\phi(1,1)\gamma^I(1)A + \int_0^1 \left(p^2\phi_{yy}(1,\delta) + c_1c_2\phi(1,\delta)\right)\gamma^I(\delta)\mathrm{d}\delta + p(c_1\psi(1,1) - p\phi_y(1,1))\gamma^I(1) \\
& - \left(pc_1\psi(1,0) - p^2\phi_y(1,0) - \gamma(1)AB\right)\gamma^I(0) + c_0\gamma^I(1)B\gamma^I(0) - p\phi(1,1)\gamma^I(1)B\gamma^I(0) \\
& - \gamma(1)A^2 + (p\phi(1,0) - \gamma(1)B)\gamma^I(0)^2B - (p\phi(1,0) - \gamma(1)B)\gamma^I(0)A \\
& + \int_0^1 c_0\psi^I(1,y)c_1\gamma^I(y)\mathrm{d}y - \int_0^1 p\phi(1,1)\psi^I(1,y)c_1\gamma^I(y)\mathrm{d}y - c_0\gamma^I(1)A.
\end{aligned}
$$

Proof of Lemma 1. 1. First, we transform the kernel equations (19)-(24) to integral equations using the method of characteristics.

Considering (23), (24), we can get the solution of $\gamma(x)$ expressed with ψ, ie,

$$\gamma(x) = \kappa e^{\frac{1}{p}Ax} + C\int_0^x e^{\frac{1}{p}A(x-\tau)}\psi(\tau,0)\mathrm{d}\tau. \tag{A1}$$

Then, we use the method of characteristic line[10] to give the successive approximations of $\psi(x,y), \phi(x,y)$. Along the line $x = -y + \bar{a}_1$, according to (22), (19), we have

$$\frac{\mathrm{d}\psi(s)}{\mathrm{d}s} = -c_2\phi(\bar{a}_1 - s, s), \tag{A2}$$

$$\psi\left(\frac{\bar{a}_1}{2}\right) = -\frac{c_2}{2p}, \tag{A3}$$

considering the characteristic $(-s + \bar{a}_1, s)$ that reaches to (x,y). According to the ODE (A2)-(A3), we obtain

$$\psi(x,y) = -\frac{c_2}{2p} - \int_{\frac{x+y}{2}}^{y} \frac{c_2}{p}\phi(x+y-\tau,\tau)\mathrm{d}\tau. \tag{A4}$$

Then, (A4) can be rewritten as the integral form

$$\psi(x,y) = G_0(x,y) + G[\phi](x,y), \tag{A5}$$

where

$$G_0(x,y) = -\frac{c_2}{2p}, \tag{A6}$$

$$G[\phi](x,y) = -\int_{\frac{x+y}{2}}^{y} \frac{c_2}{p}\phi(x+y-\tau,\tau)\mathrm{d}\tau. \tag{A7}$$

Substituting (A1) into (20), considering (20), (21) along the line $x = y + \bar{a}_2$, we have

$$\frac{\mathrm{d}\phi(s)}{s} = \frac{c_1}{p}\psi(\bar{a}_2 + s, s), \tag{A8}$$

$$\phi(0) = \kappa e^{\frac{1}{p}A\bar{a}_2}B + \frac{1}{p}C\int_0^{\bar{a}_2} e^{\frac{1}{p}A(\bar{a}_2-\tau)}\psi(\tau,0)\mathrm{d}\tau B + \psi(\bar{a}_2,0)q, \tag{A9}$$

with the characteristic $(s + \bar{a}_2, s)$ that reaches to (x,y). According to the ODE (A8)-(A9), we obtain

$$\phi(x,y) = \kappa e^{\frac{1}{p}A(x-y)}B + \frac{1}{p}C\int_0^{x-y} e^{\frac{1}{p}A(x-y-\tau)}\psi(\tau,0)\mathrm{d}\tau B + \psi(x-y,0)q + \frac{1}{p}\int_0^y c_1\psi(x-y+\tau,\tau)\mathrm{d}\tau. \tag{A10}$$

Substituting (A4) into (A10) yields

$$
\begin{aligned}
\phi(x,y) = {} & \kappa e^{\frac{1}{p}A(x-y)}B + \frac{1}{p}C\int_0^{x-y} e^{\frac{1}{p}A(x-y-\tau)}\psi(\tau,0)\mathrm{d}\tau B \\
& - \frac{qc_2}{2p} + q\int_0^{\frac{x-y}{2}} \frac{c_2}{p}\phi(x-y-\tau,\tau)\mathrm{d}\tau - \frac{c_1c_2}{2p^2}y \\
& - \int_0^y \int_{\frac{x-y+2\tau}{2}}^{\tau} \frac{c_1c_2}{p^2}\phi(x-y+2\tau-\mu,\mu)\mathrm{d}\mu\mathrm{d}\tau,
\end{aligned} \tag{A11}
$$

which can be rewritten as the integral form

$$\phi(x,y) = F_0(x,y) + F[\psi,\phi](x,y), \tag{A12}$$

where

$$F_0(x,y) = \kappa e^{\frac{1}{p}A(x-y)}B - \frac{qc_2}{2p} - \frac{c_1c_2}{2p^2}y. \tag{A13}$$

$$F[\psi,\phi](x,y)=\frac{1}{p}C\int_0^{x-y}e^{\frac{1}{p}A(x-y-\tau)}\psi(\tau,0)\mathrm{d}\tau B+q\int_0^{\frac{x-y}{2}}\frac{c_2}{p}\phi(x-y-\tau,\tau)\mathrm{d}\tau$$
$$-\int_0^y\int_{\frac{x-y+2\tau}{2}}^{\tau}\frac{c_1c_2}{p^2}\phi(x-y+2\tau-\mu,\mu)\mathrm{d}\mu\mathrm{d}\tau. \quad (A14)$$

2. Second, we use the method of successive approximations to construct a solution to the integral equations (A5), (A12) in the form of a converging series.

Setting

$$\psi^0(x,y)=0, \quad (A15)$$

$$\phi^0(x,y)=0, \quad (A16)$$

$$\psi^{n+1}(x,y)=G_0(x,y)+G[\phi^n](x,y), \quad (A17)$$

$$\phi^{n+1}(x,y)=F_0(x,y)+F[\psi^n,\phi^n](x,y), \quad (A18)$$

for $n=0,1\ldots$, with the definition of increments $\Delta\psi^{n+1}=\psi^{n+1}-\psi^n$ and $\Delta\phi^{n+1}=\phi^{n+1}-\phi^n$, where $\Delta\psi^0=G_0(x,y)$, $\Delta\phi^0=F_0(x,y)$, it is easy to see that

$$\Delta\psi^{n+1}=G[\Delta\phi^n](x,y), \quad (A19)$$

$$\Delta\phi^{n+1}=F[\Delta\psi^n,\Delta\phi^n](x,y). \quad (A20)$$

Define

$$\bar{a}=\max_{x\in[0,1]}\left\{Ce^{\frac{1}{p}Ax}B,\kappa e^{\frac{1}{p}Ax}B\right\}, \quad (A21)$$

$$\bar{b}=\frac{1}{p}\max\left\{c_2,qc_2,\frac{1}{p}c_1c_2\right\}, \quad (A22)$$

$$\eta=2\bar{a}+4\bar{b}. \quad (A23)$$

According to the definition of $\Delta\psi^0$ and $\Delta\phi^0$, we observe $\Delta\psi^0<\eta$ and $\Delta\phi^0<\eta$. Assume now that

$$|\Delta\psi^n|\le\eta^{n+1}\frac{(x-y)^n}{n!}, \quad (A24)$$

$$|\Delta\phi^n|\le\eta^{n+1}\frac{(x-y)^n}{n!} \quad (A25)$$

are true for some $n\in\mathbb{N}^*$.

Substituting (A24), (A25) into (A19), (A20) expressed in (A7) and (A14) through straight calculation, we obtain

$$|\Delta\psi^{n+1}|\le\eta^{n+2}\frac{(x-y)^{n+1}}{(n+1)!}, \quad (A26)$$

$$|\Delta\phi^{n+1}|\le\eta^{n+2}\frac{(x-y)^{n+1}}{(n+1)!}. \quad (A27)$$

Therefore, the series

$$\psi(x,y)=\sum_{n=0}^{\infty}\Delta\psi^n(x,y), \quad (A28)$$

$$\phi(x,y)=\sum_{n=0}^{\infty}\Delta\phi^n(x,y) \quad (A29)$$

uniformly converges to the solution $(\psi(x,y),\phi(x,y))$ of the kernel equations (19)-(24) in $D=\{(x,y)|0\le y\le x\le 1\}$, and then, the solution $\gamma(x)$ can be obtained via (A1).

Now, we show the continuity of the sum (A28), (A29). First, it is straightforward to show that for $n\in\mathbb{N}^*$, $\Delta\psi^0=G_0(x,y)$ and $\Delta\phi^0=F_0(x,y)$ are continuous on D. Indeed $\Delta\psi^0$ and $\Delta\phi^0$ are continuous on D as a composition of continuous functions. Besides, if we assume that $\Delta\psi^n$ and $\Delta\phi^n$ are continuous, then $\Delta\psi^{n+1}$ and $\Delta\phi^{n+1}$ are continuous as the integral (with continuous limits of integration) of continuous functions times $\Delta\psi^n$ and $\Delta\phi^n$ composed with continuous functions. Finally, the normal convergence ensures continuity of the solutions $\psi(x,y)$, $\phi(x,y)$ on D.[10]

The proof of uniqueness of the solutions, which directly relies on the linearity of the kernel equations, is identical to the work of Krstic and Smyshlyaev.[32] We will not detail it here for the brevity purposes.

The proof of Lemma 1 is completed.