Boundary Stabilization of Wave Equation With Velocity Recirculation

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***Abstract*—Nonlocal terms have been the mainstay of the applications of partial differential equation (PDE) backstepping methods to *parabolic* PDEs. The problem of similar nonlocal terms for wave equations is still open. For wave equations, similar nonlocal terms have not been studied. In this paper, we open the topic of exploration of control of wave PDEs with nonlocal terms. This paper is concerned with the wave equation with in-domain feedback/recirculation of a boundary velocity with a spatially constant coefficient. Due to this nonlocal term, the passivity of the wave equation is destroyed. We first design an explicit state feedback controller to achieve exponential stability for the closed-loop system. Then, we design the output feedback by using infinite-dimensional observer. The backstepping approach is adopted in investigation. It is shown that by using two measurements only, the output feedback makes the closed-loop system exponentially stable.**

***Index Terms*—Backstepping, hyperbolic, nonlocal term, wave equation.**

I. Introduction

Nonlocal terms, including both boundary terms and strict-feedback/volterra terms, have been the mainstay of the applications of partial differential equation (PDE) backstepping methods to *parabolic* PDEs [1]. For wave equations, similar nonlocal terms have not been studied, except in the context of shear beam models [2]. In this paper, we open the topic of exploration of control of wave PDEs with nonlocal terms. We focus on the simplest example that is still nontrivial—the wave equation with in-domain feedback/recirculation of a boundary velocity with a spatially constant coefficient. The notational simplicity allows us to focus on the concepts and even to derive an explicit controller.

In the past decade, the backstepping method has been successfully applied to the boundary stabilization and observer design for some PDEs. On one hand, the backstepping method displays several advantages in feedback controller design, such as the explicit gain kernel and numerical effectiveness. On the other hand, the backstepping method has been proved to be powerful in dealing with unstable or even antistable PDEs to which the traditional passive principle design cannot be applied [2]. The earlier efforts on the stabilization and observer design for a class of parabolic linear PDEs by developing a continuum backstepping approach can be found in [1] and [3], respectively. Some other efforts on using the backstepping approach to the stabilization for multiple-dimensional PDEs, such as two-dimensional (2-D) Navier–Stokes equations, have been made in [4]. Much more attention to applying backstepping method to the boundary feedback controller design for first-order hyperbolic equations [5], [6], 2×2 quasilinear hyperbolic system [5], and system of $n+1$ coupled first-order hyperbolic linear PDEs [7] has also been paid. As for other methods to study the boundary control for hyperbolic PDEs, we would like to refer the reader to [8]–[12]. In [13], the observer-based controller is designed using both the displacement and velocity measurements via the backstepping method to exponentially stabilize an unstable 1-D wave equation. A breakthrough was made in [14], where the antistable wave equation is exponentially stabilized through a novel backstepping transformation method. The backstepping approach has been extended to the design of controller and observer to additional second-order hyperbolic equations with in-domain antidamping [15], [16]. The stabilization of unstable shear beam equation is addressed in [17], where the boundary stabilization is discussed by using the backstepping method and observer-based feedback. In recent years, progress has been made in making use of the backstepping method to deal with coupled hyperbolic systems, delay systems [5], [7], [18]. This powerful approach is also applied in the design of adaptive controllers for parabolic equations and the challenging antistable wave equations with parametric uncertainties [19]–[22]. As for other methods to study the boundary control for hyperbolic PDEs, which is different from the method we use in our paper, which employs a transformation into coupled parabolic PDEs, we would like to refer the reader to [8]–[12].

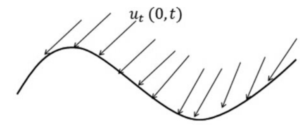

Fig. 1. Wave system with a nonlocal term.

In this paper, we consider a wave equation with a nonlocal term:

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + qu_t(0,t), \\ u_x(0,t) = 0, \\ u_x(1,t) = U(t) \end{cases} \tag{1}$$

where for each time $t \geq 0, U(t)$ is the input and $q \in \mathbb{R}$ is a constant. Also, we can verify that for $q = 0$, system (1) with $U(t) = 0$ is conservative. It is well known that system (1) with $q = 0$ can be exponentially stabilized under output feedback $U(t) = -k_1 u(1,t) - k_2 u_t(1,t), k_i > 0, i = 1,2$ or boundary state feedback $U(t) = -k_1 u(1,t) - k_2 u_t(1,t) - k_1 k_2 \int_0^1 u_t(y,t)dy, k_i > 0, i = 1,2$ (see [2, p. 83]). For $q \neq 0$, we say that $qu_t(0,t)$ is a nonlocal term, which is applied on the whole string (see Fig. 1). It is easy to verify that system (1) with $U(t) = 0$ is not conservative. The eigenvalues for the open-loop system are q and $n\pi i, n \in \mathbb{Z}$. So, when $q > 0$, the open-loop system is unstable. When $q \leq 0$, there are infinite number of

Manuscript received December 16, 2016; accepted March 11, 2017. Date of publication March 27, 2017; date of current version August 28, 2017. This work was supported in part by the National Natural Science Foundation of China under Grant 61673061 and Grant 61374088, in part by the Fundamental Research Funds for the Central Universities in UIBE (15JQ01), and in part by the China Scholarship Council. Recommended by Associate Editor C. Prieur. *(Corresponding author: Wei Guo.)*

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Digital Object Identifier 10.1109/TAC.2017.2688128

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eigenvalues for the open-loop system on the imaginary axis. So, the traditional Lyunopov design method cannot be applied. A natural idea is to adopt the backstepping approach to design boundary state feedback and output feedback to stabilize system (1). In addition, since the term $qu_t(0,t)$ is not strict feedback form, how to use the backstepping method to design the controller remains open until today.

The key characteristic of our approach is due to the relation between wave equations and coupled (2×2) first-order hyperbolic equations to find a new backstepping transformation that transforms a wave equation with nonlocal term into a cascade of two transport PDEs together with one ordinary differential equation (ODE) being driven by one of these two PDEs, which is more general than the methods in [2], [5], and [14], where control of coupled hyperbolic systems with local coupling is considered.

Although not an industrial application, a physical system giving rise to model (1) is discussed in [23, Remark 19.1]. This model is a prototype for more general wave systems with integral nonlocal terms, such as those that arise in the shear beam model [2], which are only displacement driven. We focus in this paper on a particularly simple system like (1), for which we can design a control law explicitly, in order to give the basic idea of the backstepping design for wave systems with velocity-driven nonlocal terms. Extensions with general Volterra-type nonlocal terms are fairly straightforward and merely more notationally intensive.

The rest of this paper is organized as follows. In Section II, we give the state feedback control by the backstepping method. Section III is devoted to the design of the observer. Section IV presents the output feedback control design and gives the exponentially stability of the closed-loop system. Section V presents some numerical simulations to indicate the effectiveness and credibility of our control law. Section VI gives the concluding remarks.

II. State Feedback Control Design

This section is dedicated to the design of the boundary state feedback controller by using the backstepping method. Since system (1) is not recognizable in the strict-feedback form, the first step is to change this system into a 2×2 system of first-order transport equations, which convect in opposite directions which is studied in [5] but without nonlocal terms. To this end, we define the following transformations:

$$(\xi, \eta)^\top = \mathscr{P}_1(u, u_t)^\top : \begin{cases} \xi(x,t) = u_t(x,t) + u_x(x,t), \\ \eta(x,t) = u_t(x,t) - u_x(x,t) \end{cases} \tag{2}$$

together with its inverses given by

$$(u, u_t)^\top = \mathscr{P}_1^{-1}(\xi, \eta)^\top : \begin{cases} u_t(x,t) = \frac{\xi(x,t)+\eta(x,t)}{2}, \\ u_x(x,t) = \frac{\xi(x,t)-\eta(x,t)}{2}. \end{cases} \tag{3}$$

Let

$$W(t) = u_t(1,t) + U(t). \tag{4}$$

Then, system (1) is equivalent to

$$\begin{cases} u_t(0,t) = \xi(0,t), \\ \eta_t(x,t) = -\eta_x(x,t) + q\xi(0,t), \\ \eta(0,t) = \xi(0,t), \\ \xi_t(x,t) = \xi_x(x,t) + q\xi(0,t), \\ \xi(1,t) = W(t). \end{cases} \tag{5}$$

System (5) falls outside of classes of PDE–ODE systems considered in [6] and outside of the coupled PDE classes in [5] and [7]. Observing the boundary condition $\eta(0,t) = \xi(0,t)$, we get that system (5) is equal to system (6). So, our next step is to design a controller to make system (6) exponentially stable:

$$\begin{cases} u_t(0,t) = \xi(0,t), \\ \eta_t(x,t) = -\eta_x(x,t) + q\eta(0,t), \\ \eta(0,t) = \xi(0,t), \\ \xi_t(x,t) = \xi_x(x,t) + q\xi(0,t), \\ \xi(1,t) = W(t). \end{cases} \tag{6}$$

We present a control design $U(t)$ by finding an invertible transformation to make system (6) equivalent to a stable system, which is represented as the cascade of two transport PDEs and one ODE being driven by one of the two PDEs.

First, we introduce the following backstepping transformation between (η, ξ) and (ω, ζ) :

$$\begin{cases} (\omega, \zeta)^\top = \mathscr{P}_2(\xi, \eta)^\top : \\ \omega(x,t) = \eta(x,t) - \int_0^x q e^{-q(x-y)} \eta(y,t) dy, \\ \zeta(x,t) = \xi(x,t) + \int_0^x q e^{q(x-y)} \xi(y,t) dy. \end{cases} \tag{7}$$

Its inverse backstepping transformation is given as

$$\begin{cases} (\xi, \eta)^\top = \mathscr{P}_2^{-1}(\omega, \zeta)\top : \\ \eta(x,t) = \omega(x,t) + q\int_0^x \omega(y,t) dy, \\ \xi(x,t) = \zeta(x,t) - q\int_0^x \zeta(y,t) dy. \end{cases} \tag{8}$$

Under the transformation (7), system (6) is transformed into the following intermediate system:

$$\begin{cases} u_t(0,t) = \zeta(0,t) \\ \omega_t(x,t) = -\omega_x(x,t), \\ \omega(0,t) = \zeta(0,t), \\ \zeta_t(x,t) = \zeta_x(x,t), \\ \zeta(1,t) = W_1(t) \end{cases} \tag{9}$$

where

$$W_1(t) = W(t) + \int_0^1 q e^{q(1-y)} \xi(y,t) dy. \tag{10}$$

So far, the wave phenomenon is represented as the cascade of two transport PDEs and one ODE being driven by the second PDE in (9). The ODE with state $u(0,\cdot)$ plays a central role and it has to be made exponentially stable by feedback, which is applied through the transport equation $\zeta_t(x,t) = \zeta_x(x,t)$ controlled at the boundary $x = 1$. Another transport phenomenon $\omega_t(x,t) = -\omega_x(x,t)$ is also present, in the opposite direction, accounting for the reflection of the wave at $x = 0$. Next, a backstepping transformation is introduced as

$$\begin{cases} (\omega, v)^\top = \mathscr{P}_3(\omega, \zeta)^\top : \\ \omega(x,t) = \omega(x,t) \\ v(x,t) = \zeta(x,t) + c\big[u(0,t) + \int_0^x \zeta(y,t) dy\big] \end{cases} \tag{11}$$

with its inverse transformation

$$\begin{cases} (\omega, \zeta)^\top = \mathscr{P}_3^{-1}(\omega, v)^\top : \\ \omega(x,t) = \omega(x,t) \\ \zeta(x,t) = v(x,t) - c\big[e^{-cx}u(0,t) + \int_0^x e^{-c(x-y)} v(y,t) dy\big] \end{cases} \tag{12}$$

to transform (9) into the following target system:

$$\begin{cases} u_t(0,t) = -cu(0,t) + v(0,t), \\ \omega_t(x,t) = -\omega_x(x,t), \\ \omega(0,t) = v(0,t) - cu(0,t), \\ v_t(x,t) = v_x(x,t), \\ v(1,t) = 0 \end{cases} \tag{13}$$

where $c > 0$ is a design constant, from the boundary condition of v at $x = 1$, we can get

$$W_1(t) = -cu(0,t) - c\int_0^1 \zeta(y,t) dy. \tag{14}$$

From transformations (2), (7), and (11), which transform (1) into (13), we have the following composition from $(u, u_t)\top \longrightarrow (\omega, v)\top$:

$$\begin{cases} \omega(x,t) = u_t(x,t) - u_x(x,t) + qu(x,t) - qe^{-qx}u(0,t) \\ \quad - \int_0^x qe^{-q(x-y)}u_t(y,t)dy - q^2\int_0^x e^{-q(x-y)}u(y,t)dy \\ v(x,t) = (q+c)u(x,t) + u_t(x,t) + u_x(x,t) + cu(0,t) \\ \quad -(q+c)e^{qx}u(0,t) + (q+c)\int_0^x e^{q(x-y)}u_t(y,t)dy \\ \quad +q(q+c)\int_0^x e^{q(x-y)}u(y,t)dy. \end{cases} \tag{15}$$

Remark 1: The target system (13) can also be interpreted as a wave-ODE cascade system described by

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), \\ w_x(1,t) = -w_t(1,t), \\ w_x(0,t) = \frac{c}{2}X(t), \\ \dot{X}(t) = -\frac{c}{2}X(t) + w_t(0,t). \end{cases} \tag{16}$$

By (10) and (14), we get the controller for (6):

$$W(t) = -cu(0,t) - c\int_0^1 \zeta(y,t)dy - q\int_0^1 e^{q(1-y)}\xi(y,t)dy. \tag{17}$$

Thus, by (2) and (4), the control law for (1) is found:

$$\begin{aligned} U(t) = & -u_t(1,t) - (c+q)u(1,t) + (qe^q + ce^q - c)u(0,t) \\ & - q(q+c)\int_0^1 e^{q(1-x)}u(x,t)dx \\ & - (q+c)\int_0^1 e^{q(1-x)}u_t(x,t)dx. \end{aligned} \tag{18}$$

Then, we immediately have the following lemma.

Lemma 1: Suppose that $q \in \mathbb{R}$ is a constant. Then, the closed-loop of system (1) under the state feedback control (18) is equivalent to the target system (13).

Proof: Observing the above steps in the design of controller (18), it is sufficient to prove how to find the kernel functions of the backstepping transformation $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$ and its inverse $\mathscr{P}_1^{-1}, \mathscr{P}_2^{-1}, \mathscr{P}_3^{-1}$, respectively. First, we give the proof of (7) and (8). Actually, we suppose that the transformation from $(\xi, \eta) \to (\zeta, \omega)$ is in the form of

$$\zeta(x,t) = \xi(x,t) - \int_0^x K^{uu}(x,y)\xi(y,t)dy \tag{19}$$

$$\omega(x,t) = \eta(x,t) - \int_0^x K^{vv}(x,y)\eta(y,t)dy \tag{20}$$

where K^{uu} and $K^{vv}(x,y)$ are the kernels to be chosen. Differentiate (19) and (20) with respect to t and x, respectively, equating the two expressions, substituting the PDEs for ξ and η, and integrating by parts with respect to y, we obtain conditions that the kernels $K^{uu}(x,y)$ and $K^{vv}(x,y)$ need to satisfy in order for the (ξ,η)-system and the (ζ,ω)-system to be equivalent. The kernels $K^{uu}(x,y)$ and $K^{vv}(x,y)$, respectively, satisfy the following equations:

$$\begin{cases} K_x^{uu}(x,y) + K_y^{uu}(x,y) = 0, \\ K^{uu}(x,0) - q\int_0^x K^{uu}(x,y)dy + q = 0 \end{cases} \tag{21}$$

and

$$\begin{cases} K_y^{vv}(x,y) + K_x^{vv}(x,y) = 0, \\ q - K^{vv}(x,0) - q\int_0^x K^{vv}(x,y)dy = 0. \end{cases} \tag{22}$$

Solving (21) and (22), we obtain

$$\begin{aligned} K^{uu}(x,y) = & -qe^{q(x-y)}, \\ K^{vv}(x,y) = & qe^{-q(x-y)}. \end{aligned} \tag{23}$$

Thus, (7) is verified.

We can use the same method to find the kernel functions of $\mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_1^{-1}, \mathscr{P}_2^{-1}$, and $\mathscr{P}_3^{-1}$, respectively. The detail of the proof is omitted. ∎

Now, we will study the properties of the target system (13). Let $\mathscr{H} = \mathbb{R} \times L^2(0,1) \times L^2(0,1)$ be the energy Hilbert space for system (13) with the inner product

$$\begin{aligned} \langle (h_1, f_1, g_1)\top, (h_2, f_2, g_2)^\top \rangle_{\mathscr{H}} = & \ ch_1h_2 + \int_0^1 f_1(x)f_2(x)dx \\ & + \int_0^1 g_1(x)g_2(x)dx. \end{aligned} \tag{24}$$

Define a system operator $\mathscr{A} : D(\mathscr{A}) \longrightarrow \mathscr{H}$ for (13) as follows:

$$\begin{cases} \mathscr{A}(h,f,g)^\top = (-ch + g(0), -f', g')^\top, \\ D(\mathscr{A}) = \{(h,f,g)^\top \in \mathbb{R} \times H^1(0,1) \times H^1(0,1) | \\ \qquad f(0) = g(0) - ch, g(1) = 0\}. \end{cases} \tag{25}$$

Then, (13) can be written as an evolutionary equation in $\mathscr{H}$:

$$\frac{d}{dt}(u(0,t), \omega(\cdot,t), v(\cdot,t))^\top = \mathscr{A}(u(0,t), \omega(\cdot,t), v(\cdot,t))^\top. \tag{26}$$

For system (26), we have the following lemma.

Lemma 2: Let $\mathscr{A}$ be defined by (25), then

1) $\mathscr{A}$ generates a C_0-semigroup of contractions $e^{\mathscr{A}t}$ on $\mathscr{H}$. For each initial value $(u(0,0), \omega(\cdot,0), v(\cdot,0))^\top \in \mathscr{H}$, there exists a unique solution (weak) $(u(0,t), \omega(\cdot,t), v(\cdot,t))^\top = e^{\mathscr{A}t}(u(0,0), \omega(\cdot,0), v(\cdot,0))^\top \in C((0,\infty); \mathscr{H})$ to (13). Moreover, for each initial value $(u(0,0), \omega(\cdot,0), v(\cdot,0))^\top \in D(\mathscr{A})$, there exists a unique classical solution $(u(0,t), \omega(\cdot,t), v(\cdot,t))^\top \in C((0,\infty); D(\mathscr{A}))$ to (13).
2) $e^{\mathscr{A}t}$ is exponentially stable, that is, for any initial value $(u(0,0), \omega(\cdot,0), v(\cdot,0))^\top \in \mathscr{H}$, there exist positive constants M_1, β independent of initial value such that

$$\begin{aligned} & \|e^{\mathscr{A}t}(u(0,0), \omega(\cdot,0), v(\cdot,0))^\top\|_{\mathscr{H}} \\ & \quad \le M_1e^{-\beta t}\|(u(0,0), \omega(\cdot,0), v(\cdot,0))^\top\|_{\mathscr{H}} \quad \forall t \ge 0. \end{aligned} \tag{27}$$

Proof: By definition, a simple computation shows that

$$\begin{aligned} & \mathrm{Re}\left\langle \mathscr{A}(h,f,g)^\top, (h,f,g)^\top \right\rangle_{\mathcal{H}} = \left\langle \mathscr{A}(h,f,g)^\top, (h,f,g)^\top \right\rangle_{\mathcal{H}} \\ = & \int_0^1 -f'(x)f(x) + g'(x)g(x)dx + ch(-ch + g(0)) \\ = & -\frac{1}{2}f^2(1) + \frac{1}{2}f^2(0) + \frac{1}{2}g^2(1) - \frac{1}{2}g^2(0) - c^2h^2 + chg(0) \\ = & -\frac{1}{2}f^2(1) - \frac{1}{2}c^2h^2 \le 0 \quad \forall (h,f,g)^\top \in D(\mathscr{A}). \end{aligned}$$

∎

So, $\mathscr{A}$ is dissipative. Now, we show $\mathscr{A}^{-1}$ exists. For any given $(\theta, \phi, \varphi)^\top \in \mathscr{H}$, solving $\mathscr{A}(h,f,g)^\top = (\theta, \phi, \varphi)^\top$ gives

$$\begin{cases} h = -\frac{1}{c}\left[\int_0^1 \varphi(x)dx + \theta\right], \\ f(x) = \theta - \int_0^x \phi(y)dy, \\ g(x) = \int_0^x \varphi(y)dy - \int_0^1 \varphi(x)dx. \end{cases} \tag{28}$$

From (28), we get $f(0) = \theta, g(0) = -\int_0^1 \varphi(x)dx, g(0) - ch = -\int_0^1 \varphi(x)dx + \int_0^1 \varphi(x)dx + \theta = \theta, g(1) = 0$. Hence, $(h,f,g) \in D(\mathscr{A})$ and $\mathscr{A}^{-1}$ exists. By the Sobolev embedding theorem, $\mathscr{A}^{-1}$ is compact on $\mathscr{H}$. Moreover, by the Lumer–Phillips theorem (see [24]), $\mathscr{A}$ generates a C_0-semigroup of contractions $e^{\mathscr{A}t}$ on $\mathscr{H}$. Assertion 1) is then concluded.

Now, we are in a position to prove assertion 2). By the density argument and the fact that $D(\mathscr{A})$ is dense in $\mathscr{H}$, we may assume without loss of generality that initial value $(u(0,0),\omega(\cdot,0),v(\cdot,0))^\top \in D(\mathscr{A})$. The energy of the system (13) is given as

$$E(t) = \int_0^1 v^2(x,t)dx + \int_0^1 \omega^2(x,t)dx + cu^2(0,t).$$

Define the Lyapunov–Krasovskii functional candidate

$$V(t) = \frac{1}{2}cu^2(0,t) + \frac{1}{2}\int_0^1 e^{ax}v^2(x,t)dx + \frac{1}{2}\int_0^1 e^{-bx}\omega^2(x,t)dx$$

where $a, b > 0$ are the analysis parameters. Using the Cauchy–Schwarz and Young's inequalities, one can show that

$$m_1 E(t) \le V(t) \le m_2 E(t)$$

where $m_1 = \min\{\frac{1}{2}, \frac{e^{-b}}{2}\}$ and $m_2 = \max\{\frac{1}{2}, \frac{e^{a}}{2}\}$. Taking a time derivative along the solution to (13), we get

$$\begin{aligned}\dot V(t) =& \; cu_t(0,t)u(0,t) + \int_0^1 e^{ax}v_t(x,t)v(x,t)dx \\ &+ \int_0^1 e^{-bx}\omega_t(x,t)\omega(x,t)dx \\ =& -\frac{c^2-c\delta}{2}u^2(0,t) - \frac{c\delta}{2}u^2(0,t) - \frac{a}{2}\int_0^1 e^{ax}v^2(x,t)dx \\ &-\frac{e^{-b}}{2}\omega^2(1,t) - \frac{b}{2}\int_0^1 e^{-bx}\omega^2(x,t)dx \\ \le& -mE(t) \le -\frac{m}{m_2}V(t)\end{aligned}$$

where $0 < \delta < c$ is an analysis parameter and $m = \min\{\delta, a, b\}$. It shows that system (13) is exponentially stable, which proves that $\mathscr{A}$ generates an exponentially stable C_0-semigroup. From Lemma 1, the closed-loop system of (1) under the control law (18) is governed by

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + qu_t(0,t), \\ u_x(0,t) = 0, \\ u_x(1,t) = -u_t(1,t) - (c+q)u(1,t) \\ \qquad + (qe^q - ce^q + c)u(0,t) \\ \qquad - q(q+c)\int_0^1 e^{q(1-x)}u(x,t)dx \\ \qquad - (q+c)\int_0^1 e^{q(1-x)}u_t(x,t)dx. \end{cases} \tag{29}$$

We consider system (29) in the energy Hilbert space $\mathbb{H} = H^1(0,1) \times L^2(0,1)$ with inner product induced norm :

$$\|(f,g)\|^2_{\mathbb{H}} = \int_0^1 [f'(x)^2 + g(x)^2]dx + cf(1)^2.$$

Define operator $\mathbf{A} : D(\mathbf{A}) \subset \mathbb{H} \longrightarrow \mathbb{H}$ by

$$\begin{cases} \mathbf{A}(f,g)^\top = (g, f'' + qg(0))^\top, \\ D(\mathbf{A}) = \{(f,g)^\top \in H^2(0,1)\times H^1(0,1)| \\ \quad f'(0) = 0, f'(1) = -g(1) - (c+q)f(1) \\ \quad + (qe^q + ce^q - c)f(0) - q(q+c)\int_0^1 e^{q(1-x)}f(x)dx \\ \quad - (q+c)\int_0^1 e^{q(1-x)}g(x)dx\}. \end{cases} \tag{30}$$

Then, (29) can be written as an evolutionary equation in $\mathbb{H}$:

$$\frac{d}{dt}(u(\cdot,t),u_t(\cdot,t))^T = \mathbf{A}(u(\cdot,t),u_t(\cdot,t))^T. \tag{31}$$

For system (31), we have the following result.

Theorem 1: Suppose that $q \in \mathbb{R}$, then the closed-loop system (29) is well posed: The operator $\mathbf{A}$ defined by (30) generates a C_0-semigroup $e^{\mathbf{A}t}$ on $\mathbb{H}$. Moreover, $e^{\mathbf{A}t}$ is exponentially stable: There are positive constants $M, \alpha > 0$ such that

$$\|e^{\mathbf{A}t}\| \le Me^{-\alpha t} \quad \forall t \ge 0. \tag{32}$$

Proof: This is the direct consequence of the equivalence between system (29) and system (13) claimed by Lemma 1. From (3), (8), and (12), it follows that $\mathscr{P}_1$, $\mathscr{P}_2$, and $\mathscr{P}_3$ are all inverse. So, there exists an inverse bounded operator $\mathscr{P}$ such that $\mathbf{A} = \mathscr{P}^{-1}\mathscr{A}\mathscr{P}$ and so $\mathbf{A}$ generates a C_0-semigroup on $\mathbb{H}$:

$$e^{\mathbf{A}t} = \mathscr{P}^{-1}e^{\mathscr{A}t}\mathscr{P}.$$

Since $e^{\mathscr{A}t}$ is exponentially stable on $\mathscr{H}$ and $\mathscr{P}$ is bounded, we have (32) for some constants $M, \alpha > 0$. ■

III. Observer Design

In the rest of this paper, we shall focus on output feedback control. The first problem is about the measurement. We assume that $u_t(0,t)$ and $u(1,t)$ are available for measurement. We propose the following observer for system (1):

$$\begin{cases} \widehat{u}_{tt}(x,t) = \widehat{u}_{xx}(x,t) + qu_t(0,t), \\ \widehat{u}_x(0,t) = -k_1[u_t(0,t) - \widehat{u}_t(0,t)], \\ \widehat{u}_x(1,t) = U(t) + k_2[u(1,t) - \widehat{u}(1,t)]. \end{cases} \tag{33}$$

Observer (33) is in the standard form of ¡°copy of the system plus injection of the output estimation error¡±. We will prove that system (33) is an exponential-type state observer of system (1). To this end, we consider the observer error $\varepsilon(x,t) = u(x,t) - \hat u(x,t)$ satisfies the following PDE:

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = k_1\varepsilon_t(0,t), \\ \varepsilon_x(1,t) = -k_2\varepsilon(1,t). \end{cases} \tag{34}$$

We consider system (34) in the same state-space $\mathbb{H}$ with an equivalent inner product:

$$\begin{aligned}\langle (f_1,g_1)^\top, (f_2,g_2)^\top\rangle_{\mathbb{H}} =& \int_0^1 f_1'(x)f_2'(x)dx + \int_0^1 g_1(x)g_2(x)dx \\ &+ k_2f_1(1)f_2(1) \quad \forall (f_i,g_i) \in \mathbb{H}, i = 1,2.\end{aligned}$$

Define $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{H} \to \mathbb{H}$

$$\begin{cases} \mathbb{A}(f,g)^\top = (g, f'')^\top \quad \forall (f,g) \in D(\mathbb{A}), \\ D(\mathbb{A}) = \Big\{(f,g)^\top \in H^2(0,1)\times H^1(0,1)| \\ \qquad f'(0) = k_1g(0), f'(1) = -k_2f(1)\Big\}. \end{cases} \tag{35}$$

Then, system (34) can be rewritten as an evolution equation in $\mathbb{H}$:

$$\frac{d}{dt}(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))^\top = \mathbb{A}(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))^\top.$$

It is known that $\mathbb{A}$ generates an exponentially stable C_0-semigroup for $\mathbb{H}$.

Theorem 2: For any initial value $(\varepsilon(\cdot,0),\varepsilon_t(\cdot,0))^\top \in \mathbb{H}$, system (34) admits a unique solution $(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))^\top \in C((0,\infty);\mathbb{H})$, which satisfies

$$E_\varepsilon(t) \le M_2E_\varepsilon(0)e^{-\alpha_2 t} \quad \forall t \ge 0$$

where α_2 and M_2 are two positive constants independent of initial values and $E_\varepsilon(t)$ is the energy functional of system (34) defined by

$$E_\varepsilon(t) = \frac{1}{2}\int_0^1 \left(\varepsilon_t^2(x,t) + \varepsilon_x^2(x,t)\right)dx + \frac{1}{2}k_2\varepsilon^2(1,t).$$

Moreover

$$\int_{t_1}^{t_2} |\varepsilon_s(0,s)|^2 ds \le \frac{M_2E_\varepsilon(0)}{k_1}(e^{-\alpha_2 t_1} + e^{-\alpha_2 t_2}) \quad \forall t_2 > t_1 > 0 \tag{36}$$

and especially

$$\varepsilon_t(0,t) \in L^2(0,\infty). \tag{37}$$

Proof: We only give the proof of (36) and (37). A simple computation of derivative of $E_\varepsilon(t)$ with respect to t along the solution to (34) shows that $\dot{E}_\varepsilon(t) = -k_1\varepsilon_t^2(0,t)$. Then, for any $t_1, t_2 > 0$, we have

$$\int_{t_1}^{t_2} |\varepsilon_s(0,s)|^2 ds \le \frac{M_2 E_\varepsilon(0)}{k_1}(e^{-\alpha_2 t_1} + e^{-\alpha_2 t_2}).$$

Especially, when $t_1 = 0$, $\varepsilon_t(0,t) \in L^2(0,\infty)$. ■

IV. Stability Under the Estimated State Feedback

Since by Theorem 1, the state feedback (18) stabilizes exponentially system (1), and we have the estimation $\widehat{u}$ for the state u from the observation of the original system that has been guaranteed by Theorem 2, it is naturally to design the following estimated state feedback:

$$\begin{aligned} U(t) =& -\widehat{u}_t(1,t) - (c+q)u(1,t) + (qe^q + ce^q - c)\widehat{u}(0,t) \\ & - q(q+c)\int_0^1 e^{q(1-x)}\widehat{u}(x,t)dx \\ & - (q+c)\int_0^1 e^{q(1-x)}\widehat{u}_t(x,t)dx. \end{aligned} \tag{38}$$

Combining (1), (33), and (38), the resulting closed loop is governed by

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + qu_t(0,t), \\ u_x(0,t) = 0, \\ u_x(1,t) = -\widehat{u}_t(1,t) - (c+q)u(1,t) \\ \qquad + (qe^q + ce^q - c)\widehat{u}(0,t) \\ \qquad - q(q+c)\int_0^1 e^{q(1-x)}\widehat{u}(x,t)dx \\ \qquad - (q+c)\int_0^1 e^{q(1-x)}\widehat{u}_t(x,t)dx, \\ \widehat{u}_{tt}(x,t) = \widehat{u}_{xx}(x,t) + qu_t(0,t), \\ \widehat{u}_x(0,t) = -k_1(u_t(0,t) - \widehat{u}_t(0,t)), \\ \widehat{u}_x(1,t) = -\widehat{u}_t(1,t) - (c+q)u(1,t) \\ \qquad + (qe^q + ce^q - c)\widehat{u}(0,t) \\ \qquad - q(q+c)\int_0^1 e^{q(1-x)}\widehat{u}(x,t)dx \\ \qquad - (q+c)\int_0^1 e^{q(1-x)}\widehat{u}_t(x,t)dx \\ \qquad + k_2(u(1,t) - \widehat{u}(1,t)). \end{cases} \tag{39}$$

We consider system (39) in the state-space $\mathbb{H}\times\mathbb{H}$.

Theorem 3: For each initial value $(u(\cdot,0), u_t(\cdot,0), \widehat{u}(\cdot,0), \widehat{u}_t(\cdot,0))^\top \in \mathbb{H}\times\mathbb{H}$, there exists a unique solution (weak) $(u(\cdot,t), u_t(\cdot,t), \widehat{u}(\cdot,t), \widehat{u}_t(\cdot,t))^\top \in C((0,\infty);\mathbb{H}\times\mathbb{H})$ to (39). Moreover, this closed-loop solution is exponentially stable. That is, there exist positive constants L, μ such that

$$E_c(t) \le Le^{-\mu t}E_c(0) \quad \forall t \ge 0$$

where $E_c(t) = \frac{1}{2}\int_0^1 \left[u_x^2(x,t) + u_x^2(x,t) + \widehat{u}_t^2(x,t) + \widehat{u}_x(x,t)\right]dx + \frac{1}{2}[u(1,t) + \widehat{u}(1,t)]^2$.

Proof: Since $\varepsilon(x,t) = u(x,t) - \widehat{u}(x,t)$, system (39) is equivalent to the following system:

$$\begin{cases} \varepsilon_{tt} = \varepsilon_{xx}, \\ \varepsilon_x(0,t) = k_1\varepsilon_t(0,t), \\ \varepsilon_x(1,t) = -k_2\varepsilon(1,t), \\ \widehat{u}_{tt} = \widehat{u}_{xx} + q\widehat{u}_t(0,t) + q\varepsilon_t(0,t), \\ \widehat{u}_x(0,t) = -k_1\varepsilon_t(0,t), \\ \widehat{u}_x(1,t) = -\widehat{u}_t(1,t) - (c+q)u(1,t) \\ \qquad + (qe^q + ce^q - c)\widehat{u}(0,t) \\ \qquad - q(q+c)\int_0^1 e^{q(1-x)}\widehat{u}(x,t)dx \\ \qquad - (q+c)\int_0^1 e^{q(1-x)}\widehat{u}_t(x,t)dx + k_2\varepsilon(1,t). \end{cases} \tag{40}$$

We only need to prove that for each initial value $(\varepsilon(\cdot,0), \varepsilon_t(\cdot,0), \widehat{u}(\cdot,0), \widehat{u}_t(\cdot,0)^\top \in \mathbb{H}\times\mathbb{H}$, system (40) has a unique solution $(\varepsilon(\cdot,t), \varepsilon_t(\cdot,t), \widehat{u}(\cdot,t), \widehat{u}_t(\cdot,t))^\top \in C((0,\infty);\mathbb{H}\times\mathbb{H})$ and is exponentially stable. By Theorem 2, the "ε" part of system (40) has a unique solution $(\varepsilon(\cdot,t), \varepsilon_t(\cdot,t)) \in C((0,\infty);\mathbb{H})$ and is exponentially stable. Next, it is sufficient to prove the "$\widehat{u}$" part of system (40) has a unique solution $(\widehat{u}(\cdot,t), \widehat{u}_t(\cdot,t))^\top \in C((0,\infty);\mathbb{H})$ and is exponentially stable. We rewrite it as follows:

$$\begin{cases} \widehat{u}_{tt}(x,t) = \widehat{u}_{xx}(x,t) + q\widehat{u}_t(0,t) + q\varepsilon_t(0,t), \\ \widehat{u}_x(0,t) = -k_1\varepsilon_t(0,t), \\ \widehat{u}_x(1,t) = -\widehat{u}_t(1,t) - (c+q)\widehat{u}(1,t) \\ \qquad + (qe^q + ce^q - c)\widehat{u}(0,t) \\ \qquad - q(q+c)\int_0^1 e^{q(1-x)}\widehat{u}(x,t)dx \\ \qquad - (q+c)\int_0^1 e^{q(1-x)}\widehat{u}_t(x,t)dx + (k_2 - c - q)\varepsilon(1,t). \end{cases} \tag{41}$$

By using the mapping derived from the transformations (2), (7), and (11), system (41) can be transformed into the following equivalent system:

$$\begin{cases} \hat{u}_t(0,t) = \hat{v}(0,t) - c\hat{u}(0,t) + k_1\varepsilon_t(0,t), \\ \hat{\omega}_t(x,t) = -\hat{\omega}_x(x,t) + (q - qk_1)e^{-qx}\varepsilon_t(0,t), \\ \hat{\omega}(0,t) = \hat{v}(0,t) - c\hat{u}(0,t) + 2k_1\varepsilon_t(0,t), \\ \hat{v}_t(x,t) = \hat{v}_x(x,t) + (q+c)(1+k_1)e^{qx}\varepsilon_t(0,t) \\ \qquad - c\varepsilon_t(0,t), \\ \hat{v}(1,t) = (k_2 - c - q)\varepsilon(1,t). \end{cases} \tag{42}$$

The proof will be completed if we can prove that (42) has a unique solution $(\hat{u}(0,t), \hat{\omega}(\cdot,t), \hat{v}(\cdot,t))^\top \in C((0,\infty);\mathscr{H})$ and is exponentially stable. A direct computation shows that the adjoint operator of $\mathscr{A}$ defined by (25) yields

$$\begin{cases} \mathscr{A}^*(u,w,v)^\top = (-cu - w(0), w', -v')^\top, \\ D(\mathscr{A}^*) = \{(u,w,v)^\top \in \mathbb{R}\times H^1(0,1)\times H^1(0,1) | \\ v(0) = w(0) + cu, w(1) = 0\}. \end{cases} \tag{43}$$

■

Take the inner product on both sides of (42) with $(\varphi,\phi,\psi)^\top \in D(\mathscr{A}^*)$ to get

$$\begin{aligned} &\frac{d}{dt}\left\langle \begin{pmatrix} \hat{u}(0,t) \\ \hat{\omega}(\cdot,t) \\ \hat{v}(\cdot,t) \end{pmatrix}, \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix} \right\rangle_{\mathscr{H}} \\ &= \left\langle \begin{pmatrix} \hat{u}(0,t) \\ \hat{\omega}(\cdot,t) \\ \hat{v}(\cdot,t) \end{pmatrix}, \mathscr{A}^* \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix} \right\rangle_{\mathscr{H}} \\ &+ \left\langle \begin{pmatrix} k_1\varepsilon_t(0,t) \\ (q - qk_1)e^{-qx}\varepsilon_t(0,t) \\ (q+c)(1+k_1)e^{qx}\varepsilon_t(0,t) - c\varepsilon_t(0,t) \end{pmatrix}, \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix} \right\rangle_{\mathscr{H}} \\ &+ \left\langle \begin{pmatrix} 0 \\ \delta(x) \\ 0 \end{pmatrix} 2k_1\varepsilon_t(0,t) \right. \\ &+ \left. \begin{pmatrix} 0 \\ 0 \\ \delta(x-1) \end{pmatrix} (k_2 - c - q)\varepsilon(1,t), \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix} \right\rangle_{[D(\mathscr{A}^*)]'\times D(\mathscr{A}^*)} \end{aligned} \tag{44}$$

where $[D(\mathscr{A}^*)]'$ is the dual of $D(\mathscr{A}^*)$ with the pivot space $\mathscr{H}$. Then, (42) can be written as

$$\begin{aligned} &\frac{d}{dt}(\hat{u}(0,t), \hat{\omega}(\cdot,t), \hat{v}(\cdot,t))^\top = \mathscr{A}(\hat{u}(0,t), \hat{\omega}(\cdot,t), \hat{v}(\cdot,t))^\top \\ &\quad + (k_1\varepsilon_t(0,t), (q - qk_1)e^{-qx}\varepsilon_t(0,t), \\ &\qquad (q+c)(1+k_1)e^{qx}\varepsilon_t(0,t) - c\varepsilon_t(0,t))^\top \\ &\quad + \mathscr{B}_1 2k_1\varepsilon_t(0,t) + \mathscr{B}_2(k_2 - c - q)\varepsilon(1,t) \end{aligned} \tag{45}$$

where $\mathscr{B}_1 = (0,\delta(x),0)^\top$and $\mathscr{B}_2 = (0,0,\delta(x-1))^\top$ are two unbounded operators defined from $\mathbb{R}$ to $[D(\mathscr{A})^*]'$ and $\delta(\cdot)$ is the Dirac distribution. Since $\mathscr{B}_1$ and $\mathscr{B}_2$ are two unbounded operator, we will show that they are both admissible to $\mathscr{A}$.

Lemma 3: $\mathscr{B}_1$ and $\mathscr{B}_2$ are both admissible for $e^{\mathscr{A}t}$.

Proof: Consider the dual system of (42):

$$\begin{cases} \hat{u}_t^*(0,t) = -\hat{\omega}^*(0,t) - c\hat{u}^*(0,t), \\ \hat{\omega}_t^*(x,t) = \hat{\omega}_x^*(x,t), \\ \hat{\omega}^*(1,t) = 0, \\ \hat{v}_t^*(x,t) = -\hat{v}_x^*(x,t), \\ \hat{v}^*(0,t) = \hat{\omega}^*(0,t) + c\hat{u}^*(0,t), \\ y_1(t) = \hat{\omega}^*(0,t), \\ y_2(t) = \hat{v}^*(1,t). \end{cases} \tag{46}$$

By Theorem 1, $\mathscr{A}$ generates an exponentially stable C_0-semigroup, and so does for $\mathscr{A}^*$. The energy function of system (46) is defined by

$$E^*(t) = \frac{1}{2}\int_0^1 v^{*2}(x,t)dx + \frac{1}{2}\int_0^1 \omega^{*2}(x,t)dx + \frac{1}{2}cu^{*2}(0,t). \tag{47}$$

A simple computation for the derivative of $E^*(t)$ with respect to t along the solution to (46) gives

$$\dot{E}^*(t) = -\frac{1}{2}v^{*2}(1,t) - \frac{1}{2}c^2u^{*2}(0,t). \tag{48}$$

Hence, $\dot{E}^*(t) \le 0, E^*(t) \le E^*(0)$. Let

$$\rho_1(t) = \int_0^1 -xv^{*2}(x,t) + (x-1)\omega^{*2}(x,t))dx \tag{49}$$

$$\rho_2(t) = \int_0^1 -xv^{*2}(x,t) + x\omega^{*2}(x,t)dx. \tag{50}$$

Differentiate $\rho_1(t)$ and $\rho_2(t)$ with respect to t, respectively, to yield

$$\begin{aligned} \dot{\rho_1}(t) &= \int_0^1 -xv^*(x,t)v_t^*(x,t) + (x-1)\omega^*(x,t)\omega_t^*(x,t)dx \\ &= \frac{1}{2}v^{*2}(1,t) + \frac{1}{2}\omega^{*2}(0,t) + \frac{1}{2}cu^{*2}(0,t) - E(t). \end{aligned} \tag{51}$$

$$\begin{aligned} \dot{\rho_2}(t) &= \int_0^1 -xv^*(x,t)v_t^*(x,t) + x\omega^*(x,t)\omega_t^*(x,t)dx \\ &= \frac{1}{2}v^{*2}(1,t) + \frac{1}{2}\omega^{*2}(1,t) + \frac{1}{2}cu^{*2}(0,t) - E(t). \end{aligned} \tag{52}$$

We thus have

$$\int_0^T y_1(t)^2dt = \int_0^T \hat{\omega}^*(0,t)^2dt \le 2(T+2)E(0). \tag{53}$$

$$\int_0^T y_2(t)^2dt = \int_0^T \hat{v}^*(1,t)^2dt \le 2(T+2)E(0). \tag{54}$$

On the other hand, for any given $(\theta,\phi,\varphi)\top \in \mathscr{H}$, we have

$$\mathscr{A}^{*-1}\begin{pmatrix}\theta\\ \phi(x)\\ \varphi(x)\end{pmatrix} = \begin{pmatrix} \frac{1}{c}\left(\int_0^1 \phi(y)dy - \theta\right) \\ \int_0^x \phi(y)dy - \int_0^1 \phi(y)dy \\ -\theta - \int_0^x \varphi(y)dy \end{pmatrix},$$

$$\mathscr{B}_1^*\mathscr{A}^{*-1}\begin{pmatrix}\theta\\ \phi(x)\\ \varphi(x)\end{pmatrix} = -\int_0^1 \phi(y)dy, \tag{55}$$

$$\mathscr{B}_2^*\mathscr{A}^{*-1}\begin{pmatrix}\theta\\ \phi(x)\\ \varphi(x)\end{pmatrix} = -\theta - \int_0^1 \varphi(y)dy. \tag{56}$$

So, $\mathscr{B}_1^*\mathscr{A}^{*-1}$ and $\mathscr{B}_2^*\mathscr{A}^{*-1}$ are bounded from $\mathscr{H}$ to $\mathbb{R}$. This together with (53) and (54) show that $\mathscr{B}_1$ and $\mathscr{B}_2$ are both admissible for $e^{\mathscr{A}^*t}$, so are for $e^{\mathscr{A}t}$ (see [25]).

Based on Lemma 3, there exists a unique weak solution in $\mathcal{H}$ for the evolution equation (45). That is to say, for any initial value $(\hat{u}(0,0),\hat{\omega}(\cdot,0),\hat{v}(\cdot,0))^\top \in \mathcal{H}$, there exists a unique weak solution $(\hat{u}(0,t),\hat{\omega}(\cdot,t),\hat{v}(\cdot,t))^\top \in \mathcal{H}$ to system (45). Define $F(x) = (k_1,(q-qk_1)e^{-qx},(q+c)(1+k_1)e^{qx}-c)^\top$. The solution of (45) can be written as

$$\begin{aligned} \begin{pmatrix}\hat{u}(0,t)\\ \hat{\omega}(\cdot,t)\\ \hat{v}(\cdot,t)\end{pmatrix} &= e^{\mathscr{A}t}\begin{pmatrix}\hat{u}(0,0)\\ \hat{\omega}(\cdot,0)\\ \hat{v}(\cdot,0)\end{pmatrix} \\ &\quad + \int_0^t e^{\mathscr{A}(t-s)}F(x)\varepsilon_s(0,s)ds \\ &\quad + \int_0^t e^{\mathscr{A}(t-s)}2k_1\mathscr{B}_1\varepsilon_s(0,s)ds \\ &\quad + \int_0^t e^{\mathscr{A}(t-s)}(k_2-c-q)\mathscr{B}_2\varepsilon(1,s)ds. \end{aligned} \tag{57}$$

Now, we are in a position to prove that system (45) is exponentially stable. From (27), we have

$$\begin{aligned} \left\| e^{At}\begin{pmatrix}\hat{u}(0,0)\\ \hat{\omega}(\cdot,0)\\ \hat{v}(\cdot,0)\end{pmatrix}\right\|_{\mathscr{H}} &\le \|e^{At}\| \left\|\begin{pmatrix}\hat{u}(0,0)\\ \hat{\omega}(\cdot,0)\\ \hat{v}(\cdot,0)\end{pmatrix}\right\|_{\mathscr{H}} \\ &\le M_1e^{-\beta t}\left\|\begin{pmatrix}\hat{u}(0,0)\\ \hat{\omega}(\cdot,0)\\ \hat{v}(\cdot,0)\end{pmatrix}\right\|_{\mathscr{H}}. \end{aligned} \tag{58}$$

We also have

$$\left\|\int_0^t e^{\mathscr{A}(t-s)}F(x)\varepsilon_s(0,s)ds\right\|_{\mathscr{H}} \le \int_0^t \|e^{\mathscr{A}(t-s)}\|\|F(x)\|_{\mathscr{H}}|\varepsilon_s(0,s)|ds.$$

On the other hand, by Theorem 2, we have

$$\begin{aligned} &\int_0^t e^{-\beta(t-s)}|\varepsilon_s(0,s)|ds \\ &= \int_0^{\frac{t}{2}} e^{-\beta(t-s)}|\varepsilon_s(0,s)|ds + \int_{\frac{t}{2}}^t e^{-\beta(t-s)}|\varepsilon_s(0,s)|ds \\ &= \int_{\frac{t}{2}}^t e^{-\beta\tau}|\varepsilon_\tau(0,t-\tau)|d\tau + \int_{\frac{t}{2}}^t e^{-\beta(t-s)}|\varepsilon_s(0,s)|ds \\ &\le \left[\int_{\frac{t}{2}}^t e^{-2\beta\tau}d\tau\right]^{\frac{1}{2}}\left[\int_{\frac{t}{2}}^t |\varepsilon_\tau(0,t-\tau)|^2d\tau\right]^{\frac{1}{2}} \\ &\quad + \left[\int_{\frac{t}{2}}^t e^{-2\beta(t-s)}ds\right]^{\frac{1}{2}}\left[\int_{\frac{t}{2}}^t |\varepsilon_s(0,s)|^2ds\right]^{\frac{1}{2}} \\ &\le \frac{1}{2\beta}e^{-2\beta t}\left[\int_0^\infty |\varepsilon_\tau(0,\tau)|^2d\tau\right]^{\frac{1}{2}} + \frac{1}{2\beta}\frac{\sqrt{2M_2E_\varepsilon(0)}}{\sqrt{k_1}}e^{\frac{-\alpha_2 t}{4}}. \end{aligned}$$

This together with the fact $\varepsilon_t(0,t) \in L^2(0,\infty)$ yields

$$\begin{aligned} &\left\|\int_0^t e^{\mathscr{A}(t-s)}\begin{pmatrix}k_1\\ (q-qk_1)e^{-qx}\\ (q+c)(1+k_1)e^{qx}-c\end{pmatrix}\varepsilon_s(0,s)ds\right\|_{\mathscr{H}} \\ &\le \|F(x)\|_{\mathscr{H}}\int_0^t e^{-\beta(t-s)}|\varepsilon_s(0,s)|ds \\ &\le \frac{\|F(x)\|_{\mathscr{H}}}{2\beta}\left[\int_0^\infty |\varepsilon_\tau(0,\tau)|^2d\tau\right]^{\frac{1}{2}}e^{-2\beta t} \\ &\quad + \frac{\|F(x)\|_{\mathscr{H}}}{2\beta}\frac{\sqrt{2M_2E_\varepsilon(0)}}{\sqrt{k_1}}e^{\frac{-\alpha_2 t}{4}}. \end{aligned} \tag{59}$$

It follows from Theorem 2 that $\varepsilon^2(1,t) \le \frac{2}{k_2}E_\varepsilon(t) \le \frac{2M_2}{k_2}E_\varepsilon(0)e^{-\alpha_2 t}$. The admissibility of $\mathscr{B}_2$ and [25, Remark 2.6] implies that

$$\begin{aligned}
&\left\|\int_0^{\frac{t}{2}} e^{\mathscr{A}(\frac{t}{2}-s)}\mathscr{B}_2\varepsilon(1,s)ds\right\|_{\mathscr{H}}^2 \le C_t\int_0^{\frac{t}{2}}\varepsilon^2(1,s)ds\\
\le\ & \frac{2M_2}{k_2}C_tE_\varepsilon(0)\int_0^{\frac{t}{2}}e^{-\alpha_2 s}ds \le \frac{2M_2}{k_2\alpha_2}C_tE_\varepsilon(0),\\
&\left\|\int_{\frac{t}{2}}^{t} e^{\mathscr{A}(t-s)}\mathscr{B}_2\varepsilon(1,s)ds\right\|_{\mathscr{H}}^2 \le C_1\int_{\frac{t}{2}}^{t}\varepsilon^2(1,s)ds\\
\le\ & \frac{2M_2}{k_2}C_1E_\varepsilon(0)\int_{\frac{t}{2}}^{t}e^{-\alpha_2 s}ds = \frac{2M_2}{k_2\alpha_2}C_1E_\varepsilon(0)(e^{-\frac{\alpha_2 t}{2}}-e^{-\alpha_2 t})\\
\le\ & \frac{2M_2}{k_2\alpha_2}C_1E_\varepsilon(0)(e^{-\frac{\alpha_2 t}{2}}+e^{-\alpha_2 t})
\end{aligned}$$

where C_t, and C_1 are two constants that are independent of $\varepsilon(1,t)$.

Thus, we have

$$\begin{aligned}
&\left\|\int_0^{t} e^{\mathscr{A}(t-s)}\mathscr{B}_2\varepsilon(1,s)ds\right\|_{\mathscr{H}}^2\\
=\ & \left\|e^{\mathscr{A}(t-\frac{t}{2})}\int_0^{\frac{t}{2}} e^{\mathscr{A}(\frac{t}{2}-s)}\mathscr{B}_2\varepsilon(1,s)ds\right.\\
&\left.+\int_{\frac{t}{2}}^{t} e^{\mathscr{A}(t-s)}\mathscr{B}_2\varepsilon(1,s)ds\right\|_{\mathscr{H}}^2\\
\le\ & 2\left\|e^{\mathscr{A}(t-\frac{t}{2})}\int_0^{\frac{t}{2}} e^{\mathscr{A}(\frac{t}{2}-s)}\mathscr{B}_2\varepsilon(1,s)ds\right\|_{\mathscr{H}}^2\\
&+2\left\|\int_{\frac{t}{2}}^{t} e^{\mathscr{A}(t-s)}\mathscr{B}_2\varepsilon(1,s)ds\right\|_{\mathscr{H}}^2\\
\le\ & 2M_1^2e^{-\beta t}\left\|\int_0^{\frac{t}{2}} e^{\mathscr{A}(\frac{t}{2}-s)}\mathscr{B}_2\varepsilon(1,s)ds\right\|_{\mathscr{H}}^2\\
&+2\left\|\int_{\frac{t}{2}}^{t} e^{\mathscr{A}(t-s)}\mathscr{B}_2\varepsilon(1,s)ds\right\|_{\mathscr{H}}^2\\
\le\ & M_1^2e^{-\beta t}\frac{4M_2}{k_2\alpha_2}C_tE_\varepsilon(0)\\
&+\frac{4M_2}{k_2\alpha_2}C_1E_\varepsilon(0)(e^{-\frac{\alpha_2 t}{2}}+e^{-\alpha_2 t}).
\end{aligned}\tag{60}$$

The admissibility of $\mathscr{B}_1$ implies that

$$\begin{aligned}
&\left\|\int_0^{t} e^{\mathscr{A}(t-s)}\mathscr{B}_1\varepsilon_s(0,s)ds\right\|_{\mathscr{H}}^2\\
=\ & \left\|e^{\mathscr{A}(t-\frac{t}{2})}\int_0^{\frac{t}{2}} e^{\mathscr{A}(\frac{t}{2}-s)}\mathscr{B}_1\varepsilon_s(0,s)ds\right.\\
&\left.+\int_{\frac{t}{2}}^{t} e^{\mathscr{A}(t-s)}\mathscr{B}_1\varepsilon_s(0,s)ds\right\|_{\mathscr{H}}^2\\
\le\ & 2\left\|e^{\mathscr{A}(t-\frac{t}{2})}\int_0^{\frac{t}{2}} e^{\mathscr{A}(\frac{t}{2}-s)}\mathscr{B}_1\varepsilon_s(0,s)ds\right\|_{\mathscr{H}}^2\\
&+2\left\|\int_{\frac{t}{2}}^{t} e^{\mathscr{A}(t-s)}\mathscr{B}_1\varepsilon_s(0,s)ds\right\|_{\mathscr{H}}^2\\
\le\ & 2M_1^2e^{-\beta t}\left\|\int_0^{\frac{t}{2}} e^{\mathscr{A}(\frac{t}{2}-s)}\mathscr{B}_1\varepsilon_s(0,s)ds\right\|_{\mathscr{H}}^2\\
&+2\left\|\int_{\frac{t}{2}}^{t} e^{\mathscr{A}(t-s)}\mathscr{B}_1\varepsilon_s(0,s)ds\right\|_{\mathscr{H}}^2\\
\le\ & 2M_1^2e^{-\beta t}K_t\frac{M_2}{k_1}E_\varepsilon(0)(1+e^{-\frac{\alpha_2 t}{2}})+4K_1\frac{M_2}{k_1}E_\varepsilon(0)e^{-\frac{\alpha_2 t}{2}}
\end{aligned}\tag{61}$$

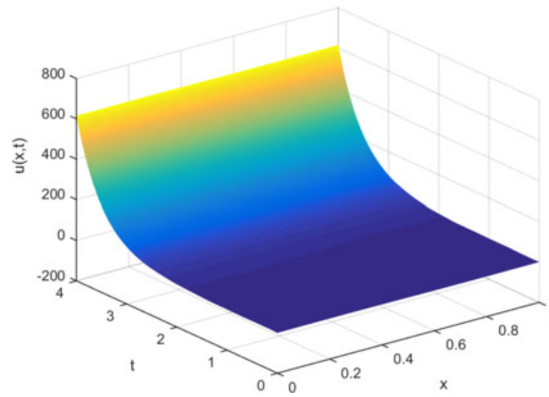

Fig. 2. Displacement $u(x,t)$ of the open-loop system (1).

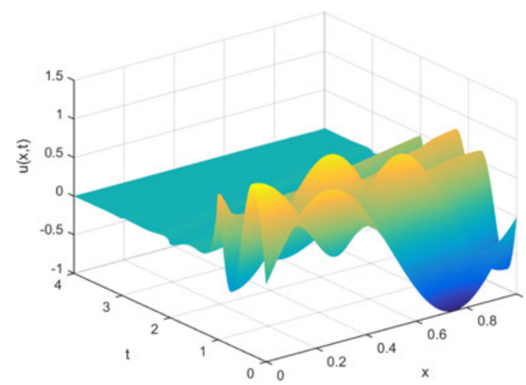

Fig. 3. Displacement $u(x,t)$ of the closed-loop system (29).

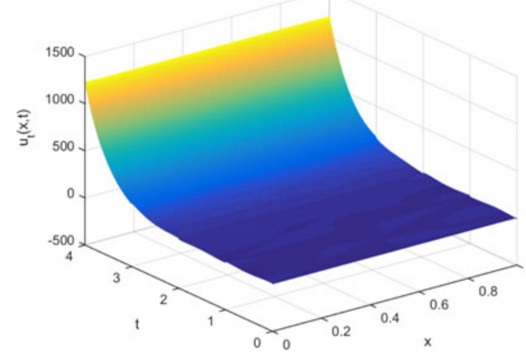

Fig. 4. Velocity $u_t(x,t)$ of the open-loop system (1).

where K_t and K_1 are two constants that are independent of $\varepsilon_t(1,t)$. It follows from (57)–(61) that

$$\left\|\begin{pmatrix}\hat{u}(0,t)\\ \hat{\omega}(\cdot,t)\\ \hat{v}(\cdot,t)\end{pmatrix}\right\|_{\mathscr{H}} \le M_3e^{-\alpha_3 t}, \forall t\ge 0,$$

where M_3 is a positive constant and $\alpha_3 = \min\{\frac{\beta}{4},\frac{\alpha_2}{4}\}$. ■

V. Numerical Simulations

In this section, we give some numerical simulation results for the open loop system (1) and the closed loop system (29). The finite difference method is adopted in both the time and the space domain for both PDEs and the boundary conditions.

The parameter values are set to be $q=1$, and $c=10$. We take the grid size $M=200$ and time step $dt=2\times10^{-3}$ for (1) and (29). The

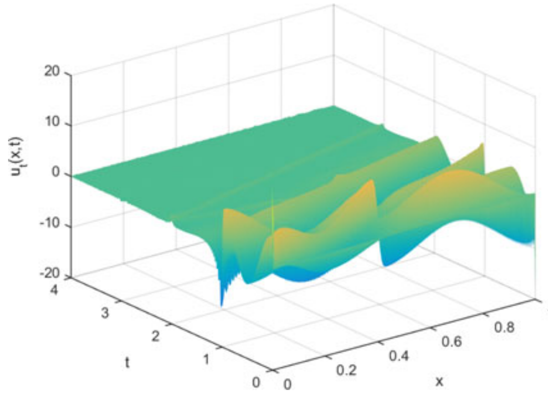

Fig. 5. Velocity $u_t(x,t)$ of the closed-loop system (29).

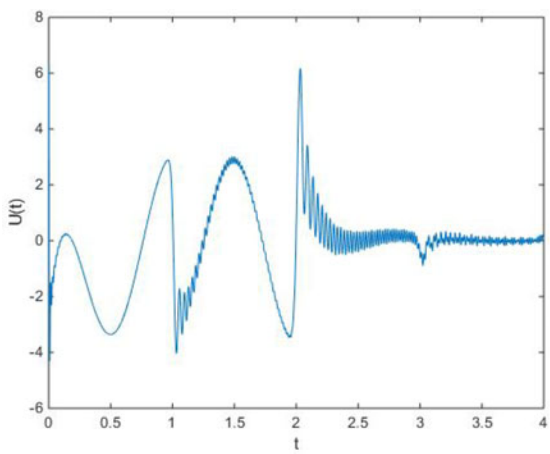

Fig. 6. Trajectory of the controller $U(t)$ (18).

initial values for system (1) and system (29) are taken to be $u(x,0) = \sin(2\pi x), u_t(x,0) = x$.

Figs. 2 and 4 show that the displacement and velocity of open-loop system (1) grow very large in a very short time and Figs. 3 and 5 show that the displacement and velocity of the closed-loop system (29) converge to zero very quickly. The control signal $U(t)$ (18) is shown in Fig. 6. It can be seen that under the control law (18), the state of the unstable wave equation with a nonlocal term is exponentially convergent to zero.

VI. Concluding Remarks

The boundary state and output feedback stabilization of velocity recirculation wave equation are considered in this paper. We first design an explicit state feedback controller to achieve exponential stability for the closed-loop system. The feedback law is found by using a new backstepping transformation that transforms the resulting closed-loop into a cascade of two transport PDEs together with one ODE being driven by one of these two PDEs. Then, we design the observer-based output feedback controller. It is shown that by using two measurements only, the output feedback can make the closed-loop system exponentially stable.

Due to the relation between wave equations and coupled (2×2) first-order hyperbolic equations, this paper is the first to also consider the control of coupled hyperbolic systems with non-local coupling, as the previous results considered only local coupling [5]. However, the coupling between the two hyperbolic PDEs in this paper is mild—it is of only feedforward character and not capable of inducing instability. Future work should explore control of hyperbolic PDEs with more general nonlocal coupling.

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