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Brief paper

Time-varying feedback for regulation of normal-form nonlinear systems in *prescribed* finite time ☆

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ABSTRACT

While non-smooth approaches (including sliding mode control) provide explicit feedback laws that ensure finite-time stabilization but in terminal time that depends on the initial condition, fixed-time optimal control with a terminal constraint ensures regulation in prescribed time but lacks the explicit character in the presence of nonlinearities and uncertainties. In this paper we present an alternative to these approaches, which, while lacking optimality, provides explicit time-varying feedback laws that achieve regulation in prescribed finite time, even in the presence of non-vanishing (though matched) uncertain nonlinearities. Our approach employs a scaling of the state by a function of time that grows unbounded towards the terminal time and is followed by a design of a controller that stabilizes the system in the scaled state representation, yielding regulation in prescribed finite time for the original state. The achieved robustness to right-hand-side disturbances is not accompanied by robustness to measurement noise, which is also absent from all controllers that are nonsmooth or discontinuous at the origin.

1. Introduction

Motivation. Regulation in finite time (Haimo, 1986) is commonly achieved using non-smooth feedback, including sliding mode control. However, regulation in *prescribed* finite time is a more demanding objective, which arises in missile guidance (Zarchan, 2007) and other applications. Two approaches to solving this problem are common—the classical (and elementary) *proportional navigation* feedback, which employs time-varying gains that go to infinity towards the terminal time, and optimal control with a terminal constraint, where such a dependency of the gains is implicit.

In this paper we present a systematic approach to regulation in prescribed finite time, which is inspired by PN for second-order missile model, but which we present for the general class of nonlinear systems in the "normal form" with a possibly non-vanishing uncertainty matched by control.

Literature on finite-time stabilization. Apart from classical sliding mode control, most finite time control results are built on the "Lyapunov differential inequality" introduced by Bhat and Bernstein (2000) and refined by Shen and Xia (2008) and Shen and Huang (2012). By using this inequality, together with other conditions, C^0 finite time feedback is presented for the double integrator by Bhat and Bernstein (1998) and for a class of planar systems by Qian and Li (2005). Homogeneous finite time local control for triangular systems and a certain class of nonlinear systems was developed by Hui, Haddad, and Bhat (2008), Hong (2002), Hong and Jiang (2006a); Hong, Wang, and Cheng (2006b). Huang, Lin, and Yang (2005) perform global finite-time stabilization of strict feedback systems; Polyakov and Poznyak (2009) present a sign function based (discontinuous) controller; Feng, Yu, and Man (2002) design a non-singular terminal sliding controller for robot systems; Shen and Huang (2009) present a global finite-time observer for globally Lipschitz systems; based on Implicit Lyapunov Functions (ILF) approach, finite-time and fixed-time stability analysis for a chain of integrators were presented in Li, Du, and Lin (2011), Polyakov, Efimov, and Perruquetti (2015), Wang, Li, and Shi (2014) and Wang and Xiao (2010) extended finite time control to consensus or containment of agents governed by single/double integrators.

The sophisticated technique of "adding power integration" introduced by Coron and Praly (1991) is employed by most authors including Huang et al. (2005), Huang et al. (2015), Li et al. (2011), Wang et al. (2014), and Wang, Song, Krstic, and Wen (2016).

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Most finite time controllers (Bhat & Bernstein, 1998, 2000; Feng et al., 2002; Hong, 2002; Hong et al., 2001; Hong & Jiang, 2006a; Hong et al., 2006b; Huang et al., 2015, 2005; Hui et al., 2008; Li et al., 2011; Miao & Xia, 2014; Qian & Li, 2005; Shen & Huang, 2009; Shen et al., 2015; Wang et al., 2014, 2016; Wang & Xiao, 2010) use fractional power feedback of the form $x^{\frac{l}{p}}$ (with p and l being some positive odd integers). Such control can only address constant unknown gains in second-order mechanical systems (Huang et al., 2015) or high-order systems with known control gains (Hong, 2002; Hong & Jiang, 2006a; Hong et al., 2006b; Huang et al., 2005; Polyakov et al., 2015; Shen & Huang, 2009).

Contributions of the paper. We introduce an entirely new methodology for solving finite-time regulation, with a prescribed regulation time, rather than a regulation time that depends on the initial condition (see Polyakov & Fridman, 2014) for differences between finite-time and fixed-time stability). We employ a *scaling of the state by a function that grows unbounded towards the terminal time* (somewhat akin to Seo et al., 2008), and then design a controller that stabilizes the system in the scaled state representation, yielding regulation in prescribed time for the original state. We develop our results for nonlinear systems diffeomorphically equivalent to the "normal form"

$$\begin{aligned}\dot{x}_i &= x_{i+1}, \quad i = 1, \ldots, n-1\\ \dot{x}_n &= f(x,t) + b(x,t)u, \end{aligned} \tag{1}$$

where $x = [x_1, \ldots, x_n]^T$ is the state, $u \in \mathbb{R}$ is control, and b, f are possibly uncertain and non-vanishing. Our result is limited to this class because non-vanishing uncertainties are impossible to reject in finite time unless they are matched by control. Since the stability proof for the class (1) for arbitrary n is rather complicated, we first present a result for the scalar case in Section 3 and then for the general case in Section 4. In addition to designs, in Section 2 we introduce new analysis tools in Lemma 1 and Corollary 1—time-varying counterparts of the lemmas by Bhat and Bernstein (2000).

The achieved robustness to right-hand-side disturbances is not accompanied by robustness to measurement noise, which is also absent from all controllers that are nonsmooth or discontinuous at the origin.

2. Assumptions and definitions

Assumption 1 (*Global Controllability*). For system (1) there exists a known $\underline{b} \neq 0$ (and w.l.o.g. $\underline{b} > 0$) such that $\underline{b} \leq |b(x,t)| < \infty$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$.

Assumption 2. (Bound on Matched but Possibly Nonvanishing Uncertainty) The nonlinearity f in (1) obeys

$$|f(x,t)| \leq d(t)\psi(x), \tag{2}$$

where $d(t)$ is a disturbance with an unknown bound

$$\|d\|_{[t_0,t]} := \sup_{\tau\in[t_0,t]} |d(\tau)|, \tag{3}$$

and $\psi(x) \geq 0$ is a known scalar-valued continuous function.

The basis of our fixed-time designs is the monotonically increasing function

$$\mu_1(t-t_0) = \frac{T}{T+t_0-t}, \quad t \in [t_0, t_0+T), \tag{4}$$

where $T > 0$, with the properties that $\mu_1(0) = 1$ and $\mu_1(T) = +\infty$. We introduce two *new* fixed-time stability definitions.

Definition 1 (*FT-ISS*). The system $\dot{x} = f(x,t,d)$ (of arbitrary dimensions of x and d) is said to be *fixed-time input-to-state stable in time T* (FT-ISS) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ, such that, for all $t \in [t_0, t_0+T)$,

$$|x(t)| \leq \beta\big(|x_0|, \mu_1(t-t_0)-1\big) + \gamma\left(\|d\|_{[t_0,t]}\right). \tag{5}$$

The function $\mu_1(t-t_0)-1 = (t-t_0)/(T+t_0-t)$ starts from zero at $t = t_0$ and grows monotonically to infinity as $t \to t_0+T$. Therefore, a system that is FT-ISS is, in particular, ISS, with the additional property that, in the absence of the disturbance d, it is fixed-time globally asymptotically stable in time T.

Definition 2 (*FT-ISS+C*). The system $\dot{x} = f(x,t,d)$ (of arbitrary dimensions of x and d) is said to be *fixed-time input-to-state stable in time T and convergent to zero* (FT-ISS+C) if there exist class $\mathcal{KL}$ functions β and β_f, and a class $\mathcal{K}$ function γ, such that, for all $t \in [t_0, t_0+T)$,

$$|x(t)| \leq \beta_f\left(\beta\big(|x_0|, t-t_0\big) + \gamma\left(\|d\|_{[t_0,t]}\right), \mu_1(t-t_0)-1\right). \tag{6}$$

Clearly a system that is FT-ISS+C is also FT-ISS, with the additional property that its state converges to zero in time T despite the presence of a disturbance.

Lemma 1. *Consider the function*

$$\mu(t-t_0) = \frac{T^{n+m}}{(T+t_0-t)^{n+m}} = \mu_1(t-t_0)^{n+m} \tag{7}$$

on $[t_0, t_0+T)$, with positive integers m, n. If a continuously differentiable function $V : [t_0, t_0+T) \to [0, +\infty)$ satisfies

$$\dot{V}(t) \leq -2k\mu(t-t_0)V(t) + \frac{\mu(t-t_0)}{4\lambda}d(t)^2 \tag{8}$$

for positive constants k and λ, then

$$V(t) \leq \zeta(t-t_0)^{2k}V(t_0) + \frac{\|d\|^2_{[t_0,t]}}{8k\lambda}, \quad \forall t \in [t_0, t_0+T), \tag{9}$$

where ζ is the monotonically decreasing (smooth "bump-like;" Fry & McManus (2002)) function

$$\zeta(t-t_0) = \exp^{\frac{T}{m+n-1}\left(1-\mu_1(t-t_0)^{m+n-1}\right)}, \tag{10}$$

with the properties that $\zeta(0) = 1$ and $\zeta(T) = 0$.

Proof. Solving the differential inequality (8) gives

$$\begin{aligned} V(t) &\leq \exp^{-2k\int_{t_0}^{t}\mu(\tau-t_0)d\tau}V(t_0)\\ &\quad + \frac{1}{4\lambda}\int_{t_0}^{t}\exp^{-2k\int_{\tau}^{t}\mu(s-t_0)ds}d(\tau)^2\mu(\tau-t_0)d\tau. \end{aligned} \tag{11}$$

We compute the second term on the right side of (11) to get

$$\begin{aligned} &\int_{t_0}^{t}\exp^{-2k\int_{\tau}^{t}\mu(s-t_0)ds}d(\tau)^2\mu(\tau-t_0)d\tau\\ &\leq \|d\|^2_{[t_0,t]}\int_{t_0}^{t}\exp^{2k\left(-\int_{t_0}^{t}\mu(s-t_0)ds+\int_{t_0}^{\tau}\mu(s-t_0)ds\right)}\mu(\tau-t_0)d\tau\\ &= \|d\|^2_{[t_0,t]}\exp^{-2k\int_{t_0}^{t}\mu(s-t_0)ds}\\ &\quad\times\int_{t_0}^{t}\exp^{2k\int_{t_0}^{\tau}\mu(s-t_0)ds}d\left(\int_{t_0}^{\tau}\mu(s-t_0)ds\right)\\ &= \|d\|^2_{[t_0,t]}\exp^{-2k\int_{t_0}^{t}\mu(s-t_0)ds}\frac{1}{2k}\exp^{2k\int_{t_0}^{\tau}\mu(s-t_0)ds}\Big|_{t_0}^{t}\\ &= \|d\|^2_{[t_0,t]}\exp^{-2k\int_{t_0}^{t}\mu(s-t_0)ds}\frac{1}{2k}\left(\exp^{2k\int_{t_0}^{t}\mu(s-t_0)ds}-1\right)\\ &= \|d\|^2_{[t_0,t]}\frac{1}{2k}\left(1-\exp^{-2k\int_{t_0}^{t}\mu(s-t_0)ds}\right)\\ &\leq \frac{\|d\|^2_{[t_0,t]}}{2k}. \end{aligned} \tag{12}$$

According to

$$\int_{t_0}^{t}\mu(\tau-t_0)d\tau=\frac{T}{n+m-1}(\mu_1(t-t_0)^{n+m-1}-1), \tag{13}$$

with (11) and (12), we obtain

$$\begin{aligned}V(t)&\le \exp^{-2k\frac{T}{m+n-1}(\mu_1(t-t_0)^{m+n-1}-1)}V(t_0)+\frac{\|d\|^2_{[t_0,t]}}{8k\lambda}\\&=\zeta(t-t_0)^{2k}V(t_0)+\frac{\|d\|^2_{[t_0,t]}}{8k\lambda}. \quad\blacksquare\end{aligned} \tag{14}$$

Corollary 1. *Under the conditions of Lemma 1, if $d(t)\equiv 0$, then* $\lim_{t\to t_0+T}V(t)=0$.

Corollary 1 is a time-varying fixed-time counterpart of the basic non-smooth finite-time result of Theorem 4.2 by Bhat and Bernstein (2000), whereas Lemma 1 is a time-varying finite-time counterpart of the robustness result in Theorem 5.2 by Bhat and Bernstein (2000).

3. The basic design idea (a scalar example)

We first consider the following first-order system,

$$\dot{x}=b(x,t)u+f(x,t),\quad x_0=x(t_0) \tag{15}$$

where $x, u\in\mathbb{R}$, under Assumptions 1 and 2.

3.1. Design and analysis

The key element in our development for finite time regulation is the time-varying scaling function (7) with $n=1$, i.e.,

$$\mu(t-t_0)=\frac{T^{1+m}}{(T+t_0-t)^{1+m}}=\mu_1(t-t_0)^{1+m}. \tag{16}$$

We employ μ in the scaling transformation

$$w(t)=\mu(t-t_0)x(t), \tag{17}$$

with which the original model (15) is converted into

$$\dot{w}=\mu\dot{x}+\dot{\mu}x=\mu\left(bu+f+\frac{\dot{\mu}}{\mu}x\right). \tag{18}$$

A feedback law is designed next both for the stabilization of this w-system and of the original x-system.

Theorem 1. *The system (15) with the controller*

$$u=-\frac{1}{\underline{b}}\left(k+\lambda\psi(x)^2+\frac{1+m}{T}\right)\mu(t-t_0)x, \tag{19}$$

where $k,\lambda>0$, is FT-ISS+C and

$$|x(t)|\le\nu(t-t_0)^{1+m}\left(\zeta(t-t_0)^k|x(t_0)|+\frac{\|d\|_{[t_0,t]}}{2\sqrt{k\lambda}}\right) \tag{20}$$

for all $t\in[t_0,t_0+T)$, where

$$\nu(t-t_0)=\mu_1(t-t_0)^{-1}=1-\frac{t-t_0}{T}, \tag{21}$$

is a monotonically decreasing linear function with the properties that $\nu(0)=1$ and $\nu(T)=0$, which means, in particular, that regulation is achieved in prescribed time T. Furthermore, the input remains uniformly bounded over $[t_0,t_0+T)$.

Proof. We start by noting that μ satisfies

$$\dot{\mu}=\frac{(1+m)T^{1+m}}{(T+t_0-t)^{1+m+1}}=\frac{1+m}{T}\mu_1^{2+m}. \tag{22}$$

We then choose the Lyapunov function candidate $V=w^2/2$, whose derivative along the solutions of (18) is

$$\dot{V}=w\dot{\mu}x+w\mu bu+w\mu f. \tag{23}$$

The last two of the three terms in this expression are uncertain and we compute bounds on them. But first,

$$\begin{aligned}w\dot{\mu}x&=w\frac{1+m}{T}\mu_1^{2+m}x=w\mu\frac{1+m}{T}\mu_1x\\&=w\mu\frac{1+m}{T}\nu^m\mu x\le w\mu b\frac{1}{\underline{b}}\frac{1+m}{T}\mu x.\end{aligned} \tag{24}$$

Second, by applying Young's inequality with $\lambda>0$,

$$\begin{aligned}w\mu f&\le\mu|w|d\psi\le\mu\lambda w^2\psi^2+\frac{\mu}{4\lambda}d^2\\&\le w\mu b\frac{1}{\underline{b}}\lambda\psi^2\mu x+\frac{\mu}{4\lambda}d^2.\end{aligned} \tag{25}$$

Inserting (24) and (25) into (23) yields

$$\dot{V}\le w\mu b\left(u+\frac{1}{\underline{b}}\mu\lambda\psi^2x+\frac{1}{\underline{b}}\frac{1+m}{T}\mu x\right)+\frac{\mu}{4\lambda}d^2. \tag{26}$$

Upon applying the control law (19), we get from (26) that

$$\dot{V}\le-\frac{b}{\underline{b}}k\mu w^2+\frac{\mu}{4\lambda}d^2\le-2k\mu V(t)+\frac{\mu}{4\lambda}d^2. \tag{27}$$

With Lemma 1 we get (9), which implies that V, as well as w, are bounded on $[t_0,t_0+T)$. Further,

$$\begin{aligned}x(t)^2&=\frac{1}{\mu(t-t_0)^2}w(t)^2=\nu(t-t_0)^{2(1+m)}2V(t)\\&\le\nu(t-t_0)^{2(1+m)}\left(\zeta(t-t_0)^{2k}x(t_0)^2+\frac{\|d\|^2_{[t_0,t]}}{4k\lambda}\right)\end{aligned} \tag{28}$$

for all $t\in[t_0,t_0+T)$, which yields (20). The claimed properties of control follow from writing (19) as

$$u=-\frac{1}{\underline{b}}\left(k+\lambda\psi(x)^2+\frac{1+m}{T}\right)w, \tag{29}$$

with the boundedness of w following from (9) and the boundedness of x established in (28). $\blacksquare$

3.2. A discussion of the merits of the design approach

(a) *Fixed-time regulation:* Fixed-time regulation is the result of employing the gain $\mu(t-t_0)$, defined in (16) and growing to infinity, inside the control law (19). The design is based on the scaling transformation (17), followed by a stabilizing control design for the scaled system, (18). Employing unbounded gain seems counterintuitive—are we achieving fixed-time regulation with unbounded inputs, a strong result at an exorbitant price? The answer is no. As illustrated by (19), the product μx in the control law is the scaled state w, which is kept bounded by (9). As μ goes to infinity, x goes to zero, resulting in the boundedness of w and hence of u. In the absence of uncertainty, the result achieved in (20) yields even more insight. Suppose, for simplicity, that b and f in (15) are known, in which case the control is chosen as $u=-(1/b)\left[f+\left(k+\frac{1+m}{T}\nu(t-t_0)\right)\mu(t-t_0)x\right]$ to result in the closed-loop $\dot{w}=-k\mu(t-t_0)w$. The scaled state w is not only bounded but goes to zero in finite time T. This results in x going to zero in finite time T at a rate that is *not* (merely) polynomially fast, namely, at a rate dictated by $\nu(t-t_0)^{1+m}$, but at a rate that is *fixed-time exponential* and governed by the function $\zeta(t-t_0)=\exp^{\frac{T}{m}\left(1-1/\nu(t-t_0)^m\right)}$, which is a function with a property that not only its value goes to zero in time T but all of the function's derivatives are zero at the final time T. This means that $x(t)$ approaches zero not only at a fast rate but it "lands smoothly" at time T. This is captured in the theorem's estimate (20), which, when $d = 0$, gives $|x(t)| \leq \nu(t-t_0)^{(1+m)}\zeta(t-t_0)^k|x(t_0)|$.

(b) *Robustness (to uncertainty in b and f):* Our controller achieves not only FT-ISS but FT-ISS+C, guaranteeing that $x(t) \to 0$ in the presence of any non-vanishing uncertainty, even when its size $\|d\|_{[t_0,t]}$ is unknown. To see things clearly, consider the special situation where $x_0 = 0$. In that case (20) yields $|x(t)| \leq \nu(t-t_0)^{1+m}\frac{\|d\|_{[t_0,t]}}{2\sqrt{k\lambda}}$, ensuring that x is regulated to zero in T under a non-vanishing disturbance. This is reminiscent of sliding mode control (which also employs a gain that goes to infinity as x reaches zero), but the result here is stronger. In sliding mode a bound on $\|d\|_{[t_0,t]}$ needs to be known, unlike here, where we employ nonlinear damping $\psi(x)^2\mu(t-t_0)x$ in the controller (19).

(c) *Consequences of employing a gain that grows unbounded and its necessity for stabilization in prescribed finite time:* Even though, ideally, the product μx of two signals, the latter of which goes to zero, while the former goes to infinity, is bounded, problems may arise either under measurement noise of x, resulting in a product of a gain μ that grows unbounded, while x does not decay fully to zero, or in the computation of the feedback μx, where a multiplication of very large and very small values creates numerical problems.

Such problems can be addressed in multiple ways. One way is employing a deadzone on x. This approach somewhat sacrifices the asymptotic performance—the regulation is not to zero but to a small neighborhood. Another way is by setting T in the gain μ to a larger value than the desired finite time of regulation, which prevents the gain from becoming infinite over the desired regulation time but, again, with some sacrifice on the regulation accuracy.

The "high-gain challenge" in our approach should be contrasted with non-smooth (or sliding) feedback, where the gain similarly becomes infinite near $x = 0$ and the problem manifests itself through chattering in "long-time" applications. Our approach is specifically intended for fixed-time applications, and the concern arises only at the terminal time, with viable ways to address it discussed above.

Any approach that is geared towards regulation in *prescribed* finite time, including finite-horizon optimal control with a *terminal constraint*, inevitably yields gains that go to infinity. So, the "high-gain challenge" is not particular to the approach presented here. The advantage of the present approach is its simplicity, transparency, and explicitness, both in terms of the feedback that needs to be implemented and in terms of the convergence estimates provided.

4. Fixed-time regulation for *n*th order systems

We consider the scaling function (7) and denote by $\mu^{(q)}$ ($q = 0,\ldots,n$) the qth derivative of μ with $\mu^{(0)} = \mu$, and denote by μ^l the lth power of $\mu(t)$. By taking the derivatives of $\mu(t)$ successively, we get

$$\mu^{(k)} = \frac{(n+m+k-1)!}{T^k(n+m-1)!}\mu_1^{n+m+k}, \quad k = 1,\ldots,n, \tag{30}$$

where μ_1 is as in (4).

4.1. Scaled state and system

To achieve fixed-time regulation for the high-order system (1), we use the scaling function μ in (7) as

$$w_1(t) = \mu(t-t_0)x_1(t),$$
$$w_q(t) = dw_{q-1}(t)/dt, \quad q = 2,\ldots,n+1. \tag{31}$$

We denote $w_{n+1} = \dot{w}_n$ and $x_{n+1} = \dot{x}_n$, with which we present the following two lemmas, whose proofs are in the Appendix.

Lemma 2 (*Scaling Transformation*). *The scaling transformation $x(t) \mapsto w(t)$ given by*

$$w = \mu_1^{m+1}P(\mu_1)x, \tag{32}$$

where the matrix $P(\mu_1)$ is a lower triangular matrix having elements $\{p_{ij}\}$ given by

$$p_{ij}(\mu_1) = \bar{p}_{ij}\mu_1^{n+i-j-1}, \qquad 1 \leq j \leq i \leq n \tag{33}$$

$$\bar{p}_{ij} = \binom{i-1}{i-j}\frac{(n+m+i-j-1)!}{T^{i-j}(n+m-1)!}, \tag{34}$$

yields the system (31).

Lemma 3 (*Inverse Transformation*). *Given the transformation $x(t) \mapsto w(t)$ defined by $w(t) = \mu_1^{m+1}P(\mu_1)x(t)$ in (32), the inverse transformation $w(t) \mapsto x(t)$ is given by*

$$x = \nu^{m+1}Q(\nu)w, \tag{35}$$

where the inverse matrix $Q(\nu) := P(\mu_1)^{-1}$ is a lower triangular matrix having elements $\{q_{ij}\}$ given by

$$q_{ij}(\nu) = \bar{q}_{ij}\nu^{n+j-i-1}, \qquad 1 \leq j \leq i \leq n \tag{36}$$

$$\bar{q}_{ij} = \binom{i-1}{i-j}\frac{(-1)^{i-j}(n+m)!}{T^{i-j}(n+m+j-i)!}. \tag{37}$$

Furthermore, $\bar{q} = \sup_{\nu\in(0,1]}|Q(\nu)|$ is finite.

4.2. A scalar system for control design

Denote

$$r_1 = [w_1,\ldots,w_{n-1}]^T = J_1 w \in \mathbb{R}^{n-1} \tag{38}$$

$$r_2 = \dot{r}_1 = [w_2,\ldots,w_n]^T = J_2 w \in \mathbb{R}^{n-1}, \tag{39}$$

where

$$J_1 = \left[I_{n-1},\ 0_{(n-1)\times 1}\right], \quad J_2 = \left[0_{(n-1)\times 1},\ I_{n-1}\right], \tag{40}$$

and $K_{n-1} = [k_1,\ldots,k_{n-1}]^T \in \mathbb{R}^{n-1}$, where K_{n-1} is an appropriately chosen coefficient vector so that the polynomial $s^{n-1} + k_{n-1}s^{n-2} + \cdots + k_1$ and the matrix

$$\Lambda = \begin{bmatrix} 0 & & I_{n-2} \\ -k_1 & -k_2 & \cdots & -k_{n-1}\end{bmatrix} \tag{41}$$

are both Hurwitz. Now we replace the state w_n by the new variable z as

$$z = w_n + K_{n-1}^T r_1. \tag{42}$$

This then results in

$$\dot{r}_1 = \Lambda r_1 + e_{n-1}z, \tag{43}$$

where $e_{n-1} = [0,\ldots,0,1]^T \in \mathbb{R}^{n-1}$. Before proceeding, we note that the linear system (43) is ISS.

The derivative of the new state (42) is

$$\dot{z} = \dot{w}_n + K_{n-1}^T J_2 w, \tag{44}$$

which, by substitution of $\dot{w}_n = w_{n+1}, \dot{x}_n = x_{n+1}$ and then (A.1), and writing out the $k = 0$ term from the sum, yields

$$\dot{z} = \mu(\dot{x}_n + L_0 + L_1) = \mu(bu + f + L_0 + L_1) \tag{45}$$

with

$$L_0 := \sum_{k=1}^{n}\binom{n}{k}\frac{\mu^{(k)}}{\mu}x_{n+1-k}, \quad L_1 := \nu^{n+m}K_{n-1}^T J_2 w. \tag{46}$$

In the following lemma, whose proof is in the Appendix, the quantity L_0 is expressed in terms of w.

Lemma 4 (*Rewriting L_0*). *The quantity L_0 is expressed as*

$$L_0 = \nu^m l_0(\nu) w, \tag{47}$$

where $l_0(\nu) = \left[l_{0,1}, l_{0,2}, \dots, l_{0,n}\right]$, and for $j = 1, 2, \dots, n$,

$$l_{0,j}(\nu) = \bar{l}_{0,j}\nu^{j-1} \tag{48}$$

with

$$\bar{l}_{0,j} = \frac{n+m}{T^{n+1-j}} \sum_{i=0}^{n-j} \binom{n}{n-i-j+1} \binom{i+j-1}{i} \times \frac{(-1)^i(2n+m-i-j)!}{(n+m-i)!}. \tag{49}$$

Furthermore, $l_0(\nu)$ is bounded.

4.3. Design without uncertainties

We consider the normal-form system (1) and first present a simple design for the case where the functions $b(x, t)$ and $f(x, t)$ are known.

Theorem 2. *The system (1) with the controller*

$$u = -\frac{1}{b}\left(f + L_0 + L_1 + kz\right), \tag{50}$$

has a globally fixed-time asymptotically stable equilibrium at the origin, with a prescribed convergence time T, and there exist $\tilde{M}, \tilde{\delta} > 0$ such that

$$|x(t)| \le \nu(t - t_0)^{m+1}\tilde{M}\mathrm{e}^{-\tilde{\delta}(t-t_0)}\,|x(t_0)| \tag{51}$$

for all $t \in [t_0, t_0 + T)$. Furthermore, the control u remains uniformly bounded over $[t_0, t_0 + T)$ and, if $f(x, t)$ is vanishing at $x = 0$, $u(t)$ also converges to zero as $t \to t_0 + T$.

Proof. By substituting (50) into (45), we get

$$\dot{z} = -k\mu z, \tag{52}$$

which, following Corollary 1, yields

$$|z(t)| \le \zeta(t - t_0)^k |z_0|, \qquad \forall t \in [t_0, t_0 + T). \tag{53}$$

At the same time, as indicated earlier, (43) is a linear system that is ISS with respect to z, which means that there exist positive constants M_1, δ_1, γ_1 such that

$$|r_1(t)| \le M_1\mathrm{e}^{-\delta_1(t-t_0)}|r_1(t_0)| + \gamma_1\|z\|_{[t_0,t]}, \qquad \forall t \in [t_0, t_0 + T). \tag{54}$$

Applying the standard cascade stability theorem to the system (43) and (52), from (10), (53) and (54) it follows that there exist positive constants $\bar{M}, \bar{\delta}$ such that

$$|\bar{w}(t)| \le \bar{M}\mathrm{e}^{-\bar{\delta}(t-t_0)}\,|\bar{w}(t_0)|\,, \quad \forall t \in [t_0, t_0 + T), \tag{55}$$

where

$$\bar{w} = \begin{bmatrix} r_1 \\ z \end{bmatrix} = \left(J_1^T + e_n K_{n-1}^T\right) r_1 + e_n w_n =: Rw, \tag{56}$$

which follows from (38) and (40), and (42), with the lower-triangular matrix R and its inverse defined as

$$R = I + e_n K_{n-1}^T J_1, \qquad R^{-1} = I - e_n K_{n-1}^T J_1. \tag{57}$$

From (35) and (56) it follows that

$$x = \nu^{m+1} Q(\nu) R^{-1} \bar{w}, \tag{58}$$

whereas from (32) and (56) it follows that

$$\bar{w}_0 = RP(0)x_0. \tag{59}$$

Invoking the last statement in Lemma 3, (58) and (59), and (55), we get, for all $t \in [t_0, t_0 + T)$,

$$|x(t)| \le \nu(t - t_0)^{m+1}\bar{q}\left|R^{-1}\right|\,|RP(0)|\bar{M}\mathrm{e}^{-\bar{\delta}(t-t_0)}\,|x(t_0)| \tag{60}$$

and arrive at (51) with $\tilde{M} = \bar{q}\left|R^{-1}\right|\,|RP(0)|\bar{M}$ and $\tilde{\delta} = \bar{\delta}$.

Now we turn our attention to proving the claims regarding the input u given in (50). From (46) and (47) we get

$$L_0 + L_1 = \nu^m\left(l_0(\nu) + \nu^n K_{n-1}^T J_2\right) w, \tag{61}$$

which is bounded and goes to zero as $t \to t_0 + T$. From (53), the term kz in (50) is also bounded and goes to zero. Finally, f is bounded and, if $f(0, t) \equiv 0$, then $f(x(t), t)$ also goes to zero as $t \to t_0 + T$, establishing the same for $u(t)$. ■

4.4. Design with uncertainties

Theorem 3. *Under Assumptions* 1 and 2, *consider the system (1) with the controller*

$$u = -\frac{1}{\underline{b}}\left(k + \theta + \lambda\psi(x)^2\right) z, \tag{62}$$

where z is defined via (38) and (42), *and Lemma 4 as*

$$z = \mu_1(t - t_0)^{m+1} K_+^T P(\mu_1(t - t_0)) x, \tag{63}$$

$$K_+^T = [k_1, \dots, k_{n-1}, 1]^T. \tag{64}$$

If the control gains are chosen such that $\rho, k, \lambda > 0$,

$$\rho k > \gamma_1/4, \tag{65}$$

where γ_1, defined in (54), depends on the choice of the gain vector K_{n-1} in (41) and (43), *and where $\theta \ge \theta_*$ and*

$$\theta_* = k_{n-1} + \bar{l}_{0,n} + \rho \max_{\nu\in[0,1]} \left|\left(\nu^n K_{n-1}^T J_2 + l_0(\nu)\right)\left(J_1^T - e_n K_{n-1}^T\right)\right|^2, \tag{66}$$

then the closed-loop system (1) with (62) is FT-ISS+C and there exist $\check{M}, \check{\delta}, \check{\gamma}$ such that

$$|x(t)| \le \nu(t - t_0)^{m+1}\left(\check{M}\mathrm{e}^{-\check{\delta}(t-t_0)}\,|x(t_0)| + \check{\gamma}\,\|d\|_{[t_0,t]}\right) \tag{67}$$

for all $\forall t \in [t_0, t_0 + T)$. Furthermore, the control u remains uniformly bounded over $[t_0, t_0 + T)$.

Proof. Consider (45). The derivative of $V = z^2/2$ is

$$\dot{V} = \mu z b u + \mu z f + \mu z(L_0 + L_1). \tag{68}$$

By using Young's inequality, we have

$$z\mu f \le \mu|z|d\psi \le \mu\lambda z^2\psi^2 + \frac{\mu}{4\lambda}d^2 \le z\mu b\frac{1}{\underline{b}}\lambda\psi^2 z + \frac{\mu}{4\lambda}d^2. \tag{69}$$

From (56) it follows that

$$w = \left(J_1^T - e_n K_{n-1}^T\right) r_1 + e_n z. \tag{70}$$

Substituting (70) into (61), we obtain

$$L_0 + L_1 = \nu^m\Big[\nu^{n-1}\left(k_{n-1}\nu + \bar{l}_{0,n}\right) z + \left(\nu^n K_{n-1}^T J_2 + l_0(\nu)\right)\left(J_1^T - e_n K_{n-1}^T\right) r_1\Big], \tag{71}$$

where, from (49), we note that

$$\bar{l}_{0,n} = \frac{n(n+m)}{T}\frac{(n+m)!}{(n+m)!} > 0. \tag{72}$$

Then, using (71) and Young's inequality, we get

$$\mu z(L_0 + L_1) \le \mu \nu^{m+n-1}\left|k_{n-1}\nu + \bar{l}_{0,n}\right| z^2 + \mu\rho\nu^{2m}$$
$$\times \left|\left(\nu^n K_{n-1}^T J_2 + l_0(\nu)\right)\left(J_1^T - e_n K_{n-1}^T\right)\right|^2 z^2 + \mu\frac{|r_1|^2}{4\rho}, \tag{73}$$

with $\rho > 0$. Denote

$$\rho_1(\rho) = \rho \max_{\nu\in[0,1]} \left|\left(\nu^n K_{n-1}^T J_2 + l_0(\nu)\right)\left(J_1^T - e_n K_{n-1}^T\right)\right|^2. \tag{74}$$

Since $k_{n-1} > 0$ and, from (72), $\bar{l}_{0,n} > 0$, with (74) it follows from (73) that

$$\mu z(L_0 + L_1) \le \mu\left[\left(k_{n-1} + \bar{l}_{0,n} + \rho_1(\rho)\right)z^2 + \frac{|r_1|^2}{4\rho}\right]. \tag{75}$$

From (68) and (69), and (75) we get

$$\dot{V} \le \mu z b\left[u + \frac{1}{\underline{b}}\left(k_{n-1} + \bar{l}_{0,n} + \rho_1(\rho) + \lambda\psi^2\right)z\right]$$
$$+ \frac{\mu}{4}\left(\frac{|r_1|^2}{\rho} + \frac{d^2}{\lambda}\right). \tag{76}$$

Substituting (62)–(66) into (76), with Lemma 1 we obtain, for $t \in [t_0, t_0 + T)$, that

$$|z(t)| \le \zeta(t - t_0)^k |z_0| + \frac{1}{2\sqrt{k}}\left(\frac{\|r_1\|_{[t_0,t]}}{\sqrt{\rho}} + \frac{\|d\|_{[t_0,t]}}{\sqrt{\lambda}}\right). \tag{77}$$

Hence, the z-system is FT-ISS w.r.t. the r_1-input with a gain of $\frac{1}{2\sqrt{k\rho}}$, and is, additionally, FT-ISS w.r.t. the d-input. By being FT-ISS, the z-system is, in particular, ISS, w.r.t. the same inputs (r_1, d). From (54) we recall that the r_1-system is ISS (though not FT-ISS) w.r.t. the z-input with a gain of γ_1. Hence, if $\frac{\gamma_1}{2\sqrt{k\rho}} < 1$, namely, if $\rho k > \gamma_1/4$, the (r_1, z)-system is ISS w.r.t. d. Note that we cannot conclude the FT-ISS property for the overall (r_1, z)-system w.r.t. d, but only the ISS property, because the r_1-subsystem is merely ISS. It is clear from (56) that $(r_1, z) = \bar{w}$, then by using the small-gain and the ISS argument that we have just completed, it follows that there exist positive constants $\hat{M}, \hat{\delta}, \hat{\gamma}$ such that

$$|\bar{w}(t)| \le \hat{M}e^{-\hat{\delta}(t-t_0)}|\bar{w}(t_0)| + \hat{\gamma}\|d\|_{[t_0,t]}, \quad \forall t \in [t_0, t_0 + T). \tag{78}$$

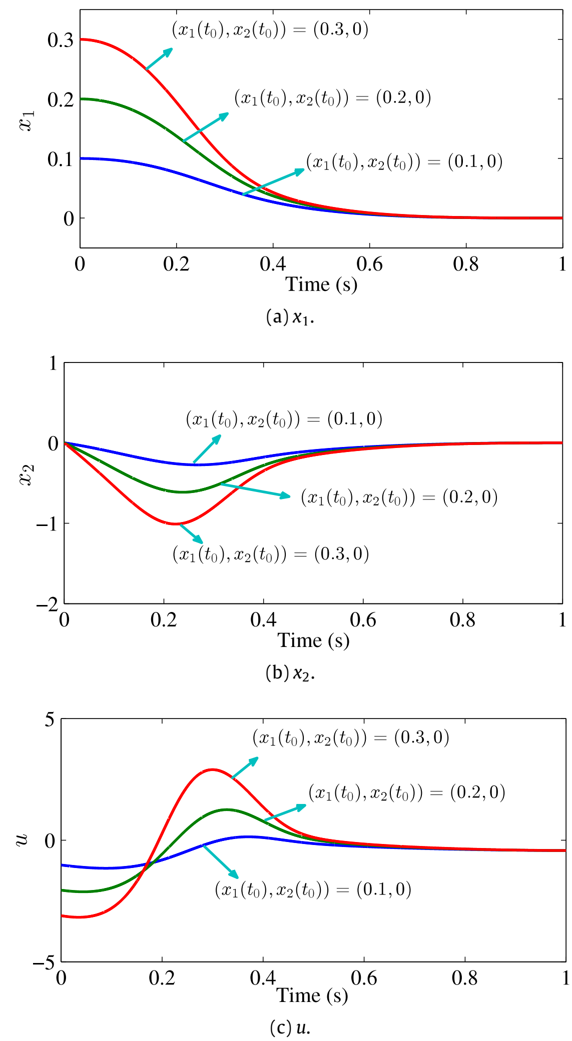
Following the same argument as in (58)–(59), we obtain

$$|x(t)| \le \nu(t - t_0)^{m+1}$$
$$\times \bar{q}\left|R^{-1}\right|\left(|RP(0)|\hat{M}e^{-\hat{\delta}(t-t_0)}|x(t_0)| + \hat{\gamma}\|d\|_{[t_0,t]}\right) \tag{79}$$

and arrive at (67) with $\check{M} = \bar{q}\left|R^{-1}\right||RP(0)|\hat{M}$, $\check{\gamma} = \bar{q}\left|R^{-1}\right|\hat{\gamma}$, $\check{\delta} = \hat{\delta}$, establishing also that the x-system is FT-ISS+C w.r.t. input d. The boundedness of u is proved as in Theorem 2. ■

4.5. Comparison with non-smooth finite-time designs

Existing non-smooth designs, such as (Huang et al., 2015, 2005; Hui et al., 2008; Li et al., 2011; Wang et al., 2014, 2016; Wang & Xiao, 2010), guarantee finite time of regulation, $T^* \le \frac{V(x(t_0))^{1-\alpha}}{c(1-\alpha)}$, which grows with the size of the initial condition. Prescribed regulation time T is achievable with such non-smooth approaches by suitably adjusting their gains based on the initial condition following the guidance by Polyakov and Poznyak (2009) and in Theorem 13 by Polyakov and Fridman (2014). Such gain adjustments are seldom simple. Polyakov (2012) developed fixed-time designs for linear controllable systems.



(a) x_1.

(b) x_2.

(c) u.

Fig. 1. Response of system (80) with control law (62) under $k = 2, \theta = 2, \lambda_1 = 0.1$, $k_1 = 0.1$ and $T = 1$s for "Wing-Rock" model (Monahemi & Krstic, 1996).

5. Numerical simulations

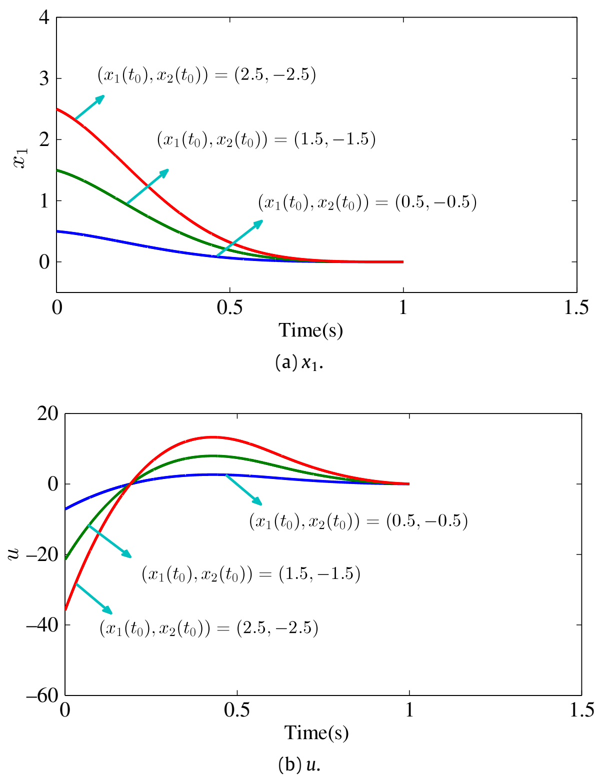
We consider the model of the "wing-rock" unstable motion in high-performance aircraft at high angle of attack (Monahemi & Krstic, 1996),

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = b(\cdot)u + f(\cdot), \tag{80}$$

where $f(\cdot) = a_0 + a_1(t)x_1 + a_2(t)x_2 + a_3|x_1|x_2 + a_4|x_2|x_2 + a_5x_1^3$ with $a_0 = 1, c_1 = 1, c_2 = 2, a_3 = 2, a_4 = 3, a_5 = 1, a_1(t) = c_1\cos(\varpi_1 t)$ and $a_2(t) = c_2\sin(\varpi_2 t)$, and $b(\cdot) = 2 + 0.4\sin(t)$. We use three sets of initial conditions for simulation: $t_0 = 0$, $x_1(t_0) = 0.1$, 0.2 and 0.3, and $x_2(t_0) = 0$, 0 and 0, respectively. For the system under consideration, it is readily verified that all the assumptions and conditions are satisfied, thus the control scheme as given in (62) is directly applicable, which takes the form,

$$u = -\frac{1}{\underline{b}}\left(k + \theta + \lambda\psi(x)^2\right)z,$$

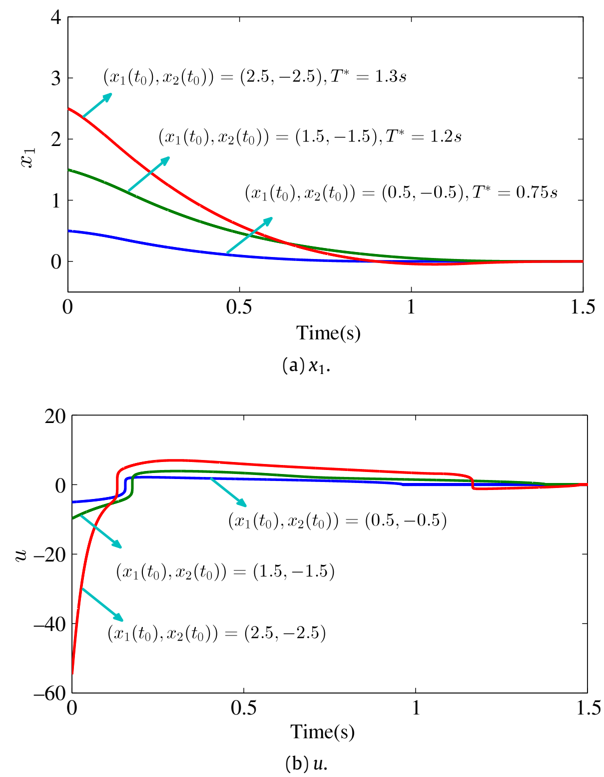
where z is defined as $z = w_2 + k_1w_1$ according to (42), in which $w_1 = \mu_1^{n+m}x_1$ and $w_2 = \mu_1^5\frac{4}{T}\mu_1x_1 + \mu_1^4x_2$ according to (A.4) (here $k_1 = 0.1$, $n = 2$, $m = 2$ and $T = 1$s in μ_1), and $\psi(x)$ is obtained as $\psi(x) = 1 + |x_1| + |x_2| + |x_1x_2| + x_2^2 + |x_1|^3$ according to $|f(\cdot)| \le d\psi(x)$ in Assumption 2. In the simulation the following control



(a) x_1.

(b) u.

Fig. 2. Response of double integrator system with the proposed control law (50) under $k = 1, k_1 = 0.1$ and $T = 1$s.



(a) x_1.

(b) u.

Fig. 3. Response of double integrator system with control law (81) in Shen and Huang (2012) under $q_2 = 3/5, c_1 = 2.5$ and $c_2 = 4.5$.

parameters are used: $\underline{b} = 1.6, k = 4, \theta = 8, \lambda = 0.1$. The results are shown in Fig. 1, which confirm the appealing performance of the proposed fixed-time control method.

To further compare our method with the improved version of the commonly used finite-time control by Shen and Huang (2012), we conduct simulation on the double integrator system (because the existing method by Shen and Huang (2012) is not applicable to systems with time-varying and unknown control gain and non-vanishing uncertainty). The fixed-time controller (50) is used with the design parameters: $k = 1$ and $k_1 = 0.1$, with $T = 1$s and $m = 2$. According to the algorithm given in Shen and Huang (2012), the corresponding finite time control law is

$$u = -\xi_2^{q_3}\left[c_2 + l/3\left(1 + \xi_2^2\right)\right], \tag{81}$$

with $\xi_2 = x_2^{1/q_2} - x_2^{*1/q_2}, x_2^* = -(c_1x_1)^{q_2}(1 + l(1 + x_1^2))$, in which the parameters are set as $q_2 = 3/5, q_3 = 1/5, c_1 = 2.5, c_2 = 4.5$, and $l = 0.18$ as in Shen and Huang (2012).

The evolution (versus time) of x_1 of the double integrator under the control of the proposed scheme (50) and the scheme (81) by Shen & Huang is presented in Fig. 2a and Fig. 3a respectively. It is seen that our control (50) achieves fixed-time regulation in $T = 1$s, whereas the convergence time of the controller by Shen and Huang (2012) depends on initial conditions. Besides, from Fig. 2b and Fig. 3b it is observed that our scheme demands a lower overall (initial) control effort and exhibits smoother control action, avoiding the sharp transitions (nearly jumps) at sign changes as reflected in Fig. 3b.

6. Conclusions

We introduced a time-varying approach for fixed-time regulation in the presence of non-vanishing uncertainties. Our design is another option in the control designer's toolbox and we do not claim its superiority with respect to the existing designs but highlight the tradeoffs. As in fixed-time optimal control with a terminal penalty, our gains grow unbounded towards the terminal time. Such a high gain character (near the set point) is similar to sliding mode control (SMC). Unlike SMC and other non-smooth methods, our time-varying approach achieves regulation in *prescribed* time and, when the nonlinearity is vanishing, the control goes to zero.

The main shortcoming of the results that we have presented is that they are restricted to matched uncertainties. This limitation is shared by SMC and is fundamental to the problem of completely rejecting non-vanishing uncertainties—this can be done only when the uncertainties are matched and requires infinite gain.

Any finite-time feedback, including the non-smooth ones and our time-varying one, exhibit a steep deterioration of asymptotic performance under measurement noise. The practitioner's simple solution (for example, in missile guidance) is to slightly lengthen the control horizon, which prevents the gains from going to infinity, while degrading the regulation slightly from perfect regulation to an acceptably small neighborhood of the target state.

Extensions to trajectory tracking are straightforward. We consider the output $y = x_1$, with a reference output $y_r(t)$, and denote the tracking error state components $\varepsilon_i = x_i - y_r^{(i-1)}(t)$, for which $\dot{\varepsilon}_i = \varepsilon_{i+1}$ and $\dot{\varepsilon}_n = f + bu + y_r^{(n-1)}(t)$. If b is known, $y_r^{(n-1)}(t)$ is canceled by control, whereas, if only b's lower bound is known, $y_r^{(n-1)}(t)$ is treated in the same fashion as the disturbance d in the paper.

Appendix

A.1. Proof of Lemma 2

Since $w_1 := \mu x_1$, the generalized Leibniz rule gives

$$w_{i+1} = (\mu x_1)^{(i)} = \sum_{k=0}^{i} \binom{i}{k} \mu^{(k)} x_1^{(i-k)} = \sum_{k=0}^{i} \binom{i}{k} \mu^{(k)} x_{i+1-k}, \qquad i = 0, \ldots, n-1, n. \tag{A.1}$$

Shifting the i index gives

$$w_i = \sum_{k=0}^{i-1} \binom{i-1}{k} \mu^{(k)} x_{i-k}, \quad i = 1, \ldots, n, n+1. \tag{A.2}$$

We next rewrite the summation using the substitution $j := i - k$, where $j = 1, 2, \ldots, i$, and arrive at

$$w_i = \sum_{j=1}^{i} \binom{i-1}{i-j} \mu^{(i-j)} x_j, \quad i = 1, \ldots, n, n+1. \tag{A.3}$$

Substituting the expression for $\mu^{(i-j)}$ in (30) results in

$$w_i = \mu_1^{n+m} \sum_{j=1}^{i} \binom{i-1}{i-j} \frac{(n+m+i-j-1)!}{T^{i-j}(n+m-1)!} \mu_1^{i-j} x_j, \tag{A.4}$$

from which we find the elements $\{p_{ij}\}$ by inspection.

A.2. Proof of Lemma 3

Since $x_1 = \frac{1}{\mu} w_1$, the generalized Leibniz rule gives

$$x_{i+1} = x_1^{(i)} = \left(\frac{1}{\mu} w_1\right)^{(i)} = \sum_{k=0}^{i} \binom{i}{k} \left(\frac{1}{\mu}\right)^{(k)} w_1^{(i-k)} = \sum_{k=0}^{i} \binom{i}{k} \left(\frac{1}{\mu}\right)^{(k)} w_{i+1-k}, \; i = 0, \ldots, n-1, n. \tag{A.5}$$

Taking the kth derivative of $1/\mu$, we have

$$\left(\frac{1}{\mu}\right)^{(k)} = \frac{(-1)^k (n+m)!}{T^k (n+m-k)!} \nu^{n+m-k}. \tag{A.6}$$

Substituting (A.6) into (A.5), we obtain

$$x_{i+1} = \nu^{n+m} \sum_{k=0}^{i} \binom{i}{k} \frac{(-1)^k (n+m)!}{T^k (n+m-k)!} \nu^{-k} w_{i+1-k}, \quad i = 0, \ldots, n-1, n, \tag{A.7}$$

which after shifting the i index can be written as

$$x_i = \nu^{n+m} \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{(-1)^k (n+m)!}{T^k (n+m-k)!} \nu^{-k} w_{i-k}. \tag{A.8}$$

We next rewrite the summation using the substitution $j := i - k$, where $j = 1, 2, \ldots, i$ to arrive at

$$x_i = \nu^{n+m} \sum_{j=1}^{i} \binom{i-1}{i-j} \frac{(-1)^{i-j} (n+m)!}{T^{i-j} (n+m+j-i)!} \nu^{j-i} w_j, \tag{A.9}$$

from which we find the elements $\{q_{ij}\}$ by inspection. The finiteness of $\bar{q}$ follows from the fact that $|Q(\nu)|$ is a continuous function of a bounded argument $\nu \in (0, 1]$.

A.3. Proof of Lemma 4

By substituting in (A.5) for x_{n+1-k} in (46), we obtain

$$\begin{aligned} L_0 &= \sum_{k=1}^{n} \binom{n}{k} \frac{\mu^{(k)}}{\mu} \left[\sum_{i=0}^{n-k} \binom{n-k}{i} \left(\frac{1}{\mu}\right)^{(i)} w_{n-k+1-i} \right] \\ &= \sum_{k=1}^{n} \sum_{i=0}^{n-k} \binom{n}{k} \binom{n-k}{i} \frac{\mu^{(k)}}{\mu} \left(\frac{1}{\mu}\right)^{(i)} w_{n-k+1-i}. \end{aligned} \tag{A.10}$$

The double sum can be viewed as the sum of function evaluations $f_{ki}(n)$ defined on a triangle in (k, i) space. Viewed this way, the double sum can be reordered as

$$\sum_{k=1}^{n} \sum_{i=0}^{n-k} f_{ki}(n) = \sum_{i=0}^{n-1} \sum_{k=1}^{n-i} f_{ki}(n). \tag{A.11}$$

This allows us to write (A.10) as

$$L_0 = \sum_{i=0}^{n-1} \sum_{k=1}^{n-i} \binom{n}{k} \binom{n-k}{i} \frac{\mu^{(k)}}{\mu} \left(\frac{1}{\mu}\right)^{(i)} w_{n-k+1-i}. \tag{A.12}$$

Next define $j := n - k + 1 - i$, where $j = n - i, n - i - 1, \ldots, 1$. Rewriting (A.12) in terms of j rather than k results in

$$L_0 = \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \binom{n}{n-i-j+1} \binom{i+j-1}{i} \times \frac{\mu^{(n-i-j+1)}}{\mu} \left(\frac{1}{\mu}\right)^{(i)} w_j. \tag{A.13}$$

We now reorder the double sum in (A.13) using (A.11) in reverse order to obtain

$$L_0 = \sum_{j=1}^{n} \sum_{i=0}^{n-j} \binom{n}{n-i-j+1} \binom{i+j-1}{i} \times \frac{\mu^{(n-i-j+1)}}{\mu} \left(\frac{1}{\mu}\right)^{(i)} w_j. \tag{A.14}$$

Then after substituting in (30) for $\mu^{(n-i-j+1)}$, (A.6) for $(1/\mu)^{(i)}$, and $1/\mu = \nu^{n+m}$, we arrive at

$$\begin{aligned} L_0 = \sum_{j=1}^{n} \sum_{i=0}^{n-j} & \binom{n}{n-i-j+1} \binom{i+j-1}{i} \nu^{n+m} \\ & \times \left[\frac{(2n+m-i-j)!}{T^{n-i-j+1}(n+m-1)!} \mu_1^{2n+m-i-j+1} \right] \\ & \times \left[\frac{(-1)^i (n+m)!}{T^i (n+m-i)!} \nu^{n+m-i} \right] w_j, \end{aligned} \tag{A.15}$$

which reduces to

$$L_0 = \sum_{j=1}^{n} \frac{n+m}{T^{n-j+1}} \sum_{i=0}^{n-j} \binom{n}{n-i-j+1} \binom{i+j-1}{i} \times \frac{(-1)^i (2n+m-i-j)!}{(n+m-i)!} \nu^{m+j-1} w_j, \tag{A.16}$$

which proves the first part of the lemma. The boundedness of the $l_{0,j}(\nu)$ follows by inspection, since $\nu(t - t_0) \le 1$ for $t \in [t_0, t_0 + T)$, and $\bar{l}_{0,j}$ are bounded since they are finite sums of real numbers.

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