Automatica 79 (2017) 17–26

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Adaptive rejection of harmonic disturbance anticollocated with control in 1D wave equation ☆

Wei Guo [a], Zhi-Chao Shao [a], Miroslav Krstic [b]

[a] School of Statistics, University of International Business and Economics, Beijing 100029, PR China

[b] Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA

ARTICLE INFO

Article history:
Received 7 April 2016
Received in revised form
12 September 2016
Accepted 9 January 2017
Available online 2 March 2017

Keywords:
Distributed parameter system
Disturbance rejection
Output regulation

ABSTRACT

In this paper, we consider the output regulation problem for a one dimensional wave equation with harmonic disturbance anti-collocated with control. We first design an adaptive observer by the measured output to estimate unknown parameters of the disturbance and recover the system state. Then by using the observer system and estimator of unknown harmonic disturbance, we construct an auxiliary system in which the control and the anti-collocated disturbance become collocated. By applying the Backstepping method for infinite dimensional system, we design an output feedback adaptive controller which regulates the output to zero and keeps all the states bounded.

1. Introduction

The output regulation problem, or alternatively the servomechanism, is one of the central problems in control theory, which addresses designing of a feedback controller to achieve asymptotic tracking of prescribed reference signals and asymptotic rejection of undesired disturbances in an uncertain system while maintaining closed-loop stability. In the finite-dimensional system (linear or nonlinear) setting, there are many classical results which include internal model principle to address this problem such as Callier and Desoer (1980), Davison (1976), Desoer and Lin (1985), Francis (1977), Francis and Wonham (1976) and Isidori and Byrnes (1990), among many others. Much work has been done to extend these classical results to infinite-dimensional systems, see, for instance Byrnes, Laukó, Gilliam, and Shubov (2000), Deutscher (2011), Hämäläinen and Pohjolainen (2010), Paunonen and Pohjolainen (2010) and Rebarber and Weiss (2003). Most of the above works about output regulation problem focus on the extension of internal model principle theory to infinite-dimensional systems where reference signal and disturbance are generated by finite-dimensional or infinite-dimensional exosystem.

Also some attempts concerned with adaptive servomechanism design for infinite-dimensional systems have been made. Earlier work on applying adaptive servomechanism design for infinite-dimensional systems is reported in Logemann and Ilchmann (1994). In paper Kobayashi and Oya (2002), an adaptive servomechanism control is designed based on passivity principle for a class of distributed parameter system where the input and output operators are collocated and the disturbance is collocated with control. Those works build a theoretical framework which covers a large class of real systems. However, many real control systems are not included in those abstract frameworks, such as the boundary control PDE (Partial Differential Equation) system which is anticollocated, or is unstable or even antistable itself. In this situation, the input–output operator is typically no longer passive or uncontrolled system has real positive eigenvalues, which results in that the passivity principle cannot be directly applied anymore. In the last few years, there have been a few works contributed to the boundary feedback stabilization and observer design of these systems. An observer-based compensator which exponentially stabilizes the string system with an anticollocated actuator/ sensor configuration is proposed in Guo and Xu (2007). A lot of papers show that the backstepping method introduced in PDEs Krstic and Smyshlyaev (2008a) is very powerful in dealing with unstable or even antistable PDE system. In Krstic, Guo, Balogh, and Smyshlyaev (2008b), an observer-based controller is designed using both the displacement and velocity

☆ This work was supported by the National Natural Science Foundation of China (61374088), the Fundamental Research Funds for the Central Universities in UIBE (15JQ01). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Oswaldo Luiz V. Costa under the direction of Editor Richard Middleton.

E-mail addresses: guowei74@126.com (W. Guo), zcshao@amss.ac.cn (Z.-C. Shao), krstic@ucsd.edu (M. Krstic).

http://dx.doi.org/10.1016/j.automatica.2017.01.034
0005-1098/

measurement via the backstepping method to exponentially stabilize an unstable one-dimensional wave equation. A breakthrough was made recently in Smyshlyaev and Krstic (2009), where the anti-stable wave equation is exponentially stabilized through a novel backstepping transformation method. The stabilization of unstable shear beam equation is addressed in Krstic, Guo, Balogh, and Smyshlyaev (2008a), where the non-collocated boundary stabilization is discussed by using the backstepping method and observer-based feedback.

Almost at the same time, much more attention has also been paid to the adaptive boundary control by using backstepping method for these anticollocated, unstable or even antistable PDE systems. The early efforts on applying the backstepping method to the design of adaptive boundary state feedback controllers or output feedback controllers for partial differential equation control systems have been made in Krstic and Smyshlyaev (2008b) and Smyshlyaev and Krstic (2007a,b), particularly for parabolic PDEs with boundary control and unknown parameters that may cause instability of the system and affect the interior of the domain. Adaptive boundary state feedback stabilization and recent output feedback stabilization for the most challenging anti-stable wave equation with unknown anti-damping in the uncontrolled boundary can be found in Krstic (2010) and Bresch-Pietri and Krstic (2014) respectively. A recent result on adaptive anti-collocated feedback stabilization for general first-order hyperbolic partial integro-differential equations with unknown functional coefficients is addressed in Bernard and Krstic (2014). Other effort on adaptive parameter identification for unstable parabolic PDEs is made in Smyshlyaev, Orlov, and Krstic (2009). Recently, an adaptive observer design for a semilinear heat equation can be found in Ahmed-Ali, Giri, Krstic, Lamnabhi-Lagarrigue, and Burlion (2016), where online estimates of the system state and unknown parameters based on sampled data are provided. As for adaptive feedback stabilization for one-dimensional unstable wave equations with collocated harmonic disturbances, we refer to Guo and Guo (2013a,b,c).

However, to the best knowledge of the authors, there are few works so far to solve the output regulation problem for these anticollocated, unstable or even antistable PDE systems. Recently in Guo and Guo (2016), an adaptive servomechanism is constructed for one-dimensional wave equation to reject the disturbances and track the reference signal.

In this paper, we are concerned with wave equation:

$$\begin{cases} y_{tt}(x,t) = y_{xx}(x,t), \quad x \in (0,1), \quad t>0, \\ y_x(0,t) = d(t), \quad t \geq 0, \\ y_x(1,t) = U(t), \quad t \geq 0, \\ y(x,0) = y_0(x), \qquad y_t(x,0) = y_1(x), \quad 0 \leq x \leq 1, \\ y_{\text{out}}(t) = (y(0,t), y_t(0,t)) \end{cases} \tag{1}$$

where (and henceforth) y' or y_x denotes the derivative of y with respect to x and $\dot{y}$ or y_t the derivative with respect to t. $U(t)$ is the control input; y_{out} is the measured output; and (y_0, y_1) is the initial value; d represents the general harmonic disturbance which has the following form:

$$d(t) = \sum_{j=1}^{m} [a_j \cos \omega_j t + b_j \sin \omega_j t].$$

Though our approach applies to this general class of disturbances, for simplicity of writing, we take $m = 1$, namely,

$$d(t) = a \cos \omega t + b \sin \omega t.$$

Here the frequency $\omega \neq 0$ is assumed to be known, whereas a, b are unknown constants. We consider the case where ω is to be known, which is same as the requirement for finite harmonic signals considered in internal model principle. There are references on disturbance cancellation which incorporate estimation of the unknown frequencies (Liu, Chen, & Huang, 2009; Pyrkin, Smyshlyaev, Bekiaris-Liberis, & Krstic, 2010).

The objective of this paper is to find an adaptive feedback control law for system (1) such that $\lim_{t\to\infty} y(0,t) = 0$ and the unknown harmonic disturbance is rejected.

System (1) is a typical anti-collocated Neumann control problem: control is on one end and the output is on the other end. In addition, the disturbance and control, in fact, are anticollocated. This paper is the first to perform adaptive cancellation of anti-collocated disturbances and its contribution should be viewed in relation to Guo and Guo (2016) and Kobayashi and Oya (2002) where only collocated disturbances are cancelled. It takes adaptive *disturbance cancellation* in the same direction as Bresch-Pietri and Krstic (2014) and Krstic (2010) have taken adaptive *stabilization*. The applications of adaptive cancellation of anti-collocated disturbances arise in many applications, from oil drilling to atomic force microscopy, where disturbances appear on the opposite boundary relative to the actuator.

The key characteristic of our approach is the use of the measurement $(y(0,t), y_t(0,t))$ to first give a backstepping-based adaptive observer of the state and the estimator of the parameters of the disturbances and then construct an auxiliary system in which the control and the anti-collocated disturbance become collocated. Our adaptive observer and controller do not seem to employ the internal model principle and the regulator equations associated with the output regulation approach.

The paper is organized as follows. In next section, Section 2, we give the adaptive disturbance cancellation controller design. The asymptotic stability of the error system is given in Section 3. In Section 4, we give the convergence analysis of the auxiliary system. We present some illustrative simulation results in Section 5. Conclusions are made in Section 6. ■

2. Adaptive disturbance cancellation controller design

We now design an adaptive observer for system (1) using the measurement of $y_{\text{out}}(t) = (y(0,t), y_t(0,t))$. Since the plant is second-order in time, $y_t(0,t)$ is not to be thought as a derivative of the displacement $y(0,t)$ but as a measured part of the overall state (y, y_t). Indeed it is common to virtually all results in boundary control of flexible structures to employ measurements of the velocity at the boundary.

Our adaptive observer is given as:

$$\begin{cases} \widehat{y}_{tt}(x,t) = \widehat{y}_{xx}(x,t), \\ \widehat{y}_x(0,t) = \widehat{a}(t)\cos\omega t + \widehat{b}(t)\sin\omega t \\ \qquad + k_1(\widehat{y}_t(0,t) - y_t(0,t)) + k_2(\widehat{y}(0,t) - y(0,t)), \\ \widehat{y}_x(1,t) = U(t), \\ \dot{\widehat{a}}(t) = -r(y_t(0,t) - \widehat{y}_t(0,t))\cos\omega t, \\ \dot{\widehat{b}}(t) = -r(y_t(0,t) - \widehat{y}_t(0,t))\sin\omega t, \\ \widehat{a}(0) = \widehat{a}_0, \qquad \widehat{b}(0) = \widehat{b}_0, \\ \widehat{y}(x,0) = \widehat{y}_0(x), \qquad \widehat{y}_t(x,0) = \widehat{y}_1(x), \end{cases} \tag{2}$$

where $k_1, k_2, r > 0$ are design parameters. In the rest of the paper, we omit the (obvious) domains for t and x.

Remark 2.1. The observer design for the case $d(t) = \sum_{j=1}^{m}[a_j \cos\omega_j t + b_j \sin\omega_j t]$ is still valid because we only need to write the term $\widehat{a}(t)\cos\omega t + \widehat{b}(t)\sin\omega t$ in (2) to be $\sum_{j=1}^{m}\widehat{a}_j(t)\cos\omega_j t + \widehat{b}_j(t)\sin\omega_j t$ and the update law to be $\dot{\widehat{a}}_j(t) = -r_j(y_t(0,t) - \widehat{y}_t(0,t))\cos\omega_j t$, $\dot{\widehat{b}}_j(t) = -r_j(y_t(0,t) - \widehat{y}_t(0,t))\sin\omega_j t$, $j = 1, 2, \ldots, m$.

Let $\varepsilon = y - \widehat{y}$ and $\widetilde{a} = a - \widehat{a}(t)$, $\widetilde{b} = b - \widehat{b}(t)$ be parameter estimation error, then from (1) and (2), ε is governed by

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = k_1\varepsilon_t(0,t) + k_2\varepsilon(0,t) + \widetilde{a}(t)\cos\omega t + \widetilde{b}(t)\sin\omega t, \\ \varepsilon_x(1,t) = 0, \\ \dot{\widetilde{a}}(t) = r\varepsilon_t(0,t)\cos\omega t, \\ \dot{\widetilde{b}}(t) = r\varepsilon_t(0,t)\sin\omega t, \\ \varepsilon(x,0) = \varepsilon_0(x) \qquad \varepsilon_t(x,0) = \varepsilon_1(x), \end{cases} \tag{3}$$

where

$$\varepsilon(x,0) = y_0(x) - \widehat{y}_0(x), \qquad \varepsilon_t(x,0) = y_1(x) - \widehat{y}_1(x). \tag{4}$$

Define the energy function for system (3) as follows:

$$E_\varepsilon(t) = \frac{1}{2}\int_0^1 [\varepsilon_t^2(x,t) + \varepsilon_x^2(x,t)]dx + \frac{k_2}{2}[\varepsilon(0,t)]^2 + \frac{1}{2r}[\widetilde{a}^2(t) + \widetilde{b}^2(t)]. \tag{5}$$

A simple computation of the derivative of $E_\varepsilon(t)$ with respect to t along the solution to (3) shows that

$$\dot{E}_\varepsilon(t) = -k_1[\varepsilon_t(0,t)]^2 \le 0, \tag{6}$$

from which we obtain the update law of $\widehat{a}(t)$ and $\widehat{b}(t)$ in system (2). By the update law of $\widehat{a}(t)$ and $\widehat{b}(t)$ in system (2), a formal computation gives

$$\begin{aligned} &\frac{d}{dt}[\widehat{a}(t)\cos\omega t + \widehat{b}(t)\sin\omega t + r\varepsilon(0,t)] \\ &\quad = \omega[\widehat{b}(t)\cos\omega t - \widehat{a}(t)\sin\omega t], \\ &\frac{d^2}{dt^2}[\widehat{a}(t)\cos\omega t + \widehat{b}(t)\sin\omega t + r\varepsilon(0,t)] \\ &\quad = -\omega^2[\widehat{a}(t)\cos\omega t + \widehat{b}(t)\sin\omega t]. \end{aligned} \tag{7}$$

Let

$$\widehat{z}(x,t) = \widehat{y}(x,t) - \frac{1}{\omega}\sin\omega x[\widehat{a}(t)\cos\omega t + \widehat{b}(t)\sin\omega t + r\varepsilon(0,t)], \quad (x,t)\in[0,1]\times[0,\infty). \tag{8}$$

Then from (2) and (7), we can get the following auxiliary system:

$$\begin{cases} \widehat{z}_{tt}(x,t) = \widehat{z}_{xx}(x,t) - \omega r\sin\omega x\varepsilon(0,t), \\ \widehat{z}_x(0,t) = -k_1\varepsilon_t(0,t) - (k_2+r)\varepsilon(0,t), \\ \widehat{z}_x(1,t) = U(t) - \cos\omega[\widehat{a}(t)\cos\omega t + \widehat{b}(t)\sin\omega t] \\ \qquad - r\cos\omega\varepsilon(0,t), \\ \widehat{z}(x,0) = \widehat{z}_0(x), \widehat{z}_t(x,0) = \widehat{z}_1(x), \end{cases} \tag{9}$$

where

$$\begin{aligned} \widehat{z}_0(x) &= \widehat{y}_0(x) - \frac{\widehat{a}_0 + r\varepsilon_1(0)}{\omega}\sin\omega x, \\ \widehat{z}_1(x) &= \widehat{y}_1(x) - \widehat{b}_0\sin\omega x. \end{aligned} \tag{10}$$

Moreover,

$$\widehat{z}(0,t) = \widehat{y}(0,t).$$

Remark 2.2. (a) The motivation of constructing auxiliary system (9) is to make the control and the anticollocated disturbance to be collocated and obtain $\widehat{z}(0,t) = \widehat{y}(0,t)$.

(b) Here we consider the auxiliary system (9) without the dynamic equations for $\widehat{a}(t)$ and $\widehat{b}(t)$ since they have been determined by the error system (3) already.

(c) This construction is still valid for the case $d(t) = \sum_{j=1}^m[a_j \cos\omega_j t + b_j\sin\omega_j t]$ if we let $\widehat{z}(x,t) = \widehat{y}(x,t) - \sum_{j=1}^m \frac{1}{\omega_j}\sin\omega_j x [\widehat{a}_j(t)\cos\omega_j t + \widehat{b}_j(t)\sin\omega_j t + r_j\varepsilon(0,t)]$.

We present the controller for (9) as follows:

$$U(t) = \cos\omega[\widehat{a}(t)\cos\omega t + \widehat{b}(t)\sin\omega t] - c_0\widehat{z}(1,t) - c_1\widehat{z}_t(1,t) - c_0c_1\int_0^1 \widehat{z}_t(\xi,t)d\xi + r\cos\omega\varepsilon(0,t), \tag{11}$$

where c_0, c_1 are positive design parameters. The closed-loop system of (8) corresponding to controller (11) is

$$\begin{cases} \widehat{z}_{tt}(x,t) = \widehat{z}_{xx}(x,t) - \omega r\sin\omega x\varepsilon(0,t), \\ \widehat{z}_x(0,t) = -k_1\varepsilon_t(0,t) - (k_2+r)\varepsilon(0,t), \\ \widehat{z}_x(1,t) = -c_0\widehat{z}(1,t) - c_1\widehat{z}_t(1,t) - c_0c_1\displaystyle\int_0^1 \widehat{z}_t(\xi,t)d\xi, \\ \widehat{z}(x,0) = \widehat{z}_0(x), \widehat{z}_t(x,0) = \widehat{z}_1(x). \end{cases} \tag{12}$$

Introduce the following transformation (see Krstic et al., 2008a or page 83 in Krstic & Smyshlyaev, 2008a)

$$\widetilde{z}(x,t) = \widehat{z}(x,t) + c_0\int_0^x \widehat{z}(\xi,t)d\xi, \tag{13}$$

which is invertible. The inverse is given by

$$\widehat{z}(x,t) = \widetilde{z}(x,t) - c_0\int_0^x e^{-c_0(x-\xi)}\widetilde{z}(\xi,t)d\xi.$$

It is seen that transformation (13) converts system (12) into

$$\begin{cases} \widetilde{z}_{tt}(x,t) = \widetilde{z}_{xx}(x,t) - [\omega r\sin\omega x + c_0k_2 \\ \qquad + 2c_0r\cos\omega x]\varepsilon(0,t) + c_0k_1\varepsilon_t(0,t), \\ \widetilde{z}_x(0,t) = c_0\widetilde{z}(0,t) - k_1\varepsilon_t(0,t) - (k_2+r)\varepsilon(0,t), \\ \widetilde{z}_x(1,t) = -c_1\widetilde{z}_t(1,t), \\ \widetilde{z}(x,0) = \widetilde{z}_0(x), \widetilde{z}_t(x,0) = \widetilde{z}_1(x), \end{cases} \tag{14}$$

where

$$\begin{aligned} \widetilde{z}_0(x) &= \widehat{z}_0(x) + c_0\int_0^x \widehat{z}_0(\xi)d\xi, \\ \widetilde{z}_1(x) &= \widehat{z}_1(x) + c_0\int_0^x \widehat{z}_1(\xi)d\xi. \end{aligned} \tag{15}$$

Then controller (11) is obtained in the process of transforming (12) into (14) under the Backstepping transformation (13). Notice that controller (11) is expressed by variable $\widehat{z}$. In order to get the closed-loop of system (1), it is necessary to write controller (11) to be expressed by variable $\widehat{y}$. Then by (8), we rewrite the controller (11) to be

$$\begin{aligned} U(t) &= -c_0\widehat{y}(1,t) - c_1\widehat{y}_t(1,t) \\ &\quad - c_0c_1\int_0^1 \widehat{y}_t(\xi,t)d\xi + \left[\frac{c_0\sin\omega}{\omega} + \cos\omega\right] \\ &\quad \times [\widehat{a}(t)\cos\omega t + \widehat{b}(t)\sin\omega t] \\ &\quad + r\left[\frac{c_0}{\omega}\sin\omega + \cos\omega\right][y(0,t) - \widehat{y}(0,t)] \\ &\quad + c_1\left[\sin\omega + \frac{c_0(1-\cos\omega)}{\omega}\right][\widehat{b}(t)\cos\omega t - \widehat{a}(t)\sin\omega t]. \end{aligned} \tag{16}$$

Combined by (1), (2) and (16), the resulting closed-loop is governed by

$$\begin{cases} y_{tt}(x,t)=y_{xx}(x,t),\\ y_x(0,t)=a\cos\omega t+b\sin\omega t,\\ y_x(1,t)=\widehat{y}_x(1,t),\\ \widehat{y}_{tt}(x,t)=\widehat{y}_{xx}(x,t),\\ \widehat{y}_x(0,t)=\widehat{a}(t)\cos\omega t+\widehat{b}(t)\sin\omega t\\ \qquad +k_1(\widehat{y}_t(0,t)-y_t(0,t))+k_2(\widehat{y}(0,t)-y(0,t)),\\ \widehat{y}_x(1,t)=-c_0\widehat{y}(1,t)-c_1\widehat{y}_t(1,t)\\ \qquad -c_0c_1\int_0^1\widehat{y}_t(\xi,t)d\xi+\left[\frac{c_0\sin\omega}{\omega}+\cos\omega\right]\\ \qquad \times[\widehat{a}(t)\cos\omega t+\widehat{b}(t)\sin\omega t]\\ \qquad +r\left[\frac{c_0}{\omega}\sin\omega+\cos\omega\right][y(0,t)-\widehat{y}(0,t)]\\ \qquad +c_1\left[\sin\omega+\frac{c_0(1-\cos\omega)}{\omega}\right][\widehat{b}(t)\cos\omega t-\widehat{a}(t)\sin\omega t],\\ \dot{\widehat{a}}(t)=-r(y_t(0,t)-\widehat{y}_t(0,t))\cos\omega t,\\ \dot{\widehat{b}}(t)=-r(y_t(0,t)-\widehat{y}_t(0,t))\sin\omega t,\\ \widehat{a}(0)=\widehat{a}_0,\qquad \widehat{b}(0)=\widehat{b}_0,\\ \widehat{y}(x,0)=\widehat{y}_0(x),\qquad \widehat{y}_t(x,0)=\widehat{y}_1(x),\\ y(x,0)=y_0(x),\qquad y_t(x,0)=y_1(x). \end{cases} \tag{17}$$ ■

3. Stability of the error system

Let $L^2(0,1)$ be the usual Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the inner product induced norm $\|\cdot\|$. Since system (3) is non-autonomous linear system, we introduce some additional variables to make it time invariant. Let

$$\xi(t)=\sin\omega t,\qquad \eta(t)=\cos\omega t.$$

Obviously (ξ,η) satisfies the following equation

$$\begin{cases}\frac{d}{dt}\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix}=\begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix},\\ \begin{pmatrix}\xi(0)\\ \eta(0)\end{pmatrix}=\begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}.\end{cases} \tag{18}$$

We consider (3) and (18) in the energy state space $\mathscr{H}=\mathscr{V}\times\mathbb{R}^2$:

$$\begin{aligned}&\langle(u_1,v_1,\theta,\vartheta,\psi,\omega),(u_2,v_2,\hat\theta,\hat\vartheta,\hat\psi,\hat\omega)\rangle_{\mathscr{H}}\\ &=\int_0^1u_1'(x)u_2'(x)dx+\int_0^1v_1(x)v_2(x)dx+k_2u_1(1)u_2(1)\\ &\quad+\left(\frac{\theta\hat\theta}{r}+\frac{\vartheta\hat\vartheta}{r}+\psi\hat\psi+\omega\hat\omega\right),\\ &\quad\forall\,(u_1,v_1,\theta,\vartheta,\psi,\omega),(u_2,v_2,\hat\theta,\hat\vartheta,\hat\psi,\hat\omega)\in\mathscr{H},\end{aligned}$$

where $\mathscr{V}=H^1(0,1)\times L^2(0,1)\times\mathbb{R}^2$. Hence (3) and (18) can be written as a nonlinear autonomous evolution equation in the state space $\mathscr{H}$

$$\frac{d}{dt}z(\cdot,t)=\mathscr{A}z(\cdot,t),z(\cdot,0)=z_0(\cdot)\in\mathscr{H}, \tag{19}$$

where

$$\begin{cases}z(x,t)=\left(\varepsilon(x,t),\varepsilon_t(x,t),\widetilde{a}(t),\widetilde{b}(t),\xi(t),\eta(t)\right),\\ z_0(x)=(\varepsilon_0(x),\varepsilon_1(x),\widetilde{a}_0,\widetilde{b}_0,\xi_0,\eta_0),\end{cases}$$

and

$$\begin{cases}\mathscr{A}(u,v,\widetilde{a},\widetilde{b},\xi,\eta)=(v,u'',rv(0)\xi,-rv(0)\eta,-\omega\eta,-\omega\xi),\\ D(\mathscr{A})=\{(u,v,\widetilde{a},\widetilde{b},\xi,\eta)\in H^2(0,1)\times H^1(0,1)\times\mathbb{R}^4|\\ \quad u'(0)=k_1v(0)+k_2u(0)+\widetilde{a}\xi+\widetilde{b}\eta,\ u'(1)=0\}.\end{cases}$$

Eq. (19) is a nonlinear autonomous evolution system. However, same as Guo and Guo (2013a), it seems hard to use nonlinear semigroup to prove its well-posedness due to the lack of dissipativity of $\mathscr{A}$ or any other kind of $\mathscr{A}+\mu I$ for constant $\mu\in\mathbb{R}$. Hence we invoke the Galerkin method to establish the existence and uniqueness for the solution of Eq. (3). To do this, we need a basis to construct a Galerkin approximation, which can be realized by the operator A defined in $L^2(0,1)$ as follows:

$$\begin{cases}A\phi=-\phi'',\quad\forall\phi\in D(A),\\ D(A)=\{\phi\in L^2(0,1)|\phi'(0)=0,\phi(1)=0\}.\end{cases}$$

It is seen that A is unbounded self-adjoint positive definite in $L^2(0,1)$ with compact resolvent. A simple computation shows that the eigenpairs $\{(\lambda_n,\phi_n)\}_{n=1}^\infty$ of A are

$$\begin{cases}\lambda_n=-\omega_n^2,\qquad\omega_n=i\left(n+\frac12\right)\pi,\\ \phi_n(x)=2\cos\omega_nx=2\cos\left(n+\frac12\right)\pi x.\end{cases} \tag{20}$$

Since $\{\phi_n(x)\}_{n=1}^\infty$ defined by (20) forms an orthogonal basis for $L^2(0,1)$, we can then follow the steps as those in Guo and Guo (2013a) to construct a Galerkin scheme to prove the existence and uniqueness for the classical solution to error system (3).

Let $V=H^3(0,1)\bigcap D(A)$.

Theorem 3.1. *Suppose that $(\varepsilon_0,\varepsilon_1,\widetilde{a}_0,\widetilde{b}_0)\in V\times V\times\mathbb{R}^2$, and they satisfy the following compatible condition:*

$$k_1\varepsilon_1(0)+k_2\varepsilon_0(0)+\widetilde{a}_0=0 \tag{21}$$

and

$$k_1\varepsilon_0''(0)-\omega\widetilde{b}_0=0. \tag{22}$$

Then system (3) admits a unique classical solution ε. That is to say, for any time $T>0$,

$$\begin{cases}\varepsilon\in L^\infty(0,T;H^3(0,1)),\qquad\varepsilon_t\in L^\infty(0,T;H^2(0,1)),\\ \varepsilon_{tt}\in L^\infty(0,T;H^1(0,1)),\qquad\widetilde{a}\in C^1[0,T],\widetilde{b}\in C^1[0,T],\\ \varepsilon_{tt}(x,t)-\varepsilon_{xx}(x,t)=0\quad\text{in }L^\infty(0,T;L^2(0,1)),\\ \varepsilon_x(0,t)=k_1\varepsilon_t(0,t)+k_2\varepsilon(0,t)+\widetilde{a}(t)\cos\omega t+\widetilde{b}(t)\sin\omega t,\\ \varepsilon_x(1,t)=0,\\ \dot{\widetilde{a}}(t)=r\varepsilon_t(0,t)\cos\omega t,\\ \dot{\widetilde{b}}(t)=r\varepsilon_t(0,t)\sin\omega t,\\ \widetilde{a}(0)=\widetilde{a}_0,\qquad\widetilde{b}(0)=\widetilde{b}_0,\\ \varepsilon(x,0)=\varepsilon_0(x),\qquad\varepsilon_t(x,0)=\varepsilon_1(x).\end{cases}$$

By the Sobolev embedding theorem, it follows that $\varepsilon\in C([0,1]\times[0,T])$.

Remark 3.1. In Theorem 3.1, condition (21) is the natural compatible condition for the classical solution of (3), and condition (22) is for the existence of the more smoother solution that we shall need in the proof of Theorem 3.2.

Next, we establish the convergence of error system (3). To do this, we need the weak solution of (3).

Definition 3.1. For any initial data $(\varepsilon_0,\varepsilon_1,\widetilde{a}_0,\widetilde{b}_0)\in\mathscr{V}$,the weak solution $(\varepsilon,\varepsilon_t,\widetilde{a},\widetilde{b})$ of Eq. (3) is defined as the limit of any convergent subsequence of $(\varepsilon^n,\varepsilon_t^n,\widetilde{a}^n,\widetilde{b}^n)$ in the space $L^\infty(0,\infty;\mathscr{V})$ where $(\varepsilon^n,\varepsilon_t^n,\widetilde{a}^n,\widetilde{b}^n)$ is the classical solution (ensured by Theorem 3.1) with the initial condition (for all $x\in(0,1)$)

$$\begin{aligned}&\left(\varepsilon^n(x,0),\varepsilon_t^n(x,0),\widetilde{a}^n(0),\widetilde{b}^n(0)\right)\\ &\quad=\left(\varepsilon_0^n(x),\varepsilon_1^n(x),\widetilde{a}_0^n,\widetilde{b}_0^n\right)\in V\times V\times\mathbb{R}^2,\end{aligned}$$

which satisfies

$$\lim_{n\to\infty} \left\| \left(\varepsilon_0^n(x), \varepsilon_1^n(x), \widetilde{a}_0^n, \widetilde{b}_0^n\right) - (\varepsilon_0, \varepsilon_1, \widetilde{a}_0, \widetilde{b}_0) \right\|_{\mathscr{V}} = 0.$$

By (5) and (6), the above weak solution is well defined, since it does not depend on the choice of initial sequence $(\varepsilon^n(x,0), \varepsilon_t^n(x,0), \widetilde{a}^n(0), \widetilde{b}^n(0))$.

Theorem 3.2. *Suppose that*

$$\omega \neq 0, n\pi + \frac{\pi}{2}, \quad n \in \mathbb{Z}. \tag{23}$$

Then for any initial value $(\varepsilon_0, \varepsilon_1, \widetilde{a}_0, \widetilde{b}_0) \in \mathscr{V}$, *the solution of system* (3) *is asymptotically stable in the sense that*

$$\lim_{t\to\infty} \left[\frac{1}{2} \int_0^1 [\varepsilon_t^2(x,t) + \varepsilon_x^2(x,t)]dx + k_2 \varepsilon^2(0,t) \right] = 0$$

and

$$\lim_{t\to\infty} \widehat{a}(t) = a, \qquad \lim_{t\to\infty} \widehat{b}(t) = b.$$

Proof. By density argument, we may assume without loss of generality that the initial value $(\varepsilon_0, \varepsilon_1, \widetilde{a}_0, \widetilde{b}_0)$ belongs to $V \times V \times \mathbb{R}^2$ and satisfies compatible conditions (5) and (6). Construct the Lyapunov functional $V_\varepsilon(t)$ for system (19) as follows:

$$\begin{aligned} V_\varepsilon(t) &= \frac{1}{2} \int_0^1 [\varepsilon_t^2(x,t) + \varepsilon_x^2(x,t)]dx + \frac{k_2}{2} \varepsilon^2(0,t) \\ &\quad + \frac{1}{2r} \left[\widetilde{a}^2(t) + \widetilde{b}^2(t)\right] + [\xi^2(t) + \eta^2(t)]. \end{aligned} \tag{24}$$

A simple computation of the time derivative of $V_\varepsilon(t)$ along the solution of system (19) shows

$$\dot{V}_\varepsilon(t) = -k_1 [\varepsilon_t(0,t)]^2.$$

This concludes that $V_\varepsilon(t) \leq V_\varepsilon(0)$, hence

$$\begin{aligned} &\sup_{t\geq 0} \left[\frac{1}{2} \int_0^1 [\varepsilon_t^2(x,t) + \varepsilon_x^2(x,t)]dx + k_2 \varepsilon^2(0,t) \right. \\ &\quad \left. + |\widetilde{a}(t)| + |\widetilde{b}(t)| \right] < \infty. \end{aligned} \tag{25}$$

In particular, one has

$$\varepsilon_t(0,t) \in L^2(0,\infty). \tag{26}$$

Similarly, define

$$U(t) = \frac{1}{2} \int_0^1 [\varepsilon_{xx}^2(x,t) + \varepsilon_{tx}^2(x,t)]dx + \frac{k_2 + r}{2} \varepsilon_t^2(0,t).$$

The time derivative of $U(t)$ along the solution of (3) can be found as

$$\dot{U}(t) = -k_1 [\varepsilon_{tt}(0,t)]^2 + \omega \varepsilon_{tt}(0,t)[\widetilde{a}(t)\sin\omega t - \widetilde{b}(t)\cos\omega t]. \tag{27}$$

Integrating over $[0,t]$ on both sides of (27) gives

$$\begin{aligned} U(t) &= -k_1 \int_0^t [\ddot{\varepsilon}(0,s)]^2 ds + \omega \int_0^t \ddot{\varepsilon}(0,s)[\widetilde{a}\sin\omega s \\ &\quad - \widetilde{b}(s)\cos\omega s]ds + U(0) \\ &= -k_1 \int_0^t [\ddot{\varepsilon}(0,s)]^2 ds + \omega\dot{\varepsilon}(0,t)[\widetilde{a}(t)\sin\omega t \\ &\quad - \widetilde{b}(t)\cos\omega t] - \omega\varepsilon_1(0)\widetilde{b}_0 + \omega \int_0^t \dot{\varepsilon}(0,s)[\widetilde{a}(t)\cos\omega s \\ &\quad + \widetilde{b}(s)\sin\omega s]ds = -k_1 \int_0^t [\ddot{\varepsilon}(0,s)]^2 ds \\ &\quad + \omega\dot{\varepsilon}(0,t)[\widetilde{a}(t)\sin\omega t - \widetilde{b}(t)\cos\omega t] - \omega\varepsilon_1(0)\widetilde{b}_0 \\ &\quad + \frac{\omega}{2r}[\widetilde{a}^2(t) + \widetilde{b}^2(t)] - \frac{\omega}{2r}[\widetilde{a}_0^2 + \widetilde{b}_0^2] + U(0). \end{aligned} \tag{28}$$

Use Young Inequality in (28) to obtain

$$\begin{aligned} U(t) &\leq \frac{1}{\delta_2 k_2}[\widetilde{a}(t)\sin\omega t - \widetilde{b}(t)\cos\omega t]^2 + \delta_2 k_2 \varepsilon_t^2(0,t) \\ &\quad + |\omega\varepsilon_1(0)\widetilde{b}_0| + \frac{\omega}{2r}[\widetilde{a}^2(t) + \widetilde{b}^2(t)] + U(0). \end{aligned} \tag{29}$$

Taking $0 < \delta_2 < \frac{1}{4}$ in (29) gives

$$U(t) - \delta k_2 \varepsilon_t^2(0,t) \geq \frac{1}{2} U(t),$$

which implies

$$\begin{aligned} U(t) &\leq \frac{2}{\delta_2 k_2}[\widetilde{a}(t)\sin\omega t - \widetilde{b}(t)\cos\omega t]^2 \\ &\quad + 2|\omega\varepsilon_1(0)\widetilde{b}_0| + \frac{\omega}{r}[\widetilde{a}^2(t) + \widetilde{b}^2(t)] + 2U(0). \end{aligned} \tag{30}$$

It follows from (25), (26) and (30) that

$$\sup_{t\geq 0} U(t) < \infty,$$

which implies that the trajectory of system (3)

$$\gamma(\varepsilon_0) = \{(\varepsilon, \varepsilon_t, , \widetilde{a}(t), \widetilde{b}(t), \xi(t), \eta(t)) | t \geq 0\}$$

is precompact in $\mathscr{H}$. In the light of Lasalle's invariance principle (Walker, 1980), any solution of system (3) tends to the maximal invariant set of the following:

$$S = \{(\varepsilon, \varepsilon_t, \widetilde{a}(t), \widetilde{b}(t), \eta_1(t), \eta_2(t)) \in \mathscr{H} | \dot{V}_\varepsilon(t) = 0\}.$$

Now, since $\dot{V}_\varepsilon(t) = 0$, it follows that $\varepsilon_t(0,t) = 0$, which implies $\dot{\widetilde{a}}(t) \equiv 0$ and $\dot{\widetilde{b}}(t) \equiv 0$. So $\widetilde{a} \equiv \widetilde{a}_0$ and $\widetilde{b} \equiv \widetilde{b}_0$. Thus the solution reduces to

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t), \\ \varepsilon_x(0,t) = k\varepsilon_0(0) + \widetilde{a}_0\cos\omega t + \widetilde{b}_0\sin\omega t, \\ \varepsilon_x(1,t) = 0, \qquad \varepsilon_t(0,t) = 0. \end{cases} \tag{31}$$

The proof will be completed if we can prove that (31) admits zero solution only. To this end, we first consider the equation

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}, \\ \varepsilon_x(1,t) = 0, \qquad \varepsilon_t(0,t) = 0. \end{cases} \tag{32}$$

Introduce a Hilbert space $\mathbb{H} = H_L^1(0,1) \times L^2(0,1)$ with the inner product

$$\langle (y_1, z_1), (y_2, z_2) \rangle_{\mathbb{H}} = \int_0^1 [y_1'(x)\overline{y_2'(x)} + z_1(x)\overline{z_2(x)}]dx,$$

where $H_L^1(0,1) = \{u \in H^1(0,1) | u(0) = 0\}$. Define a linear operator $\mathbb{A}$ in $\mathbb{H}$ associated to system (32)

$$\begin{cases} \mathbb{A}(y,z) = (z, y''), \\ D(\mathbb{A}) = \{(y,z) \in H^2(0,1) \times H^1(0,1) | y'(1) = 0, z(0) = 0\}. \end{cases} \tag{33}$$

It is a simple exercise to show that $(\mu I - \mathbb{A})^{-1}$ is compact on $\mathbb{H}$ for some $\mu > 0$. Hence $\mathbb{A}$ is a skew-adjoint operator with compact resolvent on $\mathbb{H}$. Consequently, the spectrum of $\mathbb{A}$ consists of isolated eigenvalues on the imaginary axis only, and from a general result of functional analysis, the algebraic multiplicity of each eigenvalue of

$\mathbb{A}$ is equal to its geometric multiplicity. Solve the eigenvalue problem

$$\mathbb{A}(\phi,\psi)=\lambda(\phi,\psi)$$

for any $\lambda\in\sigma_p(\mathbb{A})$. The solution is $\psi=\lambda\phi$ with $\phi\neq 0$ satisfying

$$\begin{cases}\lambda^2\phi(x)-\phi''(x)=0,\\ \phi'(1)=0,\qquad \lambda\phi(0)=0.\end{cases}\tag{34}$$

Solve (34) in the case where $\lambda=0$ to give

$$\phi(x)=c,\tag{35}$$

where c is a constant. When $\lambda\neq 0$,

$$\phi(x)=e^{\lambda x}-e^{-\lambda x}\tag{36}$$

with

$$e^{2\lambda}=-1.\tag{37}$$

So λ is geometrically simple. So each λ is algebraically simple.

Furthermore, from (36), we can obtain eigen-pairs of $\mathbb{A}$:

$$\begin{cases}\lambda_n=\left(n\pi+\dfrac{\pi}{2}\right)i,\qquad \lambda_{-n}=\overline{\lambda}_n,\\ \Phi_n=(\lambda_n^{-1}\phi_n,\phi_n),\ \Phi_{-n}=(\lambda_{-n}^{-1}\phi_n,\phi_n),\end{cases}\tag{38}$$

where

$$\phi_n(x)=\sin\left(n+\frac{1}{2}\right)\pi x,\quad n\in\mathbb{Z}.$$

By general theory of functional analysis, $\{\Phi_n\}_{n\in\mathbb{Z}}$ forms an orthogonal basis for $\mathbb{H}$. Therefore, the solution of (32) can be represented as

$$(\varepsilon(\cdot,t),\dot{\varepsilon}(\cdot,t))=k_2d_0(c,0)+\sum_{n=1}^{\infty}d_ne^{\lambda_nt}\Phi_n+\sum_{n=1}^{\infty}d_{-n}e^{\lambda_{-n}t}\Phi_{-n},$$

where the constants $\{d_n\}_{n\in\mathbb{Z}}$ are determined by the initial condition. That is,

$$\varepsilon_0=d_0c+\sum_{n=1}^{\infty}\frac{d_n}{\lambda_n}\phi_n+\sum_{n=1}^{\infty}\frac{d_{-n}}{\lambda_{-n}}\phi_n,$$

$$\varepsilon_1=\sum_{n=1}^{\infty}d_n\phi_n+\sum_{n=1}^{\infty}d_{-n}\phi_n.$$

Hence

$$\begin{aligned}\varepsilon_x(0,t)&=\sum_{n=1}^{\infty}d_n\frac{\phi_n'(0)}{\lambda_n}e^{\lambda_nt}+\sum_{n=1}^{\infty}d_{-n}\frac{\phi_n'(0)}{\lambda_{-n}}e^{\lambda_{-n}t}\\&=k_2d_0c+\widetilde{a}_0\cos\omega t+\widetilde{b}_0\sin\omega t.\end{aligned}$$

Therefore,

$$\begin{aligned}&-k_2d_0c+\sum_{n=1}^{\infty}d_n\frac{\phi_n'(0)}{\lambda_n}e^{\lambda_nt}\\&+\sum_{n=1}^{\infty}d_{-n}\frac{\phi_n'(0)}{\lambda_{-n}}e^{\lambda_{-n}t}-\frac{1}{2}[\widetilde{a}_0-\mathrm{i}\widetilde{b}_0]e^{\mathrm{i}\omega t}\\&-\frac{1}{2}[\widetilde{a}_0+\mathrm{i}\widetilde{b}_0]e^{-\mathrm{i}\omega t}=0.\end{aligned}\tag{39}$$

We now show that $d_{\pm n}=0$, for all $n\geq 1$. Since otherwise, if there exists $n_0\geq 1$ such that $\left|d_{n_0}\frac{\phi_{n_0}'(0)}{\lambda_{n0}}\right|\neq 0$, then $d_{n_0}\neq 0$ due to the fact $\phi_n'(0)\neq 0$ for all n. Furthermore, the smoothness of the initial value guarantees that $\sum_{n\in\mathbb{Z},n\neq 0}\left|d_n\frac{\phi_n'(0)}{\lambda_n}\right|<\infty$, which implies that there exists an integer $N>n_0$ such that

$$\begin{aligned}&\sum_{n=N}^{\infty}\left|d_n\frac{\phi_n'(0)}{\lambda_n}\right|<\frac{1}{4}\left|d_{n_0}\frac{\phi_{n_0}'(0)}{\lambda_{n0}}\right|,\\&\sum_{n=N}^{\infty}\left|d_{-n}\frac{\phi_{-n}'(0)}{\lambda_{-n}}\right|<\frac{1}{4}\left|d_{n_0}\frac{\phi_{n_0}'(0)}{\lambda_{n0}}\right|.\end{aligned}\tag{40}$$

Since $\lambda_n\neq\lambda_m$ for any $n,m\in\mathbb{Z}$, $n\neq m$ and $|\lambda_{n+1}-\lambda_n|=\pi$, $n\in\mathbb{Z}$, one has, for $t>0$,

$$\begin{aligned}&d_{n_0}\frac{\phi_{n_0}'(0)}{\lambda_{n0}}+\sum_{n=N+1}^{\infty}d_n\frac{\phi_n'(0)}{\lambda_n}e^{(\lambda_n-\lambda_{n_0})t}\\&\quad+\sum_{n=1,n\neq n_0}^{N}d_n\frac{\phi_n'(0)}{\lambda_n}e^{(\lambda_n-\lambda_{n_0})t}+\sum_{n=N+1}^{\infty}d_{-n}\frac{\phi_{-n}'(0)}{\lambda_{-n}}e^{(\lambda_{-n}-\lambda_{n_0})t}\\&\quad+\sum_{n=1}^{N}d_{-n}\frac{\phi_{-n}'(0)}{\lambda_{-n}}e^{(\lambda_{-n}-\lambda_{n_0})t}\\&\quad-k_2d_0ce^{-\lambda_{n_0}t}-\frac{1}{2}[\widetilde{a}_0-\mathrm{i}\widetilde{b}_0]e^{(\mathrm{i}\omega-\lambda_{n0})t}\\&\quad-\frac{1}{2}[\widetilde{a}_0+\mathrm{i}\widetilde{b}_0]e^{-(\mathrm{i}\omega+\lambda_{n0})t}=0.\end{aligned}\tag{41}$$

Integrating over $[0,t]$ on both sides of (41) and using (40), and the fact $\mathrm{Re}\lambda_n=0$, we have

$$\begin{aligned}&\left|d_{n_0}\frac{\phi_{n_0}'(0)}{\lambda_{n0}}\right|t\leq 2\left|\int_0^t\sum_{n=1,n\neq n_0}^{N}d_n\frac{\phi_n'(0)}{\lambda_n}e^{(\lambda_n-\lambda_{n_0})s}ds\right|\\&\quad+2\left|\int_0^t\sum_{n=1}^{N}d_{-n}\frac{\phi_{-n}'(0)}{\lambda_{-n}}e^{(\lambda_{-n}-\lambda_{n_0})s}ds\right|\\&\quad+2\left|\int_0^t k_2d_0ce^{-\lambda_{n_0}s}ds\right|+\left|\int_0^t[\widetilde{a}_0-\mathrm{i}\widetilde{b}_0]e^{(\mathrm{i}\omega-\lambda_{n0})s}ds\right|\\&\quad+\left|\int_0^t[\widetilde{a}_0+\mathrm{i}\widetilde{b}_0]e^{-(\mathrm{i}\omega+\lambda_{n0})s}ds\right|.\end{aligned}$$

Since the right side of the above equation has an upper bound for all $t\geq 0$, we conclude that $d_{n_0}=0$, which is a contradiction. Hence $d_{\pm n}=0$, $n=1,2,\ldots$ and by (39), $d_0=\widetilde{a}_0=\widetilde{b}_0=0$. We have thus proved that $S=\{(0,0,0,0,1,0)\}$, that is

$$\begin{aligned}&\lim_{t\to\infty}\left[\frac{1}{2}\int_0^1[\varepsilon_t^2(x,t)+\varepsilon_x^2(x,t)]dx+c_0\varepsilon^2(0,t)\right.\\&\quad\left.+\frac{1}{2r}\widetilde{a}^2(t)+\frac{1}{2r}\widetilde{b}^2(t)\right]=0.\end{aligned}$$

The proof is complete. ■

4. Convergence of the auxiliary system

Define $\mathbf{H}=H^1(0,1)\times L^2(0,1)$, which is a Hilbert Space with the two following equivalent norms induced by the inner product:

$$\|(p,q)\|^2_{(\mathbf{H};\|\cdot\|_1)}=\int_0^1[|p'(x)|^2dx+|q(x)|^2]dx+c_0|p(1)|^2,\quad\forall(p,q)\in\mathbf{H}$$

and

$$\|(p,q)\|^2_{(\mathbf{H};\|\cdot\|_2)}=\int_0^1[|p'(x)|^2dx+|q(x)|^2]dx+c_0|p(0)|^2,\quad\forall\,(p,q)\in\mathbf{H}.$$

In the rest of the paper, we write norm $\|\cdot\|_{\mathbf{H}}$ without discrimination.

Theorem 4.1. *For any initial value* $(\widehat{z}_0,\widehat{z}_1)\in\mathbf{H}$, *then there exists a unique (weak) solution to* (12) *such that* $(\widehat{z},\widehat{z}_t)\in C(0,\infty;\mathbf{H})$, *Moreover, the solution of* (12) *is asymptotically stable in the sense that*

$$\lim_{t\to\infty}E_{\widehat{z}}(t)=\lim_{t\to\infty}\left[\frac{1}{2}\int_0^1[\widehat{z}_x^2(x,t)+\widehat{z}_t^2(x,t)]dx+\frac{1}{2}c_0[\widehat{z}(1,t)]^2\right]=0.$$

Proof. For any initial value $(\widehat{z}_0,\widehat{z}_1)\in\mathbf{H}$, it follows from (15) that $(\widetilde{z}_0,\widetilde{z}_1)\in\mathbf{H}$. By the transformation (13), it is sufficient to prove system (14) has a unique (weak) solution $(\widetilde{z},\widetilde{z}_t)\in C(0,\infty;\mathbf{H})$ and asymptotically stabilization of system (14) in the sense that

$$\lim_{t\to\infty}E_{\widetilde{z}}(t)=\lim_{t\to\infty}\left[\frac{1}{2}\int_0^1[\widetilde{z}_x^2(x,t)+\widetilde{z}_t^2(x,t)]dx+\frac{1}{2}c_0[\widetilde{z}(0,t)]^2\right]=0. \tag{42}$$

Define an operator $\mathbf{A}:\mathbf{D}(\mathbf{A})\to\mathbf{H}$ by

$$\begin{cases}\mathbf{A}(u,v)^\top=(v,u'')^\top,\quad\forall(u,v)\in\mathbf{D}(\mathbf{A})\\ \mathbf{D}(\mathbf{A})=\{(u,v)^\top\in\mathbf{H}|\ \mathbf{A}(u,v)^\top\in\mathbf{H},\\ \qquad f'(0)=c_0f(0),\ f'(1)=-c_1g(1)\}.\end{cases} \tag{43}$$

Then system (14) can be written as

$$\frac{d}{dt}\begin{pmatrix}\widetilde{z}(\cdot,t)\\ \widetilde{z}_t(\cdot,t)\end{pmatrix}=\mathbf{A}\begin{pmatrix}\widetilde{z}(\cdot,t)\\ \widetilde{z}_t(\cdot,t)\end{pmatrix}+\begin{pmatrix}0\\ f(\cdot,t)\end{pmatrix}+\mathbf{B}[-k_1\varepsilon_t(0,t)-(k_2+r)\varepsilon(0,t)],$$

where $f(x,t)=-\varepsilon(0,t)[\omega r\sin\omega x-c_0r\cos\omega x+2c_0r+c_0k_2]+c_0k_1\varepsilon_t(0,t)$, $\mathbf{B}=(0,-\delta(x))^\top$.

It is well-known that $\mathbf{A}$ generates an exponential stable C_0-semigroup. Then there exist $K,\ \mu>0$ such that

$$\|e^{\mathbf{A}t}\|\le Ke^{-\mu t}. \tag{44}$$

It is a routine exercise

that $\mathbf{B}$ are admissible for $\mathbf{A}$. By Weiss (1989), it concludes that for any initial value $(\widetilde{z}_0,\widetilde{z}_1)\in\mathbf{H}$, there exists an unique solution $(\widetilde{z},\widetilde{z}_t)(\in C(0,\infty;\mathbf{H}))$ to system (14) in $\mathbf{H}$, which takes the form

$$\begin{aligned}\begin{pmatrix}\widetilde{z}(\cdot,t)\\ \widetilde{z}_t(\cdot,t)\end{pmatrix}&=e^{\mathbf{A}t}\begin{pmatrix}\widetilde{z}(\cdot,0)\\ \widetilde{z}_t(\cdot,0)\end{pmatrix}\\ &+\int_0^te^{\mathbf{A}(t-s)}\begin{pmatrix}0\\ f(\cdot,s)\end{pmatrix}ds-k_1\int_0^te^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon_t(0,s)ds\\ &-(k_2+r)\int_0^te^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon(0,s)ds.\end{aligned} \tag{45}$$

The first part is proved.

Now we are in a position to prove system (14) is asymptotically stable. By (44), the norm of the first term on the right side of (45) can be estimated as

$$\begin{aligned}\left\|e^{\mathbf{A}t}\begin{pmatrix}\widetilde{z}(\cdot,0)\\ \widetilde{z}_t(\cdot,0)\end{pmatrix}\right\|_{\mathbf{H}}&\le\|e^{\mathbf{A}t}\|\left\|\begin{pmatrix}\widetilde{z}(\cdot,0)\\ \widetilde{z}_t(\cdot,0)\end{pmatrix}\right\|_{\mathbf{H}}\\ &\le Ke^{-\mu t}\left\|\begin{pmatrix}\widetilde{z}(\cdot,0)\\ \widetilde{z}_t(\cdot,0)\end{pmatrix}\right\|_{\mathbf{H}}.\end{aligned} \tag{46}$$

We rewrite the second term on the right side of (45) to be

$$\begin{aligned}\int_0^te^{\mathbf{A}(t-s)}\begin{pmatrix}0\\ f(\cdot,s)\end{pmatrix}ds&=\int_0^te^{\mathbf{A}(t-s)}\begin{pmatrix}0\\ m(\cdot)\end{pmatrix}\varepsilon(0,s)ds\\ &+\int_0^te^{\mathbf{A}(t-s)}\begin{pmatrix}0\\ c_0k_1\end{pmatrix}\varepsilon_t(0,s)ds,\end{aligned} \tag{47}$$

where $m(x)=\omega r\sin\omega x-c_0r\cos\omega x+2c_0r+c_0k_2$. For the first term on the right side of (47), one has

$$\begin{aligned}&\left\|\int_0^te^{\mathbf{A}(t-s)}\begin{pmatrix}0\\ m(\cdot)\end{pmatrix}\varepsilon(0,s)ds\right\|_{\mathbf{H}}\\ &\le\int_0^t\|e^{\mathbf{A}(t-s)}\|\|m(\cdot)\|_{L^2(0,1)}|\varepsilon(0,s)|ds\\ &\le K\|m(\cdot)\|_{L^2(0,1)}\int_0^te^{-\mu(t-s)}|\varepsilon(0,s)|ds.\end{aligned} \tag{48}$$

By the fact

$$\varepsilon^2(0,t)\le\frac{2}{k_2}E_\varepsilon(t)\to0,\quad\text{as } t\to\infty.$$

For any given $\eta>0$, there exists $t_0>0$, such that $|\varepsilon(0,t)|<\frac{\mu\eta}{2}$, for all $t>t_0$. Then

$$\begin{aligned}\int_0^te^{-\mu(t-s)}|\varepsilon(0,s)|ds&\le\int_0^{t_0}e^{-\mu(t-s)}|\varepsilon(0,s)|ds\\ &\quad+\int_{t_0}^te^{-\mu(t-s)}|\varepsilon(0,s)|ds\\ &\le\sqrt{\frac{2}{k_2}E_\varepsilon(0)}\mu^{-1}e^{-\mu(t-t_0)}+\frac{\eta}{2}.\end{aligned}$$

Choosing $t>t_0$ large enough, the first term on the right-hand side above will be less than $\frac{\eta}{2}$, and thus for t large enough $\int_0^te^{-\mu(t-s)}|\varepsilon(0,s)|ds<\eta$, which implies that

$$\lim_{t\to\infty}\int_0^te^{-\mu(t-s)}|\varepsilon(0,s)|ds=0.$$

This together with (48) concludes that

$$\lim_{t\to\infty}\left\|\int_0^te^{\mathbf{A}(t-s)}\begin{pmatrix}0\\ m(\cdot)\end{pmatrix}\varepsilon(0,s)ds\right\|_{\mathbf{H}}=0. \tag{49}$$

On one hand, for the second term on righthand of (47), we have

$$\begin{aligned}&\left\|\int_0^te^{\mathbf{A}(t-s)}\begin{pmatrix}0\\ c_0k_1\end{pmatrix}\varepsilon_t(0,s)ds\right\|_{\mathbf{H}}\\ &\le c_0k_1\int_0^te^{-\mu(t-s)}|\varepsilon_t(0,s)|ds.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_0^te^{-\mu(t-s)}|\varepsilon_t(0,s)|ds&=\int_0^{\frac{t}{2}}e^{-\mu(t-s)}|\varepsilon_t(0,s)|ds\\ &\quad+\int_{\frac{t}{2}}^te^{-\mu(t-s)}|\varepsilon_t(0,s)|ds=\int_{\frac{t}{2}}^te^{-\mu\tau}|\varepsilon_t(0,t-\tau)|d\tau\\ &\quad+\int_{\frac{t}{2}}^te^{-\mu(t-s)}|\varepsilon_t(0,s)|ds\\ &\le\sqrt{\frac{t}{2}}e^{-\frac{\mu t}{2}}\left[\int_{\frac{t}{2}}^t[\varepsilon_t(0,t-\tau)]^2d\tau\right]^{\frac{1}{2}}\\ &\quad+\left[\int_{\frac{t}{2}}^te^{-2\mu(t-s)}ds\right]^{\frac{1}{2}}\left[\int_{\frac{t}{2}}^t[\varepsilon_t(0,s)]^2ds\right]^{\frac{1}{2}}\\ &\le\sqrt{\frac{t}{2}}e^{-\frac{\mu t}{2}}\left[\int_0^\infty[\varepsilon_t(0,\tau)]^2d\tau\right]^{\frac{1}{2}}+\left[\int_{\frac{t}{2}}^{+\infty}[\varepsilon_t(0,s)]^2ds\right]^{\frac{1}{2}}.\end{aligned}$$

This together with the fact $\varepsilon_t(0,t)\in L^2(0,\infty)$ yields

$$\left\|\int_0^t e^{\mathbf{A}(t-s)}\begin{pmatrix}0\\ c_0k_1\end{pmatrix}\varepsilon_t(0,s)ds\right\|_{\mathbf{H}}$$
$$\le c_0k_1\int_0^t e^{-\mu(t-s)}|\varepsilon_t(0,s)|ds\to 0,\quad \text{as } t\to\infty. \tag{50}$$

The admissibility of $\mathbf{B}$ implies that

$$\left\|\int_0^{t_0} e^{\mathbf{A}(t_0-s)}\mathbf{B}\varepsilon(0,s)ds\right\|_{\mathbf{H}}^2\le C_{t_0}\|\varepsilon(0,s)\|^2_{L^2_{loc}(0,t_0)}$$
$$\le t_0^2C_{t_0}\|\varepsilon(0,s)\|^2_{L^\infty(0,t_0)},$$

where C_{t_0} is a positive constant that is independent of $\varepsilon(0,s)$. From the fact that $e^{\mathbf{A}t}$ is exponentially stable, it follows from Proposition 2.5 of Weiss (1989) that

$$\left\|\int_{t_0}^t e^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon(0,s)ds\right\|_{\mathbf{H}}=\left\|\int_0^t e^{A(t-s)}\mathbf{B}(0\diamond_{t_0}\varepsilon)(s)ds\right\|_{\mathbf{H}}$$
$$\le K_0\|\varepsilon(0,s)\|_{L^\infty(0,\infty)}<\frac{\eta}{2}, \tag{51}$$

where K_0 is a constant is independent of $\varepsilon(0,s)$, and

$$(u\diamond_\tau v)(t)=\begin{cases}u(t),0\le t\le\tau\\ v(t-\tau),t>\tau.\end{cases}$$

Thus one has

$$\left\|\int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon(0,s)ds\right\|_{\mathbf{H}}$$
$$=\left\|e^{\mathbf{A}(t-t_0)}\int_0^{t_0}e^{\mathbf{A}(t_0-s)}\mathbf{B}\varepsilon(0,s)ds+\int_{t_0}^t e^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon(0,s)ds\right\|_{\mathbf{H}}$$
$$\le\left\|e^{\mathbf{A}(t-t_0)}\int_0^{t_0}e^{\mathbf{A}(t_0-s)}\mathbf{B}\varepsilon(0,s)ds\right\|_{\mathbf{H}}+\left\|\int_{t_0}^t e^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon(0,s)\right\|_{\mathbf{H}}$$
$$\le Ke^{-\mu(t-t_0)}\sqrt{C_{t0}}t_0\|\varepsilon(0,s)\|_{L^\infty(0,t_0)}+\frac{\eta}{2}. \tag{52}$$

Choose $t>t_0$ large enough to let the first term on the right-hand side above be less than $\frac{\eta}{2}$, and thus for t large enough $\left\|\int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon(0,s)ds\right\|_{\mathbf{H}}<\eta$, which implies that

$$\lim_{t\to\infty}\left\|\int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon(0,s)ds\right\|_{\mathbf{H}}=0. \tag{53}$$

By the fact that $\varepsilon_t(1,t)\in L^2(0,\infty)$, it follows that for any given $\sigma>0$,

$$\int_{t_1}^\infty|\varepsilon_t(1,s)|^2ds\le\sigma. \tag{54}$$

The admissibility of $\mathbf{B}$ implies that

$$\left\|\int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon_t(0,s)ds\right\|_{\mathbf{H}}$$
$$\le\left\|e^{\mathbf{A}(t-t_1)}\int_0^{t_1}e^{\mathbf{A}(t_1-s)}\mathbf{B}\varepsilon_t(1,s)ds\right\|_{\mathbf{H}}+\left\|\int_{t_1}^t e^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon_t(1,s)ds\right\|_{\mathbf{H}}$$
$$\le Ke^{-\omega(t-t_1)}\sqrt{K_{t_1}}t_1\|\varepsilon_t(1,s)\|_{L^2(0,t_1)}+\left\|\int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}(0\diamond_{t_1})\varepsilon_t(1,s)ds\right\|_{\mathbf{H}}$$
$$\le Ke^{-\omega(t-t_1)}\sqrt{K_{t_1}}t_1\|\varepsilon_t(1,s)\|_{L^2(0,t_1)}+K_1\|\varepsilon_t(1,s)\|_{L^2(t_1,\infty)}$$
$$\le Ke^{-\omega(t-t_1)}\sqrt{K_{t_1}}t_1\|\varepsilon_t(1,s)\|_{L^2(0,t_1)}+K_1\sqrt{\sigma}. \tag{55}$$

Same to (52), we have

$$\lim_{t\to\infty}\left\|\int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\varepsilon_t(1,s)ds\right\|_{\mathbf{H}}=0. \tag{56}$$

It follows from (45), (46), (49), (50), (53) and (56) that

$$\lim_{t\to\infty}\left\|\begin{pmatrix}\widetilde{z}(\cdot,t)\\ \widetilde{z}_t(\cdot,t)\end{pmatrix}\right\|_{\mathbf{H}}=0.$$

(42) is then proved. ■

5. Main result

We now go back to the closed-loop (17). Define $\mathscr{X}=\mathbf{H}\times\mathscr{V}$. Let us consider system (17) in space $\mathscr{X}$.

Theorem 5.1. *Suppose that $\omega\ne 0, n\pi+\frac{\pi}{2},\ n\in\mathbb{Z}$. For any initial value $(y_0,y_1,\widehat{y}_0,\widehat{y}_1,\widehat{a}_0,\widehat{b}_0)\in\mathscr{X}$, there exists a unique (weak) solution to (17) such that $(y(\cdot,t),y_t(\cdot,t),\widehat{y}(\cdot,t),\widehat{y}_t(\cdot,t),\widehat{a}(t),\widehat{b}(t)\in C([0,\infty]);\mathscr{X})$. Moreover, this closed-loop solution has the following properties:* (i). $\sup_{t\ge0}[\int_0^1[y_t^2(x,t)+y_x^2(x,t)+\widehat{y}_t^2(x,t)+\widehat{y}_x^2(x,t)]dx+\widehat{a}^2(t)+\widehat{b}^2(t)]<\infty$.

(ii). $\lim_{t\to\infty}\left[\int_0^1\{[y_x(x,t)-\widehat{y}_x(x,t)]^2+[y_t(x,t)-\widehat{y}_t(x,t)]^2\}dx+(y(0,t)-\widehat{y}(0,t))^2\right]=0$.

(iii). $\lim_{t\to\infty}\widehat{a}(t)=a,\ \lim_{t\to\infty}\widehat{b}(t)=b$.

(iv). $\lim_{t\to\infty}y(0,t)=0$.

(v). *When* $a=b=0$, $\int_0^1[y_t^2(x,t)+y_x^2(x,t)]dx+c_0y^2(0,t)\to 0$ *as* $t\to\infty$.

Proof. For any initial value $(y_0,y_1,\widehat{y}_0,\widehat{y}_1,\widehat{a}_0,\widehat{b}_0)\in\mathscr{X}$, it is easy to verify from (4) and (10) that $(\varepsilon_0,\varepsilon_1,\widetilde{a}_0,\widetilde{b}_0)\in\mathscr{V}$ and $(\widetilde{z}_0,\widetilde{z}_1)\in\mathbf{H}$, which implies that there exist a unique solution $(\varepsilon,\varepsilon_t,\widetilde{a}(t),\widetilde{b}(t))\in C[0,\infty;\mathscr{V}]$ to (3) and a unique solution $(\widehat{z},\widehat{z}_t)\in C[0,\infty;\mathbf{H}]$ to (9), respectively. It follows from (8) and $y=\widehat{y}+\varepsilon$ that system (17) has a unique solution (weak) $(y(\cdot,t),y_t(\cdot,t),\widehat{y}(\cdot,t),\widehat{y}_t(\cdot,t),\widehat{a}(t),\widehat{b}(t))\in C([0,\infty);\mathscr{X})$. The first part is proved.

From (8), we have

$$\left\|\begin{pmatrix}\widehat{y}(\cdot,t)\\ \widehat{y}_t(\cdot,t)\end{pmatrix}\right\|_{\mathbf{H}}\le\left\|\begin{pmatrix}\widehat{z}(\cdot,t)\\ \widehat{z}_t(\cdot,t)\end{pmatrix}\right\|_{\mathbf{H}}$$
$$+\|\cos\omega x\|_{L^2(0,1)}|\widehat{a}(t)\cos\omega t+\widehat{b}(t)\sin\omega t+r\varepsilon(0,t)|$$
$$+\|\sin\omega x\|_{L^2(0,1)}|\widehat{a}(t)\sin\omega t-\widehat{b}(t)\cos\omega t|.$$

This together with Theorems 3.2, 4.1 and the fact $\widehat{y}(0,t)=\widehat{z}(0,t)$ gives property (i)–(iv). (v) directly follows from (ii) and the stability of ε and $\widehat{y}$.

Remark 5.1. The technique in this paper can be used to track general sinusoidal and cosinusoid reference signals. For tracking the reference $y_{\rm ref}(t)=A\cos(\alpha t+\phi)+B\sin(\beta t+\psi)+C$ with the output $y(0,t)$, we only need $\widehat{z}$ in (8) to take the form: $\widehat{z}=\widehat{y}-\frac{1}{\omega}\sin\omega x[\widehat{a}(t)\cos\omega t+\widehat{b}(t)\sin\omega t+r\varepsilon(0,t)]-\cos\alpha xA\cos(\alpha t+\phi)-\cos\beta xB\sin(\beta t+\psi)-C$. ■

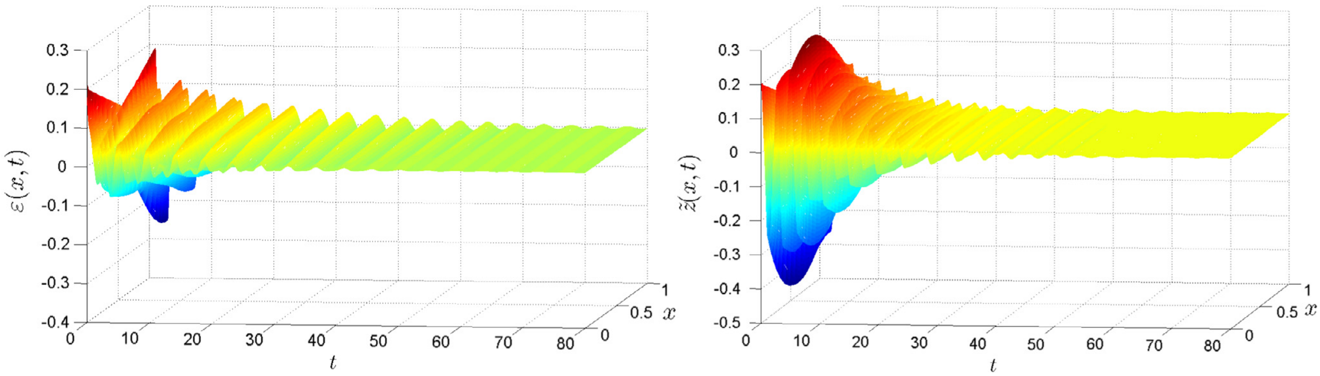

Fig. 1. Displacement of $\varepsilon(x,t)$ (left) and $\widetilde{z}(x,t)$ (right).

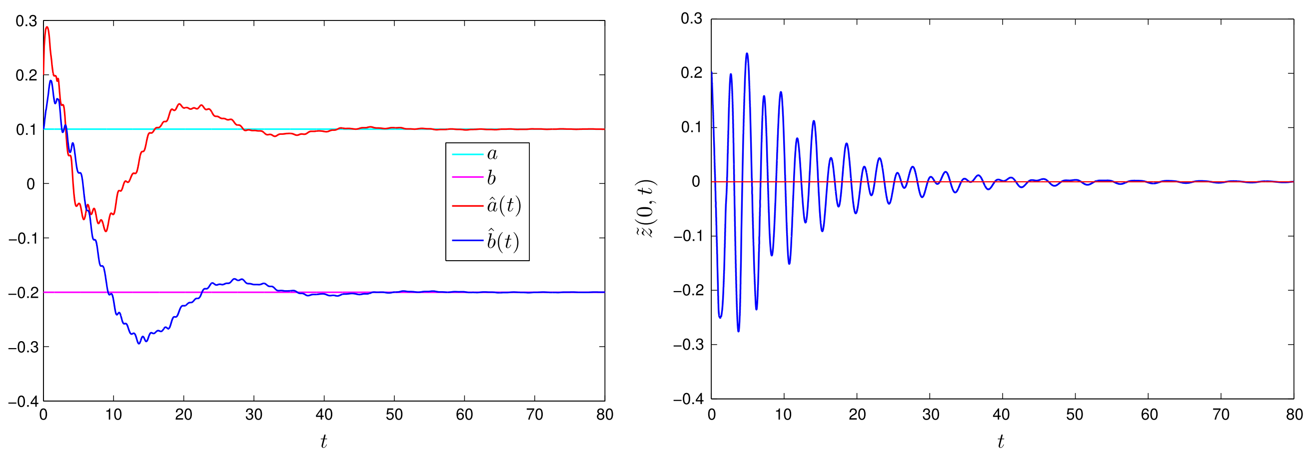

Fig. 2. Parameters estimations $\widehat{a}(t)$ and $\widehat{b}(t)$ (left) and the convergence of $\widetilde{z}(0,t)$ (right).

6. Numerical simulation

In this section, we present some numerical simulations to illustrate the theory results. In the simulation, the second order equations in time are firstly converted into a system of two first order equations, and then the backward Euler method in time and the Chebyshev spectral method in space are used. The grid size is taken as $N = 20$ and time step $dt = \times 10^{-3}$. We choose $k_1 = 0.9$, $k_2 = 4$, $c_0 = 2$, $c_1 = 0.9$ and $r = 1$. The other parameters are taken as: $a = 0.1$, $b = -0.2$, and $\omega = 3$. We can choose the proper initial conditions which are not listed here for simplicity to give the following simulation results. The numerical results for $\varepsilon(x,t)$ and $\widetilde{z}(x,t)$ are presented in Fig. 1. It is can be seen that system (3) and (14) are indeed asymptotically stable, which implies that the convergence of $\varepsilon(0,t)$ and $\widetilde{z}(0,t)$. Fig. 2(left) shows approximation of the parameters. It is seen that the estimates $\widehat{a}(t)$ and $\widehat{b}(t)$ with initial values $\widehat{a}_0 = 0.2$ and $\widehat{b}_0 = 0.1$ approximate, respectively, the system parameters $a = 0.1$ and $b = -0.2$. Fig. 2(right) shows that the convergence of $\widetilde{z}(0,t)$. The convergence of $\varepsilon(0,t)$ and $\widetilde{z}(0,t)$ together with the fact $\varepsilon(0,t) = y(0,t) - \widehat{y}(0,t)$, $\widehat{y}(0,t) = \widehat{z}(0,t) = \widetilde{z}(0,t)$ shows that $y(0,t)$ is convergent to zero as time goes to infinity. ■

7. Concluding remarks

This paper is concerned with output regulation and disturbance rejection for a wave equation with external harmonic disturbance anticollocated with control. An adaptive observer by the measured output is designed to estimate unknown parameters of the disturbance and recover the system state. An auxiliary system is constructed by using the observer and parameter estimators to make the control and the anticollocated disturbance to be collocated. By applying the Backstepping method for infinite dimensional system, we design an adaptive output feedback controller which regulates the output to zero and keep all the states bounded. This design method used in this paper also can be applied to an unstable wave equation presented in Krstic et al. (2008a). In future works, applying our approach to beam equation seems interesting, and relaxing the harmonic to general bounded disturbance is also interesting problem.

References

Ahmed-Ali, T., Giri, F., Krstic, M., Lamnabhi-Lagarrigue, F., & Burlion, L. (2016). Adaptive observer for a class of parabolic PDEs. *IEEE Transactions on Automatic Control*, *61*, 3083–3090.

Bernard, P., & Krstic, M. (2014). Adaptive output-feedback stabilization of non-local hyperbolic PDEs. *Automatica*, *50*, 2692–2699.

Bresch-Pietri, D., & Krstic, M. (2014). Output-feedback adaptive control of a wave PDE with boundary anti-damping. *Automatica*, *50*, 1407–1415.

Byrnes, C. I., Laukó, I. G., Gilliam, D. S., & Shubov, V. I. (2000). Output regulation problem for linear distributed parameter systems. *IEEE Transactions on Automatic Control*, *45*, 2236–2252.

Callier, F. M., & Desoer, C. A. (1980). Stabilization, tracking and disturbance rejection in multivariable convolution systems. *Annales de la Societe Scientifique de Bruxelles, Serie I (Sciences Mathematiques, Astronomiques et Physiques)*, *94*, 7–51.

Davison, E. J. (1976). The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Transactions on Automatic Control*, *21*, 25–34.

Desoer, C. A., & Lin, C. A. (1985). Tracking and disturbance rejection of MIMO nonlinear systems with PI controller. *IEEE Transactions on Automatic Control*, *30*, 861–867.

Deutscher, J. (2011). Output regulation for linear distributed-parameter systems using finite-dimensional dual observers. *Automatica*, *47*, 2468–2473.

Francis, B. A. (1977). The linear multivariable regulator problem. *SIAM Journal on Control and Optimization*, *15*, 486–505.

Francis, B. A., & Wonham, W. M. (1976). The internal model principle of control theory. *Automatica*, *12*, 457–465.

Guo, W., & Guo, B. Z. (2013a). Stabilization and regulator design for a one-dimensional unstable wave equation with input harmonic disturbance. *International Journal of Robust and Nonlinear Control*, *23*, 514–533.

Guo, Wei, & Guo, B. Z. (2013b). Parameter estimation and non-collocated adaptive stabilization for a wave equation with general boundary harmonic disturbance. *IEEE Transactions on Automatic Control*, *58*, 1631–1643.

Guo, W., & Guo, B. Z. (2013c). Adaptive output feedback stabilization for one-dimensional wave equation with corrupted observation by harmonic disturbance. *SIAM Journal on Control and Optimization*, *51*, 1679–1706.

Guo, W., & Guo, B. Z. (2016). Performance output tracking for a wave equation subject to unmatched general boundary harmonic disturbance. *Automatica*, *68*, 194–202.

Guo, B. Z., & Xu, C. Z. (2007). The stabilization of a one-dimensional wave equation by boundary feedback with non-collocated observation. *IEEE Transactions on Automatic Control*, *52*, 371–377.

Hämäläinen, T., & Pohjolainen, S. (2010). Robust regulation of distributed parameter systems with infinite-dimensional exosystems. *SIAM Journal on Control and Optimization*, *48*, 4846–4873.

Isidori, A., & Byrnes, C. I. (1990). Output regulation of nonlinear systems. *IEEE Transactions on Automatic Control*, *35*, 131–140.

Kobayashi, T., & Oya, M. (2002). Adaptive servomechanism design for boundary control system. *IMA Journal of Mathematical Control and Information*, *19*, 279–295.

Krstic, M. (2010). Adaptive control of an anti-stable wave PDE. *Dynamics of Continuous, Discrete, and Impulsive Systems*, *17*, 853–882.

Krstic, M., Guo, B. Z., Balogh, A., & Smyshlyaev, A. (2008a). Output-feedback stabilization of an unstable wave equation. *Automatica*, *44*, 63–74.

Krstic, M., Guo, B. Z., Balogh, A., & Smyshlyaev, A. (2008b). Control of a tip-force destabilized shear beam by non-collocated observer-based boundary feedback. *SIAM Journal on Control and Optimization*, *47*, 553–574.

Krstic, M., & Smyshlyaev, A. (2008a). *Boundary control of PDEs: A course on backstepping designs*. Philadelphia, PA: SIAM.

Krstic, M., & Smyshlyaev, A. (2008b). Adaptive boundary control for unstable parabolic PDEs — part I: Lyapunov design. *IEEE Transactions on Automatic Control*, *53*, 1575–1591.

Liu, L., Chen, Z., & Huang, J. (2009). Parameter convergence and minimal internal model with an adaptive output regulation problem. *Automatica*, *45*, 1306–1311.

Logemann, H., & Ilchmann, A. (1994). An adaptive servomechanism for a class of infinite-dimensional systems. *SIAM Journal on Control and Optimization*, *32*, 917–936.

Paunonen, L., & Pohjolainen, S. (2010). Internal model theory for distributed parameter systems. *SIAM Journal on Control and Optimization*, *48*, 4753–4775.

Pyrkin, A., Smyshlyaev, A., Bekiaris-Liberis, N., & Krstic, M. (2010). Rejection of sinusoidal disturbance of unknown frequency for linear system with input delay, In *2010 American control conference*, (pp. 5688–5692).

Rebarber, R., & Weiss, G. (2003). Internal model based tracking and disturbance rejection for stable well-posed systems. *Automatica*, *39*, 1555–1569.

Smyshlyaev, A., & Krstic, M. (2007a). Adaptive boundary control for unstable parabolic PDEs — part II: Estimation-based designs. *Automatica*, *43*, 1543–1556.

Smyshlyaev, A., & Krstic, M. (2007b). Adaptive boundary control for unstable parabolic PDEs - Part III: Output-feedback examples with swapping identifiers. *Automatica*, *43*, 1557–1564.

Smyshlyaev, A., & Krstic, M. (2009). Boundary control of an anti-stable wave equation with anti-damping on the uncontrolled boundary. *Systems & Control Letters*, *58*, 617–623.

Smyshlyaev, A., Orlov, Y., & Krstic, M. (2009). Adaptive identification of two ustable PDEs with boundary sensing and actuation. *International Journal of Adaptive Control and Signal Processing*, *23*, 131–149.

Walker, J. A. (1980). *Dynamical systems and evolution equations, theory and applications*. New York: Plenum Press.

Weiss, G. (1989). Admissibility of unbounded control operators. *SIAM Journal on Control and Optimization*, *27*, 527–545.

Wei Guo received the Ph.D. degree from the Academy of Mathematics and Systems Science, the Chinese Academy of Science, Beijing, China, in 2004. He worked in the School of Computational and Applied Mathematics, University of the Witwatersrand, South Africa in 2007 as a Postdoctoral Fellow. He is currently a Professor with the University of International Business and Economics, Beijing. He was a visiting scholar at University of California, San Diego from September 2015 to September 2016. His research interests include control of systems described by partial differential equations, and the adaptive control of infinite-dimensional systems.

Zhi-Chao Shao received his Ph.D. in Cybernetics from the Academy of Mathematics and Systems Science, the Chinese Academy of Science, Beijing, China, in 2006. He worked as a Postdoctoral Fellow in the School of Computational and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa from 2006 to 2008. He is currently a Professor with the University of International Business and Economics, Beijing. His research interests include the theory of control and application of infinite-dimensional systems.

Miroslav Krstic holds the Alspach endowed chair and is the founding director of the Cymer Center for Control Systems and Dynamics at UC San Diego. He also serves as Associate Vice Chancellor for Research at UCSD. As a graduate student, Krstic won the UC Santa Barbara best dissertation award and student best paper awards at CDC and ACC. Krstic is Fellow of IEEE, IFAC, ASME, SIAM, and IET (UK), Associate Fellow of AIAA, and foreign member of the Academy of Engineering of Serbia. He has received the PECASE, NSF Career, and ONR Young Investigator awards, the Axelby and Schuck paper prizes, the Chestnut textbook prize, the ASME Nyquist Lecture Prize, and the first UCSD Research Award given to an engineer. Krstic has also been awarded the Springer Visiting Professorship at UC Berkeley, the Distinguished Visiting Fellowship of the Royal Academy of Engineering, the Invitation Fellowship of the Japan Society for the Promotion of Science, and the Honorary Professorships from the Northeastern University (Shenyang), Chongqing University, and Donghua University, China. He serves as Senior Editor in IEEE Transactions on Automatic Control and Automatica, as editor of two Springer book series, and has served as Vice President for Technical Activities of the IEEE Control Systems Society and as chair of the IEEE CSS Fellow Committee. Krstic has coauthored eleven books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.