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Nonlinear stabilization through wave PDE dynamics with a moving uncontrolled boundary[☆]

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ABSTRACT

In deep oil drilling, the length of the domain over which the wave equation models the torsional dynamics of the drill string keeps changing with time, and it also depends on the drill bit speed. Moreover, the drill bit speed cannot be controlled directly. In this context, we consider predictor-based design for the cascade system of a nonlinear ODE and a wave PDE with a moving uncontrolled boundary. In comparison with prior results on wave PDE–ODE cascades, this work differs by giving rise to a prediction horizon that is not given explicitly but has to be found from an implicit relationship involving the delay function and the future solution of the system. Stability analysis of the closed-loop system is conducted by constructing infinite-dimensional backstepping transformations and a Lyapunov functional. An explicit feedback law for compensating the wave actuator dynamics is obtained. For the moving boundary that depends on both the ODE's state and time, a region of attraction is estimated. For the moving boundary that depends on time, a global stabilization for the closed-loop system is achieved. Finally, an example is given to illustrate the effectiveness of the proposed design technique.

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1. Introduction

Predictor-based techniques have been developed for compensating constant input delays in linear plants (Artstein, 1982; Bekiaris-Liberis & Krstic, 2011; Krstic, 2008; Manitius & Olbrot, 1979; Mondie & Michiels, 2003). These designs are extended for unknown system parameters in Bresch-Pietri and Krstic (2009), an unknown actuator delay in Bresch-Pietri and Krstic (2010), and time-varying input delays in linear systems in Krstic (2010a) and Nihtila (1989, 1991). Over the last ten years, many works have been done on control designs and stability analysis for nonlinear systems with input delays, for instance Cai, Han, and Zhang (2011), Cai and Krstic (2014), Karafyllis (2010), Karafyllis and Krstic (2012), Krstic (2009a), Mazenc and Bliman (2006), Mazenc, Mondie, and Francisco (2004) and Teel (1998). Predictor control for nonlinear systems with arbitrarily large input delays was presented in Krstic (2010b), where a Lyapunov functional is provided to the stability analysis of the closed-loop system. For nonlinear systems with time-varying input delay, asymptotic stability was achieved based on a backstepping transformation of the actuator state and a Lyapunov functional (Bekiaris-Liberis & Krstic, 2012). The results in Bekiaris-Liberis and Krstic (2012) were further extended to state-dependent input delay in Bekiaris-Liberis and Krstic (2013a), and time and state-dependent input delay in Bekiaris-Liberis and Krstic (2013b). An explicit feedback law for compensating the wave partial differential equation (PDE) dynamics at the input of a linear ordinary differential equation (ODE) can be found in Krstic (2009b). Based on an infinite-dimensional backstepping–forwarding transformation, a feedback law for a multi-input linear system which compensates the wave PDE dynamics was given in Bekiaris-Liberis and Krstic (2013c).

In the drilling application, the torsional dynamics of a drill string are modeled as a wave PDE, which is coupled with a nonlinear ODE that describes the dynamics of the angular velocity of the drill bit at the bottom of the drill string (Saldivar, Mondie, Loiseau, & Rasvan, 2011). The stick–slip phenomenon is a common type of instability for drilling, which is an undesirable limit cycle of the drill string velocity yielding potentially significant damages on oil production facilities (Bresch-Pietri & Krstic, 2014; Sagert, Di Meglio, Krstic, & Rouchon, 2013). Based on the linearization of its dynamics, a control scheme for the stabilization of the drilling instability was acquired in Sagert et al. (2013). Furthermore,

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by the fact that the friction force is nonlinear, a general result for compensating wave PDE dynamics at the input of a general nonlinear ODE was presented in Bekiaris-Liberis and Krstic (2014). The result of Bekiaris-Liberis and Krstic (2014) was extended to nonlinear system under wave actuator dynamics with time- and state-dependent moving boundary in Cai and Krstic (2015).

However, in oil drilling, the length of the domain over which the wave equation models the torsional dynamics of the drill string keeps changing with time, and it also depends on the drill bit speed. Moreover, the drill bit speed cannot be controlled directly. In this context, it is of interest to study nonlinear stabilization through wave PDE dynamics with a moving uncontrolled boundary.

Similar other applications exist where a wave equation of varying length cascades into a nonlinear ODE. One example is hoists/elevators in coal mining, which run as deep as 2 km, and where a platform at the bottom end of the cables, as well as the cables themselves, must be kept from pendulating, while the cable length is changing with time. Another example, where the cable length goes through a large and fast change, and the cable and its load at the end must be kept from oscillating, is arresting gear for aircraft landing on aircraft carriers.

In this paper, a predictor-based design for nonlinear systems through wave PDE dynamics with a moving uncontrolled boundary is studied. The result in Cai and Krstic (2015) cannot be adapted to the case of a moving uncontrolled boundary because, while in Cai and Krstic (2015) the prediction horizon is given explicitly, in the present problem the prediction horizon is given implicitly, through an equation that involves the known delay function and the system solution, which is not known a priori. Stability analysis of the closed-loop system is conducted by constructing infinite-dimensional backstepping transformations and a Lyapunov functional. For the moving boundary that depends on both the ODE's state and time, an estimate of the region of attraction is acquired. For the moving boundary that depends on time, a global stabilization for the closed-loop system is achieved. In addition, the assumption that extended closed-loop system is backward complete in Cai and Krstic (2015) is removed in this paper.

The paper is organized as follows: System description and control design are in Section 2. Standing assumptions are in Section 3. Local stability is analysed in Section 4. Global stability is proved in Section 5. An illustrative example is given in Section 6. Some conclusions are drawn in Section 7.

Notation. We use the common definitions of class $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions from Krstic (2009a). We say that a function $\rho : R_+ \times (0, 1) \to R_+$ belongs to class $\mathcal{K}\iota$ if it is increasing with respect to its first argument and continuous with respect to its second argument. It belongs to class $\mathcal{K}\iota_\infty$ if it is in $\mathcal{K}\iota$ and in $\mathcal{K}_\infty$ with respect to its first argument. For an n-vector, the norm $|\cdot|$ denotes the usual Euclidean norm. Let $\delta : R^n \times R_+ \to R$. For a scalar function $u(\cdot, t) \in L^\infty[\delta(X, t), 1]$, we denote with $\|u(t)\|_\infty$ its supremum norm, i.e. $\|u(t)\|_\infty = \sup_{x\in[\delta(X,t),1]} |u(x, t)|$. Similarly, for $w(\cdot, t) \in L^\infty[0, 1 - \delta(X, t)]$, denote $\|w(t)\|_{\infty 1} = \sup_{x\in[0,1-\delta(X,t)]} |w(x, t)|$. ∇ stands for the gradient.

The argument of the functions and of the functionals will be omitted or simplified whenever no confusion can arise from the context. For example, one may denote a function $\delta(X(t), t)$ by simply $\delta(X, t)$.

2. System description and control design

Consider the nonlinear system under wave actuator dynamics given by

$$\dot{X} = f(X, u(\delta(X, t), t)) \quad (1)$$

$$u_{tt}(x, t) = u_{xx}(x, t) \quad (2)$$

$$u_x(\delta(X, t), t) = 0 \quad (3)$$

$$u_x(1, t) = U(t) \quad (4)$$

where $X \in R^n$ is the state vector and U is the scalar input, $f : R^n \times R \to R^n$ is locally Lipschitz with $f(0, 0) = 0$, and $u(x, t)$ is the state of the PDE dynamics of the wave actuator, and $\delta : R^n \times R_+ \to R$ is continuously differentiable.

For system (1)–(4), let

$$\zeta(x, t) = u_t(x, t) + u_x(x, t), \quad (5)$$

$$\eta(x, t) = u_t(x, t) - u_x(x, t), \quad (6)$$

which in reverse gives

$$u_t(x, t) = \frac{\zeta(x, t) + \eta(x, t)}{2}, \quad (7)$$

$$u_x(x, t) = \frac{\zeta(x, t) - \eta(x, t)}{2}. \quad (8)$$

Noting that

$$\dot{u}(\delta(X, t), t) = \frac{\zeta(\delta(X, t), t) - \eta(\delta(X, t), t)}{2} \times \left(\nabla\delta(X, t)f(X, u(\delta(X, t), t)) + \frac{\partial\delta(X, t)}{\partial t}\right) + \frac{\zeta(\delta(X, t), t) + \eta(\delta(X, t), t)}{2} \quad (9)$$

with (3), (5) and (6), yields $\dot{u}(\delta(X, t), t) = \zeta(\delta(X, t), t)$. So system (1)–(4) can be represented as

$$\dot{X} = f(X, u(\delta(X, t), t)) \quad (10)$$

$$\dot{u}(\delta(X, t), t) = \zeta(\delta(X, t), t) \quad (11)$$

$$\zeta_t(x, t) = \zeta_x(x, t) \quad (12)$$

$$\eta_t(x, t) = -\eta_x(x, t) \quad (13)$$

$$\eta(\delta(X, t), t) = \zeta(\delta(X, t), t) \quad (14)$$

$$\zeta(1, t) = \eta(1, t) + 2U(t). \quad (15)$$

If there exists a control law $\kappa : R^n \to R$ such that the system $\dot{X} = f(X, \kappa(X))$ is globally asymptotically stable, then a feedback law μ for system (10), (11) can be constructed as

$$\mu(X, u(\delta(X, t), t)) = -c_1(u(\delta(X, t), t) - \kappa(X)) + \frac{\partial\kappa(X)}{\partial X}f(X, u(\delta(X, t), t)) \quad (16)$$

where $c_1 > 0$ is arbitrary.

Remark 1. The design objective of feedback law μ is to make the extended closed-loop system $(X, u(\delta(X, t), t))$ input-to-state stable (ISS) with respect to the transformed state of the transport PDE which affects this system as an external disturbance.

Let

$$\phi(t) = t - (1 - \delta(X, t)), \quad (17)$$

$$\sigma(\theta) = \phi^{-1}(\theta), \quad \phi(t) \le \theta \le t. \quad (18)$$

Note that the input to the system (10), (11) is the delayed signal $\zeta(1, t) = U(t)$, we employ the prediction of X and $u(\delta(X, t), t)$. The control law for system (10)–(15) is given by

$$U(t) = -0.5(u_t(1, t) - u_x(1, t)) - 0.5c_1(P_2(t) - \kappa(P_1(t))) + 0.5\frac{\partial\kappa(P_1(t))}{\partial P_1}f(P_1(t), P_2(t)) \quad (19)$$

where $c_1 > 0$, and for all $\phi(t) \le \theta \le t$,

$$P_1(\theta) = X(t) + \int_{\phi(t)}^{\theta} \dot{\sigma}(s)f(P_1(s), P_2(s))ds, \quad (20)$$

$$P_2(\theta) = u(\delta(X,t),t) + \int_{\phi(t)}^{\theta} \dot\sigma(s)\zeta(\delta(P_1(s),\sigma(s)),\ \sigma(s))ds, \tag{21}$$

$$\dot\sigma(s) = 1/(1+\nabla\delta(X(\sigma(s)),\sigma(s))) \times f(X(\sigma(s)),u(\delta(X(\sigma(s)),\sigma(s)),\sigma(s))) + \frac{\partial(\delta(X(\sigma(s)),\ \sigma(s)))}{\partial\sigma(s)}, \tag{22}$$

for $t \ge 0$. The initial predictor $P_1(\theta)$, $P_2(\theta)$, $\theta \in [\phi(0), 0]$ are given by (20), (21) for $t = 0$, respectively.

From (20)–(21), it is easy to show that

$$P_1(\phi(t)) = X(t), \qquad P_2(\phi(t)) = u(\delta(X,t),t) \tag{23}$$

for all $t \ge 0$. It can be deduced

$$P_1(t) = X(\sigma(t)), \qquad P_2(t) = u(\delta(X(\sigma(t)),\ \sigma(t)),\ \sigma(t)), \tag{24}$$

for all $t \ge 0$.

Remark 2. The design objective of control law (19)–(22) is such that $\omega(1-\delta(X,t),t) = 0$ in the target system $(X, u((X,t),t), \omega, \varpi)$ which is transformed by system (10)–(15) under the backstepping transformations.

Remark 3. In Cai and Krstic (2015), $p_1(x,t) \in R^n$, $p_2(x,t) \in R$ are x-time-units-ahead predictions of $X(t)$ and $u(0,t)$ respectively, the prediction horizon is given explicitly. In the present case, we need $\sigma(t) - t$-time-units-ahead predictions of $X(t)$ and $u(\delta(X,t),t)$ respectively. The prediction horizon is given implicitly, through an equation that involves the known delay function and the system solution, which is not known a priori. Further, the assumption that extended closed-loop system is backward complete in Cai and Krstic (2015) is removed in this paper. So the result in Cai and Krstic (2015) cannot be adapted to this case.

3. Standing assumptions

Note that (10), (11) can be expressed as follows

$$\dot Z = \varphi(Z, \zeta(\delta(X,t),t)) \tag{25}$$

where

$$Z = \begin{bmatrix} X \\ u(\delta(X,t),t) \end{bmatrix}, \tag{26}$$

and

$$\varphi(Z,\zeta(\delta(X,t),t)) = \begin{bmatrix} f(X,u(\delta(X,t),t)) \\ \zeta(\delta(X,t),t) \end{bmatrix}. \tag{27}$$

Assumption 1. For the system $\dot Z = \varphi(Z, v)$, there exist smooth positive definite functions R_1, R_2 and class $\mathscr{K}_\infty$ functions $\alpha_1, \ldots, \alpha_6$ such that

$$\alpha_1(|Z|) \le R_1(Z) \le \alpha_2(|Z|) \tag{28}$$

$$\frac{\partial R_1(Z)}{\partial Z}\varphi(Z,v) \le R_1(Z) + \alpha_3(|v|) \tag{29}$$

$$\alpha_4(|Z|) \le R_2(Z) \le \alpha_5(|Z|) \tag{30}$$

$$-\frac{\partial R_2(Z)}{\partial Z}\varphi(Z,v) \le R_2(Z) + \alpha_6(|v|) \tag{31}$$

for all $Z \in R^{n+1}$ and for all $v \in R$.

Assumption 2. The system $\dot X = f(X, \kappa(X) + v)$ satisfies input-to-state stability property with respect to v and the function $\kappa : R^n \to R$ is continuously differentiable with locally Lipschitz derivative $\frac{\partial\kappa(X)}{\partial X}$ and it satisfies $\kappa(0) = 0$.

4. Local stability

With the assumptions on the nonlinear function $f(X, w)$ stated below (4), the following holds

$$|f(X,w)| \le \vartheta_1(|X| + |w|) \tag{32}$$

for a class $\mathscr{K}_\infty$ function ϑ_1.

Throughout this section, we consider the solutions which are such that

$$F_c : 0 \le \nabla\delta(X,\ t)f(X,u(\delta(X,t),t)) + \frac{\partial\delta(X,\ t)}{\partial t} \le c \tag{33}$$

for $0 < c < 1$, and $t \ge 0$. We refer to F_c as the feasibility condition.

The feasibility condition (33) ensures that the transport velocities have the correct sign; if not, the boundary conditions are not in the correct side of the equations. In addition, we make the following assumption on the moving boundary.

Assumption 3. The moving boundary $\delta(X,t)$ is continuously differentiable and satisfies

$$0 \le \delta(X,t) \le 1 \tag{34}$$

for all $t \ge 0$, and $\nabla\delta(X,t)$, $\delta_t(X,t)$ are locally Lipschitz, and there exist class $\mathscr{K}_\infty$ functions ϑ_2, ϑ_3 such that

$$|\nabla\delta(X,t)| \le |\nabla\delta(0,0)| + \vartheta_2(|X|), \tag{35}$$

$$|\delta_t(X,t)| \le |\delta_t(0,0)| + \vartheta_3(|X|), \tag{36}$$

for all $t \ge 0$.

Denote

$$\pi_1^* = \sup_{\theta \ge \delta^{-1}(X(0),\,0)} \left(\nabla\delta(X(\theta),\theta)f(X(\theta),u(\delta(X,\theta),\theta)) + \frac{\partial\delta(X(\theta),\theta)}{\partial\theta}\right), \tag{37}$$

$$\pi_2^* = \inf_{\theta \ge \delta^{-1}(X(0),\,0)} \left(\nabla\delta(X(\theta),\theta)f(X(\theta),u(\delta(X,\theta),\theta)) + \frac{\partial\delta(X(\theta),\theta)}{\partial\theta}\right). \tag{38}$$

Remark 4. With the condition (33), it holds $0 \le \nabla\delta(X,\ t) f(X,u(\delta(X,t),t)) + \frac{\partial\delta(X,t)}{\partial t} < 1$, for $t \ge 0$. So there exist a unique supremum π_1^* defined by (37) and satisfying $\pi_1^* < 1$, and a unique infimum π_2^* defined by (38) and satisfying $\pi_2^* \ge 0$.

4.1. Backstepping transformations and inverse backstepping transforms

Denote

$$g(t) = t + 1 - \delta(X,t), \tag{39}$$

and

$$h(\theta) = g^{-1}(\theta), \quad t \le \theta \le g(t). \tag{40}$$

Lemma 1 (*Backstepping Transforms*). *The backstepping transformations of ζ, η are defined as*

$$\omega(x-\delta(X,t),t) = \zeta(x,t) - \mu(l(x,t)), \tag{41}$$

$$\varpi(x-\delta(X,t),t) = \eta(x,t) - \mu(r(x,t)), \tag{42}$$

where

$$l(x,t) = Z(t) + \int_{\delta(X,t)}^{x} \dot\sigma(s+t-1)\varphi(l(s,t), \zeta(\delta(l_1(s,t),\sigma(s+t-1)),\ \sigma(s+t-1)))ds, \tag{43}$$

$$r(x,t) = Z(t) - \int_{\delta(X,t)}^{x} \dot{h}(t+1-s)\varphi(r(s,t), \eta(\delta(r_1(s,t), h(t+1-s)), h(t+1-s)))ds, \quad (44)$$

for all $\delta(X,t) \le x \le 1, t \ge 0$ *and* μ *is defined in* (16) *and* $l(x,t) = [l_1^T(x,t), l_2(x,t)]^T$, $r(x,t) = [r_1^T(x,t), r_2(x,t)]^T$, *and the control law* (19)–(22) *transform system* (25), (12)–(15) *to the target system given by*

$$\dot{Z} = \varphi(Z, \mu(Z) + \omega(0,t)) \quad (45)$$

$$\omega_t(x,t) = \left(1 + \nabla\delta(X,t)f(X, u(\delta(X,t),t)) + \frac{\partial\delta(X,t)}{\partial t}\right)\omega_x(x,t) \quad (46)$$

$$\varpi_t(x,t) = -\left(1 - \nabla\delta(X,t)f(X, u(\delta(X,t),t)) - \frac{\partial\delta(X,t)}{\partial t}\right)\varpi_x(x,t) \quad (47)$$

$$\varpi(0,t) = \omega(0,t) \quad (48)$$

$$\omega(1-\delta(X,t),t) = 0. \quad (49)$$

Proof. By setting $x = \delta(X,t)$ into (41), with (25), (43), we have (45). Using (12), it can be deduced that ζ is a function of $x+t$, that is, $\zeta(x,t) = \chi_1(x+t)$ for some function χ_1 and by (20), (21) and (43), we get $l(x,t) = [P_1^T(x+t-1), P_2(x+t-1)]^T$, in view of (41), so $\omega(x,t)$ is a function of $x + \delta(X,t) + t$. It can be deduced $\omega_t(x,t) = (1 + \nabla\delta(X,t)f(X, u(\delta(X,t),t)) + \frac{\partial\delta(X,t)}{\partial t})\omega_x(x,t)$, that is, (46). Using (13), we know that η is a function of $t-x$. With the help of (44), it can be deduced that $r(x,t) = Z(h(t+1-x))$, so (47) holds. In view of $l(\delta(X,t),t) = Z(t)$ and $r(\delta(X,t),t) = Z(t)$, and with the help of (14), (41) and (42), we get (48). Last, we prove (49). Noting that $\mu(l(1,t)) = \mu(P_1(t), P_2(t)) = -c_1(P_2(t) - \kappa(P_1(t))) + \frac{\partial\kappa(P_1(t))}{\partial P_1}f(P_1(t), P_2(t))$, and with (15), (19) and (41), we have $\omega(1-\delta(X,t),t) = \zeta(1,t) - \mu(l(1,t)) = \eta(1,t) + 2U(t) - \mu(l(1,t)) = 0$.

Lemma 2 (*Inverse Backstepping Transforms*)**.** *The inverse backstepping transformations of* ω, ϖ *are defined as*

$$\zeta(x,t) = \omega(x - \delta(X,t), t) + \mu(\iota(x,t)), \quad (50)$$

$$\eta(x,t) = \varpi(x - \delta(X,t), t) + \mu(\lambda(x,t)), \quad (51)$$

where

$$\iota(x,t) = Z(t) + \int_{\delta(X,t)}^{x} \dot{\sigma}(s+t-1)\varphi(\iota(s,t), \omega(0, \sigma(s+t-1)) + \mu(\iota(s,t)))ds, \quad (52)$$

$$\lambda(x,t) = Z(t) - \int_{\delta(X,t)}^{x} \dot{h}(t+1-s)\varphi(\lambda(s,t), \varpi(0, h(t+1-s)) + \mu(\lambda(s,t)))ds, \quad (53)$$

for all $\delta(X,t) \le x \le 1, t \ge 0$ *and* μ *is defined in* (16) *and* $\iota(x,t) = [\iota_1^T(x,t), \iota_2(x,t)]^T$, $\lambda(x,t) = [\lambda_1^T(x,t), \lambda_2(x,t)]^T$, *and the control law* (19)–(22) *transform the target system* (45)–(49) *to system* (25), (12)–(15).

Proof. The proof follows from straightforward computations.

4.2. Stability of the target system

Lemma 3 (*Extended Closed-loop System is ISS*)**.** *Under Assumption* 2, *consider the following system*

$$\dot{Z} = \varphi(Z, \mu(Z) + v) = \begin{bmatrix} f(X, u(\delta(X,t),t)) \\ \mu(Z) + v \end{bmatrix} \quad (54)$$

where the control law μ *given as* (16) *and* $Z = [X^T, u(\delta(X,t),t)]^T$. *Then there exist a class* $\mathcal{KL}$ *function* $\widehat{\beta}$ *and a class* $\mathcal{K}_\infty$ *function* $\widehat{\gamma}$ *such that* $|Z(t)| \le \widehat{\beta}(|Z(t_0)|, t - t_0) + \widehat{\gamma}(\sup_{t_0 \le \tau \le t} |v(\tau)|)$, *for* $t \ge t_0$.

Proof. The proof is similar to that of Cai and Krstic (2015), so it is omitted.

Lemma 4 (*Stability Estimate for Target System*)**.** *Under Assumptions* 2 *and* 3, *consider system* (45)–(49), *there exists a class* $\mathcal{KL}$ *function* β, *such that for all solutions of the system satisfying* (33) *for* $0 < c < 1$, *the following holds*

$$|X(t)| + |u(\delta(X,t),t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1} \le \beta(|X(0)| + |u(\delta(X(0),0),0)| + \|\omega(0)\|_{\infty 1} + \|\varpi(0)\|_{\infty 1}, t) \quad (55)$$

for all $t \ge 0$.

Proof. Based on Assumption 2, from Lemma 3, there exist a smooth function $S : R^{n+1} \to R_+$ and class $\mathcal{K}_\infty$ functions $\alpha_7, \alpha_8, \alpha_9, \alpha_{10}$ such that

$$\alpha_7(|Z|) \le S(Z) \le \alpha_8(|Z|), \quad (56)$$

$$\frac{\partial S(Z)}{\partial Z}\varphi(Z, \mu(Z) + \omega(0,t)) \le -\alpha_9(|Z|) + \alpha_{10}(|\omega(0,t)|) \quad (57)$$

where $Z = [X^T, u(\delta(X,t),t)]^T$. The new variable $v(x,t), x \in [-1+\delta(X,t), 1-\delta(X,t)]$ is defined as

$$v(x,t) = \begin{cases} \omega(x,t), & \text{for all } x \in [0, 1-\delta(X,t)], \\ \varpi(-x,t), & \text{for all } x \in [-1+\delta(X,t), 0]. \end{cases} \quad (58)$$

By (46), (47), (49), we get $v_t(x,t) = (1 + \nabla\delta(X,t) \times f(X, u(\delta(X,t),t)) + \frac{\partial\delta(X,t)}{\partial t})v_x(x,t)$ for all $x \in [0, 1-\delta(X,t)]$, and $v_t(x,t) = (1 - \nabla\delta(X,t)f(X, u(\delta(X,t),t)) - \frac{\partial\delta(X,t)}{\partial t})v_x(x,t)$ for all $x \in [-1+\delta(X,t), 0]$, and $v(1-\delta(X,t),t) = 0$. Let $\Gamma(t)$ denote the following norm

$$\Gamma(t) = \sup_{x \in [-1+\delta(X,t), 1-\delta(X,t)]} \left|e^{g_1(1+x)}v(x,t)\right| = \lim_{n\to\infty}\left(\int_{-1+\delta(X,t)}^{1-\delta(X,t)} e^{2ng_1(1+x)}v(x,t)^{2n}dx\right)^{\frac{1}{2n}} \quad (59)$$

where $g_1 > 0$, and n is a positive integer. The derivative of $\Gamma(t)$ is given by

$$\begin{aligned}\dot{\Gamma}(t) &= \lim_{n\to\infty}\frac{d}{dt}\left(\int_{-1+\delta(X,t)}^{1-\delta(X,t)} e^{2ng_1(1+x)}v(x,t)^{2n}dx\right)^{\frac{1}{2n}} \\ &= \lim_{n\to\infty}\frac{1}{2n}\left(\int_{-1+\delta(X,t)}^{1-\delta(X,t)} e^{2ng_1(1+x)}v(x,t)^{2n}dx\right)^{\frac{1}{2n}-1} \\ &\quad\times\left(\int_{-1+\delta(X,t)}^{1-\delta(X,t)} 2ne^{2ng_1(1+x)}v(x,t)^{2n-1}v_t(x,t)dx\right. \\ &\quad - e^{2ng_1\delta(X,t)}v(-1+\delta(X,t),t)^{2n} \\ &\quad\left.\times\left(\nabla(\delta(X,t))f(X, u(\delta(X,t),t)) + \frac{\partial\delta(X,t)}{\partial t}\right)\right).\end{aligned} \quad (60)$$

With integration by parts we get

$$\begin{aligned}&\int_{-1+\delta(X,t)}^{1-\delta(X,t)} 2ne^{2ng_1(1+x)}v(x,t)^{2n-1}v_t(x,t)dx \\ &= \left(1 - \nabla\delta(X,t)f(X, u(\delta(X,t),t)) - \frac{\partial\delta(X,t)}{\partial t}\right) \\ &\quad\times\int_{-1+\delta(X,t)}^{0} 2ne^{2ng_1(1+x)}v(x,t)^{2n-1}v_x(x,t)dx\end{aligned}$$

$$
\begin{aligned}
&+\left(1+\nabla\delta(X,t)f(X,u(\delta(X,t),t))+\frac{\partial\delta(X,t)}{\partial t}\right)\\
&\times\int_0^{1-\delta(X,t)} 2ne^{2ng_1(1+x)}v(x,t)^{2n-1}v_x(x,t)dx\\
&=\left(1-\nabla\delta(X,t)f(X,u(\delta(X,t),t))-\frac{\partial\delta(X,t)}{\partial t}\right)\\
&\times\int_{-1+\delta(X,t)}^{0} e^{2ng_1(1+x)}dv(x,t)^{2n}\\
&+\left(1+\nabla\delta(X,t)f(X,u(\delta(X,t),t))+\frac{\partial\delta(X,t)}{\partial t}\right)\\
&\times\int_0^{1-\delta(X,t)} e^{2ng_1(1+x)}dv(x,t)^{2n}\\
&=\left(1-\nabla\delta(X,t)f(X,u(\delta(X,t),t))-\frac{\partial\delta(X,t)}{\partial t}\right)\\
&\times\Bigg(e^{2ng_1}v(0,t)^{2n}-e^{2ng_1\delta(X,t)}v(-1+\delta(X,t),t)^{2n}\\
&-2ng_1\int_{-1+\delta(X,t)}^{0} e^{2ng_1(1+x)}v(x,t)^{2n}dx\Bigg)\\
&+\left(1+\nabla\delta(X,t)f(X,u(\delta(X,t),t))+\frac{\partial\delta(X,t)}{\partial t}\right)\\
&\times\left(-e^{2ng_1}v(0,t)^{2n}-2ng_1\int_0^{1-\delta(X,t)} e^{2ng_1(1+x)}v(x,t)^{2n}dx\right).
\end{aligned}
\tag{61}
$$

By (60), (61), we get

$$
\begin{aligned}
\dot{\Gamma}(t) =& \lim_{n\to\infty}\frac{1}{2n}\left(\int_{-1+\delta(X,t)}^{1-\delta(X,t)} e^{2ng_1(1+x)}v(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\times\Bigg(-e^{2ng_1\delta(X,t)}v(-1+\delta(X,t),t)^{2n}\\
&-2e^{2ng_1}v(0,t)^{2n}\left(\nabla\delta(X,t)f(X,u(\delta(X,t),t))+\frac{\partial\delta(X,t)}{\partial t}\right)\\
&-2ng_1\left(1-\nabla\delta(X,t)f(X,u(\delta(X,t),t))-\frac{\partial\delta(X,t)}{\partial t}\right)\\
&\times\int_{-1+\delta(X,t)}^{0} e^{2ng_1(1+x)}v(x,t)^{2n}dx\\
&-2ng_1\left(1+\nabla\delta(X,t)f(X,u(\delta(X,t),t))+\frac{\partial\delta(X,t)}{\partial t}\right)\\
&\times\int_0^{1-\delta(X,t)} e^{2ng_1(1+x)}v(x,t)^{2n}dx\Bigg).
\end{aligned}
\tag{62}
$$

Using (33), for $0 < c < 1$, one has $\nabla\delta(X,t)f(X,u(\delta(X,t),t)) + \frac{\partial\delta(X,t)}{\partial t} \geq 0$, for $t \geq 0$, so we get

$$
\begin{aligned}
\dot{\Gamma}(t) \leq& \lim_{n\to\infty}\frac{1}{2n}\left(\int_{-1+\delta(X,t)}^{1-\delta(X,t)} e^{2ng_1(1+x)}v(x,t)^{2n}dx\right)^{\frac{1}{2n}-1}\\
&\times\Bigg(-2ng_1\left(1-\nabla\delta(X,t)f(X,u(\delta(X,t),t))-\frac{\partial\delta(X,t)}{\partial t}\right)\\
&\times\int_{-1+\delta(X,t)}^{0} e^{2ng_1(1+x)}v(x,t)^{2n}dx\\
&-2ng_1\left(1+\nabla\delta(X,t)f(X,u(\delta(X,t),t))+\frac{\partial\delta(X,t)}{\partial t}\right)\\
&\times\int_0^{1-\delta(X,t)} e^{2ng_1(1+x)}v(x,t)^{2n}dx\Bigg)\\
&\leq -g_1\lim_{n\to\infty}\left(\int_{-1+\delta(X,t)}^{1-\delta(X,t)} e^{2ng_1(1+x)}v(x,t)^{2n}dx\right)^{\frac{1}{2n}}\\
&\quad\times\left(1-\nabla\delta(X,t)f(X,u(\delta(X,t),t))-\frac{\partial\delta(X,t)}{\partial t}\right).
\end{aligned}
\tag{63}
$$

With (37), we have $\dot{\Gamma}(t) \leq -g_1(1-\pi_1^*)\Gamma(t)$, for $t \geq 0$.

Take a Lyapunov functional as

$$
V(t) = S(Z) + \frac{2}{g_1(1-\pi_1^*)}\int_0^{\Gamma(t)}\frac{\alpha_{10}(r)}{r}dr,
\tag{64}
$$

where α_{10} is a class $\mathcal{K}_\infty$ function given as (57), the derivative of $V(t)$ along the solutions of system (45)–(49) satisfies

$$
\dot{V}(t) \leq -\alpha_9(|Z|) + \alpha_{10}(|\omega(0,t)|) - 2\alpha_{10}(\Gamma(t)).
\tag{65}
$$

Noting that $|\omega(0,t)| \leq \sup_{x\in[0,1-\delta(X,t)]}|\omega(x,t)| \leq \Gamma(t)$, we have $\dot{V}(t) \leq -\alpha_9(|Z(t)|) - \alpha_{10}(\Gamma(t))$. By (56), there is a class $\mathcal{K}$ function Υ_1 such that $\dot{V}(t) \leq -\Upsilon_1(V(t))$. Using the comparison principle, there is a class $\mathcal{KL}$ function β_1 such that $V(t) \leq \beta_1(V(0),t)$ for $t \geq 0$. By some calculation, using (64), then there exists a class $\mathcal{KL}$ function β_2 such that

$$
|Z(t)| + \Gamma(t) \leq \beta_2(|Z(0)| + \Gamma(0), t).
\tag{66}
$$

It is easy to see that

$$
\begin{aligned}
&0.5(\|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1}) \leq \Gamma(t)\\
&\quad\leq e^{2g_1}(\|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1}),
\end{aligned}
\tag{67}
$$

thus we get

$$
\begin{aligned}
&|X(t)| + |u(\delta(X,t),t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1}\\
&\quad\leq \sqrt{2}|Z(t)| + 2\Gamma(t)\\
&\quad\leq 2\beta_2(|Z(0)| + \Gamma(0), t)\\
&\quad\leq 2\beta_2(e^{2g_1}(|Z(0)| + \|\omega(0)\|_{\infty 1} + \|\varpi(0)\|_{\infty 1}), t)\\
&\quad\leq 2\beta_2(e^{2g_1}(|X(0)| + |u(\delta(X(0),0),0)|\\
&\qquad + \|\omega(0)\|_{\infty 1} + \|\varpi(0)\|_{\infty 1}), t).
\end{aligned}
\tag{68}
$$

Let $\beta(s,t) = 2\beta_2(e^{2g_1}s,t)$, with $g_1 > 0$, we have $|X(t)| + |u(\delta(X,t),t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1} \leq \beta(|X(0)| + |u(\delta(X(0),0),0)| + \|\omega(0)\|_{\infty 1} + \|\varpi(0)\|_{\infty 1}, t)$ for all $t \geq 0$.

4.3. Bounds on the predictors

Lemma 5 (*Bound on Extended Forward State Predictor*). *Under Assumptions 1 and 3, there exists a class $\mathcal{K}_\infty$ function γ_1 such that for all solutions of the system satisfying (33) for $0 < c < 1$, the following holds*

$$
\sup_{\delta(X,t)\leq x\leq 1}|l(x,t)| \leq \gamma_1(|Z(t)| + \|\zeta(t)\|_\infty).
\tag{69}
$$

Proof. By differentiating (43) with respect to x, we get

$$
\begin{aligned}
l_x(x,t) =& \dot{\sigma}(x+t-1)\varphi(l(x,t), \zeta(\delta(l_1(x,t),\\
&\sigma(x+t-1)), \sigma(x+t-1))).
\end{aligned}
\tag{70}
$$

Using (29), we get

$$
\begin{aligned}
&\frac{\partial R_1(l(x,t))}{\partial l}\varphi(l(x,t),\\
&\quad\zeta(\delta(l_1(x,t), \sigma(x+t-1)), \sigma(x+t-1)))\\
&\leq R_1(l(x,t))\\
&\quad+\alpha_3(|\zeta(\delta(l_1(x,t), \sigma(x+t-1)), \sigma(x+t-1))|).
\end{aligned}
\tag{71}
$$

With (33), we have $\dot{\phi}(t) \geq 1 + \pi_2^* > 0$ for $t \geq 0$, so $0 < \dot{\sigma}(x+t-1) \leq \frac{1}{1+\pi_2^*}$ for $\delta(X,t) \leq x \leq 1$, we get

$$\begin{aligned}\frac{\partial R_1(l(x,t))}{\partial l}&\dot{\sigma}(x+t-1)\varphi(l(x,t),\\ &\zeta(\delta(l_1(x,t),\sigma(x+t-1)),\ \sigma(x+t-1)))\\ \leq&\ \frac{1}{1+\pi_2^*}(R_1(l(x,t))\\ &+\alpha_3(|\zeta(\delta(l_1(x,t),\sigma(x+t-1)),\ \sigma(x+t-1))|)).\end{aligned} \tag{72}$$

With (70), we have

$$\begin{aligned}\frac{\partial R_1(l(x,t))}{\partial x} \leq&\ \frac{1}{1+\pi_2^*}(R_1(l(x,t))\\ &+\alpha_3(|\zeta(\delta(l_1(x,t),\sigma(x+t-1)),\ \sigma(x+t-1))|)).\end{aligned} \tag{73}$$

With (34), it follows for all $\delta(X,t) \leq x \leq 1$ that

$$\begin{aligned}&R_1(l(x,t))\\ &\leq e^{\frac{1-\delta(X,t)}{1+\pi_2^*}} R_1(l(\delta(X,t),t)) + (e^{\frac{1-\delta(X,t)}{1+\pi_2^*}}-1)\\ &\quad\times \sup_{\delta(X,t)\leq x\leq 1}\alpha_3(|\zeta(\delta(l_1(x,t),\sigma(x+t-1)),\ \sigma(x+t-1))|)\\ &\leq e^{\frac{1}{1+\pi_2^*}}R_1(Z(t)) + (e^{\frac{1}{1+\pi_2^*}}-1)\alpha_3(\|\zeta(t)\|_\infty).\end{aligned} \tag{74}$$

Using (28), for all $\delta(X,t) \leq x \leq 1$, we get that

$$l(x,t) \leq \alpha_1^{-1}(e^{\frac{1}{1+\pi_2^*}}\alpha_2(|Z(t))| + (e^{\frac{1}{1+\pi_2^*}}-1)\alpha_3(\|\zeta(t)\|_\infty)). \tag{75}$$

Denote $\gamma_1(s) = \alpha_1^{-1}(e^{\frac{1}{1+\pi_2^*}}\alpha_2(s) + (e^{\frac{1}{1+\pi_2^*}}-1)\alpha_3(s))$, we have $\sup_{\delta(X,t)\leq x\leq 1}|l(x,t)| \leq \gamma_1(|Z(t)| + \|\zeta(t)\|_\infty)$.

Lemma 6 (*Bound on Extended Backward State Predictor*). *Under Assumptions 2 and 3, there exists a class $\mathscr{K}\iota_\infty$ function ρ_1 such that for all solutions of the system satisfying (33) for $0 < c < 1$, the following holds*

$$\sup_{\delta(X,\,t)\leq x\leq 1}|r(x,t)| \leq \rho_1(|Z(t)| + \|\eta(t)\|_\infty, c). \tag{76}$$

Proof. By differentiating (44) with respect to x, we get

$$\begin{aligned}r_x(x,t) =&\ -\dot{h}(t+1-x)\\ &\times\varphi(r(x,t),\eta(\delta(r_1(x,\ t),h(t+1-x)),\ h(t+1-x))).\end{aligned} \tag{77}$$

Using (31), we get

$$\begin{aligned}&-\frac{\partial R_2(r(x,t))}{\partial r}\\ &\quad\times\varphi(r(x,t),\eta(\delta(r_1(x,t),h(t+1-x)),\ h(t+1-x)))\\ &\leq R_2(r(x,t))\\ &\quad+\alpha_6(|\eta(\delta(r_1(x,t),h(t+1-x)),\ h(t+1-x))|).\end{aligned} \tag{78}$$

With the help of (33), we have $\dot{g}(t) \geq 1-c > 0$ for $t \geq 0$. So it can be deduced $0 < \dot{h}(t+1-x) \leq \frac{1}{1-c}$, for all $\delta(X,t) \leq x \leq 1$. Using (78), we arrive at

$$\begin{aligned}&\frac{\partial R_2(r(x,t))}{\partial r}(-\dot{h}(t+1-x))\\ &\quad\times\varphi(r(x,t),\eta(\delta(r_1(x,t),h(t+1-x)),\ h(t+1-x)))\\ &\leq\frac{1}{1-c}(R_2(r(x,t))\\ &\quad+\alpha_6(|\eta(\delta(r_1(x,t),h(t+1-x)),\ h(t+1-x))|)).\end{aligned} \tag{79}$$

By (77), one has

$$\begin{aligned}\frac{\partial R_2(r(x,t))}{\partial x} \leq&\ \frac{1}{1-c}(R_2(r(x,t))\\ &+\alpha_6(|\eta(\delta(r_1(x,t),h(t+1-x)),\ h(t+1-x))|)).\end{aligned} \tag{80}$$

With the comparison principle, using (34), it follows that

$$\begin{aligned}&R_2(r(x,t)) \leq e^{\frac{1-\delta(X,t)}{1-c}}R_2(r(\delta(X,t),t)) + (e^{\frac{1-\delta(X,t)}{1-c}}-1)\\ &\quad\times\sup_{\delta(X,t)\leq x\leq 1}\alpha_6(|\eta(\delta(r_1(x,t),h(t+1-x)),\ h(t+1-x))|)\\ &\leq e^{\frac{1}{1-c}}R_2(Z(t)) + (e^{\frac{1}{1-c}}-1)\alpha_6(\|\eta(t)\|_\infty).\end{aligned} \tag{81}$$

With (30), for all $\delta(X,t) \leq x \leq 1$, we get that

$$|r(x,t)| \leq \alpha_4^{-1}(e^{\frac{1}{1-c}}\alpha_5(|Z(t)|) + (e^{\frac{1}{1-c}}-1)\alpha_6(\|\eta(t)\|_\infty)). \tag{82}$$

Denote $\rho_1(s,c) = \alpha_4^{-1}(e^{\frac{1}{1-c}}\alpha_5(s) + (e^{\frac{1}{1-c}}-1)\alpha_6(s))$, we have $\sup_{\delta(X,t)\leq x\leq 1}|r(x,t)| \leq \rho_1(|Z(t)| + \|\eta(t)\|_\infty, c)$.

Lemma 7 (*Bound on Forward Predictor*). *Under Assumptions 2 and 3, there exists a class $\mathscr{K}$ function γ_2 such that for all solutions of the system satisfying (33) for $0 < c < 1$, the following holds*

$$\sup_{\delta(X,\,t)\leq x\leq 1}|\iota(x,t)| \leq \gamma_2(|Z(t)| + \|\omega(t)\|_{\infty 1}). \tag{83}$$

Proof. Under Assumption 2, from Lemma 3, there exist a class $\mathscr{K}\mathscr{L}$ function $\widehat{\beta}$ and a class $\mathscr{K}_\infty$ function $\widehat{\gamma}$ such that

$$|Z(t)| \leq \widehat{\beta}(|Z(t_0)|, t-t_0) + \widehat{\gamma}(\sup_{t_0\leq\tau\leq t}|\upsilon(\tau)|) \tag{84}$$

where $Z(t)$ is the solution of system (54). By differentiating (52) with respect to x, we get

$$\begin{aligned}\iota_x(x,t) =&\ \dot{\sigma}(x+t-1)\varphi(\iota(x,t),\omega(0,\ \sigma(x+t-1))\\ &+\mu(\iota(x,t)))\end{aligned} \tag{85}$$

with $\iota = [\iota_1^T, \iota_2]^T$. Denote $\theta = x+t-1$, $\iota(x,t) = \Pi(\theta)$, we arrive at

$$\dot{\Pi}(\theta) = \dot{\sigma}(\theta)\varphi(\Pi(\theta),\omega(0,\ \sigma(\theta)) + \mu(\Pi(\theta))) \tag{86}$$

with $\Pi = [\Pi_1^T, \Pi_2]^T$. Using the change of variable $y = \sigma(\theta)$, we have

$$\frac{d\Pi(\phi(y))}{dy} = \varphi(\Pi(\phi(y)),\omega(0,y) + \mu(\Pi(\phi(y)))), \tag{87}$$

for $t \leq y \leq \sigma(t)$. With (84) and (87), one has

$$|\Pi(\theta)| \leq \widehat{\beta}(|Z(t)|,\theta-\phi(t)) + \widehat{\gamma}(\sup_{\phi(t)\leq\tau\leq t}|\omega(0,\ \sigma(\tau))|) \tag{88}$$

for $\phi(t) \leq \theta \leq t$. It can be deduced that

$$|\iota(x,t)| \leq \widehat{\beta}(|Z(t)|,0) + \widehat{\gamma}(\|\omega(t)\|_{\infty 1}). \tag{89}$$

Denote $\gamma_2(s) = \widehat{\beta}(s,0) + \widehat{\gamma}(s)$, we have (83).

Lemma 8 (*Bound on Backward Predictor*). *Under Assumptions 2 and 3, there exists a class $\mathscr{K}$ function γ_3 such that for all solutions of the system satisfying (33) for $0 < c < 1$, the following holds*

$$\sup_{\delta(X,t)\leq x\leq 1}|\lambda(x,t)| \leq \gamma_3(|Z(t)| + \|\varpi(t)\|_{\infty 1}). \tag{90}$$

Proof. Under Assumption 2, from Lemma 3, there exist a class $\mathcal{KL}$ function $\widehat{\beta}$ and a class $\mathcal{K}_\infty$ function $\widehat{\gamma}$ such that (84) holds. By differentiating (53) with respect to x, one has

$$\lambda_x(x,t) = -\dot{h}(t+1-x) \times \varphi(\lambda(x,t), \varpi(0,\ h(t+1-x)) + \mu(\lambda(x,t))) \tag{91}$$

with $\lambda = [\lambda_1^T, \lambda_2]^T$. Denote $\vartheta = t+1-x$, $\lambda(x,t) = \Lambda(\vartheta)$, we get

$$\dot{\Lambda}(\vartheta) = \dot{h}(\vartheta)\varphi(\Lambda(\vartheta), \varpi(0,\ h(\vartheta)) + \mu(\Lambda(\vartheta))) \tag{92}$$

with $\Lambda = [\Lambda_1^T, \Lambda_2]^T$. Using the change of variable $y = h(\vartheta)$, we have

$$\frac{d\Lambda(g(y))}{dy} = \varphi(\Lambda(g(y)), \varpi(0,y) + \mu(\Lambda(g(y)))), \tag{93}$$

for $t \le y \le h(t)$. With (84) and (93), we arrive at

$$|\Lambda(\vartheta)| \le \widehat{\beta}(|Z(t)|, \vartheta - g(t)) + \widehat{\gamma}(\sup_{t\le\tau\le g(t)} |\varpi(0,h(\tau))|) \tag{94}$$

for $t \le \vartheta \le g(t)$. It can be deduced that

$$|\lambda(x,t)| \le \widehat{\beta}(|Z(t)|, 0) + \widehat{\gamma}(\|\varpi(t)\|_{\infty 1}). \tag{95}$$

Denote $\gamma_3(s) = \widehat{\beta}(s,0) + \widehat{\gamma}(s)$, we have (90).

4.4. Stability of the closed-loop system

Lemma 9 (*Original PDE State Bounded by Target PDE State*). *Under Assumptions 2 and 3, consider system (45)–(49), and the output maps are (50), (51). Then there exists a class $\mathcal{K}_\infty$ function γ_4 such that for all solutions of the system satisfying (33) for $0 < c < 1$, the following holds*

$$|Z(t)| + \|\zeta(t)\|_\infty + \|\eta(t)\|_\infty \le \gamma_4(|Z(t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1}). \tag{96}$$

Proof. The proof is similar to that of Cai and Krstic (2015), so it is omitted.

Lemma 10 (*Target PDE State Bounded by Original PDE State*). *Under Assumptions 1 and 3, consider system (25), (12)–(15), and the output maps are (41), (42). Then there exists a class $\mathcal{K}\iota_\infty$ function ρ_2 such that for all solutions of the system satisfying (33) for $0 < c < 1$, the following holds*

$$|Z(t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1} \le \rho_2(|Z(t)| + \|\zeta(t)\|_\infty + \|\eta(t)\|_\infty, c). \tag{97}$$

Proof. The proof is similar to that of Cai and Krstic (2015), so it is omitted.

Lemma 11 (*Ball Around the Origin Within the Feasibility Region*). *Under Assumption 3, there exists a class $\mathcal{K}\iota$ function $\overline{\varrho}_c$ such that all of the solutions that satisfy*

$$B_1(c): |X(t)| + |u(\delta(X,t),t)| + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty \le \overline{\varrho}_c(c,c), \tag{98}$$

for $t \ge 0$ and $0 < c < 1$, also satisfy (33).

Proof. With (32), (35) and (36), we know if a solution satisfies

$$(|\nabla\delta(0,0)| + \vartheta_2(|X(t)|))\vartheta_1(|X(t)| + |u(\delta(X,t),t)|) + |\delta_t(0,0)| + \vartheta_3(|X(t)|) \le c, \tag{99}$$

for $t \ge 0$, and $0 < c < 1$, then it also satisfies (33). We conclude that (99) is satisfied for $0 < c < 1$ as long as (98) holds where the class $\mathcal{K}\iota$ function ϱ_c is defined as

$$\varrho_c(s,c) = |\delta_t(0,0)| + \vartheta_3(s) + (|\nabla\delta(0,0)| + \vartheta_2(s))\vartheta_1(s) \tag{100}$$

and with $\overline{\varrho}_c$, we denote the inverse function of ϱ_c with respect to ϱ_c's first argument.

Lemma 12 (*Estimate of the Region of Attraction*). *There exists a class $\mathcal{K}$ function ψ_{ROA} such that for all initial conditions of the closed-loop system (1)–(4), (19)–(22) that satisfy*

$$B_0(c):\ \Omega(0) \le \psi_{ROA}(c) \tag{101}$$

for some $0 < c < 1$, where

$$\Omega(t) = |X(t)| + |u(\delta(X,t),t)| + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty, \tag{102}$$

the solutions of the system satisfy (98) for $0 < c < 1$ and, hence, satisfy (33).

Proof. With the help of (5)–(8), we have

$$\|\zeta(t)\|_\infty + \|\eta(t)\|_\infty \le 2(\|u_t(t)\|_\infty + \|u_x(t)\|_\infty), \tag{103}$$

$$\|u_t(t)\|_\infty + \|u_x(t)\|_\infty \le \|\zeta(t)\|_\infty + \|\eta(t)\|_\infty. \tag{104}$$

Using Lemmas 4, 9, 10, we get

$$\begin{aligned}
&|X(t)| + |u(\delta(X,t))| + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty \\
&\le |X(t)| + |u(\delta(X,t))| + \|\zeta(t)\|_\infty + \|\eta(t)\|_\infty \\
&\le \sqrt{2}(|Z(t)| + \|\zeta(t)\|_\infty + \|\eta(t)\|_\infty) \\
&\le \sqrt{2}\gamma_4(|Z(t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1}) \\
&\le \sqrt{2}\gamma_4(|X(t)| + |u(\delta(X,t),t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1}) \\
&\le \sqrt{2}\gamma_4(\beta(|X(0)| + |u(\delta(X(0),0),0)| \\
&\quad + \|\omega(0)\|_{\infty 1} + \|\varpi(0)\|_{\infty 1}, t)) \\
&\le \sqrt{2}\gamma_4(\beta(\sqrt{2}(|Z(0)| + \|\omega(0)\|_{\infty 1} + \|\varpi(0)\|_{\infty 1}), t)) \\
&\le \sqrt{2}\gamma_4(\beta(\sqrt{2}\rho_2(|Z(0)| + \|\zeta(0)\|_\infty + \|\eta(0)\|_\infty, c), t)) \\
&\le \sqrt{2}\gamma_4(\beta(\sqrt{2}\rho_2(|X(0)| + |u(\delta(X(0),0),0)| \\
&\quad + \|\zeta(0)\|_\infty + \|\eta(0)\|_\infty, c), t)) \\
&\le \sqrt{2}\gamma_4(\beta(\sqrt{2}\rho_2(|X(0)| + |u(\delta(X(0),0),0)| \\
&\quad + 2(\|u_t(0)\|_\infty + \|u_x(0)\|_\infty), c), t)) \\
&\le \sqrt{2}\gamma_4(\beta(\sqrt{2}\rho_2(2(|X(0)| + |u(\delta(X(0),0),0)| \\
&\quad + \|u_t(0)\|_\infty + \|u_x(0)\|_\infty), c), t))
\end{aligned} \tag{105}$$

where $Z(t) = [X^T(t), u(\delta(X,t),t)]^T$. By defining the class $\mathcal{K}_\infty$ function $\widetilde{\alpha}(s)$ as $\widetilde{\alpha}(s) = \sqrt{2}\gamma_4(\beta(s,0))$, we obtain

$$\Omega(t) \le \widetilde{\alpha}(\sqrt{2}\rho_2(2\Omega(0),c)). \tag{106}$$

Hence, for all initial conditions that satisfy the bound (101) with any class $\mathcal{K}$ function choice $\psi_{ROA}(c) \le \overline{\psi}^*_{ROA}(\overline{\varrho}_c(c,c),c)$, where $\overline{\psi}^*_{ROA}(s,c)$ is the inverse of the class $\mathcal{K}\iota_\infty$ function $\psi^*_{ROA}(s,c) = \widetilde{\alpha}(\sqrt{2}\rho_2(2s,c))$ with respect to ψ^*_{ROA}'s first argument, the solutions satisfy (98) for $0 < c < 1$ and, hence, satisfy (33).

Theorem 1. *Consider system (1)–(4), together with the control law (19)–(22). Under Assumptions 1–3, there exist a class $\mathcal{K}\iota_\infty$ function ϱ, and a class $\mathcal{KL}$ function $\widetilde{\beta}$ such that for all initial conditions which are compatible with the feedback law (19) and satisfy (101), there exists a unique solution to the closed-loop system with the ODE component $X(t)$ that is continuously differentiable on $[0,\infty)$ and the PDE component $(u(x,t), u_t(x,t))$ that is continuously differentiable on $[\delta(X,t),1] \times [0,\infty)$, and such* *that*

$$\Omega(t) \le \widetilde{\beta}(\varrho(\Omega(0), c), t), \quad \text{for all } t \ge 0, \tag{107}$$

where $\Omega(t)$ is given by (102).

Proof. Using (105), we obtain (107) with $\widetilde{\beta}(s,t) = \sqrt{2}\gamma_4(\beta(s,t))$ and $\varrho(s,c) = \sqrt{2}\rho_2(2s,c)$. With (46), (49), we have

$$\omega(x,t) = \begin{cases} \omega_0(t+x+\delta(X,t)), & \delta(X,t) \le t+x+2\delta(X,t) < 1 \\ 0, & t+x+2\delta(X,t) \ge 1 \end{cases} \tag{108}$$

where $\omega_0(x)$ is given by (41) with $t = 0$. With the help of (5), for any initial condition $u(x,0) \in C^1[\delta(X(0),0),1]$, $u_t(x,0) \in C^1[\delta(X(0),0),1]$ which is compatible with the feedback law (19) and satisfies (101), we have $\zeta(x,0) \in C^1[\delta(X(0),0),1]$, and hence using

$$l_x(x,t) = \dot{\sigma}(x+t-1) \times \varphi(l(x,t), \zeta(\delta(l_1(x,t),\sigma(x+t-1)), \sigma(x+t-1))), \tag{109}$$

$$l(\delta(X,t),t) = Z(t), \tag{110}$$

and the Lipschitzness of φ, we conclude the existence and uniqueness of $l(x,0) \in C^1[\delta(X(0),0),1]$. Thus, with (41) and the compatibility condition we get $\omega_0(x) \in C^1[0, 1-\delta(X(0),0)]$. With (45), (108), and the Lipschitzness of φ and μ we conclude the existence and uniqueness of $(X(t), u(\delta(X,t),t)) \in C^1[0,\infty)$. Using the fact $\omega_0(x) \in C^1[0, 1-\delta(X(0),0)]$, the compatibility condition and (108), guarantee the existence of $\omega(x,t) \in C^1([0, 1-\delta(X(t),t)] \times [0,\infty))$. The uniqueness of this solution follows from the uniqueness of the solution to (46), (49).

With the similar arguments as above and using (6), the following ODE

$$r_x(x,t) = -\dot{h}(t+1-x) \times \varphi(r(x,t), \eta(\delta(r_1(x,t), h(t+1-x)), h(t+1-x))) \tag{111}$$

$$r(\delta(X,t)), t = Z(t), \tag{112}$$

relations (47), (48), and the fact that

$$\varpi(x,t) = \begin{cases} \varpi_0(x+\delta(X,t)-t), & 0 \le t < x+\delta(X,t), \\ \omega_0(t-x-\delta(X,t)), & \delta(X,t) \le t-x < 1, \\ 0, & t-x > 1, \end{cases} \tag{113}$$

with $\varpi_0(x)$ given by (42) with $t = 0$ and the compatibility condition and $u_x(\delta(X(0),0),0) = 0$, we obtain the existence and uniqueness of $\varpi \in C^1([0, 1-\delta(X,t)] \times [0,\infty))$. With the inverse backstepping transformation (50), (51) and $\iota(x,t) \equiv l(x,t)$, $\lambda(x,t) \equiv r(x,t)$, we have the existence and uniqueness of $\zeta(x,t), \eta(x,t) \in C^1([\delta(X(t),t),1] \times [0,\infty))$. So by (7), (8), there exists a unique solution $(u(x,t), u_t(x,t)) \in C^1([\delta(X,t),1] \times [0,\infty))$.

5. Global stability

Throughout this section, we consider time-dependent moving boundary $\delta(t)$, simplify the feasibility condition (33) as

$$0 \le \dot{\delta}(t) < 1, \tag{114}$$

for all $t \ge 0$, and make the following assumption:

Assumption 4. The moving boundary $\delta : R_+ \to R$ is continuously differentiable and satisfies

$$0 \le \delta(t) \le 1 \tag{115}$$

for all $t \ge 0$, and $\dot{\delta}(t)$ is locally Lipschitz.

Denote

$$\pi_1 = \sup_{\theta \ge \delta^{-1}(0)} \dot{\delta}(X(\theta)), \tag{116}$$

$$\pi_2 = \inf_{\theta \ge \delta^{-1}(0)} \dot{\delta}(X(\theta)). \tag{117}$$

Remark 5. Under the condition (114), it is easy to know $\pi_1 < 1$ and $\pi_2 \ge 0$.

5.1. Backstepping transformations and inverse backstepping transforms

Denote

$$g(t) = t + 1 - \delta(t), \tag{118}$$

and

$$h(\theta) = g^{-1}(\theta), \quad t \le \theta \le g(t). \tag{119}$$

Lemma 13 (*Backstepping Transforms*). *The backstepping transformations of ζ, η are defined as*

$$\omega(x-\delta(t),t) = \zeta(x,t) - \mu(l(x,t)), \tag{120}$$

$$\varpi(x-\delta(t),t) = \eta(x,t) - \mu(r(x,t)), \tag{121}$$

where

$$l(x,t) = Z(t) + \int_{\delta(t)}^{x} \dot{\sigma}(s+t-1)\varphi(l(s,t), \zeta(\delta(l_1(s,t),\sigma(s+t-1)), \sigma(s+t-1)))ds, \tag{122}$$

$$r(x,t) = Z(t) - \int_{\delta(t)}^{x} \dot{h}(t+1-s)\varphi(r(s,t), \eta(\delta(r_1(s,t),h(t+1-s)), h(t+1-s)))ds, \tag{123}$$

for all $\delta(t) \le x \le 1$, $t \ge 0$ and μ is defined in (16) and $\phi(t) = t-1+\delta(t)$, $\sigma(\theta) = \phi^{-1}(\theta)$, $\phi(t) \le \theta \le t$, $l(x,t) = [l_1^T(x,t), l_2(x,t)]^T$, $r(x,t) = [r_1^T(x,t), r_2(x,t)]^T$, and the control law (19)–(22) transform system (25), (12)–(15) to the target system given by

$$\dot{Z} = \varphi(Z, \mu(Z) + \omega(0,t)) \tag{124}$$

$$\omega_t(x,t) = (1+\dot{\delta}(t))\omega_x(x,t) \tag{125}$$

$$\varpi_t(x,t) = -(1-\dot{\delta}(t))\varpi_x(x,t) \tag{126}$$

$$\varpi(0,t) = \omega(0,t) \tag{127}$$

$$\omega(1-\delta(t),t) = 0. \tag{128}$$

Proof. Note that $\delta(X,t)$ in system (25), (12)–(15) is $\delta(t)$ now, the proof is similar to that of Lemma 1, so it is omitted.

Lemma 14 (*Inverse Backstepping Transforms*). *The inverse backstepping transformations of ω, ϖ are defined as*

$$\zeta(x,t) = \omega(x-\delta(t),t) + \mu(\iota(x,t)), \tag{129}$$

$$\eta(x,t) = \varpi(x-\delta(t),t) + \mu(\lambda(x,t)), \tag{130}$$

where

$$\iota(x,t) = Z(t) + \int_{\delta(t)}^{x} \dot{\sigma}(s+t-1)\varphi(\iota(s,t), \omega(0, \sigma(s+t-1)) + \mu(\iota(s,t)))ds, \tag{131}$$

$$\lambda(x,t) = Z(t) - \int_{\delta(t)}^{x} \dot{h}(t+1-s)\varphi(\lambda(s,t), \varpi(0, h(t+1-s)) + \mu(\lambda(s,t)))ds, \tag{132}$$

for all $\delta(t) \le x \le 1, t \ge 0$ and μ is defined in (16) and $\iota(x,t) = [\iota_1^T(x,t), \iota_2(x,t)]^T$, $\lambda(x,t) = [\lambda_1^T(x,t), \lambda_2(x,t)]^T$, and the control law (19)–(22) transform the target system (124)–(128) to system (25), (12)–(15).

Proof. The proof is similar to that of Lemma 2, so it is omitted.

5.2. Stability of the target system

Lemma 15 (*Stability Estimate for Target System*)**.** *Under the condition (114), Assumptions 2 and 3, consider system (124)–(128), there exists a class $\mathcal{KL}$ function $\overline{\beta}$, such that the following holds*

$$|X(t)| + |u(\delta(t),t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1} \le \overline{\beta}(|X(0)| + |u(\delta(0),0)| + \|\omega(0)\|_{\infty 1} + \|\varpi(0)\|_{\infty 1}, t) \quad (133)$$

for all $t \ge 0$.

Proof. Based on Assumption 2, from Lemma 3, there exist a smooth function $S : R^{n+1} \to R_+$ and class $\mathcal{K}_\infty$ functions $\alpha_7, \alpha_8, \alpha_9, \alpha_{10}$ such that (56), (57) hold. The new variable $\upsilon(x,t), x \in [-1+\delta(t), 1-\delta(t)]$ is defined as

$$\upsilon(x,t) = \begin{cases} \omega(x,t), & \text{for all } x \in [0, 1-\delta(t)], \\ \varpi(-x,t), & \text{for all } x \in [-1+\delta(t), 0]. \end{cases} \quad (134)$$

By (125), (126), (128), we get $\upsilon_t(x,t) = (1+\dot{\delta}(t))\upsilon_x(x,t)$ for all $x \in [0, 1-\delta(t)]$, and $\upsilon_t(x,t) = (1-\dot{\delta}(t))\upsilon_x(x,t)$ for all $x \in [-1+\delta(t), 0]$, and $\upsilon(1-\delta(t),t) = 0$. Let $\overline{\Gamma}(t)$ denote the following norm

$$\overline{\Gamma}(t) = \sup_{x\in[-1+\delta(t),1-\delta(t)]} \left|e^{g(1+x)}\upsilon(x,t)\right| = \lim_{n\to\infty}\left(\int_{-1+\delta(t)}^{1-\delta(t)} e^{2ng(1+x)}\upsilon(x,t)^{2n}dx\right)^{\frac{1}{2n}} \quad (135)$$

where $g > 0$, and n is a positive integer. The derivative of $\overline{\Gamma}(t)$ is given by

$$\begin{aligned}\dot{\overline{\Gamma}}(t) &= \lim_{n\to\infty}\frac{d}{dt}\left(\int_{-1+\delta(t)}^{1-\delta(t)} e^{2ng(1+x)}\upsilon(x,t)^{2n}dx\right)^{\frac{1}{2n}} \\ &= \lim_{n\to\infty}\frac{1}{2n}\left(\int_{-1+\delta(t)}^{1-\delta(t)} e^{2ng(1+x)}\upsilon(x,t)^{2n}dx\right)^{\frac{1}{2n}-1} \\ &\quad\times\left(\int_{-1+\delta(t)}^{1-\delta(t)} 2ne^{2ng(1+x)}\upsilon(x,t)^{2n-1}\upsilon_t(x,t)dx\right. \\ &\quad\left. - e^{2ng\delta(t)}\upsilon(-1+\delta(t),t)^{2n}\dot{\delta}(t)\right). \end{aligned} \quad (136)$$

With integration by parts we get

$$\begin{aligned}&\int_{-1+\delta(t)}^{1-\delta(t)} 2ne^{2ng(1+x)}\upsilon(x,t)^{2n-1}\upsilon_t(x,t)dx \\ &= \int_{-1+\delta(t)}^{0} 2ne^{2ng(1+x)}\upsilon(x,t)^{2n-1}(1-\dot{\delta}(t))\upsilon_x(x,t)dx \\ &\quad + \int_{0}^{1-\delta(t)} 2ne^{2ng(1+x)}\upsilon(x,t)^{2n-1}(1+\dot{\delta}(t))\upsilon_x(x,t)dx \\ &= (1-\dot{\delta}(t))\int_{-1+\delta(t)}^{0} e^{2ng(1+x)}d\upsilon(x,t)^{2n} \\ &\quad + (1+\dot{\delta}(t))\int_{0}^{1-\delta(t)} e^{2ng(1+x)}d\upsilon(x,t)^{2n} \\ &= (1-\dot{\delta}(t))(e^{2ng}\upsilon(0,t)^{2n} - e^{2ng\delta(t)}\upsilon(-1+\delta(t),t)^{2n} \\ &\quad - 2ng\int_{-1+\delta(t)}^{0} e^{2ng(1+x)}\upsilon(x,t)^{2n}dx) \\ &\quad + (1+\dot{\delta}(t))(-e^{2ng}\upsilon(0,t)^{2n} \\ &\quad - 2ng\int_{0}^{1-\delta(t)} e^{2ng(1+x)}\upsilon(x,t)^{2n}dx). \end{aligned} \quad (137)$$

By (136), (137), one has

$$\begin{aligned}\dot{\overline{\Gamma}}(t) &= \lim_{n\to\infty}\frac{1}{2n}\left(\int_{-1+\delta(t)}^{1-\delta(t)} e^{2ng(1+x)}\upsilon(x,t)^{2n}dx\right)^{\frac{1}{2n}-1} \\ &\quad\times\left(-e^{2ng\delta(t)}\upsilon(-1+\delta(t),t)^{2n} - 2e^{2ng}\upsilon(0,t)^{2n}\dot{\delta}(t)\right. \\ &\quad - 2ng(1-\dot{\delta}(t))\int_{-1+\delta(t)}^{0} e^{2ng(1+x)}\upsilon(x,t)^{2n}dx \\ &\quad\left. - 2ng(1+\dot{\delta}(t))\int_{0}^{1-\delta(t)} e^{2ng(1+x)}\upsilon(x,t)^{2n}dx\right) \end{aligned} \quad (138)$$

note $0 \le \dot{\delta}(t) < 1$ and (116), so we have $\dot{\overline{\Gamma}}(t) \le -g(1-\pi_1)\overline{\Gamma}(t)$, for $t \ge 0$.

Take a Lyapunov functional as

$$V(t) = S(Z) + \frac{2}{g(1-\pi_1)}\int_0^{\overline{\Gamma}(t)}\frac{\alpha_{10}(r)}{r}dr, \quad (139)$$

where α_{10} is a class $\mathcal{K}_\infty$ function given as (57), the derivative of $V(t)$ along the solutions of system (124)–(128) satisfies

$$\dot{V}(t) \le -\alpha_9(|Z|) + \alpha_{10}(|\omega(0,t)|) - 2\alpha_{10}(\overline{\Gamma}(t)). \quad (140)$$

Noting that $|\omega(0,t)| \le \sup_{x\in[0,1-\delta(t)]}|\omega(x,t)| \le \overline{\Gamma}(t)$, we have $\dot{V}(t) \le -\alpha_9(|Z(t)|) - \alpha_{10}(\overline{\Gamma}(t))$. Using the arguments as in Lemma 4, then there is a class $\mathcal{KL}$ function $\overline{\beta}$, such that $|X(t)| + |u(\delta(X,t),t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1} \le \overline{\beta}(|X(0)| + |u(\delta(X(0),0),0)| + \|\omega(0)\|_{\infty 1} + \|\varpi(0)\|_{\infty 1}, t)$ for all $t \ge 0$.

5.3. Bounds on the predictors

Lemma 16 (*Bound on Extended Forward State Predictor*)**.** *Under the condition (114), Assumptions 1 and 3, there exists a class $\mathcal{K}_\infty$ function $\overline{\gamma}_1$ such that the following holds*

$$\sup_{\delta(t)\le x\le 1}|l(x,t)| \le \overline{\gamma}_1(|Z(t)| + \|\zeta(t)\|_\infty). \quad (141)$$

Proof. Similar to the proof of Lemma 5, it can be deduced that there exists a class $\mathcal{K}_\infty$ function $\overline{\gamma}_1$ as $\overline{\gamma}_1(s) = \alpha_1^{-1}(e^{\frac{1}{1+\pi_2}}\alpha_2(s) + (e^{\frac{1}{1+\pi_2}} - 1)\alpha_3(s))$ such that $\sup_{\delta(t)\le x\le 1}|l(x,t)| \le \overline{\gamma}_1(|Z(t)| + \|\zeta(t)\|_\infty)$.

Lemma 17 (*Bound on Extended Backward State Predictor*)**.** *Under the condition (114), Assumptions 2 and 3, there exists a class $\mathcal{K}_\infty$ function $\overline{\gamma}_2$ such that the following holds*

$$\sup_{\delta(t)\le x\le 1}|r(x,t)| \le \overline{\gamma}_2(|Z(t)| + \|\eta(t)\|_\infty). \quad (142)$$

Proof. Similar to the proof of Lemma 6, it can be deduced that there exists a class $\mathcal{K}_\infty$ function $\overline{\gamma}_2(s)$ as $\overline{\gamma}_2(s) = \alpha_4^{-1}(e^{\frac{1}{1-\pi_1}}\alpha_5(s) + (e^{\frac{1}{1-\pi_1}} - 1)\alpha_6(s))$ such that $\sup_{\delta(t)\le x\le 1}|r(x,t)| \le \overline{\gamma}_2(|Z(t)| + \|\eta(t)\|_\infty)$.

Lemma 18 (*Bounds on Forward and Backward Predictor*)**.** *Under the condition (114), Assumptions 2 and 3, there exist class $\mathcal{K}$ functions*

$\overline{\gamma}_3, \overline{\gamma}_4$ such that the following hold

$$\sup_{\delta(t)\le x\le 1} |\iota(x,t)| \le \overline{\gamma}_3(|Z(t)| + \|\omega(t)\|_{\infty 1}), \tag{143}$$

$$\sup_{\delta(t)\le x\le 1} |\lambda(x,t)| \le \overline{\gamma}_4(|Z(t)| + \|\varpi(t)\|_{\infty 1}). \tag{144}$$

Proof. The proof can be adapted from Lemmas 7 and 8.

5.4. Stability of the closed-loop system

Lemma 19 (*Original PDE State Bounded by Target PDE State*)**.** *Under the condition (114), Assumptions 2 and 3, consider system (124)–(128), and the output maps are (129), (130). Then there exists a class $\mathscr{K}_\infty$ function $\overline{\gamma}_5$ such that the following holds*

$$\begin{aligned}&|Z(t)| + \|\zeta(t)\|_\infty + \|\eta(t)\|_\infty \\ &\quad\le \overline{\gamma}_5(|Z(t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1}).\end{aligned} \tag{145}$$

Proof. The proof is similar to that of Lemma 9, so it is omitted.

Lemma 20 (*Target PDE State Bounded by Original PDE State*)**.** *Under the condition (114), Assumptions 1 and 3, consider system (25), (12)–(15), and the output maps are (120), (121). Then there exists a class $\mathscr{K}_\infty$ function $\overline{\gamma}_6$ such that the following holds*

$$\begin{aligned}&|Z(t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1} \\ &\quad\le \overline{\gamma}_6(|Z(t)| + \|\zeta(t)\|_\infty + \|\eta(t)\|_\infty).\end{aligned} \tag{146}$$

Proof. The proof is similar to that of Lemma 10, so it is omitted.

Theorem 2. *Under the condition (114), Assumptions 1, 2 and 4, consider system (1)–(4), together with the control law (19)–(22) for any initial condition $u(\cdot,0) \in C^1[\delta(0),1]$, $u_t(\cdot,0) \in C^1[\delta(0),1]$ which is compatible with the feedback law (19), the closed-loop system has a unique solution*

$$X(t) \in C^1[0,\infty) \tag{147}$$

$$(u(\cdot,t), u_t(\cdot,t)) \in C^1([\delta(t),1]\times[0,\infty)) \tag{148}$$

and there exists a class $\mathscr{KL}$ function $\underline{\beta}$ such that

$$\begin{aligned}&|X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty \\ &\quad\le \underline{\beta}(|X(0)| + \|u(0)\|_\infty + \|u_t(0)\|_\infty + \|u_x(0)\|_\infty, t)\end{aligned} \tag{149}$$

for all $t \ge 0$.

Proof. Owing to

$$u(x,t) = u(\delta(t),t) + \int_{\delta(t)}^{x} u_y(y,t)dy, \tag{150}$$

so we have

$$\sup_{\delta(t)\le x\le 1} |u(x,t)| \le |u(\delta(t),t)| + \sup_{\delta(t)\le x\le 1} |u_x(x,t)|. \tag{151}$$

Using Lemmas 15, 19, 20, and with (103), (104), we get

$$\begin{aligned}&|X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty \\
&\quad\le |X(t)| + |u(\delta(t),t)| + \|u_t(t)\|_\infty + 2\|u_x(t)\|_\infty \\
&\quad\le 2(|X(t)| + |u(\delta(t),t)| + \|\zeta(t)\|_\infty + \|\eta(t)\|_\infty) \\
&\quad\le 2\sqrt{2}(|Z(t)| + \|\zeta(t)\|_\infty + \|\eta(t)\|_\infty) \\
&\quad\le 2\sqrt{2}\overline{\gamma}_5(|Z(t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1}) \\
&\quad\le 2\sqrt{2}\overline{\gamma}_5(|X(t)| + |u(\delta(t),t)| + \|\omega(t)\|_{\infty 1} + \|\varpi(t)\|_{\infty 1}) \\
&\quad\le 2\sqrt{2}\overline{\gamma}_5(\overline{\beta}(|X(0)| + |u(\delta(0),0)| + \|\omega(0)\|_{\infty 1} + \|\varpi(0)\|_{\infty 1}, t)) \\
&\quad\le 2\sqrt{2}\overline{\gamma}_5(\overline{\beta}(\sqrt{2}(|Z(0)| + \|\omega(0)\|_{\infty 1} + \|\varpi(0)\|_{\infty 1}), t)) \\
&\quad\le 2\sqrt{2}\overline{\gamma}_5(\overline{\beta}(\sqrt{2}\overline{\gamma}_6(|Z(0)| + \|\zeta(0)\|_\infty + \|\eta(0)\|_\infty), t)) \\
&\quad\le 2\sqrt{2}\overline{\gamma}_5(\overline{\beta}(\sqrt{2}\overline{\gamma}_6(|X(0)| + |u(\delta(0),0)| \\
&\qquad + \|\zeta(0)\|_\infty + \|\eta(0)\|_\infty), t)) \\
&\quad\le 2\sqrt{2}\overline{\gamma}_5(\overline{\beta}(\sqrt{2}\overline{\gamma}_6(|X(0)| + \|u(0)\|_\infty \\
&\qquad + 2(\|u_t(0)\|_\infty + \|u_x(0)\|_\infty)), t)) \\
&\quad\le 2\sqrt{2}\overline{\gamma}_5(\overline{\beta}(\sqrt{2}\overline{\gamma}_6(2(|X(0)| + \|u(0)\|_\infty \\
&\qquad + \|u_t(0)\|_\infty + \|u_x(0)\|_\infty)), t))\end{aligned} \tag{152}$$

where $Z(t) = [X^T(t), u(\delta(t),t)]^T$. Denote $\underline{\beta}(s,t) = 2\sqrt{2}\overline{\gamma}_5(\overline{\beta}(\sqrt{2}\overline{\gamma}_6(2s), t))$, we have (149).

With (125), (128), we have

$$\omega(x,t) = \begin{cases}\omega_0(t+x+\delta(t)), & \delta(t) \le t+x+2\delta(t) < 1\\ 0, & t+x+2\delta(t) \ge 1\end{cases} \tag{153}$$

where $\omega_0(x)$ is given by (120) with $t = 0$. With the help of (5), for any initial condition $u(x,0) \in C^1[\delta(0),1]$, $u_t(x,0) \in C^1[\delta(0),1]$ which is compatible with the feedback law (19), we have $\zeta(x,0) \in C^1[\delta(0),1]$, and hence using

$$\begin{aligned}l_x(x,t) &= \dot{\sigma}(x+t-1)\varphi(l(x,t), \zeta(\delta(l_1(x,t),\\ &\qquad \sigma(x+t-1)), \sigma(x+t-1))),\end{aligned} \tag{154}$$

$$l(\delta(t),t) = Z(t), \tag{155}$$

and the Lipschitzness of φ, we conclude the existence and uniqueness of $l(x,0) \in C^1[\delta(0),1]$. Thus, with (120) and the compatibility condition we get $\omega_0(x) \in C^1[0,1-\delta(0)]$.

With (124), (153), and the Lipschitzness of φ and μ we conclude the existence and uniqueness of $(X(t), u(\delta(t),t)) \in C^1[0,\infty)$. Using the fact $\omega_0(x) \in C^1[0,1-\delta(0)]$, the compatibility condition and (153), guarantee the existence of $\omega(x,t) \in C^1([0,1-\delta(t)]\times[0,\infty))$. The uniqueness of this solution follows from the uniqueness of the solution to (125), (128).

With the similar arguments as above and using (6), the following ODE

$$\begin{aligned}r_x(x,t) &= -\dot{h}(t+1-x)\varphi(r(x,t), \eta(\delta(r_1(x,t),\\ &\qquad h(t+1-x)), h(t+1-x)))\end{aligned} \tag{156}$$

$$r(\delta(t),t) = Z(t), \tag{157}$$

relations (126), (127), and the fact that

$$\varpi(x,t) = \begin{cases}\varpi_0(x+\delta(t)-t), & 0 \le t < x+\delta(t),\\ \omega_0(t-x-\delta(t)), & \delta(t) \le t-x < 1,\\ 0, & t-x > 1,\end{cases} \tag{158}$$

with $\varpi_0(x) = \varpi(x,0)$ given by (121) with $t = 0$ and the compatibility condition and $u_x(\delta(0),0) = 0$, the existence and uniqueness of $\varpi \in C^1([0,1-\delta(t)]\times[0,\infty))$ is obtained. With the inverse backstepping transformation (129), (130) and $\iota(x,t) \equiv l(x,t)$, $\lambda(x,t) \equiv r(x,t)$, we have the existence and uniqueness of $\zeta(x,t), \eta(x,t) \in C^1([\delta(t),1]\times[0,\infty))$. So by (7), (8), there exists a unique solution $(u(x,t), u_t(x,t)) \in C^1([\delta(t),1]\times[0,\infty))$.

6. Example

Consider the cascade of the benchmark system and a wave PDE actuator given by

$$\dot{X}_1 = X_2 + \epsilon X_3^2 \tag{159}$$

$$\dot{X}_2 = X_3 \tag{160}$$

$$\dot{X}_3 = u(\delta(X,t),t) \tag{161}$$

$$u_{tt}(x,t) = u_{xx}(x,t) \tag{162}$$

$$u_x(\delta(X,t),t) = 0 \tag{163}$$

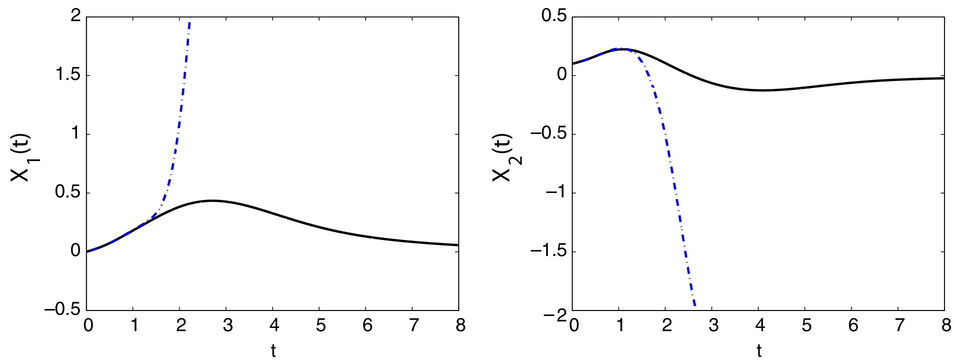

Fig. 1. Time response of the states of system (159)–(164) with the control law (166)–(171) (solid line) and with the nominal control law (165) (dashdot line) for $c_1 = 2, \epsilon = 1$ and initial condition as $X_1(0) = 0,\ X_2(0) = 0.1,\ X_3(0) = 0.1$ and $u(x, 0) = 0.1,\ u_t(x, 0) = 0.1$ for all $x \in [\delta(X(0), 0), 1]$.

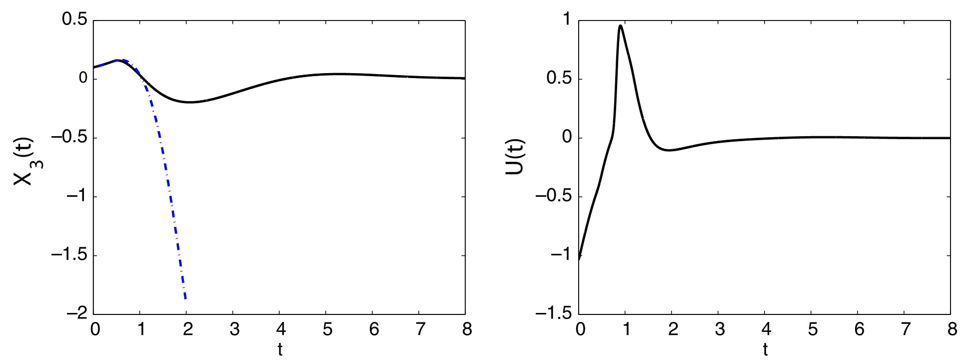

Fig. 2. Time response of the state X_3 for the proposed control law (166)–(171) and the uncompensated nominal control law (left) and the control law (right) for initial condition as $X_1(0) = 0,\ X_2(0) = 0.1, X_3(0) = 0.1$ and $u(x, 0) = 0.1,\ u_t(x, 0) = 0.1$ for all $x \in [\delta(X(0), 0), 1]$.

$$u_x(1, t) = U(t) \tag{164}$$

where $X = [X_1, X_2, X_3]^T$ and ϵ is a scalar parameter. System (159)–(161) is a representative of a larger class of nonlinear systems with parameters. A delay-free design for system (159)–(161) is (see Sepulchre, Jankovic, & Kokotovic, 1996) as follows

$$\kappa(X) = -3X_2 - 2X_3 - X_1 \\ - \epsilon(X_1X_3 + X_2X_3 + 0.5X_2^2) - 0.5\epsilon^2 X_3 X_2^2. \tag{165}$$

The closed-loop system (159)–(161) and (165) is globally asymptotically stable without any restriction on ϵ. For all $\phi(t) \le \theta \le t$, define the estimated predictors of X_1, X_2 and X_3 as

$$P_1(\theta) = X_1(t) + \int_{\phi(t)}^{\theta} \dot\sigma(s)(P_2(s) + \epsilon P_3^2(s))ds, \tag{166}$$

$$P_2(\theta) = X_2(t) + \int_{\phi(t)}^{\theta} \dot\sigma(s) P_3(s)ds, \tag{167}$$

$$P_3(\theta) = X_3(t) + \int_{\phi(t)}^{\theta} \dot\sigma(s) P_4(s)ds, \tag{168}$$

$$P_4(\theta) = u(\delta(X, t), t) + \int_{\phi(t)}^{\theta} \dot\sigma(s)\zeta(\delta(P(s), \sigma(s)),\ \sigma(s))ds, \tag{169}$$

$$\dot\sigma(s) = 1/\Bigg(1 + \nabla\delta(P(s), \sigma(s)) f(P(s), P_4(s)) \\ + \frac{\partial(\delta(P(s),\ \sigma(s)))}{\partial\sigma(s)}\Bigg) \tag{170}$$

for $t \ge 0$ with $P(s) = [P_1(s), P_2(s), P_3(s)]^T, f(P(s), P_4(s)) = [P_2(s) + \epsilon P_3^2(s), P_3(s), P_4(s)]^T$.

The control law for system (159) to (164) is given by

$$U(t) = -0.5(u_t(1, t) - u_x(1, t)) - 0.5c_1(P_4(t) - \kappa(P(t))) \\ + 0.5\frac{\partial\kappa(P(t))}{\partial P} f(P(t), P_4(t)) \tag{171}$$

where $c_1 > 0$, and P_1, P_2, P_3, P_4 are given by (166) to (169) respectively, and κ is given by (165). Assume $\delta(X, t)$ as

$$\delta(X, t) = \frac{1 + X_2^2(t) + t}{2 + X_2^2(t) + t}, \tag{172}$$

we have

$$0 < \delta(X, t) < 1, \tag{173}$$

$$|\delta_t(X, t)| \le \frac{1}{4}, \tag{174}$$

$$|\nabla\delta(X, t)| \le \frac{|X(t)|}{2}, \tag{175}$$

for all $t \ge 0$, and

$$|f(X, u(\delta(X, t), t))|^2 \\ = (X_2 + \epsilon X_3^2)^2 + X_3^2 + u^2(\delta(X, t), t) \\ \le (2 + 2\epsilon^2)(X_2^2 + X_3^4 + X_3^2 + u^2(\delta(X, t), t)) \tag{176}$$

there exists a class $\mathcal{K}_\infty$ function $\vartheta_1(\chi) = \sqrt{2 + 2\epsilon^2}|\chi| \times \sqrt{1 + |\chi|^2}$ such that $|f(X, u(\delta(X, t), t))| \le \vartheta_1(|X| + |u(\delta(X, t), t)|)$, the feasibility condition (33) is

$$0 \le \frac{1 + 2X_2X_3}{(2 + X_2^2 + t)^2} \le c \tag{177}$$

for $0 < c < 1$, and for all $t \ge 0$.

In Fig. 1, time response of the states X_1, X_2 of system (159)–(164) for the case of the proposed control law (166)–(171) and the case of the uncompensated nominal control law (165) are shown. One can observe, in the former case, the stabilization is achieved, whereas in the latter case, the closed-loop system is unstable. In Fig. 2, time response of the state X_3 of system for the proposed control law and the uncompensated nominal control law and the proposed control law (166)–(171) are shown. One can observe that the control law converges to zero.

7. Conclusion

We introduce and solve stabilization problems for nonlinear systems through wave PDE dynamics with a moving uncontrolled boundary. Stability analysis of the closed-loop system is achieved with infinite-dimensional backstepping transformations and by constructing a Lyapunov functional. An explicit feedback law for compensating the wave actuator dynamics is designed. For the moving boundary that depends on both the ODE's state and time, a region of attraction is estimated. For the moving boundary that depends on time, a global stabilization result is achieved. The feedback stabilization is illustrated by an example.

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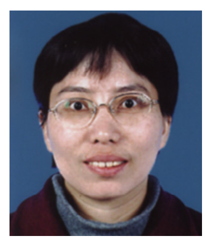

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