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Brief paper

Lyapunov–Krasovskii functionals and application to input delay compensation for linear time-invariant systems[☆]

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ABSTRACT

For linear systems with pointwise or distributed delay in the inputs which are stabilized through the reduction approach, we propose a new technique of construction of Lyapunov–Krasovskii functionals. These functionals allow us to establish the ISS property of the closed-loop systems relative to additive disturbances.

1. Introduction

Controlling dynamical systems including delays in the inputs was a problem of recurring interest in the past fifty years since it frequently arises in control applications, due to the transport and measurement delays that naturally occur (for more details, see, e.g., Michiels and Niculescu (2007)).

A number of approaches to deal with input delays have been proposed in both frequency- and time-domains. Among them, for linear systems, two of the most celebrated are the *Smith predictor* and the *reduction* technique, also known as finite spectrum assignment (FSA). To the best of the authors' knowledge, the reduction approach originates in Mayne (1968), with the well known contributions that have followed in Kwon and Pearson (1980), Manitius and Olbrot (1979) and Olbrot (1978), which have been systematized and generalized in Artstein (1982), to which we refer the reader for a pedagogical exposition. This technique is popular and frequently used in practice for stabilizing linear systems with delay in the input, due to the fact that, under an appropriate transformation, the system reduces to a finite-dimensional one. However, the control applied to the original dynamics is complicated. Alternatives to the popular reduction approach include the general observer–predictor structure in Mirkin and Raskin (2003) and the H_∞ approach in Tadmor (2000).

The reduction approach applies to cases where the delays are too large for being neglected, as done for instance in Mazenc, Malisoff, and Lin (2008): the one-dimensional system

$$\dot{X}(t) = X(t) + U(t-\tau), \tag{1}$$

where U is the input, can be exponentially stabilized through the reduction approach for any constant delay $\tau \geq 0$ although, when τ is larger than a certain value, there is no continuous function φ such that the feedback $U(t-\tau) = \varphi(X(t-\tau))$ asymptotically stabilizes (1). Moreover, this technique applies to cases where the delays are either pointwise or distributed, and significantly simplifies stabilization problems for systems with delay by reducing them to similar problems for ordinary differential equations (see, for instance Fiagbedzi and Pearson (1986) and Wang, Lee, and Tan (1998) for further discussions).

Although Lyapunov functionals are tools whose importance is more and more recognized by the researchers who work in delay area (see for instance (Bekiaris-Liberis & Krstic, 2011), Karafyllis and Jiang (2011), Pepe and Verriest (2003) and Zhou, Lin, and Duan (2010)), strict Lyapunov–Krasovskii functionals for linear systems in closed-loop with feedbacks resulting from

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the reduction approach have been constructed only recently in Bekiaris-Liberis and Krstic (2011) using a novel approach which relies on the introduction of a hyperbolic PDE. This result is motivated by the important benefits which can be derived from the knowledge of a strict Lyapunov–Krasovskii functional. In particular, strict Lyapunov–Krasovskii functionals are frequently ISS or iISS Lyapunov–Krasovskii functionals as defined and discussed in Pepe and Jiang (2006) for systems with disturbances, which straightforwardly implies that the systems possess the desirable ISS or iISS property with respect to these disturbances (see Sontag (2007) for information on the celebrated ISS notion).

In the present work, we revisit the problem of constructing Lyapunov–Krasovskii functionals for two main families of closed-loop systems with additive disturbances: the first class is associated with the classical reduction approach and the second is employing dynamic feedback to overcome the instability that arises in some implementations of control laws specific to the classical reduction approach. The new construction we propose shares some features with the one of Mazenc and Niculescu (2011) which relies on the representation of a system with delay as an ordinary differential equation interconnected with an integral equation. However, the Lyapunov functionals we propose here are by no means straightforwardly deduced from Mazenc and Niculescu (2011). Indeed, by contrast with the feedbacks resulting from the reduction approach, the control laws considered in Mazenc and Niculescu (2011) do not have distributed terms. Furthermore, our ISS Lyapunov–Krasovskii functionals do not rely on the introduction of hyperbolic PDEs and therefore are significantly different from those proposed in Bekiaris-Liberis and Krstic (2011), Krstic (2008) and Krstic and Smyshlaev (2008).

The paper is organized as follows. In Section 2, a construction of functionals for a general family of systems is presented. From the latter result, Lyapunov–Krasovskii functionals for three families of systems stabilized via control laws provided by the reduction model approach are deduced in Section 3. Finally, some conclusions are drawn in Section 4.

Notation and definitions. • The notation will be simplified whenever no confusion can arise from the context. • For any integer p, we denote by Id_p or simply Id the identity matrix in $\mathbb{R}^{p\times p}$. • We let $|\cdot|$ denote the Euclidean norm of matrices and vectors of any dimension. • Given $\phi : \mathcal{I} \to \mathbb{R}^p$ defined on an interval $\mathcal{I}$, let $|\phi|_{\mathcal{I}}$ denote its (essential) supremum over $\mathcal{I}$. • For any integer p, we let $C_{\rm in} = C([-\tau, 0], \mathbb{R}^p)$ denote the set of all continuous $\mathbb{R}^p$-valued functions defined on a given interval $[-\tau, 0]$. • For a function $x : [-\tau, +\infty) \to \mathbb{R}^k$, for all $t \geq 0$, the function x_t is defined by $x_t(\ell) = x(t+\ell)$ for all $\ell \in [-\tau, 0]$. • Let $\mathcal{K}_\infty$ denote the set of all continuous functions $\rho : [0,\infty) \to [0,\infty)$ for which (i) $\rho(0) = 0$ and (ii) ρ is strictly increasing and unbounded. • We adopt a definition of ISS Lyapunov–Krasovskii functional for coupled retarded functional differential equations and functional equations, which is an adaptation to this family of systems of the definitions given in Dashkovskiy and Naujok (2010) and Pepe, Karafyllis, and Jiang (2008).

Definition 1. We consider a system composed by a retarded functional differential equation coupled with a functional equation:

$$\begin{cases}\dot{x}_1(t) = f_1(x_{1t}, x_{2t}, u(t)),\\ x_2(t) = f_2(x_{1t}, x_{2t}),\\ (x_1(r), x_2(r)) = (x_{10}(r), x_{20}(r)), \quad \forall r \in [-\tau, 0],\end{cases} \tag{2}$$

where $t \in [0, +\infty)$, $x_1(t) \in \mathbb{R}^{N_1}$, $x_2(t) \in \mathbb{R}^{N_2}$, $u(t) \in \mathbb{R}^{N_3}$ is an essentially bounded measurable input and τ is the maximum involved delay and the functionals f_1 and f_2 are locally Lipschitz continuous on any bounded set such that all the solutions of (2) with initial function in $C_{\rm in}$ are defined and of class C^1 over $[0, +\infty)$.

A locally Lipschitz continuous functional $V : C_{\rm in} \to [0, +\infty)$ is called an ISS Lyapunov–Krasovskii functional for (2) if (i) there are functions of class $\mathcal{K}_\infty$, α_1 and α_2 such that, for all functions $(\phi_1, \phi_2) \in C_{\rm in}$ the inequalities

$$\alpha_1(|(\phi_1(0), \phi_2(0))|) \leq V(\phi_1, \phi_2) \leq \alpha_2(|(\phi_1, \phi_2)|_{[-\tau,0]}) \tag{3}$$

are satisfied,

(ii) it is continuously differentiable along the trajectories of (2) and satisfies:

$$\dot{V}(t) \leq -\alpha_3(V(x_{1t}, x_{2t})) + \alpha_4(|u(t)|), \quad \forall t \in [0, +\infty), \tag{4}$$

where α_3 and α_4 are functions of class $\mathcal{K}_\infty$.

2. Technical result

The result of this section is instrumental in establishing our main results. However, it is of interest for its own sake.

2.1. System and assumptions

We consider the system

$$\Sigma_{z,v} : \begin{cases}\dot{z}(t) = f(z(t)) + \delta(t), & \forall t \geq 0,\\ v(t) = Nz(t), & \forall t \geq 0,\end{cases} \tag{5}$$

with $z \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, where the initial conditions $\phi_z \in C_{\rm in}$ and $\phi_v \in C_{\rm in}$ are such that $\phi_v(0) = N\phi_z(0)$, where $N \in \mathbb{R}^{m\times n}$ is a constant matrix, $N \neq 0$, where f is a function of class C^1 and where δ is a continuous function. Consider also the system

$$\Sigma_x : x(t) = g(x_t, z_t, v_t), \tag{6}$$

where g is a locally Lipschitz continuous functional.

We introduce two assumptions:

Assumption H1. There exists a function S of class C^1, positive definite and radially unbounded, a function κ_1 of class $\mathcal{K}_\infty$ and a positive real number a_1 such that

$$\frac{\partial S}{\partial z}(z)[f(z) + \delta] \leq -a_1 S(z) + \kappa_1(|\delta|), \tag{7}$$

for all $z \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^n$.

Assumption H2. There exists a nonnegative function θ such that

$$\theta(Nz) \leq \frac{a_1}{2} S(z), \tag{8}$$

for all $z \in \mathbb{R}^n$, all the solutions $(z(t), x(t))$ of the system $\Sigma_{z,v} - \Sigma_x$ are defined and of class C^1 over $[0, +\infty)$ and are such that, for all $t \geq 0$, the inequality

$$\max\{|x(t)|, |\dot{x}(t)|\} \leq \sqrt{a_2\left(S(z(t)) + \overline{\theta}(v_t) + \kappa_2(|\delta(t)|)\right)}, \tag{9}$$

where a_2 is a positive real number, κ_2 is a function of class $\mathcal{K}_\infty$ and

$$\overline{\theta}(v_t) = \theta(v(t-\tau)) + \int_{t-\tau}^{t} \theta(v(m))dm, \tag{10}$$

is satisfied.

2.2. Discussion of the assumptions

1. The inequality (7) implies that the z-subsystem in (5) is ISS with respect to δ.

2. The z-subsystem is written as an ordinary differential equation, however, we regard it as a subsystem of the system with delay

$\Sigma_{z,v} - \Sigma_x$ and therefore this subsystem needs an initial condition defined over $[-\tau, 0]$. To extend the range of possible applications of the forthcoming Theorem 2.1 and with a view to the results of Section 3, we have introduced the v-subsystem. $\Sigma_{z,v}$ is a particular case of the coupled retarded functional differential equation and functional difference equation considered in Karafyllis, Pepe, and Jiang (2009), although the delay τ is not explicitly present in the equations. In order to immediately recognize in (5) the equations studied in Karafyllis et al. (2009), observe that (5) can be written as

$$\begin{cases} \dot{z}(t) = f(z(t)) + \delta(t), \\ v(t) = Mv(t-\tau) + Nz(t), \end{cases}$$

with $M = 0$.

3. The property (9) implies that $\Sigma_{z,v} - \Sigma_x$ behaves as a coupled delay differential equation and integral equation. In contrast to coupled delay differential and difference equations, which have been studied in many works and in particular in Pepe, Jiang, and Fridman (2008), Rasvan and Niculescu (2002) and Pepe and Verriest (2003), only a few works, notably (Karafyllis et al., 2009), are devoted to coupled delay differential and integral equations.

2.3. Lyapunov–Krasovskii functional

We introduce two functionals:

$$V_1(\phi_x) = \frac{1}{2}|\phi_x(0)|^2 + \int_{-\tau}^{0} e^{m}|\phi_x(m)|^2 dm, \tag{11}$$

$$V_2(\phi_z, \phi_v) = S(\phi_z(0)) + \int_{-\tau}^{0} e^{\frac{a_1}{2}m}\theta(\phi_v(m))dm \tag{12}$$

and state and prove the following result:

Theorem 2.1. *Consider the system $\Sigma_{z,v} - \Sigma_x$. Assume that it satisfies Assumptions H1 and H2. Then the derivative of functional $V_3 : C_{\rm in} \to [0, +\infty)$, defined by*

$$V_3(\phi_x, \phi_z, \phi_v) = V_1(\phi_x) + a_3 V_2(\phi_z, \phi_v), \tag{13}$$

where $a_3 = 10\frac{a_2}{a_1}e^{\frac{a_1}{2}\tau}$ and V_1 and V_2 are the functionals defined in (11) and (12), along the trajectories of $\Sigma_{z,v} - \Sigma_x$ satisfies, for all $t \geq 0$,

$$\dot{V}_3(t) \leq -\min\left\{\frac{a_1}{4}, 1\right\} V_3(x_t, z_t, v_t) + \kappa_3(|\delta(t)|), \tag{14}$$

where κ_3 is the function of class $\mathcal{K}_\infty$ defined by

$$\kappa_3(\ell) = a_3\kappa_1(\ell) + \frac{5}{2}a_2\kappa_2(\ell). \tag{15}$$

2.4. Discussion of Theorem 2.1

1. Theorem 2.1 presents an extension of the Lyapunov methodology to a special class of systems which behave as coupled retarded functional differential equations and functional equations. To the best of our knowledge, no construction of *strict Lyapunov–Krasovskii functional* for these systems is available in the literature. Many extensions of Theorem 2.1 can be established. In particular the case of multiple delays can be easily handled and since Assumption H2 is independent of the features of the function g, more general systems Σ_x can be considered. Moreover, Assumption H2 itself can be relaxed by allowing the right-hand side of (9) to depend on x_t. The latter extension is especially appealing, since it would make it possible to analyze systems which, as explained in Karafyllis et al. (2009), are of great relevance, but it is beyond the scope of the present contribution and it will be considered in a forthcoming work.

2. For the time being the advantage of having introduced v-subsystem in Theorem 2.1 is not obvious. It will become clear in the next sections why we did not replace v by Nz throughout Theorem 2.1. It is crucial to note that we *did not* impose on the initial conditions of the solutions $(z(t), v(t))$ a requirement that $\phi_v(t) = N\phi_z(t)$ for all $t \in [-\tau, 0)$: only a "point" condition $\phi_v(0) = N\phi_z(0)$ is required. Note also that, for the sake of generality, we do not assume that N is a square matrix.

3. The system $\Sigma_{z,v}$ can be interpreted as a *comparison system* for the system Σ_x in the sense that the stability properties of the latter system are deduced from the system $\Sigma_{z,v}$. The idea of comparison systems is not new (Halanay, 1966; Lakshmikantan & Leela, 1969; Verriest, 2001), but the way $|x(t)|$ and $|\dot{x}(t)|$ are estimated in (9) through functions which depend on pieces of the trajectories of $\Sigma_{z,v}$ and the Lyapunov functional (13) for the coupled system are new.

4. If the delay τ in Theorem 2.1 is unknown, but is smaller than a known constant $\tau^* > 0$, one can still construct a functional of the type of the functional U by replacing in the expression of U the constant a_3 by $10\frac{a_2}{a_1}e^{\frac{a_1}{2}\tau^*}$.

2.5. Proof of Theorem 2.1

To evaluate the time derivative of V_3 along the trajectories of $\Sigma_{z,v} - \Sigma_x$, first we observe that, for all $t \geq 0$, the derivative of

$$V_1(x_t) = \frac{1}{2}|x(t)|^2 + \int_{t-\tau}^{t} e^{m-t}|x(m)|^2 dm \tag{16}$$

satisfies

$$\begin{aligned} \dot{V}_1(t) &= x(t)^\top \dot{x}(t) - \int_{t-\tau}^{t} e^{m-t}|x(m)|^2 dm \\ &\quad + |x(t)|^2 - e^{-\tau}|x(t-\tau)|^2 \\ &\leq -V_1(x_t) + x(t)^\top\dot{x}(t) + \frac{3}{2}|x(t)|^2. \end{aligned}$$

Using (9), we obtain that

$$\dot{V}_1(t) \leq -V_1(x_t) + \frac{5}{2}a_2\left[S(z(t)) + \overline{\theta}(v_t) + \kappa_2(|\delta(t)|)\right], \tag{17}$$

for all $t \geq 0$. Now we observe that, for all $t \geq 0$, the derivative of

$$V_2(z_t, v_t) = S(z(t)) + \int_{t-\tau}^{t} e^{\frac{a_1}{2}(m-t)}\theta(v(m))dm \tag{18}$$

is given by

$$\begin{aligned} \dot{V}_2(t) &= \frac{\partial S}{\partial z}(z(t))[f(z(t)) + \delta(t)] \\ &\quad - \frac{a_1}{2}\int_{t-\tau}^{t} e^{\frac{a_1}{2}(m-t)}\theta(v(m))dm + \theta(v(t)) \\ &\quad - e^{-\frac{a_1}{2}\tau}\theta(v(t-\tau)). \end{aligned}$$

From (7) and the equality $v(t) = Nz(t)$ for all $t \geq 0$, it follows that, for all $t \geq 0$,

$$\begin{aligned} \dot{V}_2(t) &\leq -a_1 S(z(t)) + \theta(Nz(t)) \\ &\quad - \frac{a_1}{2}\int_{t-\tau}^{t} e^{\frac{a_1}{2}(m-t)}\theta(v(m))dm \\ &\quad - e^{-\frac{a_1}{2}\tau}\theta(v(t-\tau)) + \kappa_1(|\delta(t)|). \end{aligned}$$

From the inequality (8), it follows that, for all $t \geq 0$,

$$\begin{aligned} \dot{V}_2(t) &\leq -\frac{a_1}{2}S(z(t)) - \frac{a_1}{2}\int_{t-\tau}^{t} e^{\frac{a_1}{2}(m-t)}\theta(v(m))dm \\ &\quad - e^{-\frac{a_1}{2}\tau}\theta(v(t-\tau)) + \kappa_1(|\delta(t)|). \end{aligned}$$

This inequality and the definition of V_2 in (18) imply that, for all $t \geq 0$,

$$\dot V_2(t) \le -\frac{a_1}{4}S(z(t)) - e^{-\frac{a_1}{2}\tau}\theta(v(t-\tau)) - \frac{a_1 e^{-\frac{a_1}{2}\tau}}{4}\int_{t-\tau}^{t}\theta(v(m))dm - \frac{a_1}{4}V_2(z_t,v_t) + \kappa_1(|\delta(t)|). \quad (19)$$

From the definition of V_3, (17) and (19), we conclude that

$$\dot V_3(t) \le -V_1(x_t) + \frac{5}{2}a_2\left[S(z(t)) + \overline{\theta}(v_t)\right] - a_3\frac{a_1}{4}S(z(t)) - a_3 e^{-\frac{a_1}{2}\tau}\theta(v(t-\tau)) - a_3\frac{a_1 e^{-\frac{a_1}{2}\tau}}{4}\int_{t-\tau}^{t}\theta(v(m))dm - \frac{a_3 a_1}{4}V_2(z_t,v_t) + \kappa_3(|\delta(t)|),$$

with κ_3 defined in (15). Using the explicit value of a_3 in Theorem 2.1, we obtain

$$\dot V_3(t) \le -V_1(x_t) - \frac{a_3 a_1}{4}V_2(z_t,v_t) + \kappa_3(|\delta(t)|),$$

for all $t \ge 0$, which implies that (14) is satisfied.

3. Linear systems with delay in the inputs

As mentioned in the introduction, we consider three families of systems in closed-loop with control laws provided by the reduction technique. In each case, we construct ISS Lyapunov–Krasovskii functionals.

3.1. Systems with pointwise delay in the inputs

In this section, we consider the closed-loop system studied in Krstic (2009, Section 2.6):

$$\begin{cases} \dot X(t) = AX(t) + BU(t-\tau) + \lambda(t), \\ U(t) = K\left[e^{A\tau}X(t) + \int_{t-\tau}^{t} e^{A(t-\ell)}BU(\ell)d\ell\right], \end{cases} \quad (20)$$

with $X \in \mathbb{R}^n, U \in \mathbb{R}^m$, where $\tau > 0$, the matrices A, B, K are constant and λ is a continuous disturbance and introduce the assumption:

Assumption H3. There exists a symmetric and positive definite matrix Q such that

$$Q \ge Id, \qquad QH + H^\top Q \le -2Id, \quad (21)$$

where $H = A + BK$.

We recall that Assumption H3 is satisfied if and only if H is a Hurwitz matrix. Now, we state and prove the following result:

Theorem 3.1. *Assume that the system (20) satisfies Assumption H3. Then there exist three constants $c_1 > 0, c_2 > 0, c_3 > 0$ such that $W : C_{\rm in} \to [0,+\infty)$, defined by*

$$W(\Phi_{X,U}) = \frac{1}{2}|\phi_X(0)|^2 + c_1|\phi_U(0)|^2 + \int_{-\tau}^{0} e^{m}|\phi_X(m)|^2 dm + c_2\zeta(\Phi_{X,U})^\top Q\zeta(\Phi_{X,U}) + c_3\int_{-\tau}^{0} e^{\frac{1}{2|Q|}m}|\phi_U(m)|^2 dm, \quad (22)$$

with $\Phi_{X,U} = (\phi_X,\phi_U)$, with

$$\zeta(\Phi_{X,U}) = e^{A\tau}\phi_X(0) + \int_{-\tau}^{0} e^{-Am}B\phi_U(m)dm \quad (23)$$

is an ISS Lyapunov–Krasovskii functional for (20). Moreover, there are two positive constants c_4, c_5 such that the inequalities

$$c_4\beta(\Phi_{X,U}) \le W(\Phi_{X,U}) \le c_5\beta(\Phi_{X,U}), \quad (24)$$

with

$$\beta(\Phi_{X,U}) = |\Phi_{X,U}(0)|^2 + \int_{-\tau}^{0}|\Phi_{X,U}(m)|^2 dm, \quad (25)$$

hold for all $\Phi_{X,U} \in C_{\rm in}$.

Remark 2. The inequalities (24) imply that W satisfies the conditions of the Lyapunov–Krasovskii theorem (see, for instance, Hale and Verduyn Lunel (1993)). However, it is worth mentioning that, according to the type of problem which is investigated, other types of Lyapunov–Krasovskii functionals may be more convenient and they may be provided by variants of our construction. For instance, we will see in the forthcoming proof that asymptotic stability can be established via a functional W with $c_1 = 0$.

Proof. To begin with, one can easily prove that there are positive constants c_4, c_5 such that (24) is satisfied. Let

$$Z(t) = \zeta(X_t, U_t), \quad (26)$$

for all $t \ge 0$, where ζ is the functional defined in (23). Then some elementary calculations lead to

$$\begin{aligned}\dot Z(t) &= e^{A\tau}AX(t) + e^{A\tau}BU(t-\tau) + A\int_{t-\tau}^{t} e^{A(t-m)}BU(m)dm + BU(t) - e^{A\tau}BU(t-\tau) + e^{A\tau}\lambda(t) \\ &= AZ(t) + BU(t) + e^{A\tau}\lambda(t).\end{aligned}$$

Since the definition of Z and (20) imply, for all $t \ge 0$,

$$U(t) = KZ(t), \quad (27)$$

we obtain, for all $t \ge 0$,

$$\begin{cases} \dot Z(t) = HZ(t) + \delta(t), \\ U(t) = KZ(t), \\ X(t) = e^{-A\tau}Z(t) - \int_{t-\tau}^{t} e^{A(t-m-\tau)}BU(m)dm, \end{cases} \quad (28)$$

with $\delta(t) = e^{A\tau}\lambda(t)$, which are equations of the form of the system $\Sigma_{z,v} - \Sigma_x$ introduced in Section 2, with Z playing the role of z, $U(X)$ playing the role of $v(x)$ and K playing the role of N. Let us check now that Assumptions H1 and H2 are satisfied. Let $b_2 = |e^{-A\tau}|^2$, $b_3 = \int_0^\tau |e^{-Am}B|^2 dm$, $b_4 = \frac{1}{2|Q||K|}$. Then (21) implies that Assumption H1 is satisfied with $S(Z) = Z^\top QZ$, $a_1 = \frac{1}{|Q|}$, $\kappa_1(m) = |Q|m^2$.

Now, taking $\theta(\ell) = b_4|\ell|^2$ and $\overline{\theta}(U_t)$ defined as in (10) and one can prove that, for all $t \ge 0$,

$$|X(t)| \le \sqrt{b_2}|Z(t)| + \sqrt{b_3\int_{t-\tau}^{t}|U(m)|^2 dm} \le \sqrt{2\max\left\{b_2, \frac{b_3}{b_4}\right\}(S(Z(t)) + \overline{\theta}(U_t))}$$

and

$$\begin{aligned}|\dot X(t)| &\le |A||X(t)| + |B||U(t-\tau)| + |\lambda(t)| \\ &\le \sqrt{3|A|^2|X(t)|^2 + 3|B|^2|U(t-\tau)|^2 + 3b_2|\delta(t)|^2} \\ &\le \sqrt{a_2[S(Z(t)) + \overline{\theta}(U_t) + |\delta(t)|^2]},\end{aligned}$$

with $a_2 = 2\max\left\{2b_2, 2\frac{b_3}{b_4}, 6|A|^2\max\left\{b_2, \frac{b_3}{b_4}\right\} + 3\frac{|B|^2}{b_4}\right\}$. Bearing the definition of $Z(t)$ in (26) in mind, we deduce from Theorem 2.1 that there is a nonnegative real number b_5 such that the functional

$$V(\Phi_{X,U}) = \frac{1}{2}|\phi_X(0)|^2 + \int_{-\tau}^{0} e^m|\phi_X(m)|^2dm + b_5\left[\zeta(\Phi_{X,U})^\top Q\zeta(\Phi_{X,U}) + \varpi(\phi_U)\right],$$

with $\varpi(\phi_U) = b_4\int_{-\tau}^{0} e^{\frac{1}{2|Q|}m}|\phi_U(m)|^2dm$, admits a derivative along the trajectories of (28) which satisfies

$$\dot{V}(t) \le -\frac{1}{4|Q|}V(X_t, U_t) + b_6|\lambda(t)|^2,$$

for all $t \ge 0$, where b_6 is a positive real number.

Now, using again (26) and the inequality $Q \ge Id$ in Assumption H3, we deduce that, for all $t \ge 0$,

$$V(X_t, U_t) \ge b_5 Z(t)^\top QZ(t) \ge \frac{b_5}{|K|^2+1}|KZ(t)|^2 = \frac{b_5}{|K|^2+1}|U(t)|^2.$$

Consequently,

$$\dot{V}(t) \le -\frac{1}{8|Q|}V(X_t, U_t) - b_7|U(t)|^2 + b_6|\lambda(t)|^2, \tag{29}$$

with $b_7 = \frac{b_5}{8|Q|(|K|^2+1)}$, for all $t \ge 0$.

Moreover, we have, for all $t \ge 0$,

$$U(t)^\top\dot{U}(t) = Z(t)^\top K^\top KHZ(t) + Z(t)^\top K^\top Ke^{A\tau}\lambda(t) \le \frac{|K|^2(2|H|+1)}{2b_5}V(X_t, U_t) + \frac{1}{2}|K|^2|e^{A\tau}|^2|\lambda(t)|^2, \tag{30}$$

for all $t \ge 0$. The inequalities (29) and (30) allow us to conclude. □

3.2. Systems with distributed delay in the inputs

In this section, we consider the closed-loop system studied in Bekiaris-Liberis and Krstic (2011):

$$\begin{cases} \dot{X}(t) = AX(t) + \int_{t-\tau}^{t} B(t-s)U(s)ds + \lambda(t), \\ U(t) = \Big[e^{\tau A}X(t) \\ \quad + \int_{t-\tau}^{t} e^{(t-\ell)A}\int_{\ell}^{t} B(\ell+\tau-s)U(s)dsd\ell\Big], \end{cases} \tag{31}$$

with $X \in \mathbb{R}^n, U \in \mathbb{R}^m$, where $\tau > 0$, the matrices A and K are constant, B is a continuous function and λ is a continuous disturbance and introduce the assumption:

Assumption H4. There exists a symmetric and positive definite matrix Q such that

$$Q \ge Id, \qquad QH + H^\top Q \le -2Id, \tag{32}$$

where

$$H = A + \left(\int_0^\tau e^{(\tau-\ell)A}B(\ell)d\ell\right)K.$$

We state and prove the following result:

Theorem 3.2. *Assume that the system (31) satisfies Assumption H4. Then there exist three constants $e_1 > 0, e_2 > 0, e_3 > 0$ such that $W : C_{\text{in}} \to [0, +\infty)$, defined by*

$$W(\Phi_{X,U}) = \frac{1}{2}|\phi_X(0)|^2 + e_1|\phi_U(0)|^2 + \int_{-\tau}^{0} e^m|\phi_X(m)|^2dm + e_2\zeta(\Phi_{X,U})^\top Q\zeta(\Phi_{X,U}) + e_3\int_{-\tau}^{0} e^{\frac{m}{2|Q|}}|\phi_U(m)|^2dm, \tag{33}$$

with $\Phi_{X,U} = (\phi_X, \phi_U)$, where

$$\zeta(\Phi_{X,U}) = e^{\tau A}\phi_X(0) + \int_{-\tau}^{0} e^{-mA}\int_{m}^{0} B(m+\tau-s)\phi_U(s)dsdm, \tag{34}$$

is an ISS Lyapunov–Krasovskii functional for (31). Moreover, there are two positive constants e_4, e_5 such that the inequalities

$$e_4\beta(\Phi_{X,U}) \le W(\Phi_{X,U}) \le e_5\beta(\Phi_{X,U}), \tag{35}$$

with

$$\beta(\Phi_{X,U}) = |\Phi_{X,U}(0)|^2 + \int_{-\tau}^{0} |\Phi_{X,U}(m)|^2dm, \tag{36}$$

hold for all $(\phi_X, \phi_U) \in C_{\text{in}}$.

Proof. Similarly to the previous case studied, we observe that it is easy to prove that there are positive constants e_4, e_5 such that (35) is satisfied. Let

$$Z(t) = \zeta(X_t, U_t), \tag{37}$$

for all $t \ge 0$, where ζ is the functional defined in (34). Then elementary calculations give

$$\dot{Z}(t) = Ae^{\tau A}X(t) + A\int_{t-\tau}^{t} e^{(t-m)A}\int_{m}^{t} B(m+\tau-s)U(s)dsdm + \left(\int_0^\tau e^{(\tau-\ell)A}B(\ell)d\ell\right)U(t) + e^{A\tau}\lambda(t)$$
$$= AZ(t) + \left(\int_0^\tau e^{(\tau-\ell)A}B(\ell)d\ell\right)U(t) + e^{A\tau}\lambda(t).$$

Since, for all $t \ge 0$,

$$U(t) = KZ(t), \tag{38}$$

we obtain, for all $t \ge 0$,

$$\dot{Z}(t) = AZ(t) + \left(\int_0^\tau e^{(\tau-\ell)A}B(\ell)d\ell\right)KZ(t) + e^{A\tau}\lambda(t) = HZ(t) + e^{A\tau}\lambda(t).$$

Using again (38), we obtain, for all $t \ge 0$,

$$\begin{cases} \dot{Z}(t) = HZ(t) + \delta(t), \\ U(t) = KZ(t), \\ X(t) = e^{-A\tau}Z(t) \\ -\int_{t-\tau}^{t} e^{(t-m-\tau)A}\int_{m}^{t} B(m+\tau-s)U(s)dsdm, \end{cases}$$

with $\delta(t) = e^{A\tau}\lambda(t)$, which are equations of the form of the system $\Sigma_{z,v} - \Sigma_x$. Next we can conclude by arguing as we did from (28) to the end of the proof of Theorem 3.1. □

3.3. Systems with pointwise delay in the inputs endowed with a dynamic extension

The closed-loop system considered in this section comes from the stabilization approach by dynamic extension used in Mondié and Michiels (2003). More precisely, we consider the system

$$\begin{cases} \dot{X}(t) = AX(t) + BU(t-\tau) + \lambda(t), \\ \dot{U}(t) = A_f U(t) \\ \qquad + B_f \left[e^{A\tau} X(t) + \int_{t-\tau}^{t} e^{(t-\ell)A} BU(\ell) d\ell \right], \end{cases} \quad (39)$$

with $X \in \mathbb{R}^n, U \in \mathbb{R}^m$, where $\tau > 0$, the matrices A, B, A_f, B_f are constant and λ is a continuous disturbance and introduce the assumption:

Assumption H5. There exists a symmetric and positive definite matrix Q such that

$$Q \geq Id, \qquad QH + H^\top Q \leq -2Id, \quad (40)$$

where

$$H = \begin{bmatrix} A & B \\ B_f & A_f \end{bmatrix}. \quad (41)$$

We state and prove the following result:

Theorem 3.3. *Assume that the system (39) satisfies Assumption H5. Then there exist three constants $g_1 > 0, g_2 > 0, g_3 > 0$ such that $W : C_{\text{in}} \to [0, +\infty)$, defined by*

$$W(\Phi_{X,U}) = \frac{1}{2}|\phi_X(0)|^2 + g_1|\phi_U(0)|^2 + \int_{-\tau}^{0} e^m |\phi_X(m)|^2 dm + g_2 \zeta(\Phi_{X,U})^\top Q \zeta(\Phi_{X,U}) + g_3 \int_{-\tau}^{0} e^{\frac{m}{2|Q|}} |\phi_U(m)|^2 dm, \quad (42)$$

with $\Phi_{X,U} = (\phi_X, \phi_U)$, with

$$\zeta(\Phi_{X,U}) = \begin{pmatrix} e^{A\tau}\phi_X(0) + \int_{-\tau}^{0} e^{-mA} B\phi_U(m) dm \\ \phi_U(0) \end{pmatrix} \quad (43)$$

is an ISS Lyapunov–Krasovskii functional for (39). Moreover, there are two positive constants g_4, g_5 such that the inequalities

$$g_4 \beta(\Phi_{X,U}) \leq W(\Phi_{X,U}) \leq g_5 \beta(\Phi_{X,U}), \quad (44)$$

with

$$\beta(\Phi_{X,U}) = |\Phi_{X,U}(0)|^2 + \int_{-\tau}^{0} |\Phi_{X,U}(m)|^2 dm, \quad (45)$$

hold for all $\Phi_{X,U} \in C_{\text{in}}$.

Proof. To begin with, we observe that it is easy to prove that there are positive constants g_4, g_5 such that (44) is satisfied. Let

$$r(t) = e^{A\tau} X(t) + \int_{t-\tau}^{t} e^{(t-m)A} BU(m) dm, \quad (46)$$

for all $t \geq 0$. Then simple algebraic manipulations lead to

$$\begin{aligned} \dot{r}(t) &= Ae^{A\tau} X(t) + e^{A\tau} BU(t-\tau) \\ &\quad + A \int_{t-\tau}^{t} e^{(t-m)A} BU(m) dm + BU(t) \\ &\quad - e^{\tau A} BU(t-\tau) + e^{A\tau}\lambda(t) \\ &= Ar(t) + BU(t) + e^{A\tau}\lambda(t). \end{aligned}$$

Therefore we have, for all $t \geq 0$,

$$\begin{cases} \dot{r}(t) = Ar(t) + BU(t) + e^{A\tau}\lambda(t), \\ \dot{U}(t) = A_f U(t) + B_f r(t), \\ r(t) = e^{A\tau} X(t) + \int_{t-\tau}^{t} e^{(t-m)A} BU(m) dm. \end{cases}$$

Let $Z(t) = (r(t)^\top, U(t)^\top)^\top$. Then, we obtain, for all $t \geq 0$,

$$\begin{cases} \dot{Z}(t) = HZ(t) + \delta(t), \\ U(t) = P_2 Z(t), \\ X(t) = e^{-A\tau} P_1 Z(t) \\ \qquad - \int_{t-\tau}^{t} e^{(t-m-\tau)A} BU(m) dm, \end{cases} \quad (47)$$

where $P_1 = [Id_n \;\; 0_m] \in \mathbb{R}^{n \times (m+n)}$ and $P_2 = [0_n \;\; Id_m] \in \mathbb{R}^{m \times (m+n)}$, where Id_k denotes the identity matrix of dimension k and 0_k denotes the matrix of dimension k whose entries are all zero, and

$$\delta(t) = P_1 \begin{pmatrix} e^{A\tau}\lambda(t) \\ 0 \end{pmatrix}.$$

Eqs. (47) are of the form of the system $\Sigma_{z,v} - \Sigma_x$. Next we can conclude by using similar arguments to the ones proposed for Theorem 3.1. □

4. Conclusion

For linear time-invariant systems with delay and additive disturbances, we have shown how ISS Lyapunov–Krasovskii functionals can be constructed. Much remains to be done. Other types of robustness properties (for instance robustness with respect to unknown small terms with a linear growth) can be derived from our ISS Lyapunov–Krasovskii functionals and our design of Lyapunov–Krasovskii functionals may be extended to more complicated families of systems, which include nonlinear systems and time-varying systems.

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