• 3.41
(a) \[ z_2 = 1 = \dot{x}_1 \]
\[ 2x_1z_2 + 3t + 2 - 3z_1 - 2(t + 1)x_2 = 2t + 3t + 2 - 3t - 2(t + 1) = 0 = \dot{z}_2 \]
Thus, \( x(t) = \begin{bmatrix} t \\ 1 \end{bmatrix} \) is a solution.
(b) Recall from the discussion at the beginning of Section 3.4 that to show asymptotic stability of a solution we shift it to the origin and then show asymptotic stability of the origin. Let \( x_1 = x_1 - t \) and \( x_2 = x_2 - 1 \). Then:
\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = 2x_1x_2 - x_1 - 2x_2 \]
We need to show that the origin \( x = 0 \) is uniformly asymptotically stable.
\[ \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -1 + 2x_2 \\ -2x_2 & -2 + 2x_1 \end{bmatrix}, \quad A = \left. \frac{\partial f}{\partial x} \right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \]
The matrix \( A \) is Hurwitz; hence, the origin is uniformly asymptotically stable.

• 5.5 The closed-loop system is given by
\[ \dot{x} = (A - BB^T)Pz + Bg(t, x) \]
Rewrite the Riccati equation as
\[ P(A - BB^T)P + (A - BB^T)^TP + Q + PBB^TP + 2\alpha P = 0 \]
Consider \( V(x) = x^TPz \) as a Lyapunov function candidate.
\[ \dot{V}(t, x) = x^T[P(A - BB^T)P + (A - BB^T)^TP]z + 2z^TPBg(t, x) \]
\[ = -x^T[Q + PBB^TP + 2\alpha P]z + 2z^TPBg(t, x) \]
\[ \leq -k^2\|z\|^2 - \|w\|^2 - 2\alpha\lambda_{\text{min}}(P)\|z\|^2 + 2k\|w\|\|z\|, \quad \text{where} \ w = B^TPz \]
\[ = -[k\|z\|^2 + \|w\|^2 - 2\alpha\lambda_{\text{min}}(P)\|z\|^2 \leq -2\alpha\lambda_{\text{min}}(P)\|z\|^2 \]
Hence, the origin is globally exponentially stable.
5.18
(a) Let $b = 0$. Try $V(z) = \frac{1}{3}z_1^2 + \frac{1}{2}z_2^2$.

$$
\dot{V}(z) = -z_1^2 + z_1z_2(z_1 + a) - z_1z_2(z_1 + a) = -z_1^2
$$

Thus, the origin is globally asymptotically stable. To investigate exponential stability, linearize at $z = 0$.

$$
A = \frac{\partial f}{\partial z} \bigg|_{z=0} = \begin{bmatrix} -1 + z_2 & z_1 + a \\ -2z_1 - a & 0 \end{bmatrix}_{z=0} = \begin{bmatrix} -1 & a \\ -a & 0 \end{bmatrix}
$$

The characteristic equation of $A$ is $\lambda^2 + \lambda + a^2 = 0$. Hence, $A$ is Hurwitz and the origin is exponentially stable.

(b) Let $b > 0$. The linearization at the origin is given by

$$
A = \frac{\partial f}{\partial z} \bigg|_{z=0} = \begin{bmatrix} -1 & a \\ -a & b \end{bmatrix}
$$

The characteristic equation of $A$ is $\lambda^2 + (1 - b)\lambda - a^2 - b = 0$. Hence, $A$ is Hurwitz if $b < \min\{1, a^2\}$.

(c) For $b > 0$, the equilibrium points are

$$(0, 0), \left(-a + \sqrt{b}, -\frac{a + \sqrt{b}}{\sqrt{b}}\right), \left(-a - \sqrt{b}, \frac{a + \sqrt{b}}{\sqrt{b}}\right)$$

Since the system has multiple equilibria, the origin is not globally asymptotically stable.

5.21
(1) Let $V(z) = \frac{1}{2}z^2$.

$$
\dot{V} = -z^4 + z^4u
$$

For $|u| \leq r_u < 1$, we have

$$
\dot{V} \leq -(1 - r_u)z^4, \forall z
$$

By Theorem 5.2, the system is locally input-to-state stable. It is not input-to-state stable since with $u(t) \equiv c > 1$ and $z(0) > 0$, $z(t) \rightarrow \infty$ as $t \rightarrow \infty$.

(2) Let $V(z) = \frac{1}{2}z^2$.

$$
\dot{V} = -z^4 + uz^4 - z^4 \leq -z^4, \forall |z| \geq \sqrt{u}
$$

By Theorem 5.2, the system is input-to-state stable.

(3) Let $V(z) = \frac{1}{2}z^2$. For $|u| \leq r_u$ and $|z| < r < 1$, we have

$$
\dot{V} = -z^2 + z^2u \leq -(1 - \theta)z^2 - \theta z^2 + r^2|z| |u| \leq -(1 - \theta)z^2, \forall |z| \geq \frac{r^2|u|}{\theta}
$$

The preceding inequality is valid provided $rr_u < \theta < 1$. By Theorem 5.2, the system is locally input-to-state stable. It is not input-to-state stable since with $u(t) \equiv 1$ and $z(0) > 0$, $z(t) \rightarrow \infty$ as $t \rightarrow \infty$. 
5.22

(1) The $x_1$-system is ISS w.r.t. $x_2$, and the $x_2$-system is ISS w.r.t. $u$. Then by [KKK, Lemma C.4], the $(x_1, x_2)$-system is ISS. To see that the $x_1$-system is ISS, note that

$$\frac{x_1^2}{2} \leq -\frac{1}{2} x_1^2 + \frac{1}{2} x_2^4,$$

which implies that

$$|x_1(t)| \leq |x_0| e^{-t/2} + \left( \sup_{[0,t]} |x_2(t)| \right)^2.$$

(4) Take $\gamma = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$. Then

$$\gamma = -x_1^2 - x_2^4 + x_1 u_1 + x_2 u_2$$

$$\leq -x_1^2 - x_2^4 + \frac{1}{2} x_1^2 + \frac{1}{2} u_1^2 + \frac{1}{4} x_2^4 + \frac{3}{4} u_2^{4/3}$$

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$$= -\frac{1}{2} \left( x_1^2 + \frac{3}{2} x_2^4 \right) + \frac{1}{2} \left( u_1^2 + \frac{3}{2} u_2^{4/3} \right)$$

$$= -\frac{1}{2} \left( |x_1|^2 + \frac{3}{2} |x_2|^4 \right) + \frac{1}{2} \left( |u_1|^2 + \frac{3}{2} \left( \frac{3}{4} u_2 \right)^4 \right)$$

$$\triangleq -Q(|x_1|, |x_2|) + Q(|u_1|, \sqrt[4]{u_2})$$

Note that $Q(y_1, y_2)$ is pdf, smooth on $\mathbb{R}^2$ and radially unbd. Then there exist class $\mathcal{K}$ functions $\varphi_1$ and $\varphi_2$ s.t.

$$\varphi_1(|y_1|) \leq Q(y_1, y_2) \leq \varphi_2(|y_1|)$$

Hence

$$\gamma \leq -\varphi_1(|x_1|) + \varphi_2(|u_1|^2 + u_2^{3/2})$$

$$\leq -\varphi_2(|x_1|) + \varphi_3(|u_1|),$$

where $\varphi_3 \in \mathcal{K}$. Then by [KKK, Thm C.3], the system is ISS w.r.t. $u$. 