1. Using averaging theory, analyze the following system:

\[
\begin{align*}
\dot{x} &= \epsilon \left[-x + 1 - 2(y + \sin(t))^2\right] \\
\dot{y} &= \epsilon z \\
\dot{z} &= \epsilon \left[-z - \sin(t) \left(\frac{1}{2} x + (y + \sin(t))^2\right)\right].
\end{align*}
\]

Solution 1
First we calculate the average system

\[
\begin{align*}
f_{av} &= \frac{1}{2\pi} \int_0^{2\pi} f(t, x, 0)dt \\
&= \begin{bmatrix}
-x + 1 - 2y^2 - \frac{1}{\pi} \int_0^{2\pi} \sin^2(t)dt \\
-z - y \int_0^{2\pi} \sin^2(t)dt \\
-x - 2y^2 - z - y
\end{bmatrix}.
\end{align*}
\]

The Jacobian

\[
J_{av} = \frac{\partial f_{av}}{\partial x} \bigg|_{(0, 0, 0)} = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & -1
\end{bmatrix}
\]

is Hurwitz. Note that \((0, 0, 0)\) is not an equilibrium of (1)-(3). Hence, there exists \(\epsilon^* > 0\) such that for all \(\epsilon \in (0, \epsilon^*)\) there exists a locally exponentially stable solution \((x^{2\pi}(t), y^{2\pi}(t), z^{2\pi}(t))\) of period \(2\pi\) and such that \(|(x^{2\pi}(t), y^{2\pi}(t), z^{2\pi}(t))| < O(\epsilon), \forall t \geq 0.\)

2. Analyze the following system using the method of averaging for large \(\omega:\)

\[
\begin{align*}
\dot{x}_1 &= (x_2 \sin(\omega t) - 2)x_1 - x_3 \\
\dot{x}_2 &= -x_2 + (x_2^2 \sin(\omega t) - 2x_3 \cos(\omega t)) \cos(\omega t) \\
\dot{x}_3 &= 2x_2 - \sin(x_3) + (4x_2 \sin(\omega t) + x_3) \sin(\omega t)
\end{align*}
\]

Solution 2 Let \(\tau = \omega t,\)

\[
\frac{d}{d\tau} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix}
(x_2 \sin(\tau) - 2)x_1 - x_3 \\
-x_2 + (x_2^2 \sin(\tau) - 2x_3 \cos(\tau)) \cos(\tau) \\
2x_2 - \sin(x_3) + (4x_2 \sin(\tau) + x_3) \sin(\tau)
\end{bmatrix}
\]

\[
f_{av} = \frac{1}{2\pi} \int_0^{2\pi} f(\tau, x, 0)d\tau
\]

\[
= \begin{bmatrix}
-x_1 - x_3 \\
-x_2 - x_3 \\
2x_2 - \sin(x_3) + 2x_2
\end{bmatrix}
\]
The Jacobian

\[ J_{av} = \left. \frac{\partial f_{av}}{\partial x} \right|_{(0,0,0)} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 4 & -1 \end{bmatrix} \]

is Hurwitz and \( f(\tau, 0, \epsilon) = 0 \). Hence, there exists \( \omega^* > 0 \) such that for all \( \omega > \omega^* \) the origin is exponentially stable.

3. Consider the second-order system

\[
\begin{align*}
\dot{x}_1 &= \sin(\omega t)y_1 \\
\dot{x}_2 &= \cos(\omega t)y_2 \\
y_1 &= \left[ x_1 + \sin(\omega t) \right] \left[ x_2 + \cos(\omega t) - x_1 - \sin(\omega t) \right] \\
y_2 &= \left[ x_2 + \cos(\omega t) \right] \left[ x_1 + \sin(\omega t) - x_2 - \cos(\omega t) \right].
\end{align*}
\]

Show that for sufficiently large \( \omega \) there exists an exponentially stable periodic orbit in an \( O(1/\omega) \) neighborhood of the origin \( x_1 = x_2 = 0 \).

**Hint:** The following functions have a zero mean over the interval \([0, 2\pi]\): \( \sin(\tau), \cos(\tau), \sin(\tau) \cos(\tau), \sin^3(\tau), \cos^3(\tau), \sin^2(\tau) \cos(\tau), \) and \( \sin(\tau) \cos^2(\tau) \).

**Solution 3** Let \( \tau = \omega t \),

\[
\begin{align*}
\frac{dx_1}{d\tau} &= \frac{1}{\omega} y_1 \sin(\tau) \\
\frac{dx_2}{d\tau} &= \frac{1}{\omega} y_2 \cos(\tau) \\
f_{av} &= \frac{1}{2\pi} \begin{bmatrix} \int_{0}^{2\pi} y_1 \sin(\tau) d\tau \\
\int_{0}^{2\pi} y_2 \cos(\tau) d\tau \end{bmatrix} \\
&= \begin{bmatrix} -x_1 + \frac{1}{2} x_2 \\
\frac{1}{2} x_1 - x_2 \end{bmatrix}.
\end{align*}
\]

Note that the origin is not an equilibrium of (7)-(10). Hence there exist \( \omega^* > 0 \) such that for all \( \omega > \omega^* \) there exist a locally exponentially stable solution \( (x^{2\pi/\omega}(t), y^{2\pi/\omega}(t)) \) of period \( 2\pi/\omega \) and such that \( \|(x^{2\pi/\omega}(t), y^{2\pi/\omega}(t))\| \leq O(1/\omega), \forall t \geq 0 \).

4. Consider Rayleigh’s equation

\[ m \frac{d^2 u}{dt^2} + ku = \lambda \left[ 1 - \alpha \left( \frac{du}{dt} \right)^2 \right] \frac{du}{dt} \]

where \( m, k, \lambda, \) and \( \alpha \) are positive constants.

a) Using the dimensionless variables \( y = \frac{u}{u^*}, \tau = \frac{t}{t^*}, \) and \( \epsilon = \frac{\lambda}{\lambda^*}, \) where \( (u^*)^2 \alpha k = \frac{u^*}{m}, t^* = \sqrt{\frac{m}{k}}, \) and \( \lambda^* = \sqrt{km}, \) show that the equation can be normalized to

\[ \ddot{y} + y = \epsilon \left( \dot{y} - \frac{1}{3} \dot{y}^3 \right) \]

where \( \dot{y} \) denotes the derivative of \( y \) with respect to \( \tau \).

b) Apply the averaging method to show that the normalized Rayleigh equation has a stable limit cycle. Estimate the location of the limit cycle in the plane \((y, \dot{y})\).
Solution 4  By applying the chain rule

\[ m \frac{d}{d\tau} \left( \frac{du}{d\tau} \right) \frac{d\tau}{dt} + ku = \lambda \left[ 1 - a \left( \frac{du}{d\tau} \right)^2 \right] \frac{du}{d\tau} \frac{d\tau}{dt} \]

we get equation (12) from (11).

Assume \( x_1 = y \) and \( x_2 = \dot{y} \), and consider following transformation

\[ \phi = \tan^{-1} \left( \frac{x_1}{x_2} \right) \]
\[ r = \sqrt{x_1^2 + x_2^2} \]

then we have

\[ \dot{\phi} = 1 - \epsilon \left( \sin(\phi) \cos(\phi) - \frac{1}{3} r^2 \sin(\phi) \cos^3(\phi) \right) \]
\[ \dot{r} = \epsilon \left( r \cos^2(\phi) - \frac{1}{3} r^3 \cos^4(\phi) \right) \]
\[ \frac{dr}{d\phi} = \epsilon \frac{r \cos^2(\phi) - \frac{1}{3} r^3 \cos^4(\phi)}{1 - \epsilon \left( \sin(\phi) \cos(\phi) - \frac{1}{4} r^2 \sin(\phi) \cos^3(\phi) \right)} = \epsilon f(r, \phi, \epsilon) \]
\[ f_{av} = \frac{1}{2\pi} \int_{0}^{2\pi} f(r, \phi, 0) d\phi \]
\[ = \frac{1}{2} r - \frac{1}{8} r^3 = 0 \Rightarrow r = 0 \text{ or } r = 2. \]

Since \( \dot{r} > 0 \) for \( r < 2 \) and \( \dot{r} < 0 \) for \( r > 0 \), there is a stable limit cycle with \( r = 2 \).