Chetaev’s Theorem

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, and let $V$ be a functional of $x$. Then, $x = 0$ is unstable if

- $V(0) = 0$, and $V(x_0) > 0$ for some arbitrary small $|x_0|$.
- $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) > 0$ for $\forall x \in U_0$, where $U_0$ was defined as $U_0 = \{x \in B_r | V(x) > 0\}$. 

With Chetaev’s theorem, show that the equilibrium at the origin of the following three systems is unstable:

a) \[
\dot{x} = x^3 + xy^3, \quad (1) \\
\dot{y} = -y + x^2 \quad (2)
\]

Solution

The system (1) and (2) has an equilibrium at the origin \((x, y) = (0, 0)\). Let \(V(x)\) be a functional s.t.

\[
V(x, y) = \frac{1}{2}x^2 - \frac{1}{4}y^4. \quad (3)
\]

Then, \(V(0, 0) = 0\) and \(V(x, y) > 0\) for \(2x^2 > y^4\). Taking time derivative and from (1) (2), we obtain

\[
\dot{V}(x, y) = x\dot{x} - y^3\dot{y} = x(x^3 + xy^3) - y^3(-y + x^2) = x^4 + y^4. \quad (4)
\]

Defining \(U_0 = \{(x, y)|V(x, y) > 0\} = \{2x^2 > y^4\}\), we can state that \(\dot{V}(x, y) > 0\) for \(\forall x, y \in U_0\).
Therefore, by Chetaev’s theorem, the origin \((x, y) = (0, 0)\) is unstable.
b)

\[ \dot{\xi} = \eta + \xi^3 + 3\xi\eta^2, \quad (5) \]
\[ \dot{\eta} = -\xi + \eta^3 + 3\eta\xi^2 \quad (6) \]

**Solution**

The system (5) and (6) has an equilibrium at the origin \((\xi, \eta) = (0, 0)\). Let \(V(\xi, \eta)\) be a functional s.t.

\[ V(\xi, \eta) = \frac{1}{2}\xi^2 + \frac{1}{2}\eta^2. \quad (7) \]

Then, \(V(0, 0) = 0\) and \(V(\xi, \eta) > 0\) for \(\forall (\xi, \eta) \in \mathbb{R}^2/\{(0, 0)\}\). Taking time derivative and from (5) (6), we obtain

\[ \dot{V}(\xi, \eta) = \xi \dot{\xi} + \eta \dot{\eta} = \xi^4 + 6\xi^2\eta^2 + \eta^4. \quad (8) \]

Defining \(U_0 = \{(\xi, \eta)|V(\xi, \eta) > 0\} = \mathbb{R}^2/\{(0, 0)\}\), we can state that \(\dot{V}(\xi, \eta) > 0\) for \(\forall (\xi, \eta) \in U_0 = \mathbb{R}^2/\{(0, 0)\}\).

Therefore, by Chetaev’s theorem, the origin \((\xi, \eta) = (0, 0)\) is unstable.
c)
\[
\begin{align*}
\dot{x} &= |x|x + xy\sqrt{|y|}, \\
\dot{y} &= -y + |x|\sqrt{|y|}.
\end{align*}
\]

(9) (10)

Solution
The system (9) and (10) has an equilibrium at the origin \((x, y) = (0, 0)\). Let \(V(x, y)\) be a functional s.t.
\[
V(x, y) = x - \frac{y^2}{2}.
\]
(11)

Then, \(V(0, 0) = 0\) and \(V(x, y) > 0\) for \(x > y^2/2\). Taking time derivative and from (9) (10), we obtain
\[
\dot{V}(x, y) = \dot{x} - y\dot{y} = 
\left(|x|x + xy\sqrt{|y|}\right) - y \left(-y + |x|\sqrt{|y|}\right)
\]
\[
= \left(|x|x + y^2\right) + (x - |x|)y\sqrt{|y|}.
\]
(12)

Define \(U_0 = \{(x, y)|V(x, y) > 0\} = \{x > y^2/2\}\). Then, for \(\forall(x, y) \in U_0\), we have \(x > 0\), which leads to \(|x| = x\) for \(\forall(x, y) \in U_0\). Thus, we have
\[
\dot{V}(x, y) = x^2 + y^2, \quad \forall(x, y) \in U_0
\]
(13)

Therefore, \(\dot{V}(x, y) > 0\) for \(\forall(x, y) \in U_0\). By Chetaev’s theorem, the origin \((x, y) = (0, 0)\) is unstable.