1. Derive the sensitivity equations for the system

\begin{align}
\dot{x}_1 &= \tan^{-1}(\alpha x_1) - x_1 x_2 \\
\dot{x}_2 &= bx_1^2 - cx_2
\end{align}

(1) (2)

as the parameters $a, b,$ and $c$ vary from their nominal values $a_0 = 1, b_0 = 0,$ and $c_0 = 1.$

2. Calculate exactly (in closed form) the sensitivity function at $\lambda_0 = 1$ for the system

\begin{equation}
\dot{x} = -\lambda x^3.
\end{equation}

(3)

What is the approximation $x(t, \lambda) \approx x(t, \lambda_0) + S(t)(\lambda - \lambda_0)$ for $\lambda = 7/2$?
\[ \dot{x}_1 = \tan^{-1}(ax_1) - x_1x_2 \]
\[ \dot{x}_2 = bx_1^2 - cx_2 \]

Let \( \lambda = [a, b, c]^T \). The nominal values are \( a_0 = 1, b_0 = 0, \) and \( c_0 = 1 \). The Jacobian matrices \( \partial f / \partial x \) and \( \partial f / \partial \lambda \), are given by

\[
\frac{\partial f}{\partial x} = \begin{bmatrix}
\frac{a}{1 + a^2} x_1^2 - x_2 & -x_1 \\
2bx_1 & -c
\end{bmatrix}, \quad \frac{\partial f}{\partial \lambda} = \begin{bmatrix}
\frac{x_1^4}{1 + a^2} x_1^2 & 0 & 0 \\
0 & x_1^2 & -x_2
\end{bmatrix}
\]

Let
\[
S = \left. \frac{\partial x}{\partial \lambda} \right|_{\text{nominal}} = \begin{bmatrix}
z_3 \\
z_4 \\
z_5 \\
z_6 \\
z_7 \\
z_8
\end{bmatrix}
\]

Then
\[
\dot{S} = \left. \frac{\partial f}{\partial x} \right|_{\text{nominal}} S + \left. \frac{\partial f}{\partial \lambda} \right|_{\text{nominal}}, \quad S(0) = 0
\]

The augmented equation (2.11) is given by

\[
\dot{x}_1 = \tan^{-1}(x_1) - x_1x_2 \\
\dot{x}_2 = -x_2 \\
\dot{x}_3 = \left( \frac{1}{1 + x_1^2} - x_2 \right) x_3 - x_1x_4 + \frac{x_1}{1 + x_1^2} \\
\dot{x}_4 = -x_4 \\
\dot{x}_5 = \left( \frac{1}{1 + x_1^2} - x_2 \right) x_5 - x_1x_6 \\
\dot{x}_6 = -x_6 + x_1^2 \\
\dot{x}_7 = \left( \frac{1}{1 + x_1^2} - x_2 \right) x_7 - x_1x_8 \\
\dot{x}_8 = -x_8 - x_2
\]

with the initial conditions
\[
x_1(0) = x_{10}, \ x_2(0) = x_{20}, \ x_3(0) = x_4(0) = x_5(0) = x_6(0) = x_7(0) = x_8(0) = 0
\]
Homework 3 Solutions, MAE281A 2014

Prepared by Greg Mills

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Calculate exactly (in closed form) the sensitivity function at $\lambda_0 = 1$ for
the system

$$\dot{x} = -\lambda x^3$$  \hspace{1cm} (1)

The closed form solution to (3) is

$$x(t) = \frac{x_0}{\sqrt{2\lambda t^2 x_0^2 + 1}}$$  \hspace{1cm} (2)

Derive the sensitivity equation

$$\dot{s}(t)_{\lambda = \lambda_0} = -3x(t)^2s(t) - x(t)^3$$  \hspace{1cm} (3)

This is a non-homogenous first-order ode of $s$. We can use the integrating
factor method here to solve.

\[
\dot{s}(t) + 3x(t)^2 s(t) = -x(t)^3
g(t) = 3x(t)^2 \int_0^t ds(t) = -e^{\int_0^t 3x(\tau)^2 d\tau} x(t)^3
\]

\[
\frac{d}{dt} \left( e^{\int_0^t 3x(\tau)^2 d\tau} s(t) \right) = -e^{\int_0^t 3x(\tau)^2 d\tau} x(t)^3
\]

\[
e^{\int_0^t 3x(\tau)^2 d\tau} s(t) = -\int_0^t e^{\int_0^\tau 3x(z)^2 dz} x(\tau)^3 d\tau + c
\]

\[
s(0) = 0 \Rightarrow c = 0
\]

\[
s(t) = -e^{\int_0^t 3x(\tau)^2 d\tau} \int_0^t e^{\int_0^\tau 3x(z)^2 dz} x(\tau)^3 d\tau
\]

\[
s(t) = -\int_0^t e^{\int_0^\tau 3x(z)^2 dz} x(\tau)^3 d\tau
\]

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\]

Finally we have the sensitivity equation

\[
s(t) = \frac{-x_0^3 t}{(2tx_0^2 + 1)^{3/2}} \quad (4)
\]

Solve for

\[
x(t, \lambda) \approx x(t, \lambda_0) + s(t)(\lambda - \lambda_0)
\]

when \( \lambda = \frac{7}{2} \)

\[
x(t, \lambda) \approx \frac{x_0}{\sqrt{2\lambda tx_0^2 + 1}} + \frac{5}{2} \frac{-x_0^3 t}{(2tx_0^2 + 1)^{3/2}} \quad (5)
\]