Adaptive Nonlinear Control—A Tutorial

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- Backstepping
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main source:
Nonlinear and Adaptive Control Design (Wiley, 1995)
M. Krstić, I. Kanellakopoulos and P. V. Kokotović
Backstepping (nonadaptive)

\[
\dot{x}_1 = x_2 + \varphi(x_1)^T \theta, \quad \varphi(0) = 0 \\
\dot{x}_2 = u
\]

where \( \theta \) is known parameter vector and \( \varphi(x_1) \) is smooth nonlinear function.

**Goal:** stabilize the equilibrium \( x_1 = 0, x_2 = -\varphi(0)^T \theta = 0 \).

**Virtual control** for the \( x_1 \)-equation:

\[
\alpha_1(x_1) = -c_1 x_1 - \varphi(x_1)^T \theta, \quad c_1 > 0
\]

**Error variables**:

\[
\begin{align*}
z_1 & = x_1 \\
z_2 & = x_2 - \alpha_1(x_1),
\end{align*}
\]
System in error coordinates:

\[
\dot{z}_1 = \dot{x}_1 = x_2 + \varphi^T \theta = z_2 + \alpha_1 + \varphi^T \theta = -c_1 z_1 + z_2 \\
\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 = u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \theta)
\]

Need to design \( u = \alpha_2(x_1, x_2) \) to stabilize \( z_1 = z_2 = 0 \).

Choose Lyapunov function

\[
V(x_1, x_2) = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2
\]

we have

\[
\dot{V} = z_1 (-c_1 z_1 + z_2) + z_2 \left[ u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \theta) \right]
\]

\[
= -c_1 z_1^2 + z_2 \left[ u + z_1 - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \theta) \right]
\]

\[
\Rightarrow \dot{V} = -c_1 z_1^2 - c_2 z_2^2
\]
$z = 0$ is globally asymptotically stable

invertible change of coordinates

$\Downarrow$

$x = 0$ is globally asymptotically stable

The closed-loop system in $z$-coordinates is linear:

$$
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
-c_1 & 1 \\
-1 & -c_2
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}.
$$
Tuning Functions Design

Introductory examples:

\[
\begin{align*}
A & : & \dot{x}_1 &= u + \varphi(x_1)^T \theta \\
& & \dot{x}_2 &= u \\
B & : & \dot{x}_1 &= x_2 + \varphi(x_1)^T \theta \\
& & \dot{x}_2 &= x_3 \\
C & : & \dot{x}_1 &= x_2 + \varphi(x_1)^T \theta \\
& & \dot{x}_2 &= x_3 \\
& & \dot{x}_3 &= u
\end{align*}
\]

where \( \theta \) is unknown parameter vector and \( \varphi(0) = 0 \).

**Degin A.** Let \( \hat{\theta} \) be the estimate of \( \theta \) and \( \tilde{\theta} = \theta - \hat{\theta} \),

Using

\[
u = -c_1 x_1 - \varphi(x_1)^T \hat{\theta}
\]

gives

\[
\dot{x}_1 = -c_1 x_1 + \varphi(x_1)^T \tilde{\theta}
\]
To find update law for $\hat{\theta}(t)$, choose

$$V_1(x, \hat{\theta}) = \frac{1}{2}x_1^2 + \frac{1}{2}\hat{\theta}^T \Gamma^{-1}\hat{\theta}$$

then

$$\dot{V}_1 = -c_1x_1^2 + x_1\varphi(x_1)\hat{\theta} - \hat{\theta}^T \Gamma^{-1}\dot{\hat{\theta}}$$

$$= -c_1x_1^2 + \hat{\theta}^T \Gamma^{-1}\left(\Gamma \varphi(x_1)x_1 - \dot{\hat{\theta}}\right)$$

$$= 0$$

Update law:

$$\dot{\hat{\theta}} = \Gamma \varphi(x_1)x_1, \quad \varphi(x_1) — \text{regressor}$$

gives

$$\dot{V}_1 = -c_1x_1^2 \leq 0.$$  

By Lasalle’s invariance theorem, $x_1 = 0, \hat{\theta} = \theta$ is stable and

$$\lim_{t\to\infty} x_1(t) = 0.$$
Design B. replace $\theta$ by $\hat{\theta}$ in the nonadaptive design:

$$z_2 = x_2 - \alpha_1(x_1, \hat{\theta}), \quad \alpha_1(x_1, \hat{\theta}) = -c_1 z_1 - \varphi^T \hat{\theta}$$

and strengthen the control law by $\nu_2(x_1, x_2, \hat{\theta})$ (to be designed)

$$u = \alpha_2(x_1, x_2, \hat{\theta}) = -c_2 z_2 - z_1 + \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \hat{\theta}) + \nu_2(x_1, x_2, \hat{\theta})$$

error system

$$\dot{z}_1 = z_2 + \alpha_1 + \varphi^T \theta = -c_1 z_1 + z_2 + \varphi^T \hat{\theta}$$
$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \theta) - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}}$$

$$= -z_1 - c_2 z_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi^T \hat{\theta} - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} + \nu_2(x_1, x_2, \hat{\theta}) ,$$

or

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -\varphi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi^T \end{bmatrix} \hat{\theta} + \begin{bmatrix} 0 \\ -\frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} + \nu_2(x_1, x_2, \hat{\theta}) \end{bmatrix}$$

remaining: design adaptive law.
Choose

\[ V_2(x_1, x_2, \hat{\theta}) = V_1 + \frac{1}{2} z_2^2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} \hat{\theta}^T \Gamma^{-1} \hat{\theta} \]

we have

\[ \dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + [z_1, z_2] \begin{bmatrix} \varphi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi \end{bmatrix} \hat{\theta} - \hat{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \]

\[ = -c_1 z_1^2 - c_2 z_2^2 + \hat{\theta}^T \Gamma^{-1} \left( \Gamma \begin{bmatrix} \varphi, -\frac{\partial \alpha_1}{\partial x_1} \varphi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \dot{\hat{\theta}} \right). \]

The choice

\[ \dot{\hat{\theta}} = \Gamma \tau_2(x, \hat{\theta}) = \Gamma \begin{bmatrix} \varphi, -\frac{\partial \alpha_1}{\partial x_1} \varphi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \Gamma \left( \begin{bmatrix} \tau_1 \\ \frac{\tau_1}{\varphi z_1} - \frac{\partial \alpha_1}{\partial x_1} \varphi z_2 \end{bmatrix} \right)_{\tau_2} \]

(\( \tau_1, \tau_2 \) are called tuning functions)

makes

\[ \dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2, \]

thus \( z = 0, \ \tilde{\theta} = 0 \) is GS and \( x(t) \to 0 \) as \( t \to \infty \).
The closed-loop adaptive system
Design C.
We have one more integrator, so we define the third error coordinate and replace $\dot{\hat{\theta}}$ in design B by potential update law,

\[ z_3 = x_3 - \alpha_2(x_1, x_2, \hat{\theta}) \]

\[ \nu_2(x_1, x_2, \hat{\theta}) = \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2(x_1, x_2, \hat{\theta}). \]

Now the $z_1, z_2$-system is

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
-c_1 & 1 \\
-1 & -c_2
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} +
\begin{bmatrix}
\varphi^T \\
-\frac{\partial \alpha_1}{\partial x_1} \varphi^T
\end{bmatrix} \tilde{\theta} +
\begin{bmatrix}
0 \\
z_3 + \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \hat{\theta})
\end{bmatrix}
\]

and

\[
\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \hat{\theta}) + \tilde{\theta}^T (\tau_2 - \Gamma^{-1} \hat{\theta}).
\]
$z_3$-equation is given by

$$
\dot{z}_3 = u - \frac{\partial \alpha_2}{\partial x_1} \left( x_2 + \varphi^T \hat{\theta} \right) - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}}
$$

$$
= u - \frac{\partial \alpha_2}{\partial x_1} \left( x_2 + \varphi^T \hat{\theta} \right) - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_2}{\partial x_1} \varphi^T \tilde{\theta}.
$$

Choose

$$
V_3(x, \hat{\theta}) = V_2 + \frac{1}{2} z_3^2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} z_3^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}
$$

we have

$$
\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}})
$$

$$
+ z_3 \left[ z_2 + u - \frac{\partial \alpha_2}{\partial x_1} \left( x_2 + \varphi^T \hat{\theta} \right) - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} \right]
$$

$$
+ \tilde{\theta}^T \left( \tau_2 - \frac{\partial \alpha_2}{\partial x_1} \varphi z_3 - \Gamma^{-1} \dot{\hat{\theta}} \right).
$$
Pick update law

\[ \dot{\theta} = \Gamma \tau_3(x_1, x_2, x_3, \hat{\theta}) = \Gamma \left( \tau_2 - \frac{\partial \alpha_2}{\partial x_1} \varphi z_3 \right) = \Gamma \left[ \varphi, \frac{\partial \alpha_1}{\partial x_1} \varphi, -\frac{\partial \alpha_2}{\partial x_1} \varphi \right] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \]

and control law

\[ u = \alpha_3(x_1, x_2, x_3, \hat{\theta}) = -z_2 - c_3 z_3 + \frac{\partial \alpha_2}{\partial x_1} (x_2 + \varphi^T \hat{\theta}) + \frac{\partial \alpha_2}{\partial x_2} x_3 + \nu_3, \]

results in

\[ \dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_2 \frac{\partial \alpha_1}{\partial \theta} (\Gamma \tau_2 - \dot{\theta}) + z_3 \left( \nu_3 - \frac{\partial \alpha_2}{\partial \theta} \dot{\theta} \right). \]

Notice

\[ \dot{\theta} - \Gamma \tau_2 = \dot{\theta} - \Gamma \tau_3 - \Gamma \frac{\partial \alpha_2}{\partial x_1} \varphi z_3 \]

we have

\[ \dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 \left( \nu_3 - \frac{\partial \alpha_2}{\partial \theta} \Gamma \tau_3 + \frac{\partial \alpha_1}{\partial \theta} \Gamma \frac{\partial \alpha_2}{\partial x_1} \varphi z_2 \right). \]

Stability and regulation of \( x \) to zero follows.
Further insight:

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\end{bmatrix} = \begin{bmatrix}
-c_1 & 1 & 0 \\
-1 & -c_2 & 1 \\
0 & -1 & -c_3 \\
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
\end{bmatrix} + \begin{bmatrix}
\phi^T \\
-\frac{\partial \alpha_1}{\partial x_1} \phi^T \\
-\frac{\partial \alpha_2}{\partial x_1} \phi^T \\
\end{bmatrix} \tilde{\theta} + \begin{bmatrix}
0 \\
\nu_3 - \frac{\partial \alpha_2}{\partial \theta} \Gamma_3 \\
\end{bmatrix}.
\]

\[
\downarrow \hat{\theta} - \Gamma \tau_2 = \hat{\theta} - \Gamma \tau_3 - \Gamma \frac{\partial \alpha_2}{\partial x_1} \phi z_3
\]

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\end{bmatrix} = \begin{bmatrix}
-c_1 & 1 & 0 \\
-1 & -c_2 & 1 + \frac{\partial \alpha_1}{\partial \theta} \Gamma \frac{\partial \alpha_2}{\partial x_1} \phi \\
0 & -1 & -c_3 \\
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
\end{bmatrix} + \begin{bmatrix}
\phi^T \\
-\frac{\partial \alpha_1}{\partial x_1} \phi^T \\
-\frac{\partial \alpha_2}{\partial x_1} \phi^T \\
\end{bmatrix} \tilde{\theta} + \begin{bmatrix}
0 \\
\nu_3 - \frac{\partial \alpha_2}{\partial \theta} \Gamma_3 \\
\end{bmatrix}
\]

\[
\downarrow \text{selection of } \nu_3
\]

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\end{bmatrix} = \begin{bmatrix}
-c_1 & 1 & 0 \\
-1 & -c_2 & 1 + \frac{\partial \alpha_1}{\partial \theta} \Gamma \frac{\partial \alpha_2}{\partial x_1} \phi \\
0 & -1 & -c_3 \\
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
\end{bmatrix} + \begin{bmatrix}
\phi^T \\
-\frac{\partial \alpha_1}{\partial x_1} \phi^T \\
-\frac{\partial \alpha_2}{\partial x_1} \phi^T \\
\end{bmatrix} \tilde{\theta}.
\]
General Recursive Design Procedure

parametric strict-feedback system:

\[
\begin{align*}
\dot{x}_1 &= x_2 + \varphi_1(x_1)^T \theta \\
\dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^T \theta \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \varphi_{n-1}(x_1, \ldots, x_{n-1})^T \theta \\
\dot{x}_n &= \beta(x)u + \varphi_n(x)^T \theta \\
y &= x_1
\end{align*}
\]

where $\beta$ and $\varphi_i$ are smooth.

Objective: asymptotically track reference output $y_r(t)$, with $y_r^{(i)}(t), i = 1, \ldots, n$ known, bounded and piecewise continuous.
Tuning functions design for tracking \((z_0 \triangleq 0, \alpha_0 \triangleq 0, \tau_0 \triangleq 0)\)

\[
\begin{align*}
    z_i &= x_i - y_r^{(i-1)} - \alpha_{i-1} \\
    \alpha_i(\bar{x}_i, \bar{\theta}, \bar{y}_r^{(i-1)}) &= -z_{i-1} - c_i z_i - w_i^T \hat{\theta} + \sum_{k=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) + \nu_i \\
    \nu_i(\bar{x}_i, \bar{\theta}, \bar{y}_r^{(i-1)}) &= +\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i + \sum_{k=2}^{i-1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma w_i z_k \\
    \tau_i(\bar{x}_i, \bar{\theta}, \bar{y}_r^{(i-1)}) &= \tau_{i-1} + w_i z_i \\
    w_i(\bar{x}_i, \bar{\theta}, \bar{y}_r^{(i-2)}) &= \phi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_k \\
    &\quad i = 1, \ldots, n \\
    \bar{x}_i &= (x_1, \ldots, x_i), \quad \bar{y}_r^{(i)} = (y_r, y_r, \ldots, y_r^{(i)})
\end{align*}
\]

Adaptive control law:

\[
u = \frac{1}{\beta(x)} \left[ \alpha_n(x, \hat{\theta}, \bar{y}_r^{(n-1)}) + y_r^{(n)} \right]
\]

Parameter update law:

\[
\dot{\hat{\theta}} = \Gamma \tau_n(x, \hat{\theta}, \bar{y}_r^{(n-1)}) = \Gamma W z
\]
Closed-loop system

\[ \dot{z} = A_z(z, \hat{\theta}, t)z + W(z, \hat{\theta}, t)^T \tilde{\theta} \]
\[ \dot{\hat{\theta}} = \Gamma W(z, \hat{\theta}, t)z, \]

where

\[ A_z(z, \hat{\theta}, t) = \begin{bmatrix}
- c_1 & 1 & 0 & \cdots & 0 \\
-1 & - c_2 & 1 + \sigma_{23} & \cdots & \sigma_{2n} \\
0 & -1 - \sigma_{23} & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & - \sigma_{2n} & \cdots & -1 - \sigma_{n-1,n} & - c_n
\end{bmatrix} \]

\[ \sigma_{jk}(x, \hat{\theta}) = - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} w_k \]

This structure ensures that the Lyapunov function

\[ V_n = \frac{1}{2} z^T z + \frac{1}{2} \hat{\theta}^T \Gamma^{-1} \hat{\theta} \]

has derivative

\[ \dot{V}_n = - \sum_{k=1}^{n} c_k z_k^2. \]
Modular Design

Motivation: Controller can be combined with different identifiers. (No flexibility for update law in tuning function design)

Naive idea: connect a good identifier and a good controller.

Example: error system

\[ \dot{x} = -x + \varphi(x)\tilde{\theta} \]

suppose \( \tilde{\theta}(t) = e^{-t} \) and \( \varphi(x) = x^3 \), we have

\[ \dot{x} = -x + x^3e^{-t} \]

But, when \( |x_0| > \sqrt{\frac{3}{2}} \),

\[ x(t) \rightarrow \infty \quad \text{as} \quad t \rightarrow \frac{1}{3} \ln \frac{x_0^2}{x_0^2 - 3/2} \]

Conclusion: Need stronger controller.
Controller Design. nonlinear damping

\[ u = -x - \varphi(x)\hat{\theta} - \varphi(x)^2 x \]

closed-loop system

\[ \dot{x} = -x - \varphi(x)^2 x + \varphi(x)\tilde{\theta}. \]

With \( V = \frac{1}{2}x^2 \), we have

\[ \dot{V} = -x^2 - \varphi(x)^2 x^2 + x\varphi(x)\tilde{\theta} \]

\[ = -x^2 - \left[ \varphi(x)x - \frac{1}{2}\tilde{\theta} \right]^2 + \frac{1}{4}\tilde{\theta}^2 \]

\[ \leq -x^2 + \frac{1}{4}\tilde{\theta}^2. \]

bounded \( \tilde{\theta}(t) \Rightarrow \) bounded \( x(t) \)
For higher order system

\[ \dot{x}_1 = x_2 + \varphi(x_1)^T \theta \]
\[ \dot{x}_2 = u \]

set

\[ \alpha_1(x_1, \hat{\theta}) = -c_1 x_1 - \varphi(x_1)^T \hat{\theta} - \kappa_1 |\varphi(x_1)|^2 x_1, \quad c_1, \kappa_1 > 0 \]

and define

\[ z_2 = x_2 - \alpha_1(x_1, \hat{\theta}) \]

error system

\[ \dot{z}_1 = -c_1 z_1 - \kappa_1 |\varphi|^2 z_1 + \varphi^T \hat{\theta} + z_2 \]
\[ \dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \theta) - \frac{\partial \alpha_1}{\partial \hat{\theta}} \hat{\theta}. \]
Consider

\[ V_2 = V_1 + \frac{1}{2} z_2^2 = \frac{1}{2} |z|^2 \]

we have

\[
\dot{V}_2 \leq -c_1z_1^2 + \frac{1}{4\kappa_1} |\tilde{\theta}|^2 + z_1z_2 + z_2 \left[ u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \theta) - \frac{\partial \alpha_1}{\partial \tilde{\theta}} \dot{\tilde{\theta}} \right]
\]

\[
\leq -c_1z_1^2 + \frac{1}{4\kappa_1} |\tilde{\theta}|^2 + z_2 \left[ u + z_1 - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \theta) - \left( \frac{\partial \alpha_1}{\partial x_1} \varphi^T \tilde{\theta} + \frac{\partial \alpha_1}{\partial \tilde{\theta}} \right) \right] .
\]

ccontroller

\[
u = -z_1 - c_2z_2 - \kappa_2 \left| \frac{\partial \alpha_1}{\partial x_1} \varphi \right|^2 z_2 - g_2 \left| \frac{\partial \alpha_1}{\partial \tilde{\theta}} \right|^2 z_2 + \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \tilde{\theta}),
\]

achieves

\[
\dot{V}_2 \leq -c_1z_1^2 - c_2z_2^2 + \left( \frac{1}{4\kappa_1} + \frac{1}{4\kappa_2} \right) |\tilde{\theta}|^2 + \frac{1}{4g_2} |\dot{\tilde{\theta}}|^2
\]

bounded \( \tilde{\theta} \), bounded \( \dot{\tilde{\theta}} \) (or \( \in \mathcal{L}_2 \)) \( \Rightarrow \) bounded \( x(t) \)
### Controller design in the modular approach \((z_0 \triangleq 0, \alpha_0 \triangleq 0)\)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_i = x_i - y_i^{(i-1)} - \alpha_{i-1} )</td>
<td>Equation for (z_i)</td>
</tr>
<tr>
<td>(\alpha_i(x, \tilde{\theta}, \tilde{y}<em>r^{(i-1)}) = -z</em>{i-1} - c_i z_i - w_i^T \tilde{\theta} + \sum_{k=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) - s_i z_i )</td>
<td>Equation for (\alpha_i)</td>
</tr>
<tr>
<td>(w_i(x, \tilde{\theta}, \tilde{y}<em>r^{(i-2)}) = \varphi_j - \sum</em>{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_k )</td>
<td>Equation for (w_i)</td>
</tr>
<tr>
<td>(s_i(x, \tilde{\theta}, \tilde{y}_r^{(i-2)}) = \kappa_i</td>
<td>w_i</td>
</tr>
</tbody>
</table>

### Adaptive control law:

\[
u = \frac{1}{\beta(x)} \left[ \alpha_n(x, \tilde{\theta}, \tilde{y}_r^{(n-1)}) + y_r^{(n)} \right] \]

### Controller module guarantees:

If \(\tilde{\theta} \in \mathcal{L}_\infty\) and \(\dot{\tilde{\theta}} \in \mathcal{L}_2\) or \(\mathcal{L}_\infty\) then \(x \in \mathcal{L}_\infty\)
Requirement for identifier error system

\[ \dot{z} = A_z(z, \hat{\theta}, t)z + W(z, \hat{\theta}, t)^T \tilde{\theta} + Q(z, \hat{\theta}, t)^T \dot{\hat{\theta}} \]

where

\[ A_z(z, \hat{\theta}, t) = \begin{bmatrix}
    -c_1 - s_1 & 1 & 0 & \ldots & 0 \\
    -1 & -c_2 - s_2 & 1 & \ldots & \vdots \\
    0 & -1 & \ldots & \ldots & 0 \\
    \vdots & \vdots & \ldots & \ldots & 1 \\
    0 & \ldots & 0 & -1 & -c_n - s_n
\end{bmatrix} \]

\[ W(z, \hat{\theta}, t)^T = \begin{bmatrix}
    w_1^T \\
    w_2^T \\
    \vdots \\
    w_n^T
\end{bmatrix}, \quad Q(z, \hat{\theta}, t)^T = \begin{bmatrix}
    0 \\
    -\frac{\partial \alpha_1}{\partial \hat{\theta}} \\
    \vdots \\
    -\frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}
\end{bmatrix}. \]
Since

\[ W(z, \hat{\theta}, t)^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{\partial \alpha_1}{\partial x_1} & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\frac{\partial \alpha_{n-1}}{\partial x_1} & \cdots & -\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} & 1 \end{bmatrix} F(x)^T \triangleq N(z, \hat{\theta}, t)F(x)^T. \]

Identifier properties:

(i) \( \tilde{\theta} \in L_\infty \) and \( \dot{\hat{\theta}} \in L_2 \) or \( L_\infty \),
(ii) if \( x \in L_\infty \) then \( F(x(t))^T\tilde{\theta}(t) \to 0 \) and \( \dot{\hat{\theta}}(t) \to 0 \).
Identifier Design

Passive identifier

\[ \dot{x} = f + F^T \theta \]

\[ \dot{\hat{x}} = \left( A_0 - \lambda F^T F P \right) (\hat{x} - x) + f + F^T \hat{\theta} \]
\[ \dot{\epsilon} = \left[ A_0 - \lambda F(x, u)^T F(x, u) P \right] \epsilon + F(x, u)^T \tilde{\theta} \]

update law

\[ \dot{\tilde{\theta}} = \Gamma F(x, u) P \epsilon, \quad \Gamma = \Gamma^T > 0. \]

Use Lyapunov function

\[ V = \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \epsilon^T P \epsilon \]

its derivative satisfies

\[ \dot{V} \leq -\epsilon^T \epsilon - \frac{\lambda}{\lambda'(\Gamma)^2} |\dot{\theta}|^2. \]

Thus, whenever \( x \) is bounded, \( F(x(t))^T \tilde{\theta}(t) \to 0 \) and \( \dot{\theta}(t) \to 0 \).

(\( \dot{\epsilon}(t) \to 0 \) because \( \int_0^\infty \dot{\epsilon}(\tau) d\tau = -\epsilon(0) \) exists, Barbalat’s lemma...)

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Swapping identifier

\[ \dot{x} = f + F^\top \theta \]

\[ \dot{\Omega}_0 = (A_0 - \lambda F^\top FP)(\Omega_0 - x) + f \]

\[ \dot{\Omega} = (A_0 - \lambda F^\top FP)\Omega + F \]

\[ \int \dot{\theta} \]

\[ \frac{\Gamma \Omega}{1 + \nu |\Omega|^2} \]
define $\tilde{\epsilon} \equiv x + \Omega_0 - \Omega^T \theta$,

$$\dot{\tilde{\epsilon}} = [A_0 - \lambda F(x, u)^T F(x, u) P] \tilde{\epsilon}.$$

Choose

$$V = \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \tilde{\epsilon} P \tilde{\epsilon}$$

we have

$$\dot{V} \leq -\frac{3}{41 + \nu \text{tr}\{\Omega^T \Omega\}} \epsilon^T \epsilon,$$

proves identifier properties.
Output Feedback Adaptive Designs

\[ \dot{x} = Ax + \phi(y) + \Phi(y)a + \begin{bmatrix} 0 \\ b \end{bmatrix} \sigma(y)u, \quad x \in \mathbb{R}^n \]

\[ y = e_1^T x, \]

\[ A = \begin{bmatrix} 0 & I_{n-1} \\ \vdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \]

\[ \phi(y) = \begin{bmatrix} \varphi_{0,1}(y) \\ \vdots \\ \varphi_{0,n}(y) \end{bmatrix}, \quad \Phi(y) = \begin{bmatrix} \varphi_{1,1}(y) & \cdots & \varphi_{q,1}(y) \\ \vdots & \ddots & \vdots \\ \varphi_{1,n}(y) & \cdots & \varphi_{q,n}(y) \end{bmatrix}, \]

unknown constant parameters:

\[ a = [a_1, \ldots, a_q]^T, \quad b = [b_m, \ldots, b_0]^T. \]
State estimation filters

Filters:

\[
\begin{align*}
\dot{\xi} &= A_0 \xi + k y + \phi(y) \\
\dot{\Xi} &= A_0 \Xi + \Phi(y) \\
\dot{\lambda} &= A_0 \lambda + e_n \sigma(y) u \\
v_j &= A_0^j \lambda, \quad j = 0, \ldots, m \\
\Omega^T &= [v_m, \ldots, v_1, v_0, \Xi]
\end{align*}
\]
Parameter-dependent state estimate
\[ \hat{x} = \xi + \Omega^T \theta \]

The vector \( k = [k_1, \ldots, k_n]^T \) chosen so that the matrix
\[ A_0 = A - ke_1^T \]
is Hurwitz, that is,
\[ PA_0 + A_0^T P = -I, \quad P = P^T > 0 \]

The state estimation error
\[ \varepsilon = x - \hat{x} \]
satisfies
\[ \dot{\varepsilon} = A_0 \varepsilon \]
Parametric model for adaptation:

\[
\dot{y} = \omega_0 + \omega^T \theta + \varepsilon_2 \\
= b_m v_{m,2} + \omega_0 + \bar{\omega}^T \theta + \varepsilon_2,
\]

where

\[
\omega_0 = \varphi_{0,1} + \xi_2 \\
\omega = [v_{m,2}, v_{m-1,2}, \ldots, v_{0,2}, \Phi_{(1)} + \Xi_{(2)}]^T \\
\bar{\omega} = [0, v_{m-1,2}, \ldots, v_{0,2}, \Phi_{(1)} + \Xi_{(2)}]^T.
\]
Since the states \( x_2, \ldots, x_n \) are not measured, the backstepping design is applied to the system

\[
\begin{align*}
\dot{y} &= b_m v_{m,2} + \omega_0 + \bar{\omega}^T \theta + \varepsilon_2 \\
\dot{v}_{m,i} &= v_{m,i+1} - k_i v_{m,1}, \quad i = 2, \ldots, \rho - 1 \\
\dot{v}_{m,\rho} &= \sigma(y)u + v_{m,\rho+1} - k_{\rho} v_{m,1}.
\end{align*}
\]

The order of this system is equal to the relative degree of the plant.
Extensions

Pure-feedback systems.

\[
\dot{x}_i = x_{i+1} + \varphi_i(x_1, \ldots, x_{i+1})^T \theta, \quad i = 1, \ldots, n - 1
\]

\[
\dot{x}_n = \left(\beta_0(x) + \beta(x)^T \theta\right) u + \varphi_0(x) + \varphi_n(x)^T \theta,
\]

where \( \varphi_0(0) = 0, \ \varphi_1(0) = \cdots = \varphi_n(0) = 0, \ \beta_0(0) \neq 0. \)

Because of the dependence of \( \varphi_i \) on \( x_{i+1} \), the regulation or tracking for pure-feedback systems is, in general, not global, even when \( \theta \) is known.
Unknown virtual control coefficients.

\[ \dot{x}_i = b_i x_{i+1} + \varphi_i(x_1, \ldots, x_i)^T \theta, \quad i = 1, \ldots, n - 1 \]
\[ \dot{x}_n = b_n \beta(x) u + \varphi_n(x_1, \ldots, x_n)^T \theta, \]

where, in addition to the unknown vector \( \theta \), the constant coefficients \( b_i \) are also unknown.

The unknown \( b_i \)-coefficients are frequent in applications ranging from electric motors to flight dynamics. The signs of \( b_i, \ i = 1, \ldots, n \), are assumed to be known. In the tuning functions design, in addition to estimating \( b_i \), we also estimate its inverse \( \varrho_i = 1/b_i \). In the modular design we assume that in addition to \( \text{sgn} b_i \), a positive constant \( \varsigma_i \) is known such that \( |b_i| \geq \varsigma_i \). Then, instead of estimating \( \varrho_i = 1/b_i \), we use the inverse of the estimate \( \hat{b}_i \), i.e., \( 1/\hat{b}_i \), where \( \hat{b}_i(t) \) is kept away from zero by using parameter projection.
Multi-input systems.

\[
\dot{X}_i = B_i(\bar{X}_i)X_{i+1} + \Phi_i(\bar{X}_i)^T \theta, \quad i = 1, \ldots, n - 1
\]
\[
\dot{X}_n = B_n(X)u + \Phi_n(X)^T \theta,
\]

where \( X_i \) is a \( \nu_i \)-vector, \( \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n \), \( \bar{X}_i = \begin{bmatrix} X_1^T, \ldots, X_i^T \end{bmatrix}^T \), \( X = \bar{X}_n \), and the matrices \( B_i(\bar{X}_i) \) have full rank for all \( \bar{X}_i \in \mathbb{R}^{\sum_{j=1}^i \nu_j} \). The input \( u \) is a \( \nu_n \)-vector.

The matrices \( B_i \) can be allowed to be unknown provided they are constant and positive definite.
Block strict-feedback systems.

\[
\begin{align*}
\dot{x}_i &= x_{i+1} + \varphi_i(x_1, \ldots, x_i, \zeta_1, \ldots, \zeta_i)^T\theta, \quad i = 1, \ldots, \rho - 1 \\
\dot{x}_\rho &= \beta(x, \zeta)u + \varphi_\rho(x, \zeta)^T\theta \\
\dot{\zeta}_i &= \Phi_{i,0}(\bar{x}_i, \bar{\zeta}_i) + \Phi_i(\bar{x}_i, \bar{\zeta}_i)^T\theta, \quad i = 1, \ldots, \rho
\end{align*}
\]

with the following notation: \( \bar{x}_i = [x_1, \ldots, x_i]^T, \bar{\zeta}_i = [\zeta_1^T, \ldots, \zeta_i^T]^T \), \( x = \bar{x}_\rho \), and \( \zeta = \bar{\zeta}_\rho \).

Each \( \zeta_i \)-subsystem is assumed to be bounded-input bounded-state (BIBS) stable with respect to the input \( (\bar{x}_i, \bar{\zeta}_{i-1}) \). For this class of systems it is quite simple to modify the procedure in the tables. Because of the dependence of \( \varphi_i \) on \( \bar{\zeta}_i \), the stabilizing function \( \alpha_i \) is augmented by the term \( + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \zeta_k} \Phi_{k,0} \), and the regressor \( w_i \) is augmented by \( - \sum_{k=1}^{i-1} \Phi_i \left( \frac{\partial \alpha_{i-1}}{\partial \zeta_k} \right)^T \).
**Partial state-feedback systems.** In many physical systems there are unmeasured states as in the output-feedback form, but there are also states other than the output $y = x_1$ that are measured. An example of such a system is

\[
\begin{align*}
\dot{x}_1 &= x_2 + \varphi_1(x_1)^T \theta \\
\dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^T \theta \\
\dot{x}_3 &= x_4 + \varphi_3(x_1, x_2)^T \theta \\
\dot{x}_4 &= x_5 + \varphi_4(x_1, x_2)^T \theta \\
\dot{x}_5 &= u + \varphi_5(x_1, x_2, x_5)^T \theta.
\end{align*}
\]

The states $x_3$ and $x_4$ are assumed not to be measured. To apply the adaptive backstepping designs presented in this chapter, we combine the state-feedback techniques with the output-feedback techniques. The subsystem $(x_2, x_3, x_4)$ is in the output-feedback form with $x_2$ as a measured output, so we employ a state estimator for $(x_2, x_3, x_4)$ using the filters introduced in the section on output feedback.
Example of Adaptive Stabilization in the Presence of a Stochastic Disturbance

\[ \frac{dx}{dt} = u dt + x dw \]

\( w \): Wiener process with \( E\{dw^2\} = \sigma(t)^2 dt \), no a priori bound for \( \sigma \)

Control laws:

Disturbance Attenuation: \( u = -x - x^3 \)
Adaptive Stabilization: \( u = -x - \dot{\theta}x, \quad \dot{\theta} = x^2 \)
Major Applications of Adaptive Nonlinear Control

- **Electric Motors Actuating Robotic Loads**
  

- **Marine Vehicles** (ships, UUVs; dynamic positioning, way point tracking, maneuvering)
  
  *Marine Control Systems*, Fossen, 2002

- **Automotive Vehicles** (lateral and longitudinal control, traction, overall dynamics)
  
  The groups of Tomizuka and Kanellakopoulos.

Dozens of other occasional applications, including: aircraft wing rock, compressor stall and surge, satellite attitude control.
Other Books on Adaptive NL Control Theory Inspired by KKK

1. Marino and Tomei (1995),
   *Nonlinear Control Design: Geometric, Adaptive, and Robust*

2. Freeman and Kokotovic (1996),
   *Robust Nonlinear Control Design: State Space and Lyapunov Techniques*

3. Qu (1998),
   *Robust Control of Nonlinear Uncertain Systems*

4. Krstic and Deng (1998),
   *Stabilization of Nonlinear Uncertain Systems*

5. Ge, Hang, Lee, Zhang (2001),
   *Stable Adaptive Neural Network Control*

   *Stable Adaptive Control and Estimation for Nonlinear Systems: Neural and Fuzzy Approximation Techniques*

7. French, Szepesvari, Rogers (2003),
   *Performance of Nonlinear Approximate Adaptive Controllers*
Adaptive NL Control/Backstepping Coverage in Major Texts

   *Nonlinear Systems*

2. Isidori (1995),
   *Nonlinear Control Systems*

3. Sastry (1999),
   *Nonlinear Systems: Analysis, Stability, and Control*

4. Astrom and Wittenmark (1995),
   *Adaptive Control*