BOUNDARY CONTROL OF THE LINEARIZED GINZBURG-LANDAU MODEL OF VORTEX SHEDDING

OLE MORTEN AAMO†, ANDREY SMYSHTYAEV‡, AND MIROSLAV KRSTIĆ§

Abstract. In this paper, we continue the development of state feedback boundary control laws based on the backstepping methodology, for the stabilization of unstable, parabolic partial differential equations. We consider the linearized Ginzburg-Landau equation, which models, for instance, vortex shedding in bluff body flows. Asymptotic stabilization is achieved by means of boundary control via state feedback in the form of an integral operator. The kernel of the operator is shown to be twice continuously differentiable, and a series approximation for its solution is given. Under certain conditions on the parameters of the Ginzburg-Landau equation, compatible with vortex shedding modelling on a semi-infinite domain, the kernel is shown to have compact support, resulting in partial state feedback. Simulations are provided in order to demonstrate the performance of the controller. In summary, the paper extends previous work in two ways: 1) it deals with two coupled partial differential equations, and; 2) under certain circumstances handles equations defined on a semi-infinite domain.

Key words. partial differential equations, boundary control, stabilization, flow control

AMS subject classifications. 35B37, 35B65, 93D15

1. Introduction. In this paper, we continue the development of state feedback boundary control laws based on the backstepping methodology [6], for the stabilization of unstable, parabolic partial differential equations [3, 2, 10, 14]. We consider the linearized Ginzburg-Landau equation given by

$$\frac{\partial A(\bar{x},t)}{\partial t} = a_1 \frac{\partial^2 A(\bar{x},t)}{\partial \bar{x}^2} + a_2(\bar{x}) \frac{\partial A(\bar{x},t)}{\partial \bar{x}} + a_3(\bar{x}) A(\bar{x},t)$$  (1.1)

for $\bar{x} \in (0,x_d)$, with boundary conditions

$$A(0,t) = u(t),$$  (1.2)

$$A(x_d,t) = 0,$$  (1.3)

and where $A : [0,x_d] \times \mathbb{R}^+ \to \mathbb{C}$, $a_2 \in C^2([0,x_d]; \mathbb{C})$, $a_3 \in C^1([0,x_d]; \mathbb{C})$, $a_1 \in \mathbb{C}$, $x_d > 0$, and $u : \mathbb{R}^+ \to \mathbb{C}$ is the control input. $a_1$ is assumed to have strictly positive real part. In order to achieve asymptotic stabilization of the equilibrium at $A \equiv 0$, backstepping is applied resulting in a boundary control law that essentially cuts the term $a_3(\bar{x}) A(\bar{x},t)$ from equation (1.1). The result extends the work of [10, 14] in two ways: 1) it deals with two coupled partial differential equations, and; 2) under certain circumstances handles equations defined on a semi-infinite domain ($x_d \to \infty$). The theory is supplemented with a case study involving control of vortex shedding in bluff body flows. Controllers for this problem have previously been designed for finite dimensional approximations of equation (1.1) [7, 8, 1, 9]. In [12, 13], it was shown numerically that the Ginzburg-Landau model for Reynolds numbers close to $R_c$ can

*This work was supported by the National Science Foundation and the Norwegian Research Council.
†Department of Engineering Cybernetics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway (Tel: +47 73594386, Fax: +47 73594399, E-mail: aamo@iitk.ntnu.no).
‡Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093-0411, USA.
§Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093-0411, USA.
be stabilized using proportional feedback from a single measurement downstream of the cylinder, to local forcing at the location of the cylinder. In [5], an optimal solution to a boundary control problem formulated for a stationary Ginzburg-Landau model of superconductivity defined on a bounded domain was shown to exist, and the optimality system of equations was solved by employing the finite element method.

The paper is organized as follows. In Section 2, equation (1.1) is rewritten in terms of real variables and coefficients, and the problem statement is given. The main result is stated in Section 2. In Section 4 partial differential equations governing the feedback kernel are derived, and in Section 5 they are transformed to corresponding integral equations. We find a unique solution to the integral equations in Section 6.1, and show that the solution also yields the unique feedback kernel. Stability properties of the chosen target system are established in Section 7.1. In Section 9, the results are applied to a model of vortex shedding behind a bluff body immersed in a moving fluid, and it is shown that stabilizing feedback kernels that have compact support can be found even when the domain is semi-infinite. Concluding remarks are offered in Section 10.

2. Problem Statement. We now rewrite equation (1.1) to obtain two coupled partial differential equations in real variables and coefficients by defining

\[ \rho(x,t) = \Re(B(x,t)) = \frac{1}{2} (B(x,t) + \bar{B}(x,t)), \]

\[ \iota(x,t) = \Im(B(x,t)) = \frac{1}{2i} (B(x,t) - \bar{B}(x,t)), \]

where

\[ x = \frac{x_d - \bar{x}}{x_d}, \text{ and } B(x,t) = A(\bar{x},t) \exp \left( \frac{1}{2a_1} \int_0^\bar{x} a_2(\tau) d\tau \right). \]

\( i \) denotes the imaginary unit, and \( \bar{\cdot} \) denotes complex conjugation. Equation (1.1) becomes

\[ \rho_t = a_R \rho_{xx} + b_R(x) \rho - a_I \iota_{xx} - b_I(x) \iota, \]

\[ \iota_t = a_I \rho_{xx} + b_I(x) \rho + a_R \iota_{xx} + b_R(x) \iota, \]

for \( x \in (0,1) \), with boundary conditions

\[ \rho(0,t) = 0, \ \iota(0,t) = 0, \]

\[ \rho(1,t) = u_R(t), \ \iota(1,t) = u_I(t), \]

and where

\[ a_R \triangleq \frac{1}{x_d} \Re(a_1), \ a_I \triangleq \frac{1}{x_d} \Im(a_1), \]

\[ b_R(x) \triangleq \Re \left( a_3(\bar{x}) - \frac{1}{2} a_2'(\bar{x}) - \frac{1}{4a_1} a_2^2(\bar{x}) \right), \]

\[ b_I(x) \triangleq \Im \left( a_3(\bar{x}) - \frac{1}{2} a_2'(\bar{x}) - \frac{1}{4a_1} a_2^2(\bar{x}) \right). \]
Notice that transformation (2.3) serves two purposes: It normalizes the domain, and removes the convective term in (1.1). The control input will be in the form
\[ u_R(t) = \int_0^1 [k(1,y) \rho(y,t) + k_c(1,y) \iota(y,t)] dy, \]  
(2.11)
\[ u_I(t) = \int_0^1 [-k_c(1,y) \rho(y,t) + k(1,y) \iota(y,t)] dy. \]  
(2.12)
It is the objective of this paper to find stabilizing feedback gain kernels \( k \) and \( k_c \).

3. Main Result. Our main result states well posedness and stability properties of system (2.4)–(2.7) in closed loop with (2.11)–(2.12). The proof of the theorem is given in Section 8, following intermediate results.

**Theorem 3.1 (Main Result).** There exist feedback gain kernels, \( k(1,\cdot), k_c(1,\cdot) \in C^2(0,1) \), such that for arbitrary initial data \( \rho_0, \iota_0 \in L_\infty(0,1) \), system (2.4)–(2.7) in closed loop with (2.11)–(2.12) has a unique classical solution \( \rho, \iota \in C^2(0,1) \times (0,\infty) \).

The solution satisfies
\[ \| (\rho, \iota) \|_{H_1} \leq M \| (\rho_0, \iota_0) \|_{H_1} e^{-ct}, \]  
(3.1)
where \( M > 0 \), and \( c \) is a prescribable positive constant.

The basic idea of the control design is to show that the dynamics of the original system (2.4)–(2.7), with an appropriate choice for (2.11)–(2.12), is equivalent to the dynamics of a target system that has a specified structure, but whose zero-solution can be assigned desired stability properties by the choice of coefficients. This is achieved by finding an invertible coordinate transformation that transforms solutions of the original system into solutions of the target system, establishing equivalence of norms of solutions of the two systems. The transformation is found as the unique solution to a hyperbolic partial differential equation, as stated in the next Section. The proof of existence and uniqueness of solutions to this equation offers a constructive procedure to compute the transformation.

4. Derivation of PDE for the Kernels. We want to find a coordinate transformation
\[ \tilde{\rho}(x,t) = \rho(x,t) - \int_0^x [k(x,y) \rho(y,t) + k_c(x,y) \iota(y,t)] dy, \]  
(4.1)
\[ \tilde{\iota}(x,t) = \iota(x,t) - \int_0^x [-k_c(x,y) \rho(y,t) + k(x,y) \iota(y,t)] dy, \]  
(4.2)
transforming (2.4)–(2.7) into the exponentially stable system (under appropriate conditions on \( f_R(x) \) and \( f_I(x) \) that are stated in Section 7.1)
\[ \tilde{\rho}_t = a_R \tilde{\rho}_{xx} + f_R(x) \tilde{\rho} - a_I \tilde{\iota}_{xx} - f_I(x) \tilde{\iota}, \]  
(4.3)
\[ \tilde{\iota}_t = a_I \tilde{\rho}_{xx} + f_I(x) \tilde{\rho} + a_R \tilde{\iota}_{xx} + f_R(x) \tilde{\iota}, \]  
(4.4)
for \( x \in (0,1) \), with boundary conditions
\[ \tilde{\rho}(0,t) = 0, \ \tilde{\iota}(0,t) = 0, \ \tilde{\rho}(1,t) = 0, \ \tilde{\iota}(1,t) = 0, \]  
(4.5)
and where $f_R, f_I \in C^1([0, 1])$. The skew-symmetric form of (4.1)–(4.2) is postulated from the skew-symmetric form of (2.4)–(2.5). Notice that once the kernels, $k(x, y)$ and $k_c(x, y)$, have been found, setting $x = 1$ in (4.1)–(4.2) and using (4.5) yields the boundary control law (2.7) with (2.11)–(2.12).

**Lemma 4.1.** If the pair of kernels, $k(x, y)$ and $k_c(x, y)$, satisfy the partial differential equation

$$k_{xx} = k_{yy} + \beta(x, y)k + \beta_c(x, y)k_c,$$

$$k_{c,xx} = k_{c,yy} - \beta_c(x, y)k + \beta(x, y)k_c,$$

for $(x, y) \in T = \{x, y : 0 < y < x < 1\}$, with boundary conditions

$$k(x, x) = -\frac{1}{2} \int_0^x \beta(\gamma, \gamma)d\gamma,$$

$$k_c(x, x) = \frac{1}{2} \int_0^x \beta_c(\gamma, \gamma)d\gamma,$$

$$k(x, 0) = 0,$$

$$k_c(x, 0) = 0,$$

where

$$\beta(x, y) = [a_R(b_R(y) - f_R(x)) + a_I(b_I(y) - f_I(x))] / (a_R^2 + a_I^2),$$

$$\beta_c(x, y) = [a_R(b_R(y) - f_R(x)) - a_I(b_R(y) - f_R(x))] / (a_R^2 + a_I^2),$$

and if $(\rho, \iota)$ satisfies (2.4)–(2.7) with (2.11)–(2.12), then $(\tilde{\rho}, \tilde{\iota})$ satisfies (4.3)–(4.5).

**Proof.** Differentiating (4.1) with respect to time and inserting (2.4)–(2.5) we have

$$\tilde{\rho}_t(x, t) = a_R\rho_{xx}(x, t) + b_R(x)\rho - a_I\iota_{xx} - b_I(x)\iota$$

$$- \int_0^x [k(x, y)(a_R\rho_{yy} + b_R(y)\rho - a_I\iota_{yy} - b_I(y)\iota)$$

$$- k_c(x, y)(a_I\rho_{yy} + b_I(y)\rho + a_R\iota_{yy} + b_R(y)\iota)] dy.$$ (4.14)

Integrating (4.14) by parts, and using (2.6), (4.10)–(4.11), and (4.1)–(4.2) yields

$$\tilde{\rho}_t(x, t) = a_R\rho_{xx}(x, t) + a_R\frac{\partial^2}{\partial x^2} \int_0^x [k(x, y)\rho(y, t) + k_c(x, y)\iota(y, t)] dy$$

$$+ b_R(x)\rho_t(x, t) + b_R(x) \int_0^x [k(x, y)\rho(y, t) + k_c(x, y)\iota(y, t)] dy$$

$$- a_I\iota_{xx}(x, t) - a_I\frac{\partial^2}{\partial x^2} \int_0^x [-k_c(x, y)\rho(y, t) + k_c(x, y)\iota(y, t)] dy$$

$$- b_I(x)\iota_t(x, t) - b_I(x) \int_0^x [-k_c(x, y)\rho(y, t) + k_c(x, y)\iota(y, t)] dy$$

$$- k(x, x)a_R\rho_{xx}(x, t) + k(x, x)a_I\iota_{xx}(x, t) - k_c(x, x)a_I\rho_{xx}(x, t) - k_c(x, x)a_R\iota_{xx}(x, t)$$

$$+ k_y(x, x)a_R\rho(x, t) - k_y(x, x)a_I\iota(x, t) + k_{c,y}(x, x)a_I\rho(x, t) + k_{c,y}(x, x)a_R\iota(x, t)$$

$$- \int_0^x [k_{yy}(x, y)(a_R\rho(y, t) - a_I\iota(y, t)) + k(x, y)(b_R(y)\rho(y, t) - b_I(y)\iota(y, t))$$

$$+ k_{c,yy}(x, y)(a_I\rho(y, t) + a_R\iota(y, t)) + k(x, y)(b_I(y)\rho(y, t) + b_R(y)\iota(y, t))] dy.$$ (4.15)
Applying the relation
\[
\frac{\partial^2}{\partial x^2} \int_0^x \kappa(x,y) v(y,t) \, dy = \int_0^x \kappa_{xx}(x,y) v(y,t) \, dy + \kappa_x(x,x) v(x,t) + v(x,t) \frac{dk(x,x)}{dx} + \kappa(x,x) v_x(x,t),
\]
(4.16)
to appropriate terms in (4.15), and using (4.1)–(4.2) again, we get
\[
\hat{\rho}_t(x,t) = a_R \hat{\rho}_{xx}(x,t) - a_I \hat{\rho}_{xx}(x,t) + b_R(x) \hat{\rho}(x,t) - b_I(x) \hat{\rho}(x,t)
+ 2a_R \frac{dk(x,x)}{dx} \hat{\rho}(x,t) + 2a_R \frac{dk_c(x,x)}{dx} \hat{\rho}(x,t) + 2a_I \frac{dk_c(x,x)}{dx} \hat{\rho}(x,t) - 2a_I \frac{dk(x,x)}{dx} \hat{\rho}(x,t)
+ \int_0^x R(x,y) \rho(y,t) \, dy + \int_0^x I(x,y) \nu(y,t) \, dy,
\]
(4.17)
where
\[
R(x,y) = a_R(k_{xx}(x,y) - k_{yy}(x,y)) + a_I(k_{c,xx}(x,y) - k_{c,yy}(x,y))
+ \left(2a_R \frac{dk(x,x)}{dx} + 2a_I \frac{dk_c(x,x)}{dx} + b_R(x) - b_R(y)\right) k(x,y)
+ \left(-2a_R \frac{dk_c(x,x)}{dx} + 2a_I \frac{dk(x,x)}{dx} + b_I(x) - b_I(y)\right) k_c(x,y),
\]
(4.18)
and
\[
I(x,y) = -a_I(k_{xx}(x,y) - k_{yy}(x,y)) + a_R(k_{c,xx}(x,y) - k_{c,yy}(x,y))
+ \left(2a_R \frac{dk_c(x,x)}{dx} - 2a_I \frac{dk(x,x)}{dx} + b_I(x) + b_I(y)\right) k(x,y)
+ \left(2a_R \frac{dk(x,x)}{dx} + 2a_I \frac{dk_c(x,x)}{dx} + b_R(x) - b_R(y)\right) k_c(x,y).
\]
(4.19)
Substituting (4.6)–(4.7) and (4.8)–(4.9) into (4.18)–(4.19) yields
\[
R(x,y) =
(a_R \beta(x,y) - a_I \beta_c(x,y) - a_R \beta_c(x,x) + a_I \beta_c(x,x)) + b_R(x) - b_R(y) k(x,y)
+ (a_R \beta_c(x,y) + a_I \beta(x,y) - a_R \beta_c(x,x) + a_I \beta(x,x)) + b_I(x) - b_I(y) k_c(x,y),
\]
(4.20)
and
\[
I(x,y) =
(-a_R \beta_c(x,y) + a_I \beta_c(x,x)) + a_R \beta_c(x,x) + a_I \beta(x,x) - b_I(x) + b_I(y) k(x,y)
+ (a_R \beta(x,y) - a_I \beta_c(x,y) - a_R \beta(x,x) - a_I \beta_c(x,x)) + b_R(x) - b_R(y) k_c(x,y).
\]
(4.21)
From (4.12)–(4.13), we see that
\[
a_R \beta(x,y) - a_I \beta_c(x,y) = b_R(y) - f_R(x),
\]
(4.22)
and
\[ a_R \beta_c (x, y) + a_I \beta (x, y) = b_I (y) - f_I (x), \quad (4.23) \]
so it follows that
\[ R(x, y) = I(x, y) \equiv 0. \quad (4.24) \]
In view of (4.24), and using (4.8)–(4.9), (4.17) becomes
\[ \tilde{\rho}_t (x, t) = a_R \tilde{\rho}_{xx} (x, t) - a_I \tilde{\iota}_{xx} (x, t) \]
\[ + ( - (a_R \beta (x, x) - a_I \beta_c (x, x)) + b_R (x) ) \tilde{\rho} (x, t) \]
\[ + (a_R \beta_c (x, x) + a_I \beta (x, x) - b_I (x)) \tilde{\iota} (x, t). \quad (4.25) \]
Equation (4.3) now follows by substituting (4.22)–(4.23) into (4.25). The boundary
conditions (4.5) follow by setting \( x = 0 \) and \( x = 1 \) in (4.1)–(4.2), and using (2.6)–
(2.7) and (2.11)–(2.12). Equation (4.4) follows similarly by starting from the time
derivative of (4.2). \[\Box\]

5. Converting the PDE into an Integral Equation. In the following Lemma,
equation (4.6)–(4.11) is converted into an integral equation, that is suitable for analy-
sis by a fixed point method to establish existence and uniqueness of solutions.

**Lemma 5.1.** Any pair of kernels, \( k(x, y) \) and \( k_c(x, y), \) satisfying (4.6)–(4.11),
also satisfy the integral equation
\[ G(\xi, \eta) = -\frac{1}{4} \int_{\eta}^{\xi} b(\tau, 0) \, d\tau \
+ \frac{1}{4} \int_{\eta}^{\xi} \int_{0}^{\eta} b(\tau, s) G(\tau, s) \, ds \, d\tau + \frac{1}{4} \int_{\eta}^{\xi} \int_{0}^{\eta} b_c(\tau, s) G_c(\tau, s) \, ds \, d\tau, \quad (5.1) \]
\[ G_c(\xi, \eta) = \frac{1}{4} \int_{\eta}^{\xi} b_c(\tau, 0) \, d\tau \
- \frac{1}{4} \int_{\eta}^{\xi} \int_{0}^{\eta} b_c(\tau, s) G(\tau, s) \, ds \, d\tau + \frac{1}{4} \int_{\eta}^{\xi} \int_{0}^{\eta} b(\tau, s) G_c(\tau, s) \, ds \, d\tau, \quad (5.2) \]
where
\[ \xi = x + y, \ \eta = x - y, \quad (5.3) \]
\[ G(\xi, \eta) = k \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2} \right), \ G_c(\xi, \eta) = k_c \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2} \right), \quad (5.4) \]
\[ b(\xi, \eta) = \beta \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2} \right), \ \ b_c(\xi, \eta) = \beta_c \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2} \right). \quad (5.5) \]
Proof. Using the relations
\[ k_x (x, y) = G_\xi \frac{\partial \xi}{\partial x} + G_\eta \frac{\partial \eta}{\partial x} = G_\xi (\xi, \eta) + G_\eta (\xi, \eta), \]  
(5.6)
\[ k_y (x, y) = G_\xi \frac{\partial \xi}{\partial y} + G_\eta \frac{\partial \eta}{\partial y} = G_\xi (\xi, \eta) - G_\eta (\xi, \eta), \]  
(5.7)
\[ k_{xx} (x, y) = \frac{\partial}{\partial \xi} (G_\xi (\xi, \eta)) + \frac{\partial}{\partial \eta} (G_\eta (\xi, \eta)) \partial \xi \partial x \]
\[ = G_\xi \xi + 2G_\eta \xi (\xi, \eta) + G_\eta \eta (\xi, \eta), \]  
(5.8)
\[ k_{yy} (x, y) = \frac{\partial}{\partial \xi} (G_\xi (\xi, \eta)) - \frac{\partial}{\partial \eta} (G_\eta (\xi, \eta)) \partial \xi \partial y \]
\[ = G_\xi \xi (\xi, \eta) - 2G_\eta \xi (\xi, \eta) + G_\eta \eta (\xi, \eta), \]  
(5.9)
equations (4.6)–(4.7) with boundary conditions (4.8)–(4.11) are transformed to
\[ G_\eta (\xi, \eta) = \frac{1}{4} [b (\xi, \eta) G_\xi (\xi, \eta) + b_c (\xi, \eta) G_c (\xi, \eta)], \]  
(5.10)
\[ G_c.\eta (\xi, \eta) = \frac{1}{4} [-b_c (\xi, \eta) G_\xi (\xi, \eta) + b (\xi, \eta) G_c (\xi, \eta)], \]  
(5.11)
\[ G (\xi, 0) = -\frac{1}{4} \int_0^\xi b (\tau, 0) d\tau, \]  
(5.12)
\[ G_c (\xi, 0) = \frac{1}{4} \int_0^\xi b_c (\tau, 0) d\tau, \]  
(5.13)
\[ G (\xi, \xi) = 0, \]  
(5.14)
\[ G_c (\xi, \xi) = 0. \]  
(5.15)
Integrating (5.10) and (5.11) with respect to \( \eta \) from 0 to \( \eta \), we obtain
\[ G_\xi (\xi, \eta) - G_\xi (\xi, 0) = \frac{1}{4} \int_0^\eta b (\xi, s) G (\xi, s) ds \]
\[ + \frac{1}{4} \int_0^\eta b_c (\xi, s) G_c (\xi, s) ds, \]  
(5.16)
\[ G_c.\xi (\xi, \eta) - G_c.\xi (\xi, 0) = -\frac{1}{4} \int_0^\eta b_c (\xi, s) G (\xi, s) ds \]
\[ + \frac{1}{4} \int_0^\eta b (\xi, s) G_c (\xi, s) ds. \]  
(5.17)
Integrating (5.16) and (5.17) with respect to \( \xi \) from \( \eta \) to \( \xi \), and using (5.12)–(5.15) we obtain (5.1)–(5.2).

6. Analysis of the Integral Equation. Theorem 6.1. The equation (4.6)–(4.7) with boundary conditions (4.8)–(4.11) has a unique \( C^2 \) solution satisfying
\[ |k (x, y)| \leq M e^{2Mx}, \]  
(6.1)
\[ |k_c (x, y)| \leq M e^{2Mx}, \]  
(6.2)
where $M$ depends only on $a_1$, $a_2(\cdot)$, $a_3(\cdot)$, $f_R(\cdot)$, $f_I(\cdot)$, and is given in (6.7).

Proof. Set

$$G_0(\xi,\eta) = -\frac{1}{4} \int_\eta^\xi b(\tau,0) d\tau,$$

(6.3)

$$G_{n+1}(\xi,\eta) = \frac{1}{4} \int_\eta^\xi \int_0^\eta b(\tau,s) G_n(\tau,s) d\tau d\sigma + \frac{1}{4} \int_\eta^\xi \int_0^\eta b_c(\tau,s) G_{c,n}(\tau,s) d\tau d\sigma,$$

(6.4)

$$G_{c,0}(\xi,\eta) = \frac{1}{4} \int_\eta^\xi b_c(\tau,0) d\tau,$$

(6.5)

and

$$G_{c,n+1} = -\frac{1}{4} \int_\eta^\xi \int_0^\eta b_c(\tau,s) G_n(\tau,s) d\tau d\sigma + \frac{1}{4} \int_\eta^\xi \int_0^\eta b(\tau,s) G_{c,n}(\tau,s) d\tau d\sigma.$$

(6.6)

Denote

$$B = \sup_{(\xi,\eta) \in T_1} |b(\xi,\eta)|, B_c = \sup_{(\xi,\eta) \in T_1} |b_c(\xi,\eta)|, M = \max \{B, B_c\},$$

(6.7)

where $T_1 \triangleq \{\xi, \eta : 0 < \xi < 2, 0 < \eta < \min (\xi, 2 - \xi)\}$. For $G_0(\xi,\eta)$ and $G_{c,0}(\xi,\eta)$ we have

$$|G_0(\xi,\eta)| \leq \frac{1}{4} \int_\eta^\xi |b(\tau,0)| d\tau \leq \frac{1}{4} B (\xi - \eta) \leq \frac{B}{2},$$

(6.8)

$$|G_{c,0}(\xi,\eta)| \leq \frac{1}{4} \int_\eta^\xi |b_c(\tau,0)| d\tau \leq \frac{1}{4} B_c (\xi - \eta) \leq \frac{B_c}{2},$$

(6.9)

where we have used the fact that $0 < \xi - \eta < 2$. Suppose that

$$|G_n(\xi,\eta)| \leq MK^n \frac{(\xi + \eta)^n}{n!},$$

(6.10)

$$|G_{c,n}(\xi,\eta)| \leq MK^n \frac{(\xi + \eta)^n}{n!},$$

(6.11)

where $K > 0$ is a constant that we will determine later. Clearly, (6.10)–(6.11) hold for $n = 0$. Noting that

$$\int_\eta^\xi \int_0^\eta |G_n(\tau,s)| ds d\sigma \leq \frac{MK^n}{n!} \int_\eta^\xi \int_0^\eta (\tau + s)^n ds d\sigma$$

$$= \frac{MK^n}{(n+1)!} \int_\eta^\xi (\tau + s)^n ds d\sigma$$

$$\leq \frac{MK^n}{(n+1)!} \int_\eta^\xi (\tau + s)^{n+1} d\tau$$

$$\leq \frac{MK^n}{(n+1)!} \int_\eta^\xi (\xi + \eta)^{n+1} d\tau$$

$$\leq 2MK^n \frac{(\xi + \eta)^{n+1}}{(n+1)!},$$

(6.12)
we obtain from (6.4) that
\[ |G_{n+1}(\xi,\eta)| \leq \frac{1}{2} M (B + B_c) K^n \frac{(\xi + \eta)^{n+1}}{(n+1)!}, \]  
(6.13)
and from (6.6) that \(|G_{c,n+1}(\xi,\eta)|\) satisfies the same bound (6.13). Therefore, setting \(K = M\), we obtain
\[ |G_{n+1}(\xi,\eta)| \leq MK^{n+1} \frac{(\xi + \eta)^{n+1}}{(n+1)!}, \]  
(6.14)
\[ |G_{c,n+1}(\xi,\eta)| \leq MK^{n+1} \frac{(\xi + \eta)^{n+1}}{(n+1)!}. \]  
(6.15)
Thus, (6.10) and (6.11) are proved by induction, and the series
\[ G(\xi,\eta) = \sum_{n=0}^{\infty} G_n(\xi,\eta), \quad \text{and} \quad G_c(\xi,\eta) = \sum_{n=0}^{\infty} G_{c,n}(\xi,\eta), \]  
(6.16)
converge uniformly in \(\mathcal{T}_1\), and is a solution of (5.1)–(5.2). \(G\) and \(G_c\) are \(C^2(\mathcal{T}_1)\) since \(b\) and \(b_c\) are \(C^1(\mathcal{T}_1)\). The bounds (6.1)–(6.2) follow from (6.10)–(6.11), (6.16) and the fact that \(K = M\). It can be shown by the method of successive approximations that if \((G_1,G_{c,1})\) and \((G_2,G_{c,2})\) are two different solutions of (5.1)–(5.2), the resulting homogeneous integral equation for \((G,G_c) = (G_1 - G_2,G_{c,1} - G_{c,2})\) has a unique solution \((G,G_c) = 0\), which proves that the solution (6.16) is unique. We can check that (6.16) satisfies (5.10)–(5.15) by direct substitution. Equations (5.10)–(5.15) have a unique solution by Lemma 5.1.

Exponential stability of the target system (4.3)–(4.5) in the \(L_2\) and \(H_1\) norms is proved in the next section. In order to be able to imply stability of the closed loop system (2.4)–(2.7) from that result, we need to establish equivalence of norms of \((\rho,\iota)\) and \((\hat{\rho},\hat{\iota})\) in \(L_2\) and \(H_1\). This is done by proving that transformation (4.1)–(4.2) is invertible. The inverse transformation has the form
\[ \rho(x,t) = \hat{\rho}(x,t) - \int_0^x [l(x,y) \hat{\rho}(y,t) + l_c(x,y) \hat{\iota}(y,t)] dy, \]  
(6.17)
\[ \iota(x,t) = \hat{\iota}(x,t) - \int_0^x [-l_c(x,y) \hat{\rho}(y,t) + l(x,y) \hat{\iota}(y,t)] dy. \]  
(6.18)
The following result holds for the kernels \(l(x,y)\) and \(l_c(x,y)\) of transformation (6.17)–(6.18).

**Theorem 6.2.** If the pair of kernels, \(l(x,y)\) and \(l_c(x,y)\), satisfy the partial differential equation
\[ l_{xx} = l_{yy} - \beta(y,x)l - \beta_c(y,x)l_c, \]  
(6.19)
\[ l_{c,xx} = l_{c,yy} + \beta_c(y,x)l - \beta(y,x)l_c, \]  
(6.20)
with boundary conditions

\[
l(x, x) = \frac{1}{2} \int_0^x \beta(\gamma, \gamma) d\gamma, \quad (6.21)
\]

\[
l_c(x, x) = -\frac{1}{2} \int_0^x \beta_c(\gamma, \gamma) d\gamma, \quad (6.22)
\]

\[
l(x, 0) = 0, \quad (6.23)
\]

\[
l_c(x, 0) = 0, \quad (6.24)
\]

and if \((\tilde{\rho}, \tilde{\iota})\) satisfies (4.3)-(4.5), then \((\rho, \iota)\) satisfies (2.4)-(2.7) with (2.11)-(2.12).

System (6.19)-(6.24) has a unique \(C^2\) solution satisfying

\[
|l(x, y)| \leq M e^{2Mx}, \quad (6.25)
\]

\[
|l_c(x, y)| \leq M e^{2Mx}, \quad (6.26)
\]

where \(M\) is given in (6.7).

**Proof.** The proof is similar to those of Lemmas 4.1 and 5.1, and Theorem 6.1. \(\Box\)

### 7. Stability Analysis

**Theorem 7.1.** Suppose \(c > 0\), and select \(f_R(x)\) and \(f_I(x)\) such that

\[
\sup_{x \in [0,1]} \left( f_R(x) + \frac{1}{2} |f'_I(x)| \right) \leq -\frac{1}{2} c. \quad (7.1)
\]

Then the solution \((\tilde{\rho}, \tilde{\iota}) \equiv (0, 0)\) of system (4.3)-(4.5) is exponentially stable in the \(L_2(0,1)\) and \(H_1(0,1)\) norms.

**Corollary 7.2.** Suppose \(c > 0\), and set \(f_R(x) = -c\) and \(f_I(x) \equiv 0\). Then the solution \((\tilde{\rho}, \tilde{\iota}) \equiv (0, 0)\) of system (4.3)-(4.5) is exponentially stable in the \(L_2(0,1)\) and \(H_1(0,1)\) norms.

**Proof.** Consider the function

\[
E(t) = \frac{1}{2} \int_0^1 \left( \dot{\rho}(\rho^2 + \dot{\iota}^2) + \dot{\iota}(\rho \dot{\rho} + \iota \dot{\iota}) \right) dx. \quad (7.2)
\]

Its time derivative along solutions of system (4.3)-(4.5) is

\[
\dot{E}(t) = \int_0^1 \left[ \dot{\rho}(a_R \rho_{xx} + f_R(x) \rho - a_I \iota_{xx} - f_I(x) \iota) + \dot{\iota}(a_I \rho_{xx} + a_R \rho_{xx} + f_R(x) \iota) \right] dx
\]

\[
= \int_0^1 (\dot{\rho}(a_R \rho_{xx} + f_R(x) \rho - a_I \iota_{xx} + a_R \dot{\iota}_{xx}) + \dot{\iota}(a_I \rho_{xx} + a_R \dot{\rho}_{xx} + f_R(x) \iota)) dx
\]

\[
= -\int_0^1 a_R (\dot{\rho}^2 + \dot{\iota}^2) dx + \int_0^1 f_R(x) (\dot{\rho}^2 + \dot{\iota}^2) dx + a_I \int_0^1 (\dot{\rho} \dot{\iota} - \iota \dot{\rho}) dx
\]

\[
\leq \int_0^1 f_R(x) (\dot{\rho}^2 + \dot{\iota}^2) dx. \quad (7.3)
\]

So, from (7.1), and the comparison principle, we have

\[
E(t) \leq E(0) e^{-ct}, \text{ for } t \geq 0. \quad (7.4)
\]
Set
\[ V(t) = \frac{1}{2} \int \left( \dot{\rho}_x^2 (x, t) + \dot{\iota}_x^2 (x, t) \right) dx. \] (7.5)

The time derivative of \( V(t) \) along solutions of system (4.3)–(4.5) is
\[
\dot{V}(t) = \int_0^1 (\dot{\rho}_x \dot{\iota}_x + \dot{\iota}_x \dot{\rho}_x) \, dx
= -\int_0^1 (\dot{\rho}_{xx} \dot{\rho}_x + \dot{\iota}_{xx} \dot{\iota}_x) \, dx
= -\int_0^1 [\ddot{\rho}_{xx} (a_R \ddot{\rho}_{xx} + f_R (x) \ddot{\rho} - a_I \ddot{\iota}_{xx} - f_I (x) \ddot{\iota})
+ \ddot{\iota}_{xx} (a_I \ddot{\rho}_{xx} + f_I (x) \ddot{\rho} + a_R \ddot{\iota}_{xx} + f_R (x) \ddot{\iota})] \, dx
= -a_R \int_0^1 (\dot{\rho}_{xx}^2 + \dot{\iota}_{xx}^2) \, dx + \int_0^1 f_R (x) (\dot{\rho}_x^2 + \dot{\iota}_x^2) \, dx
+ \int_0^1 f_I (x) (\dot{\iota}_x \dot{\rho} - \dot{\rho}_x \dot{\iota}) \, dx
= \frac{1}{2} \int_0^1 (\dot{\rho}_x^2 + \dot{\iota}_x^2) \, dx + \frac{1}{2} \int_0^1 f_R (x) (\dot{\rho}_x^2 + \dot{\iota}_x^2) \, dx
\leq \frac{1}{2} \int_0^1 (\dot{\rho}_x^2 + \dot{\iota}_x^2) \, dx + \frac{1}{2} \int_0^1 \left( |\dot{f}_I (x)| - f''_R (x) \right) (\dot{\rho}_x^2 + \dot{\iota}_x^2) \, dx
\leq \frac{1}{2} \int_0^1 (\dot{\rho}_x^2 + \dot{\iota}_x^2) \, dx + \frac{1}{2} c_2 \int_0^1 (\dot{\rho}_x^2 + \dot{\iota}_x^2) \, dx,
\]
where we have used (7.1) and defined
\[ c_2 \triangleq \max \left\{ \sup_{x \in [0,1]} (|f'_I (x)| - f''_R (x)) , 0 \right\}. \] (7.7)

From the comparison principle, we get
\[
V(t) \leq \left( V(0) + 2 \frac{c_2}{c} E(0) \right) e^{-\frac{c}{2} t} - 2 \frac{c_2}{c} E(0) e^{-ct},
\] (7.8)
so we obtain
\[
V(t) \leq \left( V(0) + 2 \frac{c_2}{c} E(0) \right) e^{-\frac{c}{2} t}, \text{ for } t \geq 0.
\] (7.9)

Since (Poincaré inequality)
\[
E(t) \leq \frac{1}{2} V(t),
\] (7.10)
we get
\[
V(t) \leq c_3 V(0) e^{-\frac{c}{2} t}, \text{ for } t \geq 0,
\] (7.11)
with \( c_3 = 1 + c_2/c. \)
8. **Proof of Theorem 3.1.** From Theorem 7.1, \((\bar{\rho}, \bar{\iota}) = 0\) is exponentially stable in the \(L_2\) and \(H_1\) norms. Since Theorems 6.1 and 6.2 establish equivalence of norms of \((\rho, \iota)\) and \((\bar{\rho}, \bar{\iota})\) in \(L_2\) and \(H_1\), the stability statements of Theorem 7.1 also hold for the solution \((\rho, \iota) \equiv (0, 0)\) of system (2.4)–(2.5). From standard results for uniformly parabolic equations (see, for instance, [4]), it follows that system (4.3)–(4.4), with Dirichlet boundary conditions (4.5) and initial data \(\rho_0, \iota_0 \in L_\infty(0, 1)\), has a unique classical solution \(\rho, \iota \in C^{2,1}((0, 1) \times (0, \infty))\). The smoothness properties of \(k, k_c, l,\) and \(\ell_c\) stated in Theorems 6.1 and 6.2 then provide well posedness of system (2.4)–(2.7) in closed loop with (2.11)–(2.12).

9. **Application to a Model of Vortex Shedding.** The objective of this section is to provide a numerical demonstration of our results applied to a fluid flow control problem. An interesting feature of the system we study in this example, is that it is defined on an infinite domain \((x_d \to \infty)\), yet, we obtain feedback gain kernels which have compact support.

9.1. **The model.** In flows past submerged obstacles, the phenomenon of vortex shedding occurs provided the Reynolds number is sufficiently large. A popular prototype model flow for studying vortex shedding, is the flow past a 2D circular cylinder, as sketched in Figure 9.1. The vortices, which are alternatively shed from the upper and lower sides of the cylinder, induce an undesirable periodic force that acts on the cylinder. The dynamics of the cylinder wake, often referred to as the von Kármán vortex street, is governed by the Navier-Stokes equation, however, in [13], a simplified model was suggested in terms of the Ginzburg-Landau equation

\[
\frac{\partial A}{\partial t} = a_1 \frac{\partial^2 A}{\partial \bar{x}^2} + a_2 (\bar{x}) \frac{\partial A}{\partial \bar{x}} + a_3 (\bar{x}) A + a_4 |A|^2 A + \delta (\bar{x}) u, \tag{9.1}
\]

where \(\bar{x} \in \mathbb{R}, A : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}, a_1, a_4 \in \mathbb{C},\) and \(a_2, a_3 : \mathbb{R} \to \mathbb{C}\). \(\delta\) denotes the Dirac distribution and \(u : \mathbb{R}_+ \to \mathbb{C}\) is the control input. Thus, actuation is in the form of local forcing at \(\bar{x} = 0\), which is the location of the cylinder. The boundary conditions are \(A(\pm \infty, t) = 0\), that is, homogeneous Dirichlet boundary conditions. \(A(x, t)\) may represent any physical variable (velocities \((u, v)\) or pressure \(p\)), or derivations thereof, along the centerline \(y = 0\), see Figure 9.1. The choice will have an impact on the performance of the Ginzburg-Landau model, and associating \(A\) with the transverse fluctuating velocity \(v(x, y = 0, t)\) seems to be a particularly good choice [11]. In order to implement the scheme in practice, transfer functions between \(A(0)\) and the physical actuation, and the physical sensing and \(A(x)\), would have to be determined, either experimentally or computationally. The physical actuation could for instance be micro/synthetic jet actuators distributed on the cylinder surface. Numerical values for the coefficients in (9.1) were determined from experiments in [13], and are reproduced in Appendix 1.

We now simplify this problem to fit into the framework of the previous analysis. We linearize around the zero solution, discard the upstream subsystem by replacing the local forcing at \(\bar{x} = 0\) with boundary input at this location, and truncate the downstream subsystem at some \(x_d > 0\). The resulting system is of the form (1.1)–(1.3), defined on the interval \([0, x_d]\). We justify the truncation of the system by noting that the upstream subsystem (the region to the left of the cylinder in Figure 9.1) is approximately uniform flow, whereas the downstream subsystem (the region to the right of the cylinder in Figure 9.1) can be approximated to any desired level

---

4System (4.3)–(4.4) is uniformly parabolic in \((0, 1)\), with module of parabolicity \(a_R\).
of accuracy by selecting $x_d$ sufficiently large. We are now in a position to apply our results, and we will do so for different choices of $x_d$.

In this numerical example, we set the Reynolds number to $R = 50$, which corresponds to supercritical flow for which vortex shedding will occur in the uncontrolled case. For this choice of Reynolds number, the numerical coefficients of (2.4)–(2.5) derived from the coefficients given in Appendix 1, are $a_R = 0.156/x_d^2$, $a_I = 0$, and $b_R(x)$ and $b_I(x)$ are plotted in Figure 9.2 for $x_d = 2.5$, $x_d = 5$, and $x_d = 7.5$.

9.2. Feedback kernels. In terms of the feedback gain kernels, $k(1, x)$ and $k_c(1, x)$, the boundary feedback (1.2) is given by

$$u(t) = \int_0^{x_d} \frac{1}{x_d} \left( k \left( 1, \frac{x_d - \hat{x}}{x_d} \right) - i k_c \left( 1, \frac{x_d - \hat{x}}{x_d} \right) \right) \times$$

$$\exp \left( \frac{1}{2a_1} \int_0^{\hat{x}} a_2(\tau) d\tau \right) A(\dot{x}, t) d\dot{x}. \quad (9.2)$$

Thus, the feedback gain kernel for the original system (1.1)–(1.3) is complex-valued, and given by

$$k_u(\hat{x}) = \frac{1}{x_d} \left( k \left( 1, \frac{x_d - \hat{x}}{x_d} \right) - i k_c \left( 1, \frac{x_d - \hat{x}}{x_d} \right) \right) \exp \left( \frac{1}{2a_1} \int_0^{\hat{x}} a_2(\tau) d\tau \right). \quad (9.3)$$

Setting $f_R(x) = -0.2$, and $f_I(x) = 0$, exponential stability is assured by Corollary 7.2, and the stabilizing feedback gain kernel (9.3) can be calculated numerically using

---

2 The Reynolds number for flow past a circular cylinder is usually defined as $R = \rho U_\infty D/\mu$, where $U_\infty$ is the free stream velocity, $D$ is the cylinder diameter, and $\rho$ and $\mu$ are density and viscosity of the fluid, respectively. Vortex shedding occurs when $R > 47$. 

---

Fig. 9.1. Vortex shedding from a cylinder visualized by passive tracer particles.

Fig. 9.2. $b_R(x)$ and $b_I(x)$ for $x_d = 2.5$, $x_d = 5$, and $x_d = 7.5$. 

---
formulas (6.3)–(6.6), (6.16), (5.4)–(5.5), and (9.3). Figure 9.3 shows the feedback gain kernel (9.3) for \( x_d = 2.5, \ x_d = 5, \) and \( x_d = 7.5. \) It is clear that the feedback gain kernels grow rather rapidly with increasing \( x_d, \) which is an undesirable feature since we want to make \( x_d \) large in order to minimize the effect of truncating the downstream subsystem. The increase can be seen in connection with Figure 9.2, which shows that the absolute value of the differences \( b_R (x) - f_R (x) \) and \( b_I (x) - f_I (x) \) increase with increasing \( x_d. \) In other words, the control effort needed to change the dynamics of system (2.4)–(2.7) into that of (4.3)–(4.5) increases with the degree to which the two systems differ. Therefore, the functions \( f_R (x) \) and \( f_I (x) \) must be chosen more intelligently than the simple case of setting them constant. Theorem 7.1 allows some flexibility in choosing \( f_R (x) \) and \( f_I (x), \) within the constraints of (7.1). In order to postpone choosing \( x_d, \) we study \( f_R, f_I, b_R, \) and \( b_I \) as functions of \( \bar{x} \) rather than \( x \) in the following. This is convenient since \( f_R, f_I, b_R, \) and \( b_I \) are invariant of \( x_d \) when treated as functions of \( \bar{x}. \) Recall that when \( x_d \) is chosen, the two domains are related by \( x = (x_d - \bar{x})/x_d. \) We propose to choose \( f_R (\bar{x}) \) and \( f_I (\bar{x}) \) as close to \( b_R (\bar{x}) \) and \( b_I (\bar{x}) \) as possible, without violating the conditions of Theorem 7.1, which we now write

\[
\sup_{\bar{x}} \left( f_R (\bar{x}) + \frac{1}{2} |f' (\bar{x})| \right) \leq -\frac{1}{2} c. \tag{9.4}
\]

Towards that end, we first set them equal, that is \( f_R (\bar{x}) = b_R (\bar{x}) \) and \( f_I (\bar{x}) = b_I (\bar{x}), \) and plot (9.4) along with \(-\frac{1}{4} c = -0.2. \) The result is shown in Figure 9.4, for \( \bar{x} \in [0, 20]. \) The figure shows that the conditions for stability are already satisfied, without control, for \( \bar{x} \in [x_s, 20] \) (in fact, the stability conditions are satisfied for \( \bar{x} \in [x_s, \infty) \)), which means that it suffices to alter \( f_R (\bar{x}) \) and \( f_I (\bar{x}) \) in \([0, x_s] \) in order to satisfy (9.4). Thus, we set\(^3\)

\[
f_R (\bar{x}) = \begin{cases} 
-\frac{1}{2} c - \frac{1}{2} |b' (\bar{x})|, & \text{for } 0 \leq \bar{x} < x_s, \\
\quad b_R (\bar{x}), & \text{for } \bar{x} \geq x_s,
\end{cases} \tag{9.5}
\]

\[
f_I (\bar{x}) = b_I (\bar{x}), \quad \text{for all } \bar{x}. \tag{9.6}
\]

With these choices of \( f_R (\bar{x}) \) and \( f_I (\bar{x}), \) we calculate numerically the stabilizing feedback gain kernel (9.3) for \( x_d = 10, \ x_d = 20, \) and \( x_d = 40. \) Figure 9.5 shows the result. As expected, the feedback gain kernels look similar, and in particular, they appear to be zero for \( \bar{x} \) larger than approximately 7.5. In fact, they are identical and have compact support, as stated formally in the next theorem.

**Theorem 9.1.** Given \( c > 0, \) suppose there exists \( x_s \in (0, \infty) \) such that

\[
b_R (\bar{x}) + \frac{1}{2} |b' (\bar{x})| \leq -\frac{1}{2} c, \quad \text{for } \bar{x} \geq x_s. \tag{9.7}
\]

Then \( f_R (\bar{x}) \) and \( f_I (\bar{x}), \) satisfying (9.4), can be chosen such that \( f_R (\bar{x}) = b_R (\bar{x}) \) and \( f_I (\bar{x}) = b_I (\bar{x}) \) for \( \bar{x} \in [x_s, x_d]. \) The resulting stabilizing feedback gain kernel (9.3) has compact support contained in \([0, 2x_s] \). Moreover, all choices of \( x_d \geq 2x_s, \) will produce the same stabilizing feedback gain kernel (9.3) in \([0, 2x_s] \).

**Proof.** The existence of \( f_R (\bar{x}) \) and \( f_I (\bar{x}) \) satisfying the criterion for stability (9.4) follows trivially from (9.7). To prove that the kernel has support contained in \([0, 2x_s] \),

---

\(^3\)By the choice of \( x_s, \) \( f_R (\bar{x}) \) is continuous. In this example, we ignore the fact that our choice of \( f_R (\bar{x}) \) may not be \( C^1, \) although this can easily be achieved by smoothing \( f_R (\bar{x}) \) in a small neighborhood of \( x_s. \)
**Fig. 9.3.** Feedback kernel (9.3) for a) $x_d = 2.5$, b) $x_d = 5$, and c) $x_d = 7.5$. $f_R(x) = -0.2$, and $f_I(x) = 0$.

**Fig. 9.4.** The stability criterion (9.4) when $f_R(\tilde{x}) = b_R(\tilde{x})$, and $f_I(\tilde{x}) = b_I(\tilde{x})$, is satisfied for $\tilde{x} \geq x_s$. 

we show that it is identically zero outside this interval. We have that
\[ b(\xi, 0) = \beta(x, x) = 0, \]  
\[ b_c(\xi, 0) = \beta_c(x, x) = 0, \]
for
\[ \xi \in \left[ 0, 2 \left( 1 - \frac{x_s}{x_d} \right) \right]. \]

It follows that
\[ G_0 (\xi, \eta) = 0, \quad G_{c,0} (\xi, \eta) = 0, \]  
for \( (\xi, \eta) \in T_1, \xi \leq 2 \left( 1 - \frac{x_s}{x_d} \right) \).

Now, suppose that
\[ G_n (\xi, \eta) = 0, \quad G_{c,n} (\xi, \eta) = 0, \]  
for \( (\xi, \eta) \in T_1, \xi \leq 2 \left( 1 - \frac{x_s}{x_d} \right). \)
Thus, (9.12) is proved by induction, and
\[ k_c (1, y) = G_c (1 + y, 1 - y) = 0 \]
\[ k (1, y) = G (1 + y, 1 - y) = 0 \]
for \( 0 \leq y \leq 2 \left( 1 - \frac{x_s}{x_d} \right) - 1. \) \hspace{1cm} (9.14)

Therefore, \( k_u (\bar{x}) = 0, \) for
\[
0 \leq \frac{x_d - \bar{x}}{x_d} \leq 2 \left( 1 - \frac{x_s}{x_d} \right) - 1,
\]
which yields
\[
x_d \geq \bar{x} \geq 2x_s.
\] \hspace{1cm} (9.15)

In order to prove the last part of the theorem, we need to show that for any \( x_{d,1}, x_{d,2} \in [2x_s, \infty), \)
\[
\frac{1}{x_{d,1}} k_{x_{d,1}} \left( 1, \frac{x_{d,1} - \bar{x}}{x_{d,1}} \right) = \frac{1}{x_{d,2}} k_{x_{d,2}} \left( 1, \frac{x_{d,2} - \bar{x}}{x_{d,2}} \right)
\]
\[
\frac{1}{x_{d,1}} k_{c,x_{d,1}} \left( 1, \frac{x_{d,1} - \bar{x}}{x_{d,1}} \right) = \frac{1}{x_{d,2}} k_{c,x_{d,2}} \left( 1, \frac{x_{d,2} - \bar{x}}{x_{d,2}} \right)
\]
\hspace{1cm} (9.17)

where the additional subscripts, \( x_{d,1} \) and \( x_{d,2}, \) on variables identify the domain of the problem from which they stem. We have
\[
k_{x_d} \left( 1, \frac{x_d - \bar{x}}{x_d} \right) = G_{x_d} \left( 2 - \frac{1}{x_d}, \frac{1}{x_d} \right). \] \hspace{1cm} (9.18)

From (9.8)–(9.10) it follows that
\[
x_dG_{0,x_d} \left( 2 - \frac{1}{x_d}, \frac{1}{x_d} \right) = -\frac{1}{4} \int_{\bar{x}}^{2x_d} b_{x_d} \left( 2 - \frac{1}{x_d}, 0 \right) \, dx_d \] \hspace{1cm} (9.19)

for \( x_d \geq 2x_s. \) From the definition of \( b, \) we have that
\[
x_{d,2}^2 b_{x_{d,2}} \left( 2 - \frac{1}{x_{d,1}}, \tau, 0 \right) = x_{d,1} b_{x_{d,1}} \left( 2 - \frac{1}{x_{d,2}}, \tau, 0 \right) \] \hspace{1cm} (9.20)

for \( \tau \in [0, 2x_s]. \) It follows from (9.19)–(9.20) that
\[
x_{d,2}G_{0,x_{d,1}} \left( 2 - \frac{1}{x_{d,1}}, \frac{1}{x_{d,1}} \right) = x_{d,1}G_{0,x_{d,2}} \left( 2 - \frac{1}{x_{d,2}}, \frac{1}{x_{d,2}} \right) \] \hspace{1cm} (9.21)

Similar arguments for \( G_{c,0}, G_n \) and \( G_{c,n} \) yield
\[
x_{d,2}G_{x_{d,1}} \left( 2 - \frac{1}{x_{d,1}}, \frac{1}{x_{d,1}} \right) = x_{d,1}G_{x_{d,2}} \left( 2 - \frac{1}{x_{d,2}}, \frac{1}{x_{d,2}} \right), \] \hspace{1cm} (9.22)

which in turn gives (9.17).

The significance of Theorem 9.1 is that it guarantees stabilization of the system evolving on an infinite domain by solving the stabilization problem on a finite domain. The procedure for verifying the conditions of the theorem was demonstrated above, but for clarity we repeat it in the following remark.
Remark 2. The key to applying Theorem 9.1 is being able to find an $x_s$ that satisfies (9.7). This is most easily done by inspecting a graph as the one shown in Figure 9.4. Once $x_s$ is found, a possible choice of $f_R(\hat{x})$ and $f_I(\hat{x})$ is given in (9.5)-(9.6). Other choices are possible, and in particular, care should be taken to ensure necessary smoothness properties of $f_R(\hat{x})$. Also, note that the estimate for the support of (9.3) is not tight, as suggested by Figures 9.4 and 9.5, which indicate that $k_u(\hat{x})$ is supported on approximately $[0, 7.5]$ while $2x_s = 11.2$. Theorem 9.1 states that $[0, 7.5] \subseteq [0, 2x_s]$, which is true.

9.3. Numerical simulations. For completeness, we include numerical simulations of the controlled and uncontrolled systems. The simulations have been performed by discretizing (1.1) on the domain $\hat{x} \in [0, 15]$, using finite differences on a grid of 200 nodes. To make the simulation study more interesting, the nonlinear term in (9.1) is accounted for in the simulations. Figures 9.6a and 9.6b show the real and imaginary parts of $A(\hat{x}, t)$ for the uncontrolled case. The system is in a periodic state reminiscent of vortex shedding. Figures 9.6c and 9.6d show the real and imaginary parts of $A(\hat{x}, t)$ for the controlled case. The figures show that $A(\hat{x}, t)$ is effectively driven to zero by the control. Figure 9.7 shows the control effort.

10. Conclusions. This paper extends previous work in two ways: 1) it deals with two coupled partial differential equations, and; 2) under certain circumstances handles equations defined on a semi-infinite domain. For the linearized Ginzburg-
Landau equation, asymptotic stabilization is achieved by means of boundary control via state feedback in the form of an integral operator. The kernel of the operator is shown to be twice continuously differentiable, and a series approximation for its solution is given. Under certain conditions (given in (9.7)) on the parameters of the Ginzburg-Landau equation, compatible with vortex shedding modelling on a semi-infinite domain, the kernel is shown to have compact support, resulting in partial state feedback. Simulations are provided in order to demonstrate the performance of the controller.

Acknowledgements. We thank Professor Peter Monkewitz for helpful explanations on relationships between Navier-Stokes and Ginzburg-Landau models of vortex shedding and on implementability of GL-based controllers on NS simulations or experiments.

REFERENCES


1. Coefficients for the Ginzburg-Landau Equation. The numerical coefficients below are taken from [13, Appendix A].

\[
R_c = 47 \\
x^t = 1.183 - 0.031i \\
\omega_0^t = 0.690 + 0.080i + (-0.00159 + 0.00447i)(R - R_c) \\
k_0^t = 1.452 - 0.844i + (0.00341 + 0.011i)(R - R_c) \\
\omega^t_{kk} = -0.292i \\
\omega^t_{xx} = 0.108 - 0.057i \\
k_x^t = 0.164 - 0.006i \\
\omega_0(\ddot{x}) = \omega_0^t + \frac{1}{2} \omega^t_{xx} (\ddot{x} - x^t)^2 \\
k_0(\ddot{x}) = k_0^t + k_x^t (\ddot{x} - x^t) \\
a_1 = \frac{1}{2} i \omega_{kk}^t \\
a_2(\ddot{x}) = \omega_{kk}^t k_0(\ddot{x}) \\
a_3(\ddot{x}) = - \left( \omega_0(\dddot{x}) + \frac{1}{2} \omega_{kk}^t k_0^2(\dddot{x}) \right) i \\
a_4 = -0.0225 + 0.0671i.
\]