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Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays

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Abstract

We consider a problem of boundary feedback stabilization of first-order hyperbolic partial differential equations (PDEs). These equations serve as a model for physical phenomena such as traffic flows, chemical reactors, and heat exchangers. We design controllers using a backstepping method, which has been recently developed for parabolic PDEs. With the integral transformation and boundary feedback the unstable PDE is converted into a "delay line" system which converges to zero in finite time. We then apply this procedure to finite-dimensional systems with actuator and sensor delays to recover a well-known infinite-dimensional controller (analog of the Smith predictor for unstable plants). We also show that the proposed method can be used for the boundary control of a Korteweg–de Vries-like third-order PDE. The designs are illustrated with simulations.

Keywords: Delays; Distributed parameter systems; Backstepping

1. Introduction

In this paper we apply the backstepping method, recently developed for parabolic PDEs [28,29] and second-order hyperbolic PDEs [18,19,17], to solve a problem of stabilization of a class of first-order hyperbolic PDEs. While hyperbolic PDEs of second order usually describe oscillatory systems such as strings and beams, the first-order hyperbolic equations describe quite a different set of physical problems, such as traffic flows, chemical reactors, and heat exchangers.

The existing results on feedback control of first-order hyperbolic PDEs include [6–8,20,26,32]. Boundary controllability, including null controllability, of these systems is studied in [1,4,5,14]. The focus in the field of control of first-order hyperbolic PDEs is on coupled systems of conservation laws, including nonlinear conservation laws. As *conservation* laws, such systems are typically neutrally stable, but with a possibility of infinitely many eigenvalues on the imaginary axis (in the case of *coupled* first-order hyperbolic PDEs). Such systems are stabilizable by static output feedbacks in the form of simple boundary conditions, however, the construction of strict Lyapunov functions is a delicate matter, and so is proving local stability of the nonlinear closed-loop PDE system (versus the easier problem of proving the stability of the linearization).

We first develop backstepping controllers for pure first-order hyperbolic PDEs. The idea of the backstepping method is to use invertible Volterra integral transformation together with the boundary feedback to convert the unstable hyperbolic PDE into a "delay line" system which converges to zero in finite time. The kernel of this transformation satisfies a certain PDE, which turns out to be also in the class of first-order hyperbolic equations. This PDE can be solved numerically, or, in certain cases, even in closed form.

We then apply the backstepping method to coupled ODE–PDE systems. First we design the controllers for a system which consists of the first-order hyperbolic PDE coupled with a second-order (in space) ODE. This system resembles the Korteweg-de Vries equation (see [3,22] and the references therein), which describes shallow water waves and ion acoustic waves in plasma.

For our second application, we consider LTI finite-dimensional systems with actuator and sensor delays, which

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can be viewed as a coupled system consisting of the ODE and the first-order hyperbolic PDE (which models delays). We combine the backstepping design for hyperbolic PDEs with the backstepping design for linear ODEs [15] to recover classical results for linear systems with actuator and sensor delay [25,21,23,2,24,9,31,14] (for a recent survey on the control of time-delay systems, see [11]). Even though the obtained controllers are well known, the backstepping procedure provides a more general and systematic way to handle time-delay systems. In particular, it can be applied to PDEs with sensor and actuator delays.

2. First-order hyperbolic PDEs

Consider the plant

$$v_t(x,t) = v_x(x,t) + \lambda(x)v(x,t) + \bar{g}(x)v(0,t) + \int_0^x \bar{f}(x,y)v(y,t)\mathrm{d}y \tag{1}$$

for $0 < x < 1$ with initial condition $v(x,t) = v_0(x)$ and boundary condition

$$v(1,t) = \bar{U}(t). \tag{2}$$

We assume that functions λ, $\bar{g}$, $\bar{f}$ are continuous. Our objective is to stabilize the zero equilibrium of this system with the boundary control $U(t)$ (when $U(t) \equiv 0$, this system is unstable for large positive $\bar{g}$ and $\bar{f}$).

We start by applying the state transformation

$$v(x,t) = \mathrm{e}^{-\int_0^x \lambda(\xi)\,\mathrm{d}\xi}u(x,t), \tag{3}$$

which results in the following plant

$$u_t(x,t) = u_x(x,t) + g(x)u(0,t) + \int_0^x f(x,y)u(y,t)\mathrm{d}y \tag{4}$$

$$u(1,t) = U(t), \tag{5}$$

where $U(t) = \bar{U}(t)\mathrm{e}^{\int_0^1 \lambda(\xi)\,\mathrm{d}\xi}$ and

$$g(x) = \bar{g}(x)\mathrm{e}^{\int_0^x \lambda(\xi)\,\mathrm{d}\xi}, \qquad f(x,y) = \bar{f}(x,y)\mathrm{e}^{\int_y^x \lambda(\xi)\,\mathrm{d}\xi}. \tag{6}$$

Following the backstepping approach [28], we use the transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\,\mathrm{d}y \tag{7}$$

along with the feedback

$$u(1,t) = \int_0^1 k(1,y)u(y,t)\,\mathrm{d}y \tag{8}$$

to convert the plant (4) into the target system

$$w_t(x,t) = w_x(x,t) \tag{9}$$

$$w(1,t) = 0. \tag{10}$$

This system is a delay line with unit delay, output $w(0,t) = w(1,t-1)$, and zero input at $w(1,t)$. Its solution is

$$w(x,t) = \begin{cases} w_0(t+x), & 0 \le x+t < 1 \\ 0, & x+t \ge 1, \end{cases} \tag{11}$$

where $w_0(x)$ is the initial condition. We see that this solution converges to zero in finite time.

To derive the condition that $k(x,y)$ should satisfy, we compute:

$$w_x(x,t) = u_x(x,t) - k(x,x)u_x(x,t) - \int_0^x k_x(x,y)u(y,t)\,\mathrm{d}y \tag{12}$$

and

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - \int_0^x k(x,y)\left(u_x(y,t) + g(y)u(0,t)\right)\mathrm{d}y \\ &\quad - \int_0^x k(x,y)\int_0^y f(y,\xi)u(\xi,t)\,\mathrm{d}\xi\,\mathrm{d}y \\ &= u_x(x,t) + u(0,t)\left(g(x) - \int_0^x k(x,y)g(y)\,\mathrm{d}y\right) \\ &\quad + \int_0^x u(y,t)\left(f(x,y) - \int_y^x k(x,\xi)f(\xi,y) \times\mathrm{d}\xi\right)\mathrm{d}y - k(x,x)u(x,t) \\ &\quad + k(x,0)u(0,t) + \int_0^x k_y(x,y)u(y)\,\mathrm{d}y. \end{aligned} \tag{13}$$

Subtracting (12) from (13) and using (9), we obtain the following set of conditions on $k(x,y)$:

$$k_x(x,y) + k_y(x,y) = \int_y^x k(x,\xi)f(\xi,y)\mathrm{d}\xi - f(x,y) \tag{14}$$

$$k(x,0) = \int_0^x k(x,y)g(y)\mathrm{d}y - g(x). \tag{15}$$

The following theorem establishes the well-posedness of the PDE (14), (15).

Theorem 1. *The PDE (14), (15) has a unique $C^1([0,1] \times [0,1])$ solution with a bound*

$$|k(x,y)| \le (\bar{g} + \bar{f})\mathrm{e}^{(\bar{g}+\bar{f})(x-y)}, \tag{16}$$

where

$$\bar{g} = \max_{x\in[0,1]} g(x), \quad \bar{f} = \max_{(x,y)\in[0,1]\times[0,1]} f(x,y). \tag{17}$$

Proof. It is easy to show that $k(x,y)$ satisfies the following integral equation:

$$k(x,y) = F_0(x,y) + F[k](x,y), \tag{18}$$

where

$$F_0(x,y) = -g(x-y) - \int_0^y f(x-y+\xi,\xi)\,\mathrm{d}\xi \tag{19}$$

$$\begin{aligned} F[k](x,y) &= \int_0^y \int_0^{x-y} k(x-y+\eta,\xi+\eta)f(\xi+\eta,\eta)\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad + \int_0^{x-y} k(x-y,\xi)g(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{20}$$

Let us solve this equation using the method of successive approximations. Set

$$k^0(x,y) = F_0(x,y),$$
$$k^{n+1}(x,y) = F_0(x,y) + F[k^n](x,y) \tag{21}$$

for $n = 0, 1, \ldots$ and consider the differences $\Delta k^{n+1} = k^{n+1} - k^n$ with $\Delta k^0 = F_0$. It is easy to see that Δk^n satisfy

$$\Delta k^{n+1}(x,y) = F[\Delta k^n](x,y). \tag{22}$$

Let us assume that

$$|\Delta k^n(x,y)| \le \frac{(\bar{g}+\bar{f})^{n+1}(x-y)^n}{n!}. \tag{23}$$

Then from (20) and (22) we get

$$\begin{aligned}|\Delta k^{n+1}| &\le \bar{g}\frac{(\bar{g}+\bar{f})^{n+1}(x-y)^{n+1}}{(n+1)!} + \bar{f}(\bar{g}+\bar{f})^{n+1} \\ &\quad\times \int_0^y \int_0^{x-y} \frac{(x-y-\xi)^n}{n!}\mathrm{d}\xi\,\mathrm{d}\eta \\ &\le \frac{(\bar{g}+\bar{f})^{n+2}(x-y)^{n+1}}{(n+1)!}.\end{aligned} \tag{24}$$

By induction (23) is proved. Therefore, the series

$$k(x,y) = \sum_{n=0}^{\infty} \Delta k^n(x,y) \tag{25}$$

uniformly converges to the solution of (21) with $n \to \infty$ and the bound (16). The fact that this solution satisfies the PDE (14), (15) is checked by simple differentiation. To show the uniqueness of this solution, consider the difference between two solutions k_1 and k_2: $\delta k = k_1 - k_2$. For δk we obtain the homogeneous integral equation

$$\delta k(x,y) = F[\delta k](x,y). \tag{26}$$

It is now easy to show by repeating the calculations above that

$$|\delta k(x,y)| \le \frac{(\bar{g}+\bar{f})^{n+1}(x-y)^n}{n!} \tag{27}$$

for any n, which implies that $\delta k(x,y) \equiv 0$, or $k_1 \equiv k_2$. □

We are ready to state the main result of this section.

Theorem 2. *For any initial condition $u(x,0) = u_0 \in H = \{f | f \in H^1(0,1), f(1) = \int_0^1 k(1,y)f(y)\mathrm{d}y\}$, the closed-loop system (4), (8) with $k(x,y)$ given by (14), (15) has a unique solution $u \in C([0,\infty), H) \cap C^1([0,\infty), L^2(0,1))$ which becomes zero in finite time.*

Proof. From the transformation (7) we see that the initial condition of the target system $w_0 \in \bar{H} = \{f | f \in H^1(0,1), f(1) = 0\}$ and therefore it immediately follows from (11) that $w \in C([0,\infty), \bar{H}) \cap C^1([0,\infty), L^2(0,1))$. One can show that the transformation, inverse to (7), has the form

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)\mathrm{d}y, \tag{28}$$

where $l(x,y)$ satisfies the following PDE

$$l_x(x,y) + l_y(x,y) = -\int_y^x f(x,\xi)l(\xi,y)\mathrm{d}\xi - f(x,y) \tag{29}$$

$$l(x,0) = -g(x). \tag{30}$$

This PDE is very similar to the PDE (14), (15) and repeating the arguments in the proof of Theorem 1, one can show that (29) and (30) has a unique continuously differentiable solution. Therefore, from (28) we obtain $u \in C([0,\infty), H) \cap C^1([0,\infty), L^2(0,1))$. The explicit form of the solution is obtained by using (11) and transformations (7) and (28):

$$\begin{aligned}u(x,t) &= u_0(x+t) - \int_0^x u_0(y+t)\Big[k(x+t, y+t) \\ &\quad - l(x,y) + \int_y^x l(x,\xi)k(\xi+t, y+t)\,\mathrm{d}\xi\Big]\mathrm{d}y, \quad x+t < 1\end{aligned} \tag{31}$$

and $u(x,t) \equiv 0$ for $x + t \ge 1$, so that the control objective is achieved for all $t \ge 1$. The uniqueness of this solution follows from the well-known uniqueness of the weak solution to (9), (10) (see, e.g., [5]). □

Remark 1. When $u_0 \in L^2(0,1)$ (without the compatibility condition), the solution (31) belongs to $C([0,\infty), L^2(0,1))$.

Since we have established the stabilizability of class (1) with boundary feedback, it may be natural to expect that this class of systems would be controllable in an appropriate sense. The null controllability for $T \ge 1$ of the special case of system (37) for $f = 1, \lambda = 0$ is established in [5] and a similar result may very well hold for the entire class (1).

We now illustrate the design with two examples.

Example 2.1. Consider the plant

$$u_t(x,t) = u_x(x,t) + g\mathrm{e}^{bx}u(0,t), \tag{32}$$

where g and b are constants. The Eq. (14) becomes

$$k_x(x,y) + k_y(x,y) = 0, \tag{33}$$

which has a general solution $k(x,y) = \phi(x-y)$. If we substitute this solution into (15), we get the integral equation for $\phi(x)$:

$$\phi(x) = \int_0^x g\mathrm{e}^{by}\phi(x-y)\mathrm{d}y - g\mathrm{e}^{bx}. \tag{34}$$

The solution to this equation can be easily obtained by applying the Laplace transform in x to both sides of (34). We get

$$\hat{\phi}(s) = -\frac{g}{s-b-g}, \tag{35}$$

and after taking the inverse Laplace transform, $\phi(x) = -g\mathrm{e}^{(b+g)x}$. Therefore, the solution to the kernel PDE is

$$k(x,y) = -g\mathrm{e}^{(b+g)(x-y)} \tag{36}$$

and the controller is given by (8).

Example 2.2. Consider the plant

$$u_t(x,t)=u_x(x,t)+\int_0^x f\mathrm{e}^{\lambda(x-y)}u(y,t)\,\mathrm{d}y, \tag{37}$$

where f and λ are constants. The kernel PDE (14), (15) takes the form

$$k_x(x,y)+k_y(x,y)=\int_y^x k(x,\xi)f\mathrm{e}^{\lambda(\xi-y)}\mathrm{d}\xi-f\mathrm{e}^{\lambda(x-y)} \tag{38}$$

$$k(x,0)=0. \tag{39}$$

After we differentiate (38) with respect to y, the integral term gets eliminated:

$$k_{xy}(x,y)+k_{yy}(x,y)=-fk(x,y)-\lambda k_x(x,y)-\lambda k_y(x,y). \tag{40}$$

Since we now increased the order of the equation, we need an extra boundary condition. We get it by setting $y=x$ in (38):

$$\frac{\mathrm{d}}{\mathrm{d}x}k(x,x)=k_x(x,x)+k_y(x,x)=-f, \tag{41}$$

which, after integration, becomes $k(x,x)=-fx$.

Introducing the change of variables

$$k(x,y)=p(z,y)\mathrm{e}^{\lambda(z-y)/2},\qquad z=2x-y, \tag{42}$$

we get the following PDE for $p(z,y)$:

$$p_{zz}(z,y)-p_{yy}(z,y)=fp(z,y) \tag{43}$$

$$p(z,0)=0 \tag{44}$$

$$p(z,z)=-fz. \tag{45}$$

This PDE has the following solution [28]:

$$p(z,y)=-2fy\frac{I_1\left(\sqrt{f(z^2-y^2)}\right)}{\sqrt{f(z^2-y^2)}}, \tag{46}$$

where I_1 is the modified Bessel function. In the original variables we obtain

$$k(x,y)=-f\mathrm{e}^{\lambda(x-y)}y\frac{I_1\left(2\sqrt{fx(x-y)}\right)}{\sqrt{fx(x-y)}}, \tag{47}$$

and the controller is given by

$$u(1,t)=-\int_0^1 f\mathrm{e}^{\lambda(1-y)}y\frac{I_1\left(2\sqrt{f(1-y)}\right)}{\sqrt{f(1-y)}}u(y,t)\,\mathrm{d}y. \tag{48}$$

3. Korteweg–de Vries-like equation

Consider the first-order hyperbolic PDE coupled with a second-order ODE:

$$\varepsilon u_t(x,t)=u_x(x,t)-v(x,t) \tag{49}$$

$$0=\varepsilon v_{xx}(x,t)+a(-v(x,t)+\gamma u_x(x,t)), \tag{50}$$

where $\varepsilon>0$, $a>0$. The boundary condition $v_x(0,t)=0$ and $u(1,t)=U_1(t)$, $v(1,t)=U_2(t)$, where U_1 and U_2 are control inputs.

The motivation for considering the system (49) and (50) comes from the fact that it can be viewed as a third-order PDE

$$u_t(x,t)-\nu u_{txx}(x,t)+\delta u_{xxx}(x,t)+\lambda u_x(x,t)=0, \tag{51}$$

which is obtained by differentiating (49) with respect to x twice, substituting the result into (50), and denoting

$$\delta=\frac{1}{a},\qquad \nu=\frac{\varepsilon}{a},\qquad \lambda=\varepsilon^{-1}(\gamma-1). \tag{52}$$

The PDE (51) resembles a linearized Korteweg–de Vries equation which serves as a model of shallow water waves and ion acoustic waves in plasma. Compared to the traditional form of the Korteweg–de Vries equation, it has an additional term $-\nu u_{txx}$, which is small when ε/a is small in the original system (49), (50). In fact, this term appears in the derivation of KdV equation, but is then dropped as it is small compared to u_t [16]. The PDE (51) is unstable when λ/δ is positive and large. Besides being related to the Korteweg–de Vries PDE, equation (51) can be obtained as an approximation of the linearized Boussinesq PDE system modeling complex water waves such as tidal bores [10].

To apply the backstepping design to the system (49)–(50), we first solve (50) with respect to v:

$$v(x,t)=\cosh(bx)v(0,t)-\gamma b\int_0^x\sinh(b(x-y))\times u_y(y,t)\mathrm{d}y, \tag{53}$$

where $b=\sqrt{a/\varepsilon}$. Setting $x=1$ in (53), we express $v(0,t)$ in terms of $v(1,t)$:

$$v(0,t)=\frac{1}{\cosh b}\left[v(1,t)-\gamma b\sinh(b)u(0,t)+\gamma b^2\int_0^1\cosh(b(1-y))u(y,t)\,\mathrm{d}y\right]. \tag{54}$$

The integral in (54) has the limits from 0 to 1 and not in the class of PDEs (4). Therefore we select the first boundary control to be

$$v(1,t)=\gamma b\sinh(b)u(0,t)-\gamma b^2\int_0^1\cosh(b(1-y))\times u(y,t)\mathrm{d}y, \tag{55}$$

which guarantees that $v(0,t)=0$. Substituting (53) into (49) we get

$$\varepsilon u_t(x,t)=u_x(x,t)-\gamma b\sinh(bx)u(0,t)+\gamma b^2\int_0^x\cosh(b(x-y))u(y,t)\,\mathrm{d}y. \tag{56}$$

Note that this PDE is exactly of the form (4). We can now use the design developed in Section 2. The second control law is

$$u(1,t)=\int_0^1 k(1,y)u(y,t)\,\mathrm{d}y, \tag{57}$$

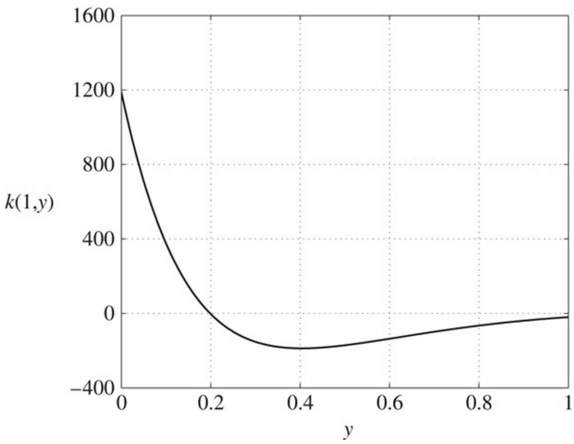

Fig. 1. Control gain for the Korteweg-de-Vries equation.

where the control kernel $k(x, y)$ is found from the PDE

$$k_x(x, y) + k_y(x, y) = \gamma b^2 \int_y^x k(x, \xi) \cosh(b(\xi - y))\, \mathrm{d}\xi - \gamma b^2 \cosh(b(x - y)) \quad (58)$$

with the boundary condition

$$k(x, 0) = \gamma b \sinh(bx) - \gamma b \int_0^x k(x, y) \sinh(by)\, \mathrm{d}y. \quad (59)$$

Using Theorem 2, we obtain the following result.

Theorem 3. *For any initial condition $u_0 \in H$, the system (49)–(50) with the controllers (55), (57) has a unique solution $u \in C([0, \infty), H) \cap C^1([0, \infty), L^2(0, 1))$ which becomes zero in finite time.*

The simulation results for $a = 1$, $\varepsilon = 0.2$, $\gamma = 4$ are presented in Figs. 1 and 2. The control gain (Fig. 1) is obtained by discretizing (58), (59) using the implicit Euler finite-difference scheme (an alternative is to use the series (21)). We can see that the open-loop plant (49)–(50) is unstable and the controller stabilizes the system.

4. ODE Systems with actuator delay

We now apply the design developed in the previous section to ODEs with the actuator delay. Consider a linear finite-dimensional system

$$\dot{X} = AX + BU(t - D), \quad (60)$$

where $X \in \mathbb{R}^n$, (A, B) is a controllable pair and the input signal $U(t)$ is delayed by D units of time. The delay can be modelled by the following first-order hyperbolic PDE

$$u_t(x, t) = u_x(x, t) \quad (61)$$

$$u(D, t) = U(t). \quad (62)$$

The solution to this equation is $u(x, t) = U(t + x - D)$ and therefore the output $u(0, t) = U(t - D)$ gives the delayed input.

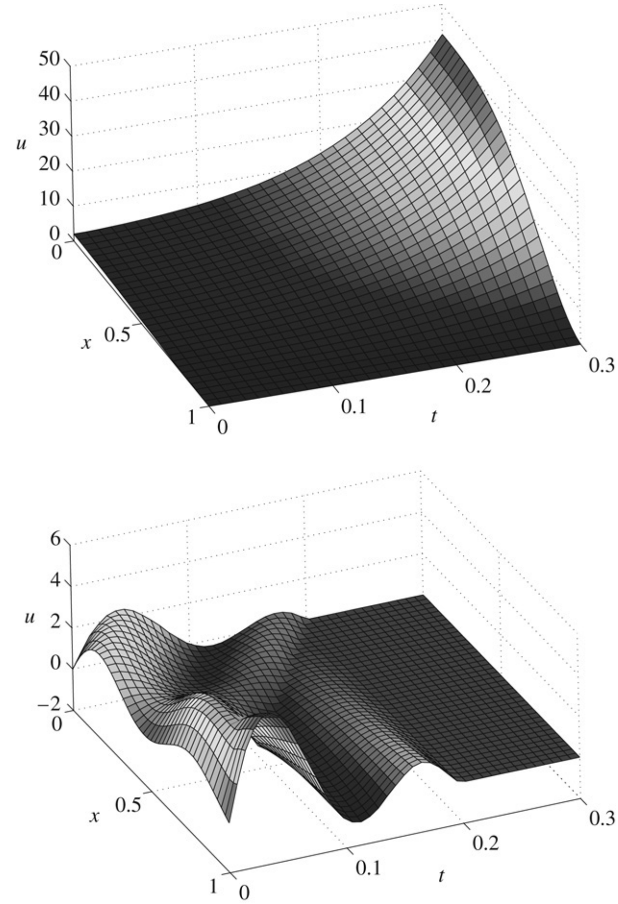

Fig. 2. The open-loop (left) and the closed-loop (right) responses of the Korteweg–de Vries-like plant (49)–(50) with backstepping controllers (55) and (57)–(59).

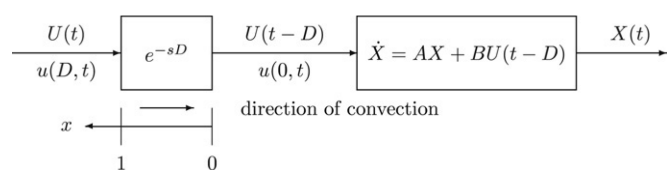

Fig. 3. Linear system $\dot{X} = AX + BU(t - D)$ with actuator delay D.

The system (60) can be now written as.[1]

$$\dot{X} = AX + Bu(0). \quad (63)$$

Eqs. (61)–(63) form an ODE–PDE cascade which is driven by the input U from the boundary of the PDE (Fig. 3).

Suppose a static state feedback control has been designed for a system with no delay (i.e. with $D = 0$) such that $U = KX$ is a stabilizing controller, i.e., the matrix $(A + BK)$ is Hurwitz. Consider the backstepping transformation

$$w(x) = u(x) - \int_0^x q(x, y)u(y)\mathrm{d}y - \gamma(x)^{\mathrm{T}} X \quad (64)$$

[1] From this point onwards, we suppress the time dependence for clarity, so that $u(0) \equiv u(0, t)$, etc.

which maps (61)–(63) into the target system

$$\dot{X} = (A + BK)X + Bw(0) \tag{65}$$

$$w_t = w_x \tag{66}$$

$$w(D) = 0. \tag{67}$$

Let us calculate the time and spatial derivatives of the transformation (64):

$$w_x = u_x - q(x,x)u(x) - \int_0^x q_x(x,y)u(y)\mathrm{d}y - \gamma'(x)^{\mathrm{T}} X \tag{68}$$

$$\begin{aligned} w_t &= u_t - \int_0^x q(x,y)u_t(y)\mathrm{d}y - \gamma(x)^{\mathrm{T}}[AX + Bu(0)] \\ &= u_x - q(x,x)u(x) + q(x,0)u(0) \\ &\quad - \int_0^x q_x(x,y)u(y)\mathrm{d}y - \gamma(x)^{\mathrm{T}}[AX + Bu(0)]. \end{aligned} \tag{69}$$

Subtracting (68) from (69) we get

$$\int_0^x (q_x(x,y) + q_y(x,y))u(y)\mathrm{d}y + \left[q(x,0) - \gamma(x)^{\mathrm{T}} B\right] u(0) + \left[\gamma'(x)^{\mathrm{T}} - \gamma(x)^{\mathrm{T}} A\right] X = 0. \tag{70}$$

This equation should be valid for all u and X, so we have three conditions:

$$q_x(x,y) + q_y(x,y) = 0 \tag{71}$$

$$q(x,0) = \gamma(x)^{\mathrm{T}} B \tag{72}$$

$$\gamma'(x) = A^{\mathrm{T}}\gamma(x). \tag{73}$$

The first two conditions form a first-order hyperbolic PDE and the third one is a simple ODE. To find the initial condition for this ODE, let us set $x = 0$ in (64), which gives $w(0) = u(0) - \gamma(0)^{\mathrm{T}} X$. Substituting this expression into (65), we get

$$\dot{X} = AX + Bu(0) + B\left(K - \gamma(0)^{\mathrm{T}}\right) X. \tag{74}$$

Comparing this equation with (63), we have $\gamma(0) = K^T$. Therefore the solution to the ODE (73) is $\gamma(x) = \mathrm{e}^{A^{\mathrm{T}}x} K^{\mathrm{T}}$ which gives

$$\gamma(x)^{\mathrm{T}} = K\mathrm{e}^{Ax}. \tag{75}$$

A general solution to (71) is $q(x,y) = \phi(x-y)$, where the function ϕ is determined from (72). We get

$$q(x,y) = K\mathrm{e}^{A(x-y)} B. \tag{76}$$

We can now plug the gains $\gamma(x)$ and $q(x,y)$ into the transformation (64) and set $x = D$ to get the control law:

$$u(D) = \int_0^D K\mathrm{e}^{A(D-y)} Bu(y)\mathrm{d}y + K\mathrm{e}^{AD} X. \tag{77}$$

The stability result is given by the following theorem.

Theorem 4. *The closed-loop system consisting of the plant (63), (61) and (62) with the controller (77) is exponentially stable at the origin in the sense of the norm* $(|X(t)|^2 + \int_0^D u(x,t)^2 \mathrm{d}x)^{1/2}$.

Proof. First we prove that the origin of the target system (65)–(67) is exponentially stable. Consider a Lyapunov function

$$V = X^T P X + \frac{a}{2}\int_0^D (1+x)w(x)^2\,\mathrm{d}x, \tag{78}$$

where $P = P^T > 0$ is the solution to the Lyapunov equation $P(A+BK) + (A+BK)^T P = -Q$ for some $Q = Q^T > 0$, and the parameter $a > 0$ is to be chosen later. We have

$$\begin{aligned} \dot{V} &= X^T((A+BK)^T P + P(A+BK))X \\ &\quad + 2X^T PBw(0) - \frac{a}{2}w(0)^2 - \frac{a}{2}\int_0^D w(x)^2\,\mathrm{d}x \\ &\le -X^T QX + \frac{2}{a}\|X^T PB\|^2 - \frac{a}{2}\int_0^D w(x)^2\,\mathrm{d}x. \end{aligned} \tag{79}$$

Let us choose $a = 4\lambda_{\max}(PBB^{\mathrm{T}}P)/\lambda_{\min}(Q)$, where $\lambda_{\min}$ and $\lambda_{\max}$ are minimum and maximum eigenvalues of the corresponding matrices. Then

$$\begin{aligned} \dot{V} &\le -\frac{\lambda_{\min}(Q)}{2}\|X\|^2 - \frac{2\lambda_{\max}(PBB^{\mathrm{T}}P)}{\lambda_{\min}(Q)}\int_0^D w(x)^2\,\mathrm{d}x \\ &\le -\min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{a}{2}\right\} V. \end{aligned} \tag{80}$$

Thus we obtain exponential stability in the sense of the full state norm in the transformed variable, $\left(|X(t)|^2 + \int_0^D w(x,t)^2\mathrm{d}x\right)^{1/2}$. To show exponential stability in the sense of the norm $\left(|X(t)|^2 + \int_0^D u(x,t)^2\mathrm{d}x\right)^{1/2}$, we need the inverse of the transformation (64). One can show with calculations similar to (69)–(76) that such a transformation is

$$u(x) = w(x) + \int_0^x K\mathrm{e}^{(A+BK)(x-y)} Bw(y)\mathrm{d}y + K\mathrm{e}^{(A+BK)x} X. \tag{81}$$

From (81) one readily obtains exponential stability in the sense of the norm $\left(|X(t)|^2 + \int_0^D u(x,t)^2\mathrm{d}x\right)^{1/2}$. □

Remark 2. The controller (77) is given in terms of the transport delay state $u(y)$. Using (61) and (62) one can also derive the representation in terms of the input signal $U(t)$:

$$U(t) = K\left[\mathrm{e}^{AD} X + \int_{t-D}^t \mathrm{e}^{A(t-\theta)} BU(\theta)\mathrm{d}\theta\right]. \tag{82}$$

The controller (82) is the analog of the Smith Predictor [27] extended to unstable plants and was first derived in 1978–1982 [21,23,2] (see also, for example, the more recent book [13]). The derivation in these references is very different and employs a transformation of the ODE state rather than the delay state. As a result, the analysis in these references does not capture the entire ODE + delay system as succinctly and completely as (65)–(67).

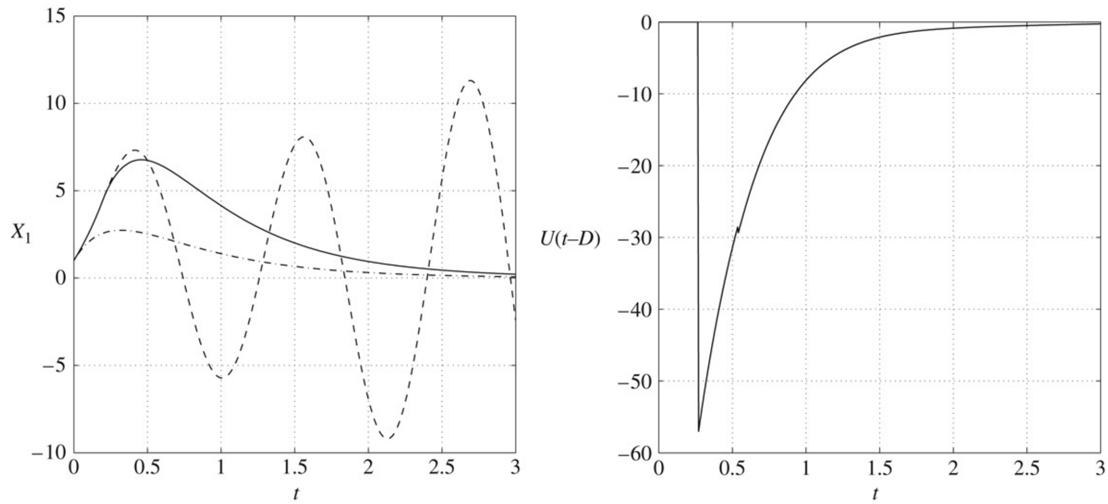

Fig. 4. The closed-loop response of the finite-dimensional system with actuator delay. Left: state evolution with nominal LQR controller in the absence of the delay (dash-dotted); with nominal LQR controller in the presence of the delay (dashed); with the backstepping controller in the presence of the delay (solid). Right: delayed control input.

In Fig. 4 the simulation results for system (60) are presented for

$$A=\begin{bmatrix}2 & 0 & 1\\ 1 & -2 & -2\\ 0 & 1 & -1\end{bmatrix},\qquad B=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

This system is unstable at the origin (eigenvalues are 2 and $-1.5 \pm 1.4j$). One can see that the nominal LQR controller (with $Q = I$ and $R = 1$) does not stabilize the system when the small delay ($D = 0.3$) is present. The controller (77) stabilizes the system. The larger transient is due to the fact that in the beginning the input to the system is zero because of the delay.

5. Observers for ODE systems with sensor delay

The procedure developed in the previous section extends to the case of sensor delay. Consider the system

$$\dot{X} = AX \tag{83}$$

$$Y(t) = CX(t-D), \tag{84}$$

where (A, C) is an observable pair. The output equation can be represented through the first-order hyperbolic PDE as

$$u_t = u_x \tag{85}$$

$$u(D) = CX \tag{86}$$

$$Y = u(0). \tag{87}$$

We have the following result for (83), (85)–(87).

Theorem 5. *The observer*

$$\dot{\hat{X}} = A\hat{X} + \mathrm{e}^{AD}L\left(Y - \hat{u}(0)\right) \tag{88}$$

$$\hat{u}_t = \hat{u}_x + C\mathrm{e}^{Ax}L\left(Y - \hat{u}(0)\right) \tag{89}$$

$$\hat{u}(D) = C\hat{X}, \tag{90}$$

where L is chosen such that $A - LC$ is Hurwitz, guarantees that $\hat{X}$, $\hat{u}$ exponentially converge to X, u, i.e., more specifically, that the observer error system is exponentially stable in the sense of the norm

$$\left(|X(t) - \hat{X}(t)|^2 + \int_0^D \left(u(x,t) - \hat{u}(x,t)\right)^2 \mathrm{d}x\right)^{1/2}.$$

Proof. Introducing the error variables $\tilde{X} = X - \hat{X}$, $\tilde{u} = u - \hat{u}$, we obtain:

$$\dot{\tilde{X}} = A\tilde{X} - \mathrm{e}^{AD}L\tilde{u}(0) \tag{91}$$

$$\tilde{u}_t = \tilde{u}_x - C\mathrm{e}^{Ax}L\tilde{u}(0) \tag{92}$$

$$\tilde{u}(D) = C\tilde{X}. \tag{93}$$

Consider the transformation

$$\tilde{w}(x) = \tilde{u}(x) - C\mathrm{e}^{A(x-D)}\tilde{X}. \tag{94}$$

We have

$$\begin{aligned}\tilde{w}_t - \tilde{w}_x &= \tilde{u}_x - C\mathrm{e}^{Ax}L\tilde{u}(0) - C\mathrm{e}^{A(x-D)}(A\tilde{X} - \mathrm{e}^{AD}L\tilde{u}(0))\\ &\quad - \tilde{u}_x + C\mathrm{e}^{A(x-D)}A\tilde{X} = 0\end{aligned} \tag{95}$$

and $\tilde{w}(D) = \tilde{u}(D) - C\tilde{X} = 0$. This means that $\tilde{w}$ converges to zero in finite time. The equation (91) can be written as

$$\begin{aligned}\dot{\tilde{X}} &= A\tilde{X} - \mathrm{e}^{AD}L\tilde{u}(0) = A\tilde{X} - \mathrm{e}^{AD}L(\tilde{w}(0) + C\mathrm{e}^{-AD}\tilde{X})\\ &= (A - \mathrm{e}^{AD}LC\mathrm{e}^{-AD})\tilde{X} - \mathrm{e}^{AD}L\tilde{w}(0).\end{aligned} \tag{96}$$

The matrix $A - \mathrm{e}^{AD}LC\mathrm{e}^{-AD}$ is Hurwitz, which can be easily seen by using a similarity transformation e^{AD}, which commutes with A. With a Lyapunov function

$$V = \tilde{X}^T\mathrm{e}^{-A^TD}P\mathrm{e}^{-AD}\tilde{X} + \frac{a}{2}\int_0^D (1+x)\tilde{w}(x)^2\,\mathrm{d}x, \tag{97}$$

where $P = P^T > 0$ is the solution to the Lyapunov equation $P(A - LC) + (A - LC)^TP = -Q$ for some $Q = Q^T > 0$, and a is sufficiently large, one can show that $\dot{V} \leq -\mu V$ for some $\mu > 0$, i.e., the $(\tilde{X}, \tilde{w})$ system is exponentially stable at the origin. From (94) we get exponential stability in the sense of $\left(|\tilde{X}(t)|^2 + \int_0^D \tilde{u}(x,t)^2 \mathrm{d}x\right)^{1/2}$. □

Remark 3. The observer (88)–(90) can be represented in terms of the output Y by taking the Laplace transform of (89), solving the resulting ODE and taking the inverse Laplace transform:

$$\dot{\hat{X}} = A\hat{X} + \mathrm{e}^{AD} L(Y - \hat{Y}) \tag{98}$$

$$\hat{Y}(t) = C\hat{X}(t-D) + C\int_{t-D}^{t} \mathrm{e}^{A(t-\theta)} L(Y(\theta) - \hat{Y}(\theta))\,\mathrm{d}\theta. \tag{99}$$

Remark 4. Unlike in the case of actuator delay, the observer presented in this section seems to be novel, i.e., it is not merely a different way of obtaining the known delay-compensating observer results in [31,14]. Those results take a different approach and use an observer of the form which is essentially

$$\dot{\hat{X}} = A\hat{X} + \mathrm{e}^{AD} L\left(Y - C\mathrm{e}^{-AD}\hat{X}\right), \tag{100}$$

where the gain vector L is selected to make the matrix $A - LC$ Hurwitz (which is equivalent to making $A - \mathrm{e}^{AD}LC\mathrm{e}^{-AD}$ Hurwitz). The observer (100) differs from our observer (88)–(90) in the way that the estimate of $Y(t)$ is introduced in the estimation error. While (100) uses $C\mathrm{e}^{-AD}\hat{X}(t)$ in lieu of an estimate of $Y(t)$, we use a distributed estimator $\hat{u}(x,t)$ of $Y(t+x), x \in [0, D]$, given by (89), (90), with output injection, which can also be viewed as the estimator of the actual plant output $CX(\theta)$ over the window $\theta \in [t-D, t]$. In other words, our observer generates not only a convergent estimate $\hat{X}(t)$ of $X(t)$, but also a (quantifiably) convergent estimate $\hat{Y}(t+x) = \hat{u}(x,t)$ of $Y(t+x) = CX(t+x-D)$ for $x \in [0, D]$. Since our observer is infinite dimensional, whereas the observer (100) is finite dimensional, it is valid to ask a question whether the additional dimensionality is of any value. One should first note that (100) is a classical *reduced-order observer* for the plant (83), (84) which treats the infinite-dimensional "sensor state" $Y(t+x), x \in [0, D]$, as known (in the future), and does not 'waste' dynamic order to estimate it. In contrast, our observer is a full-order observer, which estimates both the plant state X and the sensor state. One benefit of employing a full-order observer over a reduced-order observer is that reduced-order observers are well known to be overly sensitive to measurement noise. An additional comment in favor of our full-order observer approach is that the idea that allows the reduced-order observer (100) does not extend to more general sensor dynamics (whether finite or infinite dimensional). It works only with delays because of the special form of their dynamics (pure 'time-shift') and also thanks to the fact that the transport delay dynamics are exponentially stable, hence output injection is not necessary to stabilize their observer error system.

Remark 5. It is possible that the observer (98), (99) may be implicitly contained in the general infinite-dimensional observer form in [30, 4.1], however it is not clear that [30] contains a constructive result to obtain (98), (99).

Remark 6. The dimensionality advantage of the reduced-order observer (100) disappears the moment one adds an input into the plant (83), (84), i.e., $\dot{X}(t) = AX(t) + BU(t)$, $Y(t) = CX(t-D)$. Then, the observer (100) assumes the form

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \mathrm{e}^{AD} L\left(Y(t) - C\mathrm{e}^{-AD}\hat{X}(t) + C\int_{t-D}^{t} \mathrm{e}^{A(t-D-\theta)} BU(\theta)\mathrm{d}\theta\right). \tag{101}$$

Note that, even though infinite dimensional, this is still a reduced-order observer because it does not attempt to estimate the sensor state. Our observer (88)–(90) needs only a slight modification when the term $BU(t)$ is added to the plant and its order does not increase:

$$\dot{\hat{X}} = A\hat{X} + BU + \mathrm{e}^{AD} L\left(Y - \hat{u}(0)\right) \tag{102}$$

$$\hat{u}_t = \hat{u}_x + C\mathrm{e}^{Ax} L\left(Y - \hat{u}(0)\right) \tag{103}$$

$$\hat{u}(D) = C\hat{X}. \tag{104}$$

Remark 7. An alternative implementation of the reduced-order observer (101) is

$$\dot{\Xi}(t) = A\Xi(t) + BU(t-D) + L(Y(t) - C\Xi(t))$$

$$\hat{X}(t) = \mathrm{e}^{AD}\Xi(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)} BU(\theta)\mathrm{d}\theta. \tag{105}$$

So, the classical [14,31] observer essentially estimates the past state from D seconds back, and then advances it in an open-loop manner D seconds in the future.

6. Conclusions

We presented a new approach for the boundary stabilization of the first-order hyperbolic PDEs. Application to systems with delays seems a particularly interesting application of this result, opening many new opportunities for research. In future work, several directions can be pursued. First, various PDEs with actuator and sensor delays (resulting in a cascade of a second-order parabolic or hyperbolic PDE with the first-order hyperbolic PDE) can be investigated. Second, the (still open) control problem for finite-dimensional systems with both state and actuator delays can be addressed. Finally, the approach holds a great promise for nonlinear systems [12], for which there are currently no controllers available that compensate for arbitrarily long actuator delays.

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