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CONTROL OF A TIP-FORCE DESTABILIZED SHEAR BEAM BY OBSERVER-BASED BOUNDARY FEEDBACK*

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Abstract. We consider a model of the undamped shear beam with a destabilizing boundary condition. The motivation for this model comes from atomic force microscopy, where the tip of the cantilever beam is destabilized by van der Waals forces acting between the tip and the material surface. Previous research efforts relied on collocated actuation and sensing at the tip, exploiting the passivity property between the corresponding input and output in the beam model. In this paper we design a stabilizing output-feedback controller in a noncollocated setting, with measurements at the free end (tip) of the beam and actuation at the beam base. Our control design is a novel combination of the classical "damping boundary feedback" idea with a recently developed backstepping approach. A change of variables is constructed which converts the beam model into a wave equation (for a very short string) with boundary damping. This approach is physically intuitive and allows both an elegant stability analysis and an easy selection of design parameters for achieving desired performance. Our observer design is a dual of the similar ideas, combining the damping feedback with backstepping, adapted to the observer error system. Both stability and well-posedness of the closed-loop system are proved. The simulation results are presented.

Key words. distributed parameter systems, shear beam, backstepping, stabilization, boundary control

AMS subject classifications. 35J05, 93B07, 93D15, 93B52, 93B60

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1. Introduction.

Flexible beams constitute an important benchmark problem in many application areas ranging from aerospace to civil structures. In some of the exciting modern fields such as atomic force microscopy, the cantilever beam is more than just a prototype problem and constitutes an important application topic in its own right.

In this paper we consider a model of the undamped shear beam [3] with a destabilizing boundary condition. It consists of a wave equation coupled with a second-order-in-space ODE or can be alternatively represented as a fourth-order-in-space/second-order-in-time PDE. This makes it more complex than the Euler–Bernoulli model [3], similar in structure to the Rayleigh beam model [3], and slightly simpler than the Timoshenko model [3]. The destabilizing boundary condition is motivated by the physics of the atomic force microscopy (AFM), where the tip of the cantilever beam is destabilized by van der Waals forces acting between the tip and the material surface [19].

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Extensive literature exists on control of beam models [1, 2, 8, 4, 7, 9, 11, 17, 20, 21]. However, previous research efforts all relied on collocated actuation and sensing at the tip, exploiting in an elegant way the passivity property between the corresponding input and output in the beam model. The main drawback of this approach is that the tip of the beam is not a very convenient place to put an actuator. Therefore such feedbacks are usually implemented via passive dampers or through rather elaborate ways, such as electromagnets or small airjets at the tip of the beam.

Our objective is different—to design controllers implementable through noncollocated architecture, with actuation only at the base and sensing only at the tip of the beam. This architecture makes active control more readily implementable to several applications; for example, in AFM this allows a natural use of piezo actuation at the base of the beam. Our control design is a novel combination of the classical "damping boundary feedback" idea with a recently developed backstepping approach, which has been used to design boundary controllers [12, 14] and observers [13] for parabolic equations and for the Timoshenko beam model with a small amount of Kelvin–Voigt damping [6]. A change of variables is constructed which converts the beam model into a wave equation (for a very short string) with boundary damping. This approach is physically intuitive and allows both an elegant stability analysis and an easy selection of design parameters for achieving desired performance. Our observer design is a dual of the similar ideas, combining the damping feedback with backstepping, adapted to the observer error system.

In addition to rigorous stability and well-posedness analysis of the closed-loop system, we also present the results of simulations that illustrate the performance of the controller.

2. Model. The shear beam model can be represented in a number of equivalent ways [3]. One often used form is a single second-order-in-time fourth-order-in-space PDE

$$a w_{tt}(x,t) - \delta w_{xxtt}(x,t) + w_{xxxx}(x,t) = 0, \quad 0 < x < 1,\ t > 0, \tag{2.1}$$

with $a, \delta > 0$ and two "free end" boundary conditions $\delta w_{tt}(0,t) = w_{xx}(0,t)$ and $\delta w_{xtt}(0,t) = w_{xxx}(0,t) - q\frac{a}{\delta} w(0,t)$.

We, however, will use another common form of this model, consisting of a wave equation coupled with a second-order ODE:

$$\begin{cases} \delta w_{tt}(x,t) = w_{xx}(x,t) - \alpha_x(x,t), \quad 0 < x < 1, t > 0, \\ w_x(0,t) = \alpha(0,t) - q w(0,t), \quad t \geq 0, \\ w(1,t) = u_1(t), \quad t \geq 0, \\ 0 = \alpha_{xx}(x,t) - b^2 \alpha(x,t) + b^2 w_x(x,t), \quad 0 < x < 1, t > 0, \\ \alpha_x(0,t) = 0, \quad t \geq 0, \\ \alpha(1,t) = u_2(t), \quad t \geq 0, \\ y(t) = (w(0,t), \alpha(0,t)), \quad t \geq 0, \end{cases} \tag{2.2}$$

which is obtained from (2.1) by introducing a new state $\alpha_x = w_{xx} - \delta w_{tt}$ and denoting $b = \sqrt{a/\delta}$. The state w represents the transversal displacement of the beam, and α is the angle due to bending. The objective is to use the control input $u(t) = (u_1(t), u_2(t))$ at the base of the beam to stabilize the tip of the beam with the measurement $y(t)$ available only at the free end.

It is important to note the term $-qw(0,t)$ in the boundary condition (2.2). This type of boundary condition corresponds to situations where the tip of the beam is subject to an external force which depends on the displacement. Such a force arises in AFM as a van der Waals force acting between the atoms on the material surface and the beam tip. The term $-qw(0,t)$ is the linearized model of that force; the original nonlinear model has cubic nonlinearity [19]. Typically $q>0$ has a destabilizing effect (in that case, one can think of this parameter as "antistiffness"), whereas $q<0$ has a stabilizing effect. We stress that this force occurs on the opposite end of the beam from where the actuator is located. If the actuator were at the tip, canceling the effect of this force would be trivial. In the configuration that we pursue here, stabilization, and even vibration suppression when $q=0$, is a nontrivial problem.

We will first present control and observer designs separately to make the ideas clear and then prove the certainty equivalence principle and well-posedness of the closed-loop system in sections 5–7.

3. Controller. In order to proceed with the control design we first need to write the model (2.2) in yet another form. To this end, we solve the ODE part of (2.2) as a two point boundary value problem for α with boundary condition $\alpha_x(0,t)=0$:

$$\alpha(x,t)=\cosh(bx)\alpha(0,t)-b\int_0^x \sinh(b(x-s))w_x(s,t)\,ds. \tag{3.1}$$

Setting $x=1$ in (3.1) and using the boundary condition $\alpha(1,t)=u_2(t)$, we can express $\alpha(0,t)$ in terms of w and u_2:

$$\alpha(0,t)=\frac{1}{\cosh(b)}u_2(t)+\frac{b}{\cosh(b)}\int_0^1 \sinh(b(1-s))w_x(s,t)\,ds. \tag{3.2}$$

Next, we differentiate (3.1) in x and substitute the result into the first equation of (2.2). This way, instead of a wave equation coupled with a second-order ODE, we obtain a single hyperbolic partial integrodifferential equation for w:

$$\begin{cases} \delta w_{tt}(x,t)=w_{xx}(x,t)-b^2\cosh(bx)w(0,t)+b^3\displaystyle\int_0^x \sinh(b(x-y))w(y,t)dy \\ \qquad\qquad +\, b^2 w(x,t)-\dfrac{b\sinh(bx)}{\cosh(b)}\left[u_2(t)+b\displaystyle\int_0^1 \sinh(b(1-s))w_x(s,t)\,ds\right], \\ w_x(0,t)=\dfrac{1}{\cosh(b)}\left[u_2(t)+b\displaystyle\int_0^1 \sinh(b(1-s))w_x(s,t)\,ds\right]-qw(0,t), \\ w(1,t)=u_1(t). \end{cases} \tag{3.3}$$

Since the backstepping control design [12] needs the PDE to be in a "strict-feedback" form (in other words, its right-hand side must be "causal" in x), we are going to use the control $u_2(t)$ to cancel the definite integral both in the domain and in the boundary condition:

$$u_2(t)=-b\int_0^1 \sinh(b(1-s))w_x(s,t)\,ds. \tag{3.4}$$

We get the following PDE:

$$\left\{\begin{aligned} \delta w_{tt}(x,t) &= w_{xx}(x,t) + b^2 w(x,t) - b^2\cosh(bx)w(0,t) \\ &\qquad + b^3\int_0^x \sinh(b(x-y))w(y,t)dy, \\ w_x(0,t) &= -qw(0,t), \\ w(1,t) &= u_1(t)\,. \end{aligned}\right. \tag{3.5}$$

The basic idea of the backstepping design is to use the transformation

$$\bar{w}(x,t) = w(x,t) - \int_0^x k(x,y)w(y,t)\,dy, \tag{3.6}$$

with specially designed control kernel $k(x,y)$ along with the boundary feedback law

$$\left\{\begin{aligned} u_1(t) &= w(1,t), \\ w_x(1,t) &= k(1,1)w(1,t) - c_1 w_t(1,t) \\ &\qquad + c_1\int_0^1 k(1,y)w_t(y,t)\,dy + \int_0^1 k_x(1,y)w(y,t)\,dy \end{aligned}\right. \tag{3.7}$$

to map (3.5) into the exponentially stable target system

$$\left\{\begin{aligned} \delta\bar{w}_{tt}(x,t) &= \bar{w}_{xx}(x,t), \\ \bar{w}_x(0,t) &= c_0\bar{w}(0,t), \\ \bar{w}_x(1,t) &= -c_1\bar{w}_t(1,t), \end{aligned}\right. \tag{3.8}$$

where $c_0 > 0$ and $c_1 > 0$ are design parameters. The system (3.8) is exponentially stable at the origin iff c_0 and c_1 are positive. Note the crucial difference between the second equations of (3.5) and (3.8)—the destabilizing negative sign in the former and the stabilizing positive sign in the latter. The gain kernel $k(x,y)$ is given by the following PDE:

$$\left\{\begin{aligned} k_{xx}(x,y) &= k_{yy}(x,y) + b^2 k(x,y) - b^3\sinh(b(x-y)) \\ &\qquad + b^3\int_y^x k(x,\xi)\sinh(b(\xi-y))d\xi, \\ k(x,x) &= -\frac{b^2}{2}x - c_0 - q, \\ k_y(x,0) &= -b^2\left[\cosh(bx) - \int_0^x k(x,y)\cosh(by)dy\right] - qk(x,0), \end{aligned}\right. \tag{3.9}$$

which is obtained by substituting (3.6) into (3.8) and matching the terms. Incidentally, this equation for $k(x,y)$ is in the same class as the one obtained in the control design for parabolic PDEs [12]. As shown in [12], the PDE (3.9) has a unique solution $k \in C^2(\Omega)$.

It can be solved either numerically or by using the following symbolic recursion:

$$\text{(3.10)}\quad \begin{cases} k(x,y) = \lim_{n\to\infty} k_n(x,y), \\ \quad k_0 = -\frac{b}{2}[-\sinh(b(x-y)) + by\cosh(b(x-y))] - c_0 - q, \\ \quad k_{n+1} = k_0 + b^2 \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \int_0^{\frac{x-y}{2}} k_n(\sigma+s,\sigma-s)\,ds\,d\sigma + q\int_0^{x-y} k_n(\sigma,0)\,d\sigma \\ \qquad + b^2 \int_0^{\frac{x-y}{2}} \int_0^{\sigma} [2k_n(\sigma+s,\sigma-s) - k_n(\sigma,s)\cosh(bs)]\,ds\,d\sigma \\ \qquad + b^3 \int_{\frac{x-y}{2}}^{\frac{x+y}{2}} \int_0^{\frac{x-y}{2}} \int_{\sigma-s}^{\sigma+s} k_n(\sigma+s,\xi)\sinh(b(\xi-\sigma+s))\,d\xi\,ds\,d\sigma \\ \qquad + 2b^3 \int_0^{\frac{x-y}{2}} \int_0^{\sigma} \int_{\sigma-s}^{\sigma+s} k_n(\sigma+s,\xi)\sinh(b(\xi-\sigma+s))\,d\xi\,ds\,d\sigma. \end{cases}$$

The first step of this recursion provides approximate control gain kernels, which are explicit:

$$\text{(3.11)}\quad \begin{cases} k_0(1,y) = -\frac{b}{2}[-\sinh(b(1-y)) + by\cosh(b(1-y))] - c_0 - q, \\ k_{0x}(1,y) = -\frac{b}{2}[-\cosh(b(1-y)) + by\sinh(b(1-y))], \\ k_0(1,1) = k(1,1) = -\frac{b^2}{2} - c_0 - q. \end{cases}$$

Since $k \in C^2(\Omega)$, the transformation (3.6) is bounded invertible, and therefore the system (3.5) with the controller (3.7) dynamically behaves as (3.8). The important question is why we chose our system's "target" behavior as in (3.8). This PDE with a homogeneous Dirichlet boundary condition at $x = 0$ (i.e., for $c_0 = \infty$) has been studied in many papers on control of wave equations by "boundary damper" feedback. For a large positive c_0, our "target" system has a similar behavior to those well-studied problems. Obviously, the most desirable behavior would be with $c_0 = 0$; however, such behavior is achievable only if one could put an actuator at the tip. In that case, the end $x = 0$ would be clamped, and the end $x = 1$ would be actuated with a "boundary damper." Since we are pursuing the opposite problem, where the tip end $x = 0$ is free and the actuator is at the opposite end $x = 1$, it is only through the very sophisticated construction that we presented above that a behavior similar to the boundary damper feedback is achievable. The plant boundary condition at $x = 0$ is of Robin type, and no state transformation can change it into Dirichlet. However, we can change it into a Robin condition of favorable sign ($c_0 > 0$) and make it behave similar to a Dirichlet condition (with large c_0). To achieve all of this, we construct the change of variable (3.6) which starts at $x = 0$ and goes towards $x = 1$, collecting all of the terms in the shear beam model and converting them into a wave equation model. But it is ultimately the boundary feedback (3.7) that absorbs the effects of the transformation and results in a damping boundary condition at the end $x = 1$. Clearly such feedback has to be rather complicated because it achieves a similar effect as the boundary damper but from the opposite end. In addition to the first two terms on the right-hand side of (3.7), which arise in boundary dampers and essentially amount to PD control, our feedback law incorporates the two integral operators acting on the displacement and velocity fields (as we will show in the next section, the direct measurement of $w(x,t)$ and $w_t(x,t)$ along the whole beam is not necessary).

The control law (3.7) has to be implemented by solving for $w(1,t)$. In the frequency domain it is equivalent to employing a low pass filter acting on $w_x(1,s)$ and the integral operator:

$$u_1(s) = \frac{1}{c_1 s + \frac{b^2}{2} + c_0 + q}\left[-w_x(1,s) + \int_0^1 (c_1 s k(1,y) + k_x(1,y)) w(y,s)\, dy\right]. \tag{3.12}$$

In AFM u_1 is implemented via a piezo actuator which actuates the beam base displacement. Implementation of u_2 would involve two piezo actuators to produce a commanded u_2.

4. Observer. Before we start with the observer design, we write the model (3.3) in a slightly different form using (3.2):

$$\begin{cases} \delta w_{tt}(x,t) = w_{xx}(x,t) + b^2 w(x,t) + b^3 \displaystyle\int_0^x \sinh(b(x-y)) w(y,t) dy \\ \qquad\qquad - b^2 \cosh(bx) w(0,t) - b \sinh(bx) \alpha(0,t), \\ w_x(0,t) = \alpha(0,t) - q w(0,t), \\ w(1,t) = u_1(t). \end{cases} \tag{4.1}$$

Note that in this form u_2 is not selected as in (3.4) and the observer is designed for arbitrary inputs u_1 and u_2.

We assume that the only available measurements are of the tip displacement $w(0,t)$ and of the tip angle due to bending $\alpha(0,t)$. In AFM the displacement and the slope of the tip are routinely measured using a laser and a photodiode.

The observer is designed along the lines of the design presented in [13] for parabolic systems and follows a standard finite-dimensional approach "copy of the plant plus output injection terms":

$$\begin{cases} \delta \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t) + b^2 \widehat{w}(x,t) + b^3 \displaystyle\int_0^x \sinh(b(x-y)) \widehat{w}(y,t) dy \\ \qquad\qquad - b^2 \cosh(bx) w(0,t) - b \sinh(bx) \alpha(0,t) \\ \qquad\qquad + p_y(x,0)[w(0,t) - \widehat{w}(0,t)] - c_2 p(x,0)[w_t(0,t) - \widehat{w}_t(0,t)], \\ \widehat{w}_x(0,t) = \alpha(0,t) - q w(0,t) + p(0,0)[w(0,t) - \widehat{w}(0,t)] \\ \qquad\qquad - c_2 [w_t(0,t) - \widehat{w}_t(0,t)], \\ \widehat{w}(1,t) = u_1(t). \end{cases} \tag{4.2}$$

The constant $c_2 > 0$ is the design parameter that sets the convergence rate of the observer. Note that the output error terms are injected both in the domain and in the boundary condition. The observer gains $p(x,0)$, $p_y(x,0)$, and $p(0,0)$ in (4.2) are determined by solving the following PDE in $\Omega = \{(x,y) |\, 0 \le y \le x \le 1\}$:

$$\begin{cases} p_{yy}(x,y) = p_{xx}(x,y) + b^2 p(x,y) - b^3 \sinh(b(x-y)) \\ \qquad\qquad + b^3 \displaystyle\int_y^x p(\xi,y) \sinh(b(x-\xi)) d\xi, \\ p(x,x) = \dfrac{b^2}{2}(x-1), \\ p(1,y) = 0. \end{cases} \tag{4.3}$$

It has been shown in [13] that this equation has a unique solution $p \in C^2(\Omega)$. One can see the similarity between this PDE and the one for the control kernel. This is due to the duality between observer and control designs, a concept well known in finite-dimensional control. One can think of the gains $p(x,0)$, $p_y(x,0)$, and $p(0,0)$ as dual counterparts to the control gains $k(1,y)$, $k_x(1,y)$, and $k(1,1)$.

(4.3) can be solved numerically or symbolically using the following recursive procedure [13]:

$$\left\{\begin{aligned} p(x,y) &= \lim_{n\to\infty} p_n(x,y),\\ p_0 &= -\frac{b}{2}[-\sinh(b(x-y)) + b(1-x)\cosh(b(x-y))],\\ p_{n+1} &= p_0 + b^2 \int_{\frac{x-y}{2}}^{\frac{2-x-y}{2}} \int_0^{\frac{x-y}{2}} p_n(\sigma+s,\sigma-s)ds d\sigma\\ &\quad + 2b^2 \int_0^{\frac{x-y}{2}} \int_0^{\sigma} p_n(\sigma+s,\sigma-s) ds d\sigma\\ &\quad + b^3 \int_{\frac{x-y}{2}}^{\frac{2-x-y}{2}} \int_0^{\frac{x-y}{2}} \int_{\sigma-s}^{\sigma+s} p_n(\sigma+s,\xi)\sinh(b(\xi-\sigma+s)) d\xi ds d\sigma\\ &\quad + 2b^3 \int_0^{\frac{x-y}{2}} \int_0^{\sigma} \int_{\sigma-s}^{\sigma+s} p_n(\sigma+s,\xi)\sinh(b(\xi-\sigma+s)) d\xi ds d\sigma. \end{aligned}\right. \tag{4.4}$$

The observer gain in the boundary condition (4.2) is known exactly:

$$p(0,0) = -\frac{b^2}{2}. \tag{4.5}$$

Let us denote the observer error by $\varepsilon(x,t) = w(x,t) - \widehat{w}(x,t)$. Using (4.2) and (4.1) we obtain the observer error dynamics

$$\left\{\begin{aligned} \delta\varepsilon_{tt}(x,t) &= \varepsilon_{xx}(x,t) + b^2\varepsilon(x,t) + b^3\int_0^x \sinh(b(x-y))\varepsilon(y,t)dy\\ &\quad - p_y(x,0)\varepsilon(0,t) + c_2 p(x,0)\varepsilon_t(0,t),\\ \varepsilon_x(0,t) &= -p(0,0)\varepsilon(0,t) + c_2\varepsilon_t(0,t),\\ \varepsilon(1,t) &= 0. \end{aligned}\right. \tag{4.6}$$

The convergence of the observer is established by the following lemma.

LEMMA 4.1. *Suppose the classical solution of* (4.6) *exists. Then the invertible transformation*

$$\left\{\begin{aligned} \varepsilon(x,t) &= \widetilde{\varepsilon}(x,t) - \int_0^x p(x,y)\widetilde{\varepsilon}(y,t)dy = [(I-\mathbb{P}_1)\widetilde{\varepsilon}](x,t),\\ \widetilde{\varepsilon}(x,t) &= [(I-\mathbb{P}_1)^{-1}\varepsilon](x,t) = \varepsilon(x,t) - \int_0^x p^{\ominus}(x,y)\varepsilon(y,t)dy \end{aligned}\right. \tag{4.7}$$

converts the error system (4.6) *into the exponentially stable system*

$$\left\{\begin{aligned} \delta\widetilde{\varepsilon}_{tt}(x,t) &= \widetilde{\varepsilon}_{xx}(x,t),\\ \widetilde{\varepsilon}_x(0,t) &= c_2\widetilde{\varepsilon}_t(0,t),\\ \widetilde{\varepsilon}(1,t) &= 0. \end{aligned}\right. \tag{4.8}$$

Proof. We differentiate the transformation (4.7) with respect to t and x:

$$\begin{aligned}\delta\varepsilon_{tt}(x,t) &= \delta\widetilde{\varepsilon}_{tt}(x,t) - \int_0^x p(x,y)\delta\widetilde{\varepsilon}_{tt}(y,t)dy - \varepsilon_{xx}(x,t) + \varepsilon_{xx}(x,t) \\ &= \delta\widetilde{\varepsilon}_{tt}(x,t) - \int_0^x p_{yy}(x,y)\widetilde{\varepsilon}(y,t)dy - p(x,x)\widetilde{\varepsilon}_x(x,t) + p(x,0)\widetilde{\varepsilon}_x(0,t) \\ &\quad + p_y(x,x)\widetilde{\varepsilon}(x,t) - p_y(x,0)\widetilde{\varepsilon}(0,t) - \widetilde{\varepsilon}_{xx}(x,t) + [2p_x(x,x) + p_y(x,x)]\widetilde{\varepsilon}(x,t) \\ &\quad + p(x,x)\widetilde{\varepsilon}_x(x,t) + \int_0^x p_{xx}(x,y)\widetilde{\varepsilon}(y,t)\,dy + \varepsilon_{xx}(x,t) \\ &= \varepsilon_{xx}(x,t) + b^2\varepsilon(x,t) + c_2 p(x,0)\varepsilon_t(0,t) - p_y(x,0)\varepsilon(0,t) \\ &\quad + \int_0^x (p_{xx}(x,y) - p_{yy}(x,y) + b^2 p(x,y))\widetilde{\varepsilon}(y,t)\,dy.\end{aligned}$$

Using the observer gain PDE (4.3) in the above equation we get the governing equation of (4.8).

Next we differentiate the transformation (4.7) with respect to x and set $x = 0$:

$$\varepsilon_x(0,t) = \tilde{\varepsilon}_x(0) - p(0,0)\tilde{\varepsilon}(0,t)\,.$$

Comparing this with the boundary condition of (4.6), which can be written as

$$\varepsilon_x(0,t) = -p(0,0)\widetilde{\varepsilon}(0,t) + c_2\widetilde{\varepsilon}_t(0,t),$$

we get the boundary condition of (4.8) at $x = 0$. Finally, the boundary condition at $x = 1$ is obviously satisfied because $p(1,y) = 0$. ◻

5. Output feedback. Consider the observer (4.2) and the control (3.4), (3.7) with the observer state instead of the unmeasured plant state:

$$\text{(5.1)}\qquad \begin{cases} u_1(t) = \widehat{w}(1,t), \\ \widehat{w}_x(1,t) = k(1,1)\widehat{w}(1,t) - c_1\widehat{w}_t(1,t) \\ \qquad\qquad + c_1\displaystyle\int_0^1 k(1,y)\widehat{w}_t(y,t)dy + \int_0^1 k_x(1,y)\widehat{w}(y,t)dy, \\ u_2(t) = -b\displaystyle\int_0^1 \sinh(b(1-y))\widehat{w}_x(y,t)\,dy. \end{cases}$$

We employ an invertible state transformation

$$\text{(5.2)}\qquad \begin{cases} \widetilde{w}(x,t) = \widehat{w}(x,t) - \displaystyle\int_0^x k(x,y)\widehat{w}(y,t)dy = [(I-\mathbb{P}_2)\widehat{w}](x,y), \\ \widehat{w}(x,t) = [(I-\mathbb{P}_2)^{-1}\widetilde{w}](x,y) = \widetilde{w}(x,t) - \displaystyle\int_0^x k^{\ominus}(x,y)\widetilde{w}(x,y)dy, \end{cases}$$

where $k(x,y)$ is given by (3.10) and both $k(x,y)$ and $k^{\ominus}(x,y)$ are of C^2 in Ω [12].

LEMMA 5.1. *Suppose the classical solution of* (4.2) *with the control* (5.1) *exists. Then the transformation* (5.2) *converts* (4.2) *and* (5.1) *into*

$$(5.3)\quad \begin{cases} \delta\widetilde{w}_{tt}(x,t) = \widetilde{w}_{xx}(x,t) - b\sinh(bx)\alpha(0,t) + k(x,0)\alpha(0,t) + p_y(x,0)\varepsilon(0,t) \\ \qquad + k(x,0)p(0,0)\varepsilon(0,t) + k_y(x,0)\varepsilon(0,t) - c_2k(x,0)\varepsilon_t(0,t) \\ \qquad + b\displaystyle\int_0^x k(x,y)\sinh(by)dy\,\alpha(0,t) - \int_0^x k(x,y)p_y(y,0)dy\,\varepsilon(0,t) \\ \qquad - c_2p(x,0)\varepsilon_t(0,t) + c_2\displaystyle\int_0^x k(x,y)p(y,0)dy\,\varepsilon_t(0,t), \\ \widetilde{w}_x(0,t) = c_0\widetilde{w}(0,t) + \alpha(0,t) + p(0,0)\varepsilon(0,t) - c_2\varepsilon_t(0,t), \\ \widetilde{w}_x(1,t) = -c_1\widetilde{w}_t(1,t). \end{cases}$$

Proof. First we compute the second spatial derivative of the transformation (5.2):

$$(5.4)\quad \begin{aligned} \widetilde{w}_{xx}(x,t) &= \widehat{w}_{xx}(x,t) - [2k_x(x,x) + k_y(x,x)]\widehat{w}(x,t) - k(x,x)\widehat{w}_x(x,t) \\ &\quad - \int_0^x k_{xx}(x,y)\widehat{w}(y,t)dy. \end{aligned}$$

The next step is to compute $\tilde{w}_{tt}$:

$$(5.5)\quad \begin{aligned} \delta\widetilde{w}_{tt}(x,t) &= \delta\widehat{w}_{tt}(x,t) - \int_0^x k(x,y)\delta\widehat{w}_{tt}(y,t)dy \\ &= \widehat{w}_{xx}(x,t) + b^2\widehat{w}(x,t) + b^3\int_0^x \sinh(b(x-y))\widehat{w}(y,t)dy \\ &\quad - b^2\cosh(bx)w(0,t) - b\sinh(bx)\alpha(0,t) + p_y(x,0)\varepsilon(0,t) \\ &\quad - c_2p(x,0)\varepsilon_t(0,t) - \int_0^x k(x,y)\widehat{w}_{yy}(y,t)dy - b^2\int_0^x k(x,y)\widehat{w}(y,t)dy \\ &\quad - b^3\int_0^x\int_y^x k(x,\xi)\sinh(b(\xi-y))\widehat{w}(y,t)dy \\ &\quad + b^2\int_0^x k(x,y)\cosh(by)dyw(0,t) + b\int_0^x k(x,y)\sinh(by)dy\,\alpha(0,t) \\ &\quad - \int_0^x k(x,y)p_y(y,0)dy\varepsilon(0,t) + c_2\int_0^x k(x,y)p(y,0)dy\varepsilon_t(0,t). \end{aligned}$$

We notice that

$$(5.6)\quad \begin{aligned} \int_0^x k(x,y)\widehat{w}_{yy}(y,t)dy &= \int_0^x k_{yy}(x,y)\widehat{w}(y,t)dy \\ &\quad + k(x,x)\widehat{w}_x(x,t) - k(x,0)\widehat{w}_x(0,t) \\ &\quad - k_y(x,x)\widehat{w}(x,t) + k_y(x,0)\widehat{w}(0,t) \\ &= \int_0^x k_{yy}(x,y)\widehat{w}(y,t)dy + k(x,x)\widehat{w}_x(x,t) - k(x,0)\alpha(0,t) \\ &\quad + qk(x,0)w(0,t) + k(x,0)p(0,0)\varepsilon(0,t) + c_2k(x,0)\varepsilon_t(0,t) \\ &\quad - k_y(x,x)\widehat{w}(x,t) - k_y(x,0)\varepsilon(0,t) + k_y(x,0)w(0,t). \end{aligned}$$

Substracting (5.4) from (5.5) and using (5.6), we get (5.3).

The boundary condition at $x = 1$ is verified in the following way:

$$\begin{aligned}0 &= \widetilde{w}_x(1,t) + c_1\widetilde{w}_t(1,t)\\ &= \widehat{w}_x(1,t) - k(1,1)\widehat{w}(1,t) - \int_0^1 k_x(1,y)\widehat{w}(y,t)\\ &\quad + c_1\widehat{w}_t(1,t) - c_1\int_0^1 k(1,y)\widehat{w}_t(y,t)dy,\end{aligned}$$

which gives exactly the controller (5.1). Finally, for the boundary condition at $x = 0$ we have

$$\begin{aligned}\widetilde{w}_x(0,t) &= \widehat{w}_x(0,t) - k(0,0)\widehat{w}(0,t) = \widehat{w}_x(0,t) - k(0,0)\widetilde{w}(0,t)\\ &= \alpha(0,t) - qw(0,t) + p(0,0)\varepsilon(0,t) - c_2\varepsilon_t(0,t) - k(0,0)\widetilde{w}(0,t)\\ &= c_0\widetilde{w}(0,t) + \alpha(0,t) + p(0,0)\varepsilon(0,t) - c_2\varepsilon_t(0,t).\end{aligned}$$

The proof is complete. ∎

6. Well-posedness and stability of the transformed system. Lemmas 4.1 and 5.1 establish the following transformed system $(\tilde{\varepsilon}, \tilde{w})$ which is a cascade of two wave equations (with additional integral terms):

$$\text{(6.1)}\quad \begin{cases}\delta\widetilde{\varepsilon}_{tt}(x,t) = \widetilde{\varepsilon}_{xx}(x,t),\\ \widetilde{\varepsilon}_x(0,t) = c_2\widetilde{\varepsilon}_t(0,t),\\ \widetilde{\varepsilon}(1,t) = 0,\\ \delta\widetilde{w}_{tt}(x,t) = \widetilde{w}_{xx}(x,t) - b\sinh(bx)\alpha(0,t) + k(x,0)\alpha(0,t) - c_2k(x,0)\widetilde{\varepsilon}_t(0,t)\\ \qquad + p_y(x,0)\widetilde{\varepsilon}(0,t) + k(x,0)p(0,0)\widetilde{\varepsilon}(0,t) + k_y(x,0)\widetilde{\varepsilon}(0,t)\\ \qquad + b\int_0^x k(x,y)\sinh(by)\,dy\alpha(0,t) - \int_0^x k(x,y)p_y(y,0)\,dy\widetilde{\varepsilon}(0,t)\\ \qquad - c_2p(x,0)\widetilde{\varepsilon}_t(0,t) + c_2\int_0^x k(x,y)p(y,0)dy\widetilde{\varepsilon}_t(0,t),\\ \widetilde{w}_x(0,t) = c_0\widetilde{w}(0,t) + \alpha(0,t) + p(0,0)\widetilde{\varepsilon}_t(0,t) - c_2\widetilde{\varepsilon}_t(0,t),\\ \widetilde{w}_x(1,t) = -c_1\widetilde{w}_t(1,t).\end{cases}$$

Here $\alpha(0,t)$ is expressed in terms of $\tilde{\varepsilon}$ using (3.2) and (4.7):

$$\text{(6.2)}\quad \begin{aligned}\alpha(0,t) = &\frac{b}{\cosh(b)}\int_0^1 \sinh(b(1-x))[\widetilde{\varepsilon}_x(x,t) - p(x,x)\widetilde{\varepsilon}(x,t)]\,dx\\ &- \frac{b}{\cosh(b)}\int_0^1\int_x^1 \sinh(b(1-y))p_x(y,x)\,dy\,\widetilde{\varepsilon}(x,t)\,dx.\end{aligned}$$

From (6.2) and the fact that $\widetilde{\varepsilon}(1,t) = 0$, we know that there exists a constant $C_1 > 0$ such that

$$\text{(6.3)}\quad |\alpha(0,t)|^2 \le C_1\int_0^1 \widetilde{\varepsilon}_x^2(x,t)\,dx.$$

We consider the system (6.1) in the space $H = H^1_R(0,1)\times L^2(0,1)\times H^1(0,1)\times L^2(0,1)$,

$H_R^1(0,1) = \{f|\ f \in H^1(0,1)|\ f(1) = 0\}$, with the inner product

$$\begin{aligned}&\langle (f_1, g_1, \phi_1, \psi_1), (f_2, g_2, \phi_2, \psi_2)\rangle \\ &\quad = K\int_0^1 \left[f_1'(x)\overline{f_2'(x)} + \frac{1}{\delta}g_1(x)\overline{g_2(x)} + \delta_0(x-1)(f_1'(x)\overline{g_2(x)} + g_1(x)\overline{f_2'(x)})\right]dx \\ &\quad + \int_0^1 \left[\phi_1'(x)\overline{\phi_2'(x)} + \frac{1}{\delta}\psi_1(x)\overline{\psi_2(x)} + \delta_0(x+1)(\phi_1'(x)\overline{\psi_2(x)} + \psi_1(x)\overline{\phi_2'(x)})\right]dx \\ &\quad + c_1\phi_1(0)\overline{\phi_2(0)} \qquad \forall\, (f, g, \phi, \psi) \in H,\end{aligned}$$

where $\delta_0 > 0$ is sufficiently small so that above inner product is well-defined and $K > 0$ is large enough so that A is dissipative in H as in the proof of Lemma 5.1 below. Define the system operator $A : D(A)(\subset H) \to H$ as follows:

$$\text{(6.4)}\quad \begin{cases} D(A) = \Big\{(f, g, \phi, \psi) \in (H^2(0,1) \cap H_R^1(0,1)) \times H_R^1(0,1) \times H^2(0,1) \\ \qquad \times\ H^1(0,1)|\ f'(0) = \dfrac{c_2}{\delta}g(0),\ \phi'(1) = -\dfrac{c_1}{\delta}\psi(1) \\ \qquad\qquad \phi'(0) = c_0\phi(0) + \dfrac{1}{\delta}[p(0,0) - c_2]g(0) + \alpha(0)\Big\}, \\ A(f, g, \phi, \psi) = \Big(\dfrac{g}{\delta},\ f'',\ \dfrac{\psi}{\delta}, \phi'' - b\sinh(bx)\alpha(0) + [p_y(x,0) + k_y(x,0)]f(0) \\ \qquad +\ k(x,0)[\alpha(0) + p(0,0)f(0)] - \displaystyle\int_0^x k(x,y)p_y(y,0)dyf(0) \\ \qquad + b\displaystyle\int_0^x k(x,y)\sinh(by)dy\,\alpha(0) - \dfrac{c_2}{\delta}k(x,0)g(0) \\ \qquad + \dfrac{c_2}{\delta}\left[-p(x,0) + \displaystyle\int_0^x k(x,y)p(y,0)dy\right]g(0)\Big), \\ \alpha(0) = \dfrac{b}{\cosh(b)}\displaystyle\int_0^1 \sinh(b(1-x))[f'(x) - p(x,x)f(x)]\,dx \\ \qquad - \dfrac{b}{\cosh(b)}\displaystyle\int_0^1\int_x^1 \sinh(b(1-y))p_x(y,x)\,dyf(x)\,dx \\ \qquad \forall\, (f, g, \phi, \psi) \in D(A). \end{cases}$$

Then the system (6.1) can be written as

$$\text{(6.5)}\qquad \frac{d}{dt}(\widetilde{\varepsilon}(\cdot,t), \delta\widetilde{\varepsilon}_t(\cdot,t), \widetilde{w}(\cdot,t), \delta\widetilde{w}_t(\cdot,t)) = A(\widetilde{\varepsilon}(\cdot,t), \delta\widetilde{\varepsilon}_t(\cdot,t), \widetilde{w}(\cdot,t), \delta\widetilde{w}_t(\cdot,t)).$$

THEOREM 6.1. *Let A be defined by* (6.4). *Then A generates an exponential stable C_0-semigroup on H. For any initial value $(\widetilde{\varepsilon}(\cdot,0), \delta\widetilde{\varepsilon}_t(\cdot,0), \widetilde{w}(\cdot,0), \delta\widetilde{w}_t(\cdot,0)) \in H$, there exists a unique (mild) solution to* (6.1) *such that $(\widetilde{\varepsilon}(\cdot,t), \delta\widetilde{\varepsilon}_t(\cdot,t), \widetilde{w}(\cdot,t), \delta\widetilde{w}_t(\cdot,t)) \in C([0,\infty); H)$, and there exists a positive constant ω such that*

$$\text{(6.6)}\qquad \begin{aligned}&\|(\widetilde{\varepsilon}(\cdot,t), \delta\widetilde{\varepsilon}_t(\cdot,t), \widetilde{w}(\cdot,t), \delta\widetilde{w}_t(\cdot,t))\|_H \\ &\qquad \le e^{-\omega t}\|(\widetilde{\varepsilon}(\cdot,0), \delta\widetilde{\varepsilon}_t(\cdot,0), \widetilde{w}(\cdot,0), \delta\widetilde{w}_t(\cdot,0))\|_H.\end{aligned}$$

Moreover, if $(\widetilde{\varepsilon}(\cdot,0), \delta\widetilde{\varepsilon}_t(\cdot,0), \widetilde{w}(\cdot,0), \delta\widetilde{w}_t(\cdot,0)) \in D(A)$, then

$$\text{(6.7)}\qquad (\widetilde{\varepsilon}(\cdot,t), \delta\widetilde{\varepsilon}_t(\cdot,t), \widetilde{w}(\cdot,t), \delta\widetilde{w}_t(\cdot,t)) \in C^1([0,\infty); H)$$

is the classical solution of (6.1).

Proof. Define the Lyapunov functions

$$E_{\widetilde{\varepsilon}}(t) = \frac{1}{2}\int_0^1 [\widetilde{\varepsilon}_x^2(x,t) + \delta\widetilde{\varepsilon}_t^2(x,t)]dx + \delta_0 \int_0^1 (x-1)\widetilde{\varepsilon}_x(x,t)\delta\widetilde{\varepsilon}_t(x,t)dx \tag{6.8}$$

and

$$\begin{aligned} E_{\widetilde{w}}(t) = &\frac{1}{2}\int_0^1 [\widetilde{w}_x^2(x,t) + \delta\widetilde{w}_t^2(x,t)]dx + \frac{\widetilde{\delta}_0}{2}\widetilde{w}^2(0,t) \\ &+ \delta_0 \int_0^1 (1+x)\widetilde{w}_x(x,t)\delta\widetilde{w}_t(x,t)dx. \end{aligned} \tag{6.9}$$

Both of them are positive definite for small $\delta_0, \widetilde{\delta}_0 > 0$. The time derivatives of $E_{\widetilde{\varepsilon}}$ and $E_{\widetilde{w}}$ along the trajectory of (6.1) are, respectively,

$$\dot{E}_{\widetilde{\varepsilon}}(t) = -\left[c_2 - \frac{\delta_0}{2}(1+c_2^2)\right]\widetilde{\varepsilon}_t^2(0,t) - \frac{\delta_0}{2}\int_0^1 [\widetilde{\varepsilon}_x^2(x,t) + \delta\widetilde{\varepsilon}_t^2(x,t)]dx, \tag{6.10}$$

(6.11)

$$\begin{aligned} \dot{E}_{\widetilde{w}}(t) = &-c_1\widetilde{w}_t^2(1,t) - c_0\widetilde{w}^2(0,t) - \alpha(0,t)\widetilde{w}_t(0,t) - p(0,0)\widetilde{w}_t(0,t)\widetilde{\varepsilon}_t(0,t) \\ &+ c_2\widetilde{w}_t(0,t)\widetilde{\varepsilon}_t(0,t) + \int_0^1 [k(x,0) - b\sinh(bx)]\widetilde{w}_t(x,t)dx\,\alpha(0,t) \\ &+ \int_0^1 \widetilde{w}_t(x,t)p_y(x,0)dx\,\widetilde{\varepsilon}(0,t) + b\int_0^1 \widetilde{w}_t(x,t)dx\int_0^x k(x,y)\sinh(by)dy\,\alpha(0,t) \\ &+ \int_0^1 \widetilde{w}_t(x,t)k(x,0)dx\,p(0,0)\widetilde{\varepsilon}(0,t) + \int_0^1 \widetilde{w}_t(x,t)k_y(x,0)dx\,\widetilde{\varepsilon}(0,t) \\ &- \int_0^1 \widetilde{w}_t(x,t)dx\int_0^x k(x,y)p_y(y,0)dy\,\widetilde{\varepsilon}(0,t) - c_2\int_0^1 \widetilde{w}_t(x,t)k(x,0)dx\,\widetilde{\varepsilon}_t(0,t) \\ &- c_2\int_0^1 \widetilde{w}_t(x,t)p(x,0)dx\,\widetilde{\varepsilon}_t(0,t) + c_2\int_0^1 \widetilde{w}_t(x,t)dx\int_0^x k(x,y)p(y,0)dy\,\widetilde{\varepsilon}_t(0,t) \\ &- \frac{\delta_0}{2}\int_0^1 [\widetilde{w}_x^2(x,t) + \delta\widetilde{w}_t^2(x,t)]dx - \frac{\delta\delta_0}{2}\widetilde{w}_t^2(0,t) + \delta_0(\delta + c_1^2)\widetilde{w}_t^2(1,t) \\ &- \frac{\delta_0}{2}\left[c_0\widetilde{w}(0,t) + \alpha(0,t) + p(0,0)\widetilde{\varepsilon}_t(0,t) - c_2\widetilde{\varepsilon}_t(0,t)\right]^2 \\ &+ \delta_0\int_0^1 (1+x)\widetilde{w}_x(x,t)(k(x,0) - b\sinh(bx))dx\,\alpha(0,t) \\ &+ \delta_0\int_0^1 (1+x)\widetilde{w}_x(x,t)(p_y(x,0) + k_y(x,0) + p(0,0)k(x,0))dx\,\widetilde{\varepsilon}(0,t) \\ &+ b\delta_0\int_0^1 (1+x)\widetilde{w}_x(x,t)dx\int_0^x k(x,y)\sinh(by)dy\,\alpha(0,t) \\ &- \delta_0\int_0^1 (1+x)\widetilde{w}_x(x,t)dx\int_0^x k(x,y)p_y(y,0)dy\,\widetilde{\varepsilon}(0,t) \\ &- c_2\delta_0\int_0^1 (1+x)\widetilde{w}_x(x,t)(p(x,0) + k(x,0))dx\,\widetilde{\varepsilon}_t(0,t) \\ &+ c_2\delta_0\int_0^1 (1+x)\widetilde{w}_x(x,t)dx\int_0^x k(x,y)p(y,0)dy\,\widetilde{\varepsilon}_t(0,t) + \widetilde{\delta}_0\widetilde{w}(0,t)\widetilde{w}_t(0,t). \end{aligned}$$

Using (6.3) and the fact that $k, p \in C^2(\Omega)$, we obtain

$$\dot{E}_{\widetilde{w}}(t) \leq \left[\frac{\delta_0}{2} - \delta_1\right] \int_0^1 [\widetilde{w}_x^2(x,t) + \delta \widetilde{w}_t^2(x,t)]dx - \left[c_0 + \frac{\delta c_0^2}{2} - \delta_2\right] \widetilde{w}^2(0,t)$$
$$- \left[c_1 - \delta_0(\delta + c_1^2)\right] \widetilde{w}_t^2(1,t) - \left[\frac{\delta\delta_0}{2} - \delta_3\right] \widetilde{w}_t^2(0,t)$$
$$+ C_2\alpha^2(0,t) + C_2\widetilde{\varepsilon}^2(0,t) + C_2\widetilde{\varepsilon}_t^2(0,t),$$

where C_2 and $\delta_i, i = 1,2,3$, are some positive constants satisfying

$$\delta_1 < \frac{\delta_0}{2},\ \delta_2 < c_0 + \frac{\delta c_0^2}{2},\ \delta_3 < \frac{\delta\delta_0}{2}. \tag{6.12}$$

Now for large $K > 0$, we take the overall Laypunov function as

$$E(t) = E_{\widetilde{w}}(t) + K E_{\widetilde{\varepsilon}}(t). \tag{6.13}$$

Since from (6.3), $\alpha^2(0,t) + \widetilde{\varepsilon}^2(0,t) \leq (1 + C_1)\|\widetilde{\varepsilon}_x^2(\cdot,t)\|^2_{L^2(0,1)}$, we obtain its derivative along the solution of (6.1) that

$$\begin{aligned}\dot{E}(t) \leq& -K\left[c_2 - \frac{\delta_0}{2}(1 + c_2^2)\right]\widetilde{\varepsilon}_t^2(0,t) - K\frac{\delta_0}{2}\int_0^1[\widetilde{\varepsilon}_x^2(x,t) + \delta\widetilde{\varepsilon}_t^2(x,t)]dx\\ &- \left[\frac{\delta_0}{2} - \delta_1\right]\int_0^1[\widetilde{w}_x^2(x,t) + \delta\widetilde{w}_t^2(x,t)]dx - \left[c_0 + \frac{\delta c_0^2}{2} - \delta_2\right]\widetilde{w}^2(0,t)\\ &+ C_2\alpha^2(0,t) + C_2\widetilde{\varepsilon}^2(0,t) + C_2\widetilde{\varepsilon}_t^2(0,t),\end{aligned} \tag{6.14}$$

where we assumed that

$$\delta_0(\delta + c_1^2) < c_1. \tag{6.15}$$

Hence

$$\begin{aligned}\dot{E}(t) \leq& -\left[K\left(c_2 - \frac{\delta_0}{2}(1 + c_2^2)\right) - C_2\right]\widetilde{\varepsilon}_t^2(0,t)\\ &- \left[K\frac{\delta_0}{2} - C_2(1 + C_1)\right]\int_0^1[\widetilde{\varepsilon}_x^2(x,t) + \delta\widetilde{\varepsilon}_t^2(x,t)]dx\\ &- \left[\frac{\delta_0}{2} - \delta_1\right]\int_0^1[\widetilde{w}_x^2(x,t) + \delta\widetilde{w}_t^2(x,t)]dx - \left[c_0 + \frac{\delta c_0^2}{2} - \delta_2\right]\widetilde{w}^2(0,t).\end{aligned} \tag{6.16}$$

Choosing $K > 0$ sufficiently large, it follows from (6.16) that there exists an $\omega > 0$ such that

$$\dot{E}(t) \leq -\omega E(t). \tag{6.17}$$

The above procedure also gives the following estimate:

$$\mathrm{Re}\langle A(f,g,\phi,\psi),(f,g,\phi,\psi)\rangle_H \leq -\omega\|(f,g,\phi,\psi)\|_H^2 \quad \forall\,(f,g,\phi,\psi) \in D(A).$$

So A is dissipative in H ([10]), and if A generates a C_0-semigroup, this semigroup must be exponentially stable. By the Lumer–Phillips theorem (Theorem 4.3, p. 14 in [10]), the proof will be accomplished if we can show that A^{-1} exists and is bounded on H. Actually, a simple computation shows that

$$A^{-1}(f,g,\phi,\psi) = (f^*,g^*,\phi^*,\psi^*) \quad \forall\,(f,g,\phi,\psi) \in H,$$

where $g^* = \delta f, \psi^* = \delta\phi$ and

$$
\begin{aligned}
f^*(x) &= c_2 f(0)(x-1) + \int_0^x (x-\tau)g(\tau)d\tau - \int_0^1 (1-\tau)g(\tau)d\tau,\\
\phi^*(x) &= \int_1^x (x-\tau)\psi(\tau)d\tau + \int_1^x (x-\tau)F(\tau)d\tau - c_1\phi(1)x\\
&\quad - \int_0^1 \tau\psi(\tau)d\tau - \int_0^1 \tau F(\tau)d\tau + \phi^*(0),\\
\phi^*(0) &= -\frac{1}{c_0}\int_0^1 \psi(\tau)d\tau - \frac{1}{c_0}\int_0^1 F(\tau)d\tau - \frac{c_1}{c_0}\phi(1) - \frac{\alpha^*(0)}{c_0} - \frac{p(0,0)-c_2}{c_0}f(0),\\
F(x) &= b\sinh(bx)\alpha^*(0) - k(x,0)\alpha^*(0) - [p_y(x,0) + k(x,0)p(0,0) + k_y(x,0)]f^*(0)\\
&\quad - b\int_0^x k(x,y)\sinh(by)dy\alpha^*(0) + \int_0^x k(x,y)p_y(y,0)dyf^*(0)\\
&\quad + \left[c_2 k(x,0) + c_2 p(x,0) - c_2\int_0^x k(x,y)p(y,0)dy\right] f(0),\\
\alpha^*(0) &= \frac{b}{\cosh(b)}\int_0^1 \sinh(b(1-x)){f^*}'(x)dx\\
&\quad - \frac{b}{\cosh(b)}\int_0^1 \left[\sinh(b(1-x))p(x,x) + \int_x^1 \sinh(b(1-y))p_x(y,x)dy\right] f^*(x)dx.
\end{aligned}
$$

The proof is complete. ∎

7. Well-posedness and stability of the closed-loop system. The closed-loop system consists of the plant (3.3), the observer (4.2), and the feedback controller (5.1):

$$
\text{(7.1)}\quad
\begin{cases}
\delta w_{tt}(x,t) = w_{xx}(x,t) + b^2 w(x,t) + b^3\int_0^x \sinh(b(x-y))w(y,t)dy\\
\qquad\qquad - b^2\cosh(bx)w(0,t) - b\sinh(bx)\alpha(0,t),\\
w_x(0,t) = \alpha(0,t) - qw(0,t),\\
w(1,t) = \widehat{w}(1,t),\\
\delta\widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t) + b^2\widehat{w}(x,t) + b^3\int_0^x \sinh(b(x-y))\widehat{w}(y,t)dy\\
\qquad\qquad - b^2\cosh(bx)w(0,t) - b\sinh(bx)\alpha(0,t)\\
\qquad\qquad + p_y(x,0)[w(0,t) - \widehat{w}(0,t)] - c_2 p(x,0)[w_t(0,t) - \widehat{w}_t(0,t)],\\
\widehat{w}_x(0,t) = \alpha(0,t) - qw(0,t) + p(0,0)[w(0,t) - \widehat{w}(0,t)]\\
\qquad\qquad - c_2[w_t(0,t) - \widehat{w}_t(0,t)],\\
\widehat{w}_x(1,t) = -c_1\widehat{w}_t(1,t) + k(1,1)\widehat{w}(1,t)\\
\qquad\qquad + c_1\int_0^1 k(1,y)\widehat{w}_t(y,t)dy + \int_0^1 k_x(1,y)\widehat{w}(y,t)dy,\\
\alpha(0,t) = \frac{b}{\cosh(b)}\int_0^1 \sinh(b(1-s))[w_x(x,t) - \widehat{w}_x(x,t)]dx,
\end{cases}
$$

and

$$\alpha(x,t) = \cosh(bx)\alpha(0,t) - b\int_0^x \sinh(b(x-s))w_x(s,t)ds. \tag{7.2}$$

We consider the system (7.1) in the state space $\mathcal{H} = \{(f,g,\phi,\psi) \in (H^1(0,1)\times L^2(0,1))^2 |\ f(1) = \phi(1)\}$. Define the system operator

$$\begin{cases} D(\mathcal{A}) = \Big\{(f,g,\phi,\psi) \in \mathcal{H}|\ \mathcal{A}(f,g,\phi,\psi) \in \mathcal{H}, f'(0) = \alpha(0) - qf(0), \\ \quad \phi'(0) = \alpha(0) - qf(0) + p(0,0)[f(0)-\phi(0)] - \dfrac{c_2}{\delta}[g(0)-\psi(0)], \\ \quad \phi'(1) = k(1,1)\phi(1) - \dfrac{c_1}{\delta}\psi(1) + \dfrac{c_1}{\delta}\displaystyle\int_0^1 k(1,x)\psi(x)dx + \int_0^1 k_x(1,x)\phi(x)\Big\}, \\ [\mathcal{A}(f,g,\phi,\psi)](x) = \Big(\dfrac{g(x)}{\delta}, f''(x) + b^2 f + b^3\displaystyle\int_0^x \sinh(b(x-y))f(y)dy \\ \quad - b^2\cosh(bx)f(0) - b\sinh(bx)\alpha(0), \dfrac{\psi(x)}{\delta}, \phi''(x) + b^2\phi(x) \\ \quad + b^3\displaystyle\int_0^x \sinh(b(x-y))\phi(y)dy - b^2\cosh(bx)f(0) - b\sinh(bx)\alpha(0) \\ \quad + p_y(x,0)[f(0)-\phi(0)] - \dfrac{c_2}{\delta}p(x,0)[g(0)-\psi(0)]\Big), \\ \alpha(0) = \dfrac{b}{\cosh(b)}\displaystyle\int_0^1 \sinh(b(1-s))[f'(x)-\phi'(x)]dx \quad \forall\ (f,g,\phi,\psi) \in D(\mathcal{A}). \end{cases} \tag{7.3}$$

Then the system (7.3) can be written as an evolution equation in $\mathcal{H}$:

$$\frac{d}{dt}(w(\cdot,t), \delta w_t(\cdot,t), \widehat{w}(\cdot,t), \delta\widehat{w}_t(\cdot,t)) = \mathcal{A}(w(\cdot,t), \delta w_t(\cdot,t), \delta\widehat{w}(\cdot,t), \widehat{w}_t(\cdot,t)). \tag{7.4}$$

THEOREM 7.1. *Let $\mathcal{A}$ be defined by* (7.3). *Then $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$, which is exponentially stable:*

$$\|e^{\mathcal{A}t}\|_{\mathcal{H}} \le Me^{-\omega t} \quad \forall\ t \ge 0$$

for some positive constants M and ω independent of t. In particular,

$$E_o(t) \le Ce^{-\omega t}E_o(0) \tag{7.5}$$

for some $C > 0$, where

$$E_o(t) = \int_0^1 \left[w_x^2(x,t) + \delta w_t^2(x,t) + \widehat{w}_x^2(x,t) + \delta\widehat{w}_t^2(x,t) + \alpha^2(x,t)\right]dx. \tag{7.6}$$

Proof. For any initial value $(w(\cdot,0), \delta w_t(\cdot,0), \widehat{w}(\cdot,0), \delta\widehat{w}_t(\cdot,0)) \in D(\mathcal{A})$, let

$$\begin{cases} \widetilde{\varepsilon}(x,0) = [(I-\mathbb{P}_1)^{-1}(w(\cdot,0) - \widehat{w}(\cdot,0))](x,0), \\ \delta\widetilde{\varepsilon}_t(x,0) = [(I-\mathbb{P}_1)^{-1}(\delta w_t(\cdot,0) - \delta\widehat{w}_t(\cdot,0))](x,0), \\ \widetilde{w}(x,0) = \widehat{w}(x,0) - \displaystyle\int_0^x k(x,y)\widehat{w}(y,0)dy, \\ \delta\widetilde{w}_t(x,0) = \delta\widehat{w}_t(x,0) - \displaystyle\int_0^x k(x,y)\delta\widehat{w}_t(y,0)dy. \end{cases} \tag{7.7}$$

A direct computation shows that $(\widetilde{\varepsilon}(\cdot,0), \delta\widetilde{\varepsilon}_t(\cdot,0), \widetilde{w}(\cdot,0), \delta\widetilde{w}_t(\cdot,0)) \in D(A)$. So there exists a unique classical solution to (6.1) with this initial value. Let

$$
\left\{
\begin{array}{l}
w(x,t) = \widehat{w}(x,t) + \widetilde{\varepsilon}(x,t) - \displaystyle\int_0^x p(x,y)\widetilde{\varepsilon}(y,t)dy, \\
\widehat{w}(x,t) = [(I - \mathbb{P}_2)^{-1}\widetilde{w}](x,t).
\end{array}
\right.
\tag{7.8}
$$

Similarly to (5.5), one can show that $(w, \widehat{w})$ defined in this way satisfies (7.1) with initial value $(w(\cdot,0), \delta w_t(\cdot,0), \widehat{w}(\cdot,0), \delta\widehat{w}_t(\cdot,0))$. This solution is unique by the invertible transformation and the uniqueness of the classical solution to (6.1), where $\mathbb{T}$ is a one to one

$$
\begin{array}{l}
\begin{pmatrix} \widetilde{\varepsilon} \\ \delta\widetilde{\varepsilon}_t \\ \widetilde{w} \\ \delta\widetilde{w}_t \end{pmatrix}
=
\begin{pmatrix}
I - \mathbb{P}_1 & 0 & -I + \mathbb{P}_1 & 0 \\
0 & I - \mathbb{P}_1 & 0 & -I + \mathbb{P}_1 \\
0 & 0 & I - \mathbb{P}_2 & 0 \\
0 & 0 & 0 & I - \mathbb{P}_2
\end{pmatrix}
\begin{pmatrix} w \\ \delta w_t \\ \widehat{w} \\ \delta\widehat{w}_t \end{pmatrix}, \\
\begin{pmatrix} w \\ \delta w_t \\ \widehat{w} \\ \delta\widehat{w}_t \end{pmatrix}
=
\begin{pmatrix}
I - \mathbb{P}_1 & 0 & (I - \mathbb{P}_2)^{-1} & 0 \\
0 & I - \mathbb{P}_1 & 0 & (I - \mathbb{P}_1)^{-1} \\
0 & 0 & (I - \mathbb{P}_2)^{-1} & 0 \\
0 & 0 & 0 & (I - \mathbb{P}_2)^{-1}
\end{pmatrix}
\begin{pmatrix} \widetilde{\varepsilon} \\ \delta\widetilde{\varepsilon}_t \\ \widetilde{w} \\ \delta\widetilde{w}_t \end{pmatrix}
\end{array}
\tag{7.9}
$$

and onto operator from $\mathcal{H}$ to H. Moreover, this solution is exponentially stable by (6.6) and (7.9):

$$
\begin{aligned}
&\|(w(\cdot,t), \delta w_t(\cdot,t), \widehat{w}(\cdot,t), \delta\widehat{w}_t(\cdot,t)\|_{\mathcal{H}} \\
&\qquad \le Me^{-\omega t}\|(w(\cdot,0), \delta w_t(\cdot,0), \widehat{w}(\cdot,0), \delta\widehat{w}_t(\cdot,0)\|_{\mathcal{H}}
\end{aligned}
\tag{7.10}
$$

for some positive constant M independent of t. From transformation (7.9), we know that $\mathcal{A} = \mathbb{T}^{-1}A\mathbb{T}$, and hence $\mathcal{A} = \mathbb{T}A\mathbb{T}^{-1}$, where A is the operator defined by (6.4). Hence $\mathcal{A}^{-1}$ exists and is bounded on $\mathcal{H}$, which implies that $\rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, is not empty. Since obviously $D(\mathcal{A})$ is dense in $\mathcal{H}$, it follows from Theorem 1.3 on page 102 of [10] that $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. (7.10) shows that $e^{\mathcal{A}t}$ is exponentially stable, with $\|e^{\mathcal{A}t}\| \le Me^{-\omega t}$ for all $t \ge 0$. Finally, it follows from (7.2) that $\|\alpha(\cdot,t)\|_{L^2(0,1)} \le \widetilde{C}[\|w_x(\cdot,t)\|_{L^2(0,1)} + \|\widehat{w}_x(\cdot,t)\|_{L^2(0,1)}]$ for some constant $\widetilde{C} > 0$ independent of t. This together with (7.10) gives (7.5). The proof is complete. $\square$

8. Simulation results. In this section we demonstrate through numerical simulations the effectiveness of the control and the observer.

We use the backward Euler method in the time domain and the Chebyshev spectral method in space. For this purpose the second-order-in-time equations are first converted into first-order-in-time (evolution-type) systems of equations. The control kernel k is first approximated on a uniform grid using the iterative scheme (3.10), and then linear interpolation was used to obtain values on the nonuniform Chebyshev grid. The boundary conditions were implemented using second-order explicit discretization. In the numerical simulations we used grid size $N = 40$ in space and time step $dt = 10^{-4}$. The convergence of the numerical method was checked by varying N between $N = 30$ and $N = 70$ and varying dt between $dt = 10^{-2}$ and $dt = 10^{-5}$. The maximum variation of the solution did not exceed 10^{-3} over the whole time and space domain. The numerical code was programed in MATLAB (see, e.g., [18]).

The main system parameters are set to $\delta = 1$, $b = 0.6$, and $q = 0.9$. The design parameters are set to $c_0 = 10$, $c_1 = 1$, and $c_2 = 1$. The initial conditions are

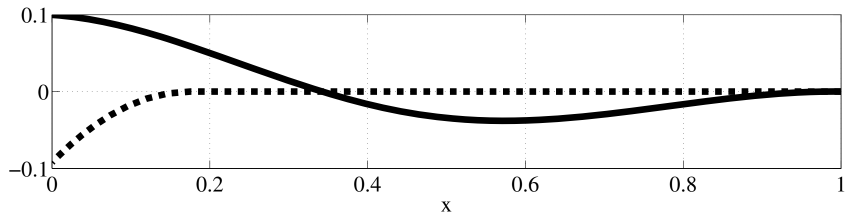

FIG. 8.1. *Initial conditions of the beam. Solid line,* $w(0,x)$*; dashed line,* $w_t(0,x)$*.*

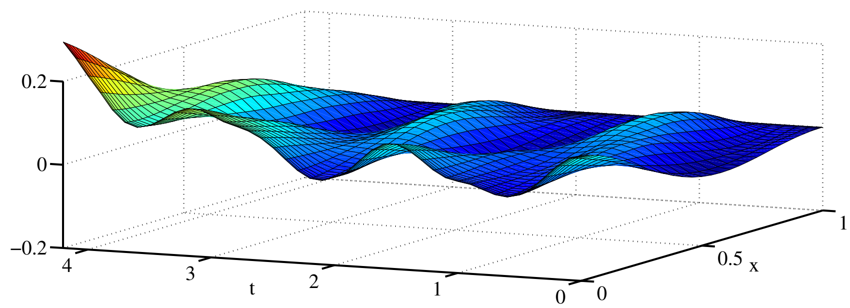

FIG. 8.2. *Beam response* $w(x,t)$*. Uncontrolled case, clamped at* $x=1$*. Note the instability that results from* $q=0.9$*.*

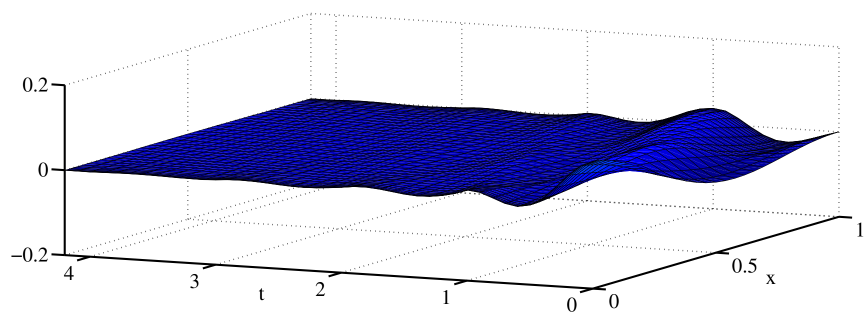

FIG. 8.3. *Beam response* $w(x,t)$ *with control* $w_x(1,t)$ *applied at* $x=1$*. Note that the instability at* $x=0$ *is stabilized.*

$w(x,0) = -0.1(1-x)\sin(1.52\pi(1-x))$ for $x \in [0,1]$ and

$$w_t(x,0) = \begin{cases} -3(x-x_0)^2 & \text{if } x \in [0,x_0], \\ 0 & \text{if } x \in [x_0,1], \end{cases}$$

with $x_0 = 0.1753$. These initial conditions correspond to hitting the tip part of an already bent beam (Figure 8.1).

8.1. No observer, full state feedback. First we consider the full state feedback case. This is equivalent to assuming that the observer starts from the same initial conditions as the plant itself, and hence it is identical with it for all time. Figures 8.2 and 8.3 show the results of our simulation for the shear beam with a zero

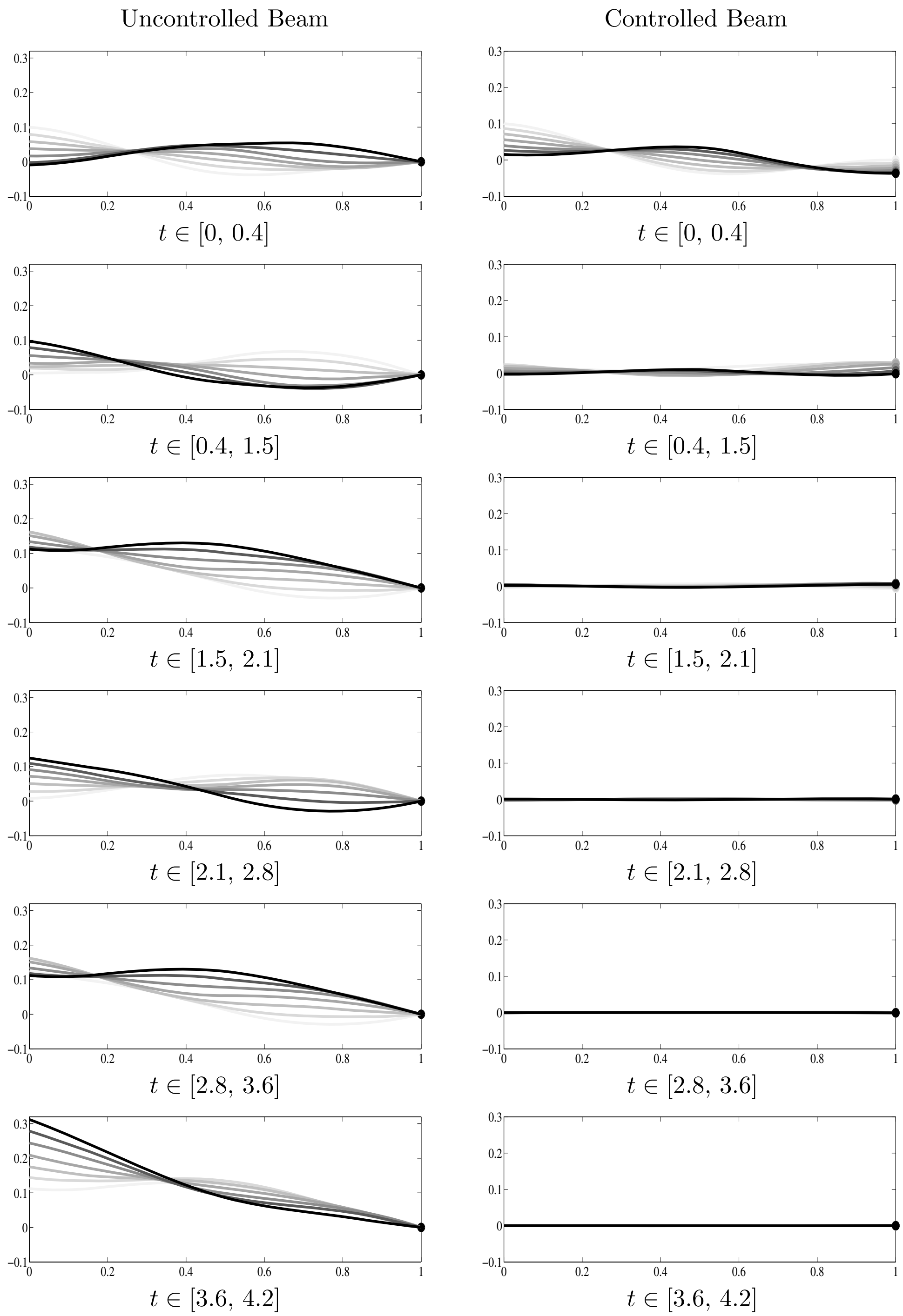

FIG. 8.4. *Snapshots of the beam movements with increasing darkness denoting increasing time in the sequences.*

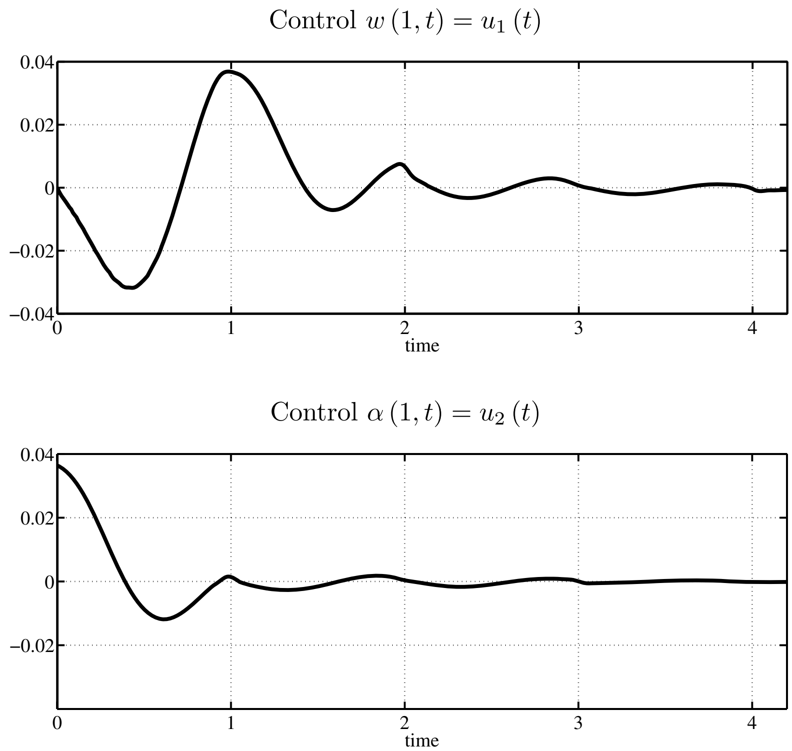

FIG. 8.5. *Time trace of controls* $w(1,t) = u_1(t)$ *and* $\alpha(1,t) = u_2(t)$.

Dirichlet boundary condition at $x = 1$ in the uncontrolled case and with control (5.1) in the controlled case. The uncontrolled case shows that the relatively large value $q = 0.9$ destabilizes the trivial zero solution. The controlled case shows asymptotic stability with small control effort. Snapshots of beam movements (uncontrolled and controlled) are depicted in Figure 8.4, where vibrations are shown in sequences of time intervals. In each sequence the time evolution is represented by increasing darkness. The control effort is shown explicitly as function of time in Figure 8.5. Notice that the maximum control effort is about one magnitude smaller than the maximum of the uncontrolled solution. Gain kernels $k(1,y)$ and $k_x(1,y)$ of the first control law (4.3) are shown in Figure 8.6 for $y \in [0,1]$. These kernel functions have small spatial variations and can be easily approximated by low order polynomials.

8.2. Observer design. We now introduce the observer (4.2) in the simulations. We assume no knowledge of the initial state of the beam, which means that the observer is started with zero initial conditions $\widehat{w}(x,0) = \widehat{w}_t(x,0)$ for $x \in [0,1]$. Figure 8.7 shows that in the uncontrolled case, although the observer starts far from the state, the observer error quickly converges to zero over a time period of $t \in [0,4]$. As expected in the controlled case (see Figure 8.8) the convergence takes place over a longer time period $t \in [0,6]$. The reason for this short delay in the convergence is that the observer has to compensate for the additional error introduced by the control feedback of the observer into the plant. Nevertheless, the closed-loop state quickly converges to zero (Figure 8.9). Finally, the observer gains $p(x,0)$ and $p_y(x,0)$ can be seen in Figure 8.10.

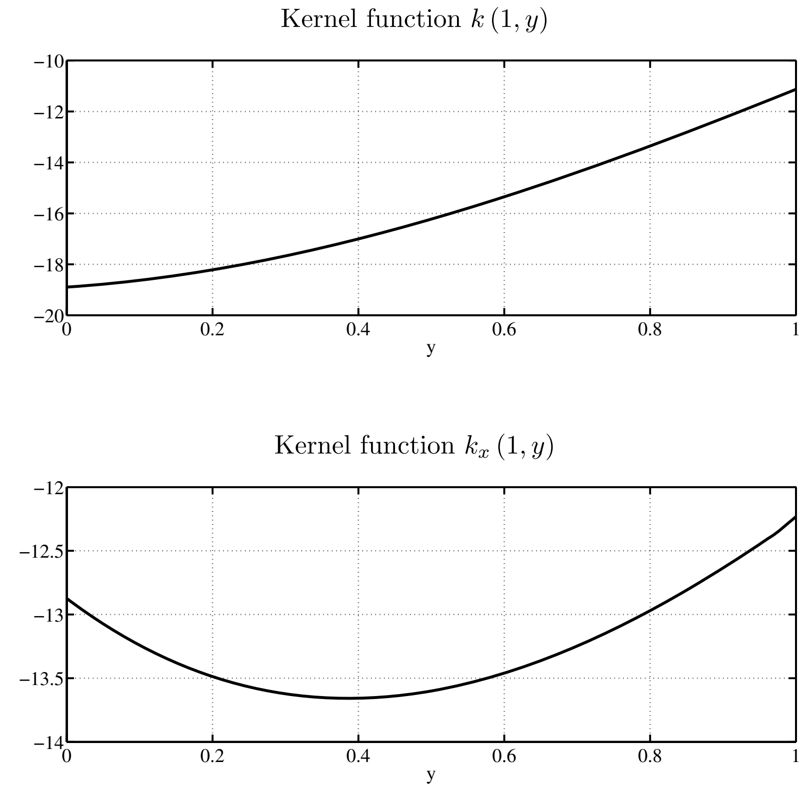

FIG. 8.6. *Kernel functions* $k\,(1,y)$ *and* $k_x\,(1,y)$ *of control law* (5.1) *for* $y \in [0,1]$.

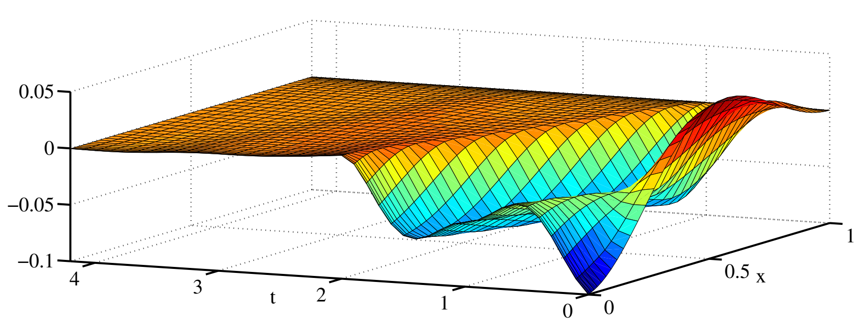

FIG. 8.7. *Observer error* $\widehat{w} - w$ *in the uncontrolled case.*

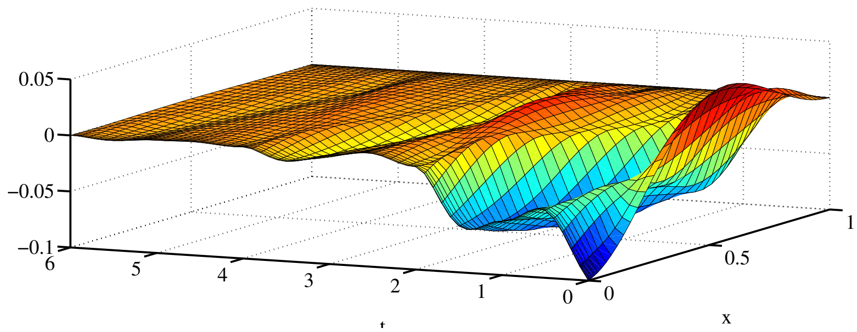

FIG. 8.8. *Observer error* $\widehat{w} - w$ *in the controlled case.*

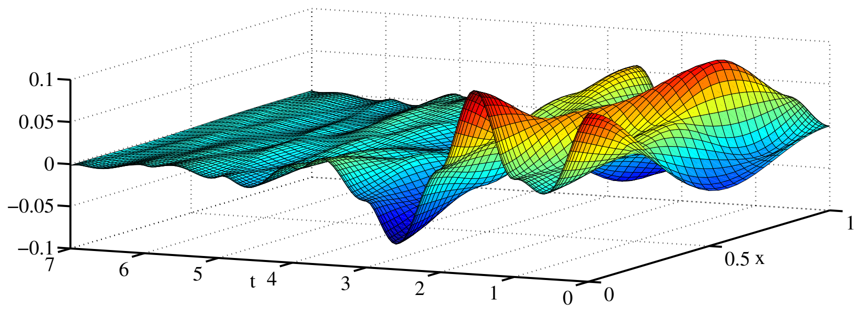

FIG. 8.9. *Plant controlled using the observer.*

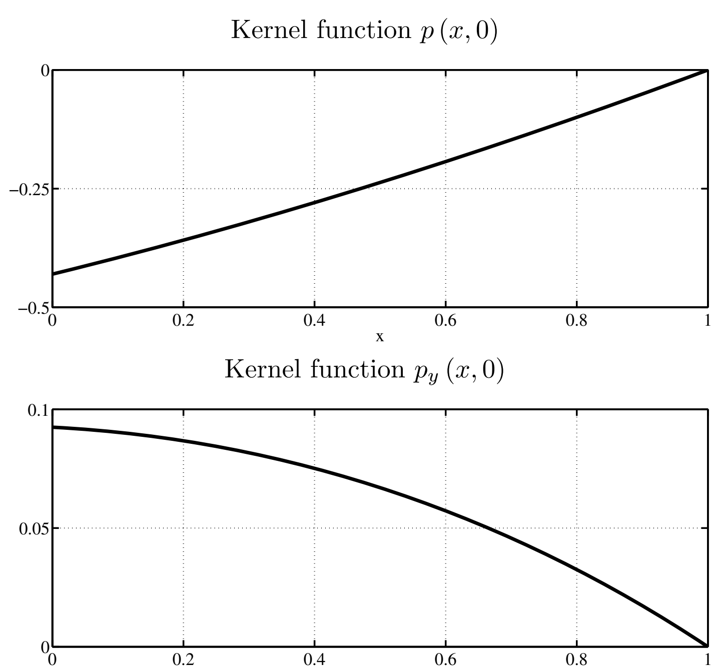

FIG. 8.10. *Kernel functions* $p\,(x, 0)$ *and* $p_y\,(x, 0)$ *of the observer* (4.2) *for* $x \in [0, 1]$.

9. Conclusion. In this paper we presented the output-feedback controller for an undamped shear beam. Future efforts will be concentrated on developing the controllers for higher-dimensional flexible structures such as plates and shells. Another interesting avenue of research is the control of beams (plates, shells) in the presence of parametric uncertainties, such as unknown structural damage. Successful backstepping boundary adaptive controllers for parabolic PDEs were recently developed in [5, 15, 16], and similar ideas could be applied to the hyperbolic equations.

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