A Closed-Form Feedback Controller for Stabilization of the Linearized 2-D Navier–Stokes Poiseuille System

Rafael Vazquez, *Student Member, IEEE*, and Miroslav Krstic, *Fellow, IEEE*

***Abstract*—We present a formula for a boundary control law which stabilizes the parabolic profile of an infinite channel flow, which is linearly unstable for high Reynolds numbers. Also known as the Poiseuille flow, this problem is frequently cited as a paradigm for transition to turbulence, whose stabilization for arbitrary Reynolds numbers, without using discretization, has so far been an open problem. Our result achieves exponential stability in the L^2, H^1, and H^2 norms, for the linearized Navier–Stokes equations. Explicit solutions are obtained for the closed loop system. This is the first time explicit formulae are produced for solutions of the linearized Navier–Stokes equations in a channel flow, with feedback in the boundary conditions used to make this possible. The result is presented for the 2-D case for clarity of exposition. An extension to 3-D is available and will be presented in a future publication.**

***Index Terms*—Backstepping, boundary control, distributed parameter systems, flow control, Lyapunov function, Navier–Stokes equations, stabilization.**

I. Introduction

We present an explicit boundary control law which stabilizes a benchmark 2-D linearized Navier–Stokes system. Despite the deceptive simplicity of the channel flow geometry, there is a number of complex issues underlying this problem [19], making it extremely hard to solve. The fact that the channel is unbounded further complicates the problem [27].

Controllability and stabilizability results for the Navier–Stokes equations are available for general geometries; for example, see [15]–[18] and references therein. However, these results do not provide the means of computing a feedback controller.

Many efforts in the design of feedback controllers for the Navier–Stokes system employ in-domain actuation, using optimal control methods [9], spectral decomposition and pole placement [11], or stochastic control [26]. For boundary feedback control, there are theoretical works in optimal control theory for general geometries [22], and more applied works that consider specific geometries, like cylinder wake [21]. In [12], optimal controllers that act only tangentially to the boundary are designed, and it is shown for the 2-D case that an stabilizing controller acting only on a arbitrarily small subset of the boundary can be found. There are also new techniques arising for specific flow control problems like separation control [3].

Optimal control has so far been the most successful technique for addressing channel flow stabilization [17], in a periodic setting, by using a discretized version of the equations and employing high-dimensional algebraic Riccati equations for computation of gains. The computational complexity of this approach is formidable if a very fine grid is necessary in the discretizations, for example if the Reynolds number is very large. Using a Lyapunov/passivity approach, another control design [1], [7] was developed for stabilization of the (periodic) channel flow; the design was simple and explicit and did not rely on discretization or linearization, but its theory was restricted to low Reynolds numbers though in simulations the approach was successful at high Reynolds numbers, above the linear instability threshold. Other works make use of nonlinear model reduction techniques to solve the problem, though they employ in-domain actuation [4]–[6]. Boundary controllers using spectral decomposition and pole-placement methods have been developed, using normal actuation [10] or tangential actuation in an arbitrarily small subset of the walls [30].

The approach we present in this paper is the first result that provides an explicit control law (with symbolically computed gains) for stabilization at an arbitrarily high Reynolds number in nondiscretized linearized Navier–Stokes equations, and it is applicable to both infinite and periodic channels with arbitrary periodic box size, and also extends to 3-D. Thanks to the explicitness of the controller, we are able to obtain approximate analytical solutions for the linearized Navier–Stokes equations. Exponential stability in the L^2, H^1, and H^2 norms is proved for the linearized Stokes system.

The method we use for solving the stabilization problem is based on the recently developed backstepping technique for parabolic systems [28], which has been successfully applied to flow control problems, for example to the vortex shedding problem [2] and to feedback stabilization of an unstable convection loop [34].

We start the paper by stating, in Section II, the mathematical model, which consists of the linearized Navier–Stokes equations for the velocity fluctuation around the (Pouisseuille) equilibrium profile. In Section III, we introduce the control law that stabilizes the equilibrium profile. Explicit solutions for the closed loop system are then stated in Section IV along with the main results of the paper. Sections V–VII deal with the

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R. Vazquez is with the Departamento de Ingeniería Aeroespacial, Universidad de Sevilla, 41092 Seville, Spain (e-mail: rvazquez1@us.es).

M. Krstic is with the Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093-0411 USA (e-mail: krstic@ucsd.edu).

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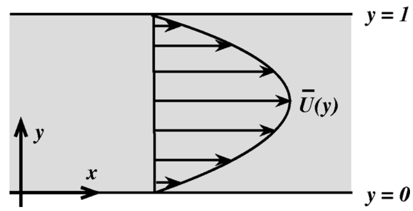

Fig. 1. Two-dimensional channel flow and equilibrium profile. Actuation is on the top wall.

proof of, respectively, L^2, H^1 and H^2 stability of the closed loop system. A Fourier transform approach allows separate analysis for each wave number. For certain wave numbers, a normal velocity controller puts the system into a form where a linear Volterra operator, combined with boundary feedback, transforms the original normal velocity PDE into a stable heat equation. The rest of wave numbers are proved to be open loop exponentially stable, and left uncontrolled. These two results are combined to prove stability of the closed loop system for all wave numbers and in physical space. In Section VIII we justify the well-posedness of the system. Section IX is devoted to study some properties of the control laws. In Section X, we finish the paper with a discussion of the results.

II. Model

Consider a 2-D incompressible channel flow evolving in a semi-infinite rectangle $(x,y) \in \Omega = (-\infty,\infty) \times [0,1]$ as in Fig. 1. The dimensionless velocity field is governed by the Navier–Stokes equations

$$U_t = \frac{1}{Re}(U_{xx}+U_{yy}) - UU_x - VU_y - P_x \tag{1}$$

$$V_t = \frac{1}{Re}(V_{xx}+V_{yy}) - UV_x - VV_y - P_y \tag{2}$$

and the continuity equation

$$U_x + V_y = 0 \tag{3}$$

where U denotes the streamwise velocity, V the wall-normal velocity, P the pressure, and Re is the Reynolds number. The boundary conditions for the velocity field are the no-penetration, no-slip boundary conditions for the uncontrolled case, i.e., $V(x,0) = V(x,1) = U(x,0) = U(x,1) = 0$. Instead of using (3), we derive a Poisson equation that P verifies, combining (1), (2), and (3)

$$P_{xx} + P_{yy} = -2(V_y)^2 - 2V_xU_y \tag{4}$$

with boundary conditions $P_y(x,0) = (1/Re)V_{yy}(x,0)$ and $P_y(x,1) = (1/Re)V_{yy}(x,1)$, which are obtained evaluating (2) at $y = 0, 1$.

The equilibrium solution of (1)–(3) is the parabolic Poiseuille profile

$$U^e = 4y(1-y) \tag{5}$$

$$V^e = 0 \tag{6}$$

$$P^e = P_0 - \frac{8}{Re}x \tag{7}$$

shown in Fig. 1. This equilibrium is unstable for high Reynolds numbers [25]. Defining the fluctuation variables $u = U - U^e$ and $p = P - P^e$, and linearizing around the equilibrium profile (5)–(7), the plant equations become the Stokes equations

$$\begin{aligned} u_t = {} & \frac{1}{Re}(u_{xx}+u_{yy}) + 4y(y-1)u_x \\ & + 4(2y-1)V - p_x \end{aligned} \tag{8}$$

$$\begin{aligned} V_t = {} & \frac{1}{Re}(V_{xx}+V_{yy}) + 4y(y-1)V_x \\ & - p_y, \end{aligned} \tag{9}$$

$$p_{xx} + p_{yy} = 8(2y-1)V_x \tag{10}$$

with boundary conditions

$$u(x,0) = 0, \tag{11}$$

$$u(x,1) = U_c(x) \tag{12}$$

$$V(x,0) = 0, \tag{13}$$

$$V(x,1) = V_c(x) \tag{14}$$

$$p_y(x,0) = \frac{V_{yy}(x,0)}{Re} \tag{15}$$

$$p_y(x,1) = \frac{V_{yy}(x,1) + (V_c)_{xx}(x)}{Re} - (V_c)_t(x). \tag{16}$$

The continuity equation is still verified

$$u_x + V_y = 0. \tag{17}$$

We have added in (12) and (14) the actuation variables $U_c(x)$ and $V_c(x)$, respectively, for streamwise and normal velocity boundary control. The actuators are placed along the top wall, $y = 1$, and we assume they can be independently actuated for all $x \in \mathbb{R}$. No actuation is done inside the channel or at the bottom wall.

Taking Laplacian in (9) and using (10), we get an autonomous equation for the normal velocity, the well-known Orr–Sommerfeld equation

$$\triangle V_t = \frac{1}{Re}\triangle^2 V + 4y(y-1)\triangle V_x - 8V_x \tag{18}$$

with boundary conditions (13)–(14), as well as $V_y(x,0) = 0$, $V_y(x,1) = -(U_c)_x$, derived from (11)–(12) and (17). This equation is numerically studied in hydrodynamic theory to determine stability of the channel flow [23].

Defining $Y = -V_y$ and using the Fourier transform, it is possible to partially solve (18) and obtain an evolution equation for Y

$$\begin{aligned} Y_t = {} & \frac{1}{Re}(Y_{xx}+Y_{yy}) + 4y(y-1)Y_x \\ & + \int_0^y \int_{-\infty}^{\infty} Y(\xi,\eta) \int_{-\infty}^{\infty} 16\pi k e^{2\pi i k(x-\xi)} \\ & \times [\pi k(2y-1) - 2\sin h(2\pi k(y-\eta)) \\ & - 2\pi k(2\eta-1)\cosh(2\pi k(y-\eta))]\, dk d\xi d\eta \\ & + \int_0^1 \int_{-\infty}^{\infty} Y(\xi,\eta) \int_{-\infty}^{\infty} 32\pi k e^{2\pi i k(x-\xi)} \\ & \times \frac{\cosh(2\pi k y)}{\sinh(2\pi k)} [\cosh(2\pi k(1-\eta)) \\ & + \pi k(2\eta-1)\sinh(2\pi k(1-\eta))]\, dk d\xi d\eta \\ & + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{Y_y(\xi,1) - (V_c)_{xx}(\xi)}{Re} + (V_c)_t(\xi) \right) \end{aligned}$$

$$\times 2\pi k e^{2\pi i k(x-\xi)} \frac{\cosh(2\pi k y)}{\sinh(2\pi k)} dk d\xi$$
$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{Y_y(\xi,0)}{Re}$$
$$\times 2\pi k e^{2\pi i k(x-\xi)} \frac{\cosh(2\pi k(1-y))}{\sinh(2\pi k)} dk d\xi \quad (19)$$

with boundary conditions $Y_y(x,0) = 0$ and $Y(x,1) = (U_c)_x$. Equation (19) governs the channel flow, since from Y and using (17), we recover both components of the velocity field

$$V(x,y) = -\int_0^y Y(x,\eta) d\eta \quad (20)$$
$$u(x,y) = \int_{-\infty}^{x} Y(\xi,y) d\xi. \quad (21)$$

Equation (19) displays the full complexity of the Navier–Stokes dynamics, which the PDE system (8)–(10) conceals through the presence of the pressure (10), and the Orr–Sommerfeld (18) conceals through the use of fourth order derivatives. Besides being unstable (for high Reynolds numbers), the Y system incorporates (on its right-hand side) the components of $Y(x,y)$ from everywhere in the domain. This is the main source of difficulty for both controlling and solving the Navier–Stokes equations. A perturbation somewhere in the flow is instantaneously felt everywhere—a consequence of the incompressible nature of the flow. Our approach to overcoming this obstacle is to use one of the two control variables (normal velocity $V_c(x)$, which is incorporated explicitly inside the equation) to prevent perturbations from propagating in the direction from the controlled boundary towards the uncontrolled boundary. This is a sort of "spatial causality" on y, which in the nonlinear control literature is referred to as the 'strict-feedback structure' [20].

III. Controller

The explicit control law consists of two parts—the normal velocity controller $V_c(x)$ and the streamwise velocity controller $U_c(x)$. $V_c(x)$ makes the integral operator in the third to fifth lines of (19) spatially causal in y,[1] which is a necessary structure for the application of a "backstepping" boundary controller for stabilization of spatially causal partial integro-differential equations [28]. $U_c(x)$ is a backstepping controller which stabilizes the spatially causal structure imposed by $V_c(x)$. The expressions for the control laws are

$$U_c(t,x) = \int_0^1 \int_{-\infty}^{\infty} Q_u(x-\xi,\eta) u(t,\xi,\eta) d\xi d\eta \quad (22)$$
$$V_c(t,x) = h(t,x) \quad (23)$$

where h verifies the equation

$$h_t = h_{xx} + g(t,x) \quad (24)$$

where

$$g = \int_0^1 \int_{-\infty}^{\infty} Q_V(x-\xi,\eta) V(t,\xi,\eta) d\xi d\eta + \int_{-\infty}^{\infty} Q_0(x-\xi)(u_y(t,\xi,0) - u_y(t,\xi,1)) d\xi \quad (25)$$

[1]The first, second, and tenth lines are already spatially causal in y.

and the kernels Q_u, Q_V and Q_0 are defined as

$$Q_u = \int_{-\infty}^{\infty} \chi(k) K(k,1,\eta) e^{2\pi i k(x-\xi)} dk \quad (26)$$
$$Q_V = \int_{-\infty}^{\infty} \chi(k) 16\pi k i(2\eta - 1)\cosh(2\pi k(1-\eta)) \times e^{2\pi i k(x-\xi)} dk \quad (27)$$
$$Q_0 = \int_{-\infty}^{\infty} \chi(k) \frac{2\pi k i}{Re} e^{2\pi i k(x-\xi)} dk. \quad (28)$$

In expressions (26)–(28), $\chi(k)$ is a truncating function in the wave number space whose definition is

$$\chi(k) = \begin{cases} 1, & m < |k| < M \\ 0, & \text{otherwise} \end{cases} \quad (29)$$

where m and M are, respectively, the low and high cutoff wave numbers, two design parameters which can be conservatively chosen as $m \leq (1)/(32\pi Re)$ and $M \geq (1/\pi)\sqrt{(Re/2)}$. The function $K(k,y,\eta)$ appearing in (26) is a (complex valued) gain kernel defined as

$$K(k,y,\eta) = \lim_{n\to\infty} K_n(k,y,\eta) \quad (30)$$

where K_n is recursively defined as[2]

$$\begin{aligned} K_0 = &-2\pi k \frac{\cosh(2\pi k(1-y+\eta)) - \cosh(2\pi k(y-\eta))}{\sinh(2\pi k)} \\ &+ 4iRe\eta(\eta-1)\sinh(2\pi k(y-\eta)) \\ &- \frac{Re}{3}\pi i k\eta(21y^2 - 6y(3+4\eta) + \eta(12+7\eta)) \\ &- 6\eta i \frac{Re}{\pi k}(1 - \cosh(2\pi k(y-\eta))) \end{aligned} \quad (31)$$

and

$$\begin{aligned} K_n = K_{n-1} &- 4\pi k i Re \int_{y-\eta}^{y+\eta} \int_0^{y-\eta} \int_{-\delta}^{\delta} \left\{ \frac{\sinh(\pi k(\xi+\delta))}{\pi k} \right. \\ &\left. -(2\xi - 1) + 2(\gamma - \delta - 1)\cosh(\pi k(\xi+\delta)) \right\} \\ &\times K_{n-1}\left(k, \frac{\gamma+\delta}{2}, \frac{\gamma+\xi}{2}\right) d\xi d\delta d\gamma \\ &+ \frac{Re}{2}\pi i k \int_{y-\eta}^{y+\eta} \int_0^{y-\eta} (\gamma-\delta)(\gamma-\delta-2) \\ &\times K_{n-1}\left(k, \frac{\gamma+\delta}{2}, \frac{\gamma-\delta}{2}\right) d\delta d\gamma \\ &+ 2\pi k \int_0^{y-\eta} \frac{\cosh(2\pi k(1-\delta)) - \cosh(2\pi k\delta)}{\sinh(2\pi k)} \\ &\times K_{n-1}(k, y-\eta, \delta)\, d\delta. \end{aligned} \quad (32)$$

The terms of this series can be computed symbolically as they only involve integration of polynomials and exponentials. In implementation, a few terms are sufficient to obtain a highly accurate approximation because the series is rapidly convergent [28].

Formulas (22)–(32) constitute the complete statement of our feedback law. Their mathematical validity is established in Theorem 2 and Proposition 1.

[2]This infinite sequence is convergent, smooth, and uniformly bounded over $(y,\eta) \in [0,1]^2$, and analytic in k. See Proposition 1 for details

Remark 1: (23) is a dynamic controller whose magnitude is determined by the variable $h(t,x)$, which evolves according to (24). The initial condition $h(0,x)$ must verify the compatibility condition for the plant to be well-posed. This amounts to setting $h(0,x) \equiv V(0,1,x)$.

Remark 2: Control kernels (27) and (28) can be explicitly expressed as

$$Q_V(\xi,\eta) = 8(2\eta-1)\frac{R_V(\xi,\eta,M)-R_V(\xi,\eta,m)}{\xi^2+(1-\eta)^2} \tag{33}$$

$$Q_0(\xi,\eta) = \frac{R_0(\xi,\eta,M)-R_0(\xi,\eta,m)}{Re\,\xi} \tag{34}$$

where $R_V(\xi,\eta,k)$ and $R_0(\xi,\eta,k)$ are defined

$$\begin{aligned} R_V = &\frac{((1-\eta)^2-\xi^2)\sin(2\pi k\xi)\cosh(2\pi k(1-\eta))}{2\pi(\xi^2+(1-\eta)^2)} \\ &+ k\xi\cos(2\pi k\xi)\cosh(2\pi k(1-\eta)) \\ &- \frac{\xi(1-\eta)\cos(2\pi k\xi)\sinh(2\pi k(1-\eta))}{\pi(\xi^2+(1-\eta)^2)} \\ &- k(1-\eta)\sin(2\pi k\xi)\sinh(2\pi k(1-\eta)) \end{aligned} \tag{35}$$

$$R_0 = k\cos(2\pi k\xi) - \frac{\sin(2\pi k\xi)}{2\pi\xi}. \tag{36}$$

IV. Main Results

Due to the explicit form of the controller, the solution of the closed loop system is also obtained in the explicit form

$$u(t,x,y) = u^*(t,x,y) + \epsilon_u(t,x,y) \tag{37}$$

$$V(t,x,y) = V^*(t,x,y) + \epsilon_V(t,x,y) \tag{38}$$

where

$$\begin{aligned} u^* = &2\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi(k)\mathrm{e}^{-t\frac{4k^2\pi^2+\pi^2j^2}{Re}+2\pi ik(x-\xi)} \\ &\times\left[\sin(\pi jy)+\int_0^y L(k,y,\eta)\sin(\pi j\eta)d\eta\right] \\ &\times\int_0^1\left[\sin(\pi j\eta)-\int_\eta^1 K(k,\sigma,\eta)\sin(\pi j\sigma)d\sigma\right] \\ &\times u(0,\xi,\eta)d\eta d\xi dk \end{aligned} \tag{39}$$

$$\begin{aligned} V^* = &-2\sum_{j=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi(k)\mathrm{e}^{-t\frac{4k^2\pi^2+\pi^2j^2}{Re}+2\pi ik(x-\xi)} \\ &\times\left[\int_0^y\left(\int_\eta^y L(k,\sigma,\eta)d\sigma\right)\sin(\pi j\eta)d\eta\right. \\ &\left.+\frac{1-\cos(\pi jy)}{\pi j}\right]\int_0^1\left[\pi j\cos(\pi j\eta)\right. \\ &+K(k,\eta,\eta)\sin(\pi j\eta)-\int_\eta^1 K_\eta(k,\sigma,\eta) \\ &\left.\times\sin(\pi j\sigma)d\sigma\right]V(0,\xi,\eta)d\eta d\xi dk. \end{aligned} \tag{40}$$

The variables $\epsilon_u(t,x,y)$ and $\epsilon_V(t,x,y)$ represent the error of approximation of the velocity field and are bounded in the following way:

$$\begin{aligned} &\|\epsilon_u(t)\|_{L^2}^2+\|\epsilon_V(t)\|_{L^2}^2 \\ &\qquad\le \mathrm{e}^{-\frac{1}{4Re}t}\left(\|\epsilon_u(0)\|_{L^2}^2+\|\epsilon_V(0)\|_{L^2}^2\right) \end{aligned} \tag{41}$$

where both $\epsilon_u(0,x,y)$ and $\epsilon_V(0,x,y)$ can be written in terms of the initial conditions of the velocity field as

$$\begin{aligned} \epsilon_u(0,x,y) = &u(0,x,y)-\int_{-\infty}^{\infty}\frac{\sin(2\pi M\xi)-\sin(2\pi m\xi)}{\pi\xi} \\ &\times u(0,x-\xi,y)\,d\xi \end{aligned} \tag{42}$$

$$\begin{aligned} \epsilon_V(0,x,y) = &V(0,x,y)-\int_{-\infty}^{\infty}\frac{\sin(2\pi M\xi)-\sin(2\pi m\xi)}{\pi\xi} \\ &\times V(0,x-\xi,y)\,d\xi. \end{aligned} \tag{43}$$

The bound on the errors is proportional to the initial kinetic energy of ϵ_u and ϵ_V, which, as made explicit in the expressions (42)–(43), is, in turn, proportional to the kinetic energy of u and V at very small and very large length scales (the integral that we are substacting from the initial conditions represents the intermediate length scale content), and decays exponentially. Therefore, this initial energy will typically be a very small fraction of the overall kinetic energy, making the errors ϵ_u and ϵ_V very small in comparison with u^* and V^*, respectively.

The kernel L in (40) is defined as a convergent, smooth sequence of fuctions

$$L(k,y,\eta) = \lim_{n\to\infty} L_n(k,y,\eta) \tag{44}$$

whose terms are recursively defined as

$$L_0 = K_0 \tag{45}$$

$$\begin{aligned} L_n = &L_{n-1}+4iRe\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}\int_{-\delta}^{\delta}\{2\pi k(\gamma+\xi-1) \\ &\times\cosh(\pi k(\xi-\delta))+\sinh(\pi k(\xi-\delta)) \\ &-\pi k(2\delta-1)\}L_{n-1}\left(k,\frac{\gamma+\xi}{2},\frac{\gamma-\delta}{2}\right)d\xi d\delta d\gamma \\ &-\frac{Re}{2}\pi ik\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}(\gamma+\delta)(\gamma+\delta-2) \\ &\times L_{n-1}\left(k,\frac{\gamma+\delta}{2},\frac{\gamma-\delta}{2}\right)d\delta d\gamma. \end{aligned} \tag{46}$$

Control laws (22)–(32) guarantee the following results.

Theorem 1: Assume $u_0(x,y)$ and $V_0(x,y)$, initial conditions for u and V, belong to $H^2(\Omega)$ and that the following compatibility conditions[3] are verified:

$$0 = \frac{\partial}{\partial x}u_0(x,y)+\frac{\partial}{\partial y}V_0(x,y) \tag{47}$$

$$u_0(x,1) = \int_0^1\int_{-\infty}^{\infty}Q_u(x-\xi,\eta)u_0(\xi,\eta)\,d\xi d\eta. \tag{48}$$

[3]The compatibility condition $\int_{-\infty}^{\infty}V_c(t,x)dx=0$ is automatically verified, see Theorem 2.

Then, the equilibrium $u(x,y) \equiv V(x,y) \equiv 0$ of system (8)–(16), with feedback law (22)–(32) where the function $h(t,x)$ in (24) verifies the initial condition $h(0,x)=V_0(x,1)$, is exponentially stable in the L^2, H^1 and H^2 norms. Moreover, the solutions for $u(t,x,y)$ and $V(t,x,y)$ belong to $L^2((0,\infty),H^2(\Omega))$ and are given explicitly by (37)–(46).

Theorem 2: Under the same assumptions of Theorem 1, control laws U_c, V_c and kernels Q_u, Q_V, Q_0, as defined by (22)–(32), have the following properties.

i) U_c and V_c are spatially invariant in x.
ii) $\int_{-\infty}^{\infty} V_c(t,\xi)d\xi = 0$ (zero net flux).
iii) $|Q| \le C/|x-\xi|$, for $Q=Q_u,Q_V,Q_0$.
iv) U_c and V_c are smooth functions of x.
v) Q_u,Q_V,Q_0 are real valued.
vi) Q_u,Q_V,Q_0 are smooth in their arguments.
vii) U_c and V_c are L^2 functions of x.
viii) All spatial derivatives of U_c and V_c are L^2 function of x.

Remark 3: By Sobolev's Embedding Theorem [31], H^2 stability suffices to establish continuity of the velocity field, which holds on bounded (e.g., periodic) domains as well as on infinite domains.

Remark 4: Theorem 2 ensures that the control laws are well behaved and their formal definition makes sense. Property i, spatial invariance, means that the feedback operators commute with translations in the x direction [8], which is crucial for implementation. Property ii ensures that we do not violate the physical restriction of zero net flux, which is derived from mass conservation. Property iii allows to truncate the integrals with respect to ξ to the vicinity of x, which allows sensing to be restricted just to a neighborhood (in the x direction) of the actuator. Properties iv to vi ensure that the control laws are well defined. Properties vii and viii prove finiteness of energy of the controllers and their spatial derivatives.

The next sections are devoted to proving these theorems. We first derive *a priori* estimates; then we prove well-posedness in a direct way using explicit closed-loop solutions.

V. L^2 STABILITY

As common for infinite channels, we use a Fourier transform in x. The transform pair (direct and inverse transform) has the following definition:

$$f(k,y) = \int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{-2\pi ikx}dx \tag{49}$$

$$f(x,y) = \int_{-\infty}^{\infty} f(k,y)\mathrm{e}^{2\pi ikx}dk. \tag{50}$$

Note that we use the same symbol f for both the original $f(x,y)$ and the image $f(k,y)$. In hydrodynamics, k is referred to as the "wave number."

One property of the Fourier transform is that the L^2 norm is the same in Fourier space as in physical space, i.e.,

$$\begin{aligned}\|f\|_{L^2}^2 &= \int_0^1\int_{-\infty}^{\infty} f^2(k,y)\,dkdy \\ &= \int_0^1\int_{-\infty}^{\infty} f^2(x,y)\,dxdy\end{aligned} \tag{51}$$

allowing us to derive L^2 exponential stability in physical space from the same property in Fourier space. This result is called Parseval's formula in the literature [13].

Remark 5: Given a state f, we define feedback operators that act on the state for each wave number k. Calling the result of the operator $Kf(k)$,

$$Kf(k) = \int_0^1 K(k,y)f(k,y)\,dy \tag{52}$$

where K is a kernel that is itself a function of k. Applying the inverse transform we can write (52) in physical space

$$Kf(x) = \int_0^1\int_{-\infty}^{\infty} K(k,y)f(k,y)\mathrm{e}^{2\pi ikx}dkdy \tag{53}$$

or in terms of f in physical space

$$Kf = \int_{-\infty}^{\infty}\int_0^1\left(\int_{-\infty}^{\infty} K(k,y)\mathrm{e}^{2\pi ik(x-\xi)}dk\right) \times f(\xi,y)\,dyd\xi. \tag{54}$$

This is known as the Convolution Theorem. Supposing that f is an L^2 function of space, and that K is bounded and has finite support in k, it follows that Expression (54) makes sense and defines an L^2 function in space.

We also define the L^2 norm of $f(k,y)$ with respect to y

$$\|f(k)\|_{\hat{L}^2}^2 = \int_0^1 |f(k,y)|^2dy. \tag{55}$$

The $\hat{L}^2$ norm as a function of k is related to the L^2 norm as

$$\|f\|_{L^2}^2 = \int_{-\infty}^{\infty}\|f(k)\|_{\hat{L}^2}^2dk. \tag{56}$$

Equations (8)–(10) written in the Fourier domain are

$$\begin{aligned}u_t = &\frac{-4\pi^2k^2u+u_{yy}}{Re} + 8k\pi iy(y-1)u \\ &+ 4(2y-1)V - 2\pi ikp\end{aligned} \tag{57}$$

$$\begin{aligned}V_t = &\frac{-4\pi^2k^2V+V_{yy}}{Re} + 8\pi kiy(y-1)V \\ &- p_y\end{aligned} \tag{58}$$

$$-4\pi^2k^2p + p_{yy} = 16\pi ki(2y-1)V \tag{59}$$

with boundary conditions

$$u(k,0) = 0 \tag{60}$$

$$u(k,1) = U_c(k) \tag{61}$$

$$V(k,0) = 0 \tag{62}$$

$$V(k,1) = V_c(k) \tag{63}$$

$$p_y(k,0) = \frac{V_{yy}(k,0)}{Re} \tag{64}$$

$$p_y(k,1) = \frac{V_{yy}(k,1) - 4\pi^2k^2V_c(k)}{Re} - (V_c)_t(k) \tag{65}$$

and the continuity (17) is now

$$2\pi k i u(k,y) + V_y(k,y) = 0. \tag{66}$$

Thanks to linearity and spatial invariance, there is no coupling between different wave numbers. This allows us to consider the equations for each wave number independently. Then, the main idea behind the design of the controller is to consider two different cases depending on the wave number k. For wave numbers $m < |k| < M$, which we will refer to as *controlled* wave numbers, we will design a backstepping controller that achieves stabilization, whereas for wave numbers in the range $|k| \geq M$ or in the range $|k| \leq m$, which we will call *uncontrolled* wave numbers, the system is left without control but is exponentially stable. This is a well-known fact from hydrodynamic stability theory [25].

Estimates of m and M are found in the paper based on Lyapunov analysis and allow us to use feedback for only the wave numbers $m < |k| < M$. This is crucial because feedback over the entire infinite range of k's would not be convergent. The truncations at $k = m, M$ are truncations in Fourier space which do not result in a discontinuity in x.

We now analyze (57)–(59) in detail, for both controlled and uncontrolled wave numbers.

A. Controlled Wave Numbers

For $m < |k| < M$ we first solve (59) in order to eliminate the pressure. The equation can be easily solved since it is just an ODE in y, for each k. Introducing its solution into (57), we are left with

$$\begin{aligned} u_t = {} & \frac{1}{Re}(u_{yy} - 4\pi^2 k^2 u) \\ & + 8\pi k i y(y-1)u + 4(2y-1)V \\ & + 16\pi k \int_0^y V(k,\eta)(2\eta - 1) \\ & \times \sinh(2\pi k(y-\eta))d\eta \\ & + i\frac{\cosh(2\pi k(1-y))}{\sinh(2\pi k)} \frac{V_{yy}(k,0)}{Re} \\ & - i\frac{\cosh(2\pi k y)}{\sinh(2\pi k)} \\ & \times \left(\frac{V_{yy}(k,1) - 4\pi^2 k^2 V_c(k)}{Re} - (V_c)_t(k) \right) \\ & - 16\pi k \frac{\cosh(2\pi k y)}{\sinh(2\pi k)} \int_0^1 V(k,\eta)(2\eta - 1) \\ & \times \cosh(2\pi k(1-\eta))d\eta. \end{aligned} \tag{67}$$

We do not need to separately write and control the V equation because, by the continuity (66) and using the fact that $V(k,0) = 0$, we can write V in terms of u

$$V(k,y) = \int_0^y V_y(k,\eta)d\eta = -2\pi k i \int_0^y u(k,\eta)d\eta. \tag{68}$$

Introducing (68) in (67), and simplifying the resulting double integral by changing the order of integration, we reduce (67) to an autonomous equation that governs the whole velocity field. This equation is

$$\begin{aligned} u_t = {} & \frac{1}{Re}(-4\pi^2 k^2 u + u_{yy}) + 8\pi k i y(y-1)u \\ & + \frac{2\pi k \cosh(2\pi k(1-y))}{\sinh(2\pi k)} \frac{u_y(k,0)}{Re} \\ & + 8i \int_0^y \{\pi k(2y-1) - 2\sinh(2\pi k(y-\eta)) \\ & - 2\pi k(2\eta - 1)\cosh(2\pi k(y-\eta))\} u(k,\eta)d\eta \\ & + 16i \frac{\cosh(2\pi k y)}{\sinh(2\pi k)} \int_0^1 \{\cosh(2\pi k(1-\eta)) \\ & \times \pi k(2\eta - 1)\sinh(2\pi k(1-\eta))\} u(k,\eta)d\eta \\ & + i\frac{\cosh(2\pi k y)}{\sinh(2\pi k)} \left(\frac{2\pi k i u_y(k,1) + 4\pi^2 k^2 V_c(k)}{Re} \right. \\ & \left. + (V_c)_t(k) \right) \end{aligned} \tag{69}$$

with boundary conditions

$$u(k,0) = 0, \tag{70}$$

$$u(k,1) = U_c(k). \tag{71}$$

Note that the relation between Y in (19) and u in (69) is that $Y(k,y) = 2\pi k i u(k,y)$.

Now, we design the controller in two steps. First, we set V_c so that (69) has a strict-feedback form in the sense previously defined

$$\begin{aligned} (V_c)_t = {} & \frac{2\pi k i(u_y(k,0) - u_y(k,1)) - 4\pi^2 k^2 V_c}{Re} \\ & - 16\pi k i \int_0^1 (2\eta - 1)V(k,\eta) \\ & \times \cosh(2\pi k(1-\eta))d\eta. \end{aligned} \tag{72}$$

This can be integrated and explicitly stated as a dynamic controller in the Laplace domain

$$\begin{aligned} V_c = {} & \frac{2\pi k i}{s + \frac{4\pi^2 k^2}{Re}} \left[\frac{u_y(s,k,0) - u_y(s,k,1)}{Re} - 8 \right. \\ & \left. \times \int_0^1 (2\eta - 1)V(s,k,\eta)\cosh(2\pi k(1-\eta))d\eta \right]. \end{aligned} \tag{73}$$

Control law (72) can be expressed in the time domain and physical space as (23)–(25) and (27), (28), by use of the convolution theorem of the Fourier transform.

Introducing V_c in (69) yields

$$\begin{aligned} u_t = {} & \frac{1}{Re}(-4\pi^2 k^2 u + u_{yy}) + 8\pi k i y(y-1)u \\ & + 8i \int_0^y \{\pi k(2y-1) - 2\sinh(2\pi k(y-\eta)) \\ & - 2\pi k(2\eta - 1)\cosh(2\pi k(y-\eta))\} u(k,\eta)d\eta \\ & - 2\pi k \frac{\cosh(2\pi k y) - \cosh(2\pi k(1-y))}{\sinh(2\pi k)} \frac{u_y(k,0)}{Re}. \end{aligned} \tag{74}$$

Equation (74) can be stabilized using the backstepping technique for parabolic partial integro-differential equations [28]. This method consists on finding an invertible Volterra transformation that maps the original unstable equation into a target system with the desired stability properties.

Using backstepping, we map u, for each wave number $m < |k| < M$, into the family of heat equations

$$\alpha_t = \frac{1}{Re}(-4\pi^2 k^2 \alpha + \alpha_{yy}) \tag{75}$$
$$\alpha(k,0) = 0, \tag{76}$$
$$\alpha(k,1) = 0 \tag{77}$$

where

$$\alpha = u - \int_0^y K(k,y,\eta) u(t,k,\eta) d\eta \tag{78}$$
$$u = \alpha + \int_0^y L(k,y,\eta)\alpha(t,k,\eta) d\eta \tag{79}$$

are respectively the direct and inverse transformation. The kernel K is found by substituting (74) and (78) into (75)–(77). Then integration by parts, following exactly the same steps as in [28], leads to the following equation that K must verify

$$\begin{aligned}\frac{1}{Re}K_{yy} = {} & \frac{1}{Re}K_{\eta\eta} + 8\pi i k\eta(\eta-1)K \\ & + 8\pi k i \int_\eta^y \left\{ (2\xi - 1) - 2\frac{\sinh(2\pi k(\xi-\eta))}{\pi k} \right. \\ & \left. -2(2\eta-1)\cosh(2\pi k(\xi-\eta)) \right\} K(k,y,\xi) d\xi \\ & - 8i\{\pi k(2y-1) - \sinh(2\pi k(y-\eta)) \\ & - 2\pi k(2\eta-1)\cosh(2\pi k(y-\eta))\}\end{aligned} \tag{80}$$

a hyperbolic partial integro-differential equation (PIDE) in the region $\mathcal{T} = \{(y,\eta) : 0 \le \eta \le y \le 1\}$ with boundary conditions

$$\begin{aligned} K(y,y) = {} & -2\pi k \left\{ \frac{Re}{3} i y^2 (2y-3) \right. \\ & \left. + \frac{\cosh(2\pi k) - 1}{\sinh(2\pi k)} \right\} \end{aligned} \tag{81}$$

$$\begin{aligned} K(y,0) = {} & \frac{2\pi k}{\sinh(2\pi k)} \left\{ \cosh(2\pi k y) \right. \\ & - \cosh(2\pi k(1-y)) \\ & + \int_0^y K(k,y,\xi)[\cosh(2\pi k(1-\xi)) \\ & \left. - \cosh(2\pi k \xi)] d\xi \right\}. \end{aligned} \tag{82}$$

Regarding (80)–(82), we have the following result.

Proposition 1: Consider (80) in the domain $(k,y,\eta) \in \mathbb{C} \times \mathcal{T}$ with boundary conditions (81)–(82). There is a solution K, given by (30)–(32), such that K belongs to $\mathcal{C}^2(\mathcal{T})$. Moreover K as a complex-valued function of k is analytic in the annulus $m < |k| < M$.

Proof: We transform (80)–(82) into an integral equation. This is done following the same steps as in [28], by defining new variables $a = y+\eta, b = y-\eta$. Then one obtains a PIDE in a and b and parameterized by k, that can be partially solved by integration, finally reaching an integral equation of Volterra type in two variables. The integral equation can be solved explicitly for each k via a successive approximation series; this explicit solution is given by (30)–(32). For each $k \in \mathbb{C}$, the same method of [28] proves convergence of the series and, hence, the existence of a solution. One gets the following estimate when k is in the annulus $m < |k| < M$

$$|K| \le N e^{2N} \tag{83}$$

where $N = Re(12\pi M + 2\sinh(2\pi M) + 2\pi M \cosh(2\pi M))$. Moreover, using the estimate and the fact that the terms in the series definition (31)–(32) of K are analytic in k, it is shown that the kernel itself is also analytic as a complex function of k, for compact subsets of the annulus $m < |k| < M$. This implies analyticity in the given annulus [24]. The required smoothness in y and η is shown by differentiating the series term by term.■

Remark 6: Proposition 1 implies that the kernel and its first and second order derivatives in y and η are bounded for $m < |k| < M$ and $(y,\eta) \in \mathcal{T}$.

Remark 7: Using Proposition 1, (78)–(79) and Remark 6, it is shown that the backstepping transformation (78) maps the spaces L^2, H^1 and H^2 back to themselves.

From the transformation (78) and the boundary condition (70), the control law is

$$U_c = \int_0^1 K(k,1,\eta) u(t,k,\eta) d\eta. \tag{84}$$

Using the convolution theorem of the Fourier transform (see Remark 5) we write the control law (84) back in physical space. The resulting expressions is (22).

The equation for the inverse kernel L in (79) is similar to the one of K and enjoys similar properties

$$\begin{aligned}\frac{1}{Re}L_{yy} = {} & \frac{1}{Re}L_{\eta\eta} - 8\pi i k y(y-1)L \\ & - 8i\{\pi k(2y-1) - 2\sinh(2\pi k(y-\eta)) \\ & - 2\pi k(2\eta-1)\cosh(2\pi k(y-\eta))\} \\ & - 8i\int_\eta^y \{\pi k(2y-1) - \sinh(2\pi k(y-\xi)) \\ & + 2\pi k(2\xi-1)\cosh(2\pi k(y-\xi))\} \\ & \times L(k,\xi,\eta) d\xi\end{aligned} \tag{85}$$

again a hyperbolic partial integro-differential equation in the region $\mathcal{T}$ with boundary conditions

$$L(y,y) = -2\pi k \left\{ \frac{Re}{3} i y^2(2y-3) + \frac{\cosh(2\pi k) - 1}{\sinh(2\pi k)} \right\} \tag{86}$$

$$L(y,0) = \frac{2\pi k\{\cosh(2\pi k y) - \cosh(2\pi k(1-y))\}}{\sinh(2\pi k)}. \tag{87}$$

The equation can be transformed into an integral equation and calculated via the successive approximation series (45)–(46). A similar result to Proposition 1 holds for L.

By using (68) and (78)–(79), V can also be expressed in terms of α

$$\alpha = i\frac{V_y - \int_0^y K(k,y,\eta)V_y(t,k,\eta)d\eta}{2\pi k} \quad (88)$$

$$V = -2\pi k i \int_0^y \left[1 + \int_\eta^y L(k,\eta,\sigma)d\sigma\right] \times \alpha(t,k,\eta)d\eta. \quad (89)$$

Since (78)–(79) map (74) into (75), stability properties of the velocity field follows from those of the α system.

Proposition 2: For any k in the range $m < |k| < M$, the equilibrium $u(t,k,y) \equiv V(t,k,y) \equiv 0$ of system (57)–(65) with control laws (72), (84) is exponentially stable in the L^2 norm, i.e.,

$$\|V(t,k)\|^2_{\hat{L}^2} + \|u(t,k)\|^2_{\hat{L}^2} \leq D_0 \mathrm{e}^{\frac{-1}{2Re}t}(\|V(0,k)\|^2_{\hat{L}^2} + \|u(0,k)\|^2_{\hat{L}^2}) \quad (90)$$

where D_0 is defined as

$$D_0 = (1+4\pi^2M^2) \times \max_{m<|k|<M}\{(1+\|L\|_\infty)^2(1+\|K\|_\infty)^2\}. \quad (91)$$

Proof: First, from the α (75), it is possible to get an L^2 estimate

$$\|\alpha(t,k)\|^2_{\hat{L}^2} \leq \mathrm{e}^{-\frac{1}{2Re}t}\|\alpha(0,k)\|^2_{\hat{L}^2} \quad (92)$$

then employing the direct and inverse transformations (78)–(79) and (89), we get (90)–(91). ■

Now, if we apply the feedback laws (72) and (84) for *all* wave numbers $m < |k| < M$, then the control laws in physical space are given by expressions (22)–(28), where the inverse transform integrals are truncated at $k = m, M$ in (26)–(28). If we define

$$V^*(t,x,y) = \int_{-\infty}^{\infty} \chi(k)V(t,k,y)\mathrm{e}^{2\pi ikx}dk \quad (93)$$

$$u^*(t,x,y) = \int_{-\infty}^{\infty} \chi(k)u(t,k,y)\mathrm{e}^{2\pi ikx}dk \quad (94)$$

which are variables that contain all velocity field information for wave numbers $m < |k| < M$, the following result holds.

Proposition 3: Consider (8)–(16) with control laws (22)–(23). Then the variables $u^*(t,x,y)$ and $V^*(t,x,y)$ defined in (93)–(94) decay exponentially

$$\|V^*(t)\|^2_{L^2} + \|u^*(t)\|^2_{L^2} \leq D_0\mathrm{e}^{\frac{-1}{2Re}t}\left(\|V^*(0)\|^2_{L^2} + \|u^*(0)\|^2_{L^2}\right). \quad (95)$$

Proof: The Fourier transform of the star variables is, by definition, the same as the Fourier transform of the original variables for $m < |k| < M$, and zero otherwise. Therefore, applying Parseval's formula and Proposition 2

$$\begin{aligned}\|V^*(t)\|^2_{L^2} + \|u^*(t)\|^2_{L^2} &= \int_{-\infty}^{\infty}\left(\|V^*(t,k)\|^2_{\hat{L}^2} + \|u^*(t,k)\|^2_{\hat{L}^2}\right)dk\\ &= \int_{-\infty}^{\infty}\chi(k)\left(\|V(t,k)\|^2_{\hat{L}^2} + \|u(t,k)\|^2_{\hat{L}^2}\right)dk\\ &\leq D_0\mathrm{e}^{\frac{-1}{2Re}t}\int_{-\infty}^{\infty}\chi(k)\left(\|V(0,k)\|^2_{\hat{L}^2} + \|u(0,k)\|^2_{\hat{L}^2}\right)dk\\ &= D_0\mathrm{e}^{-\frac{1}{2Re}t}\left(\|V^*(0)\|^2_{L^2} + \|u^*(0)\|^2_{L^2}\right)\end{aligned} \quad (96)$$

proving (95).

B. Uncontrolled Wave Number Analysis

For the uncontrolled system (57)–(58), we define, for each k, the Lyapunov functional

$$\Lambda(k,t) = \frac{1}{2}\left(\|V(t,k)\|^2_{\hat{L}^2} + \|u(t,k)\|^2_{\hat{L}^2}\right) \quad (97)$$

The time derivative of Λ is

$$\dot{\Lambda} = -\frac{8\pi^2k^2}{Re}\Lambda - \frac{1}{Re}\left(\|u_y(k)\|^2_{\hat{L}^2} + \|V_y(k)\|^2_{\hat{L}^2}\right) + 4\int_0^1(2y-1)\frac{u\bar{V}+\bar{u}V}{2}dy \quad (98)$$

where the bar denotes the complex conjugate, and the pressure term cancels out using integration by parts and the continuity (66). The second term in the first line of (98) can be bounded using the Poincare inequality, thanks to the Dirichlet boundary condition at $y = 0$

$$-\|u_y(k)\|^2_{\hat{L}^2} - \|V_y(k)\|^2_{\hat{L}^2} \leq -\frac{\Lambda}{2}. \quad (99)$$

Consider now separately the two cases $|k| \leq m$ and $|k| \geq M$. In the first case, we can bound the second line of (98) as

$$\dot{\Lambda} \leq -\frac{8\pi^2k^2}{Re}\Lambda - \frac{1}{2Re}\Lambda + 4\Lambda \quad (100)$$

so, if $|k| \geq (1/\pi)\sqrt{(Re/2)}$, then

$$\dot{\Lambda} \leq -\frac{1}{2Re}\Lambda. \quad (101)$$

Now, consider the case of small wave numbers. We bound the second line of (98) using the continuity (66)

$$\dot{\Lambda} \leq -\frac{8\pi^2k^2}{Re}\Lambda - \frac{1}{2Re}\Lambda + 8\pi|k|\Lambda \quad (102)$$

so, if $|k| \leq (1)/(32\pi Re)$, then

$$\dot{\Lambda} \leq -\frac{1}{4Re}\Lambda. \quad (103)$$

We have just proved the following result.

Proposition 4: If $m = (1)/(32\pi Re)$ and $M = (1/\pi)\sqrt{(Re/2)}$, then for both $|k| \leq m$ and $|k| \geq M$ the equilibrium $u(t,k,y) \equiv V(t,k,y) \equiv 0$ of the uncontrolled system (57)–(65) is exponentially stable in the L^2 sense

$$\|V(t,k)\|_{\hat{L}^2}^2 + \|u(t,k)\|_{\hat{L}^2}^2 \leq \mathrm{e}^{\frac{-1}{4Re}t}\left(\|V(0,k)\|_{\hat{L}^2}^2 + \|u(0,k)\|_{\hat{L}^2}^2\right). \quad (104)$$

Since the decay rate in (104) is independent of k, that allows us to claim the following result for *all* uncontrolled wave numbers.

Proposition 5: The variables $\epsilon_u(t,x,y)$ and $\epsilon_V(t,x,y)$ defined as

$$\epsilon_u(t,x,y) = \int_{-\infty}^{\infty}(1-\chi(k))u(t,k,y)\mathrm{e}^{2\pi ikx}dk \quad (105)$$

$$\epsilon_V(t,x,y) = \int_{-\infty}^{\infty}(1-\chi(k))V(t,k,y)\mathrm{e}^{2\pi ikx}dk \quad (106)$$

decay exponentially as

$$\|\epsilon_V(t)\|_{L^2}^2 + \|\epsilon_u(t)\|_{L^2}^2 \leq \mathrm{e}^{\frac{-1}{4Re}t}\left(\|\epsilon_V(0)\|_{L^2}^2 + \|\epsilon_u(0)\|_{L^2}^2\right). \quad (107)$$

Proof: As in Proposition 3. ■

C. Analysis for the Entire Wave Number Range

Using (37)–(38)

$$\begin{aligned}\|V(t)\|_{L^2}^2 &= \int_{-\infty}^{\infty}\|V(t,k)\|_{\hat{L}^2}^2 dk \\ &= \int_0^1\int_{-\infty}^{\infty}(V^*(t,k,y)+\epsilon_V(t,k,y))^2 dkdy \\ &= \int_0^1\int_{-\infty}^{\infty}\left((V^*)^2+\epsilon_V^2+2V^*\epsilon_V\right)dkdy \\ &= \|V^*(t)\|_{L^2}^2 + \|\epsilon_V(t)\|_{L^2}^2 \end{aligned} \quad (108)$$

where we have used the fact that $V^*(t,k,y)\epsilon_V(t,k,y) = \chi(k)(1-\chi(k))V(t,k,y)$ and $\chi(k)(1-\chi(k))$ is zero for all k by its definition (29).

This shows that the L^2 norm of V is the sum of the L^2 norms of $V^*(t,k,y)$ and $\epsilon_V(t,k,y)$. The same holds for u. Therefore, Theorem 1 follows from Propositions 3 and 5. Noting that D_0 as defined in (91) is greater than unity, we obtain the following estimate of the decay:

$$\|V(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 \leq D_0\mathrm{e}^{\frac{-1}{4Re}t}\left(\|V(0)\|_{L^2}^2 + \|u(0)\|_{L^2}^2\right). \quad (109)$$

VI. H^1 Stability

We define the H^1 norm of $f(x,y)$ as

$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|f_x\|_{L^2}^2 + \|f_y\|_{L^2}^2. \quad (110)$$

We also define the H^1 norm of $f(k,y)$ with respect to y as

$$\|f(k)\|_{\hat{H}^1}^2 = (1+4\pi^2k^2)\|f(k)\|_{\hat{L}^2}^2 + \|f_y(k)\|_{\hat{L}^2}^2. \quad (111)$$

The $\hat{H}^1$ norm as a function of k is related to the H^1 norm as

$$\|f\|_{H^1}^2 = \int_{-\infty}^{\infty}\|f(k)\|_{\hat{H}^1}^2 dk. \quad (112)$$

A. H^1 Stability for Controlled Wave Numbers

For each k, one has that

$$\|f(k)\|_{\hat{H}^1}^2 \leq (5+16\pi^2M^2)\|f_y(k)\|_{\hat{H}^1}^2 \quad (113)$$

where we have used (111) and Poincare's inequality. This proves the equivalence, for any k, of the $\hat{H}^1$ norm of $f(k,y)$ and the $\hat{L}^2$ norm of just $f_y(k,y)$. Therefore, we only have to show exponential decay for u_y and V_y.

Due to the backstepping transformations (78) and (79) and (88) and (89)

$$\alpha_y = u_y - K(k,y,y)u - \int_0^y K_y(k,y,\eta)u(t,k,\eta)d\eta \quad (114)$$

$$u_y = \alpha_y + L(k,y,y)\alpha + \int_0^y L_y(k,y,\eta)\alpha(t,k,\eta)d\eta \quad (115)$$

$$\alpha = \frac{-1}{2\pi ki}\left(V_y - \int_0^y K(k,y,\eta)V_y(t,k,\eta)d\eta\right) \quad (116)$$

$$V_y = -2\pi ki\left(\alpha + \int_0^y L(k,y,\eta)\alpha(t,k,\eta)d\eta\right) \quad (117)$$

and then it is possible to write the following estimates, which are derived from simple estimates on α and α_y from (75)

$$\|u_y(t,k)\|_{\hat{L}^2}^2 \leq D_1\mathrm{e}^{-\frac{2}{5Re}t}\|u_y(0,k)\|_{\hat{L}^2}^2 \quad (118)$$

$$\|V_y(t,k)\|_{\hat{L}^2}^2 \leq D_0\mathrm{e}^{-\frac{1}{2Re}t}\|V_y(0,k)\|_{\hat{L}^2}^2 \quad (119)$$

where

$$D_1 = 5\max_{m<|k|<M}\{(1+4\|L\|_\infty+4\|L_y\|_\infty)^2 \times(1+4\|K\|_\infty+4\|K_y\|_\infty)^2\}. \quad (120)$$

Using these estimates the following proposition can be stated regarding the velocity field at each k in the controlled range.

Proposition 6: For any k in the range $m<|k|<M$, the equilibrium $u(t,k,y)\equiv V(t,k,y)\equiv 0$ of the system (57)–(65) with control laws (72), (84) is exponentially stable in the H^1 sense

$$\|V(t,k)\|_{\hat{H}^1}^2 + \|u(t,k)\|_{\hat{H}^1}^2 \leq D_2\mathrm{e}^{\frac{-2}{5Re}t}\left(\|V(0,k)\|_{\hat{H}^1}^2 + \|u(0,k)\|_{\hat{H}^1}^2\right) \quad (121)$$

where D_2 is defined as

$$D_2 = (5 + 16\pi^2 M^2) \max\{D_0, D_1\}. \quad (122)$$

Thanks to the same argument as in Proposition 3, for *all* wave numbers $m < |k| < M$, the following result holds.

Propostion 7: Consider (8)–(16) with control laws (22)–(23). Then the variables $u^*(t,x,y)$ and $V^*(t,x,y)$ defined in (93)–(94) decay exponentially in the H^1 norm

$$\|u^*(t)\|_{H^1}^2 + \|V^*(t)\|_{H^1}^2 \leq D_2 e^{\frac{-2}{5Re}t} \left(\|u^*(0)\|_{H^1}^2 + \|V^*(0)\|_{H^1}^2\right). \quad (123)$$

B. H^1 Stability for Uncontrolled Wave Numbers

Following the same argument as in (97)–(103), a slightly different bound can be derived that keeps some of the $\hat{H}^1$ norm in (102)

$$\dot{\Lambda} \leq -\frac{\Lambda}{8Re} - \frac{\Lambda_H}{2Re} \quad (124)$$

where

$$\Lambda_H(k,t) = \frac{1}{2}\left(\|u_y(t,k)\|_{\hat{L}^2}^2 + \|V_y(t,k)\|_{\hat{L}^2}^2\right). \quad (125)$$

The time derivative of Λ_H can be bounded as

$$\begin{aligned}\frac{d\Lambda_H}{dt} &= \int_0^1 \frac{u_y\bar{u}_{yt} + \bar{u}_y u_{yt} + \bar{V}_y V_{yt} + V_y \bar{V}_{yt}}{2} dy \\ &= -\int_0^1 \frac{u_{yy}\bar{u}_t + \bar{u}_{yy}u_t + \bar{V}_{yy}V_t + V_{yy}\bar{V}_t}{2} dy \\ &= -\frac{1}{Re}\left(\|u_{yy}\|_{\hat{L}^2}^2 + \|V_{yy}\|_{\hat{L}^2}^2\right) \\ &\quad + 4k^2\pi^2 \int_0^1 \frac{u_{yy}\bar{u} + \bar{u}_{yy}u + \bar{V}_{yy}V + V_{yy}\bar{V}}{2Re} dy \\ &\quad + 2\pi ki \int_0^1 \frac{u_{yy}\bar{p} - \bar{u}_{yy}p}{2} dy \\ &\quad - \int_0^1 \frac{\bar{V}_{yy}p_y + V_{yy}\bar{p}_y}{2} dy + 8\pi ki \int_0^1 y(y-1) \\ &\quad \times \frac{u_{yy}\bar{u} - \bar{u}_{yy}u - \bar{V}_{yy}V + V_{yy}\bar{V}}{2} dy \\ &\quad - 4\int_0^1 (2y-1)\frac{u_{yy}\bar{V} + \bar{u}_{yy}V}{2} dy \end{aligned} \quad (126)$$

where we have used integration by parts and the Dirichlet boundary conditions of the uncontrolled wave number range. Doing further integration by parts and using the divergence free condition, we can simplify a little the previous expression

$$\begin{aligned}\frac{d\Lambda_H}{dt} = &-\frac{1}{Re}\left(\|u_{yy}\|_{\hat{L}^2}^2 + \|V_{yy}\|_{\hat{L}^2}^2\right) - \frac{8k^2\pi^2}{Re}\Lambda_h \\ &- 16\pi^2 k^2 \int_0^1 (2y-1)\frac{\bar{u}V - u\bar{V}}{2} dy \\ &- \left.\frac{\bar{V}_{yy}p + V_{yy}\bar{p}}{2}\right|_0^1. \end{aligned} \quad (127)$$

Only the last term remains to be estimated. Using (64) and (65) with V_c being zero for uncontrolled wave number, the last term in (127) can be expresssed as

$$\left.\frac{\bar{V}_{yy}p + V_{yy}\bar{p}}{2}\right|_0^1 = Re\left.\frac{\bar{p}_y p + p_y\bar{p}}{2}\right|_0^1. \quad (128)$$

This quantity can be estimated using the following lemma.

Lemma 1: If the pressure p verifies the Poisson equation (59) with boundary conditions (64)–(65), then

$$-\left.\frac{\bar{p}_y p + p_y\bar{p}}{2}\right|_0^1 \leq 16\|V(t,k)\|_{\hat{L}^2}^2. \quad (129)$$

Proof: Multiplying (59) by $\bar{p}$ and integrating from zero to one, one gets

$$-4\pi^2 k^2 \|p(t,k)\|_{\hat{L}^2}^2 + \int_0^1 \bar{p}p_{yy} dy = \int_0^1 16\pi ki(2y-1)\bar{p}V dy \quad (130)$$

which integrated by parts, becomes

$$-\bar{p}p_y|_0^1 = -4\pi^2 k^2 \|p(t,k)\|_{\hat{L}^2}^2 - \|p_y(t,k)\|_{\hat{L}^2}^2 - \int_0^1 16\pi ki(2y-1)\bar{p}V dy. \quad (131)$$

Now using Young's inequality, one finally arrives at

$$-\bar{p}p_y|_0^1 \leq 16\|V(t,k)\|_{\hat{L}^2}^2. \quad (132)$$

For the other conjugate pair one proceeds analogously, thus completing the proof. ∎

Using the lemma, the time derivative of Λ_H can be estimated as follows:

$$\frac{d\Lambda_H}{dt} \leq -\frac{8k^2\pi^2}{Re}\Lambda_H + 16\pi^2 k^2 \Lambda + 16 Re \Lambda. \quad (133)$$

We take the following Lyapunov functional:

$$\Lambda_T = \Lambda_H + (1 + 64Re^2 + 4\pi^2 k^2 + 64 Re \pi^2 k^2)\Lambda \quad (134)$$

which is *equivalent* to the H^1 norm, whose definition in terms of Λ and Λ_H is

$$\|u(t,k)\|_{\hat{H}^1}^2 + \|V(t,k)\|_{\hat{H}^1}^2 = 2(1 + 4\pi^2 k^2)\Lambda + 2\Lambda_H. \quad (135)$$

Computing the derivative of (134)

$$\frac{d\Lambda_T}{dt} \leq -\frac{\Lambda_H}{2Re} - \frac{1 + 4\pi^2 k^2}{8Re}\Lambda \leq -d_1 \Lambda_T \quad (136)$$

where d_1 is a (possible very conservative) positive constant, which depends on the Reynolds number (but *not* on k)

$$d_1 = \frac{1}{8 D_3 Re} \quad (137)$$

and where

$$D_3 = \max\{1 + 64Re^2, 1 + 16Re\}. \tag{138}$$

Deriving an estimate of the H^1 norm from this estimate for Λ_T, one reaches the following result.

Proposition 8: If $m = (1)/(32\pi Re)$ and $M = (1/\pi)\sqrt{(Re/2)}$, then for both $|k| \le m$ and $|k| \ge M$ the equilibrium $u(t,k,y) \equiv V(t,k,y) \equiv 0$ of the uncontrolled system (57)–(65) is exponentially stable in the H^1 sense

$$\|V(t,k)\|^2_{\hat{H}^1} + \|u(t,k)\|^2_{\hat{H}^1} \le D_3 e^{-d_1 t}\left(\|V(0,k)\|^2_{\hat{H}^1} + \|u(0,k)\|^2_{\hat{H}^1}\right). \tag{139}$$

Since the decay rate in (139) is independent of k, that allows us to claim the following result for *all* uncontrolled wave numbers.

Proposition 9: The variables $\epsilon_u(t,x,y)$ and $\epsilon_V(t,x,y)$ defined as in (105)–(106) decay exponentially in the H^1 norm as

$$\|\epsilon_u(t)\|^2_{H^1} + \|\epsilon_V(t)\|^2_{H^1} \le D_3 e^{-d_1 t}\left(\|\epsilon_u(0)\|^2_{H^1} + \|\epsilon_V(0)\|^2_{H^1}\right). \tag{140}$$

C. Analysis for All Wave Numbers

From Propositions 7 and 9, and using the same argument as in Section V-C, the H^1 stability part of Theorem 1 is proved. One gets that

$$\|u(t)\|^2_{H^1} + \|V(t)\|^2_{H^1} \le D_4 e^{-d_1 t}\left(\|u(0)\|^2_{H^1} + \|V(0)\|^2_{H^1}\right) \tag{141}$$

where $D_4 = \max\{D_2, D_3\}$.

VII. H^2 Stability

The H^2 norm of $f(x,y)$ is defined as

$$\|f\|^2_{H^2} = \|f\|^2_{H^1} + \|f_{xx}\|^2_{L^2} + \|f_{xy}\|^2_{L^2} + \|f_{yy}\|^2_{L^2}. \tag{142}$$

We also define the H^2 norm of $f(k,y)$ with respect to y as

$$\|f(k)\|^2_{\hat{H}^2} = \|f(k)\|^2_{\hat{H}^1} + 16\pi^4k^4\|f(k)\|^2_{\hat{L}^2} + 4\pi^2k^2\|f_y(k)\|^2_{\hat{L}^2} + \|f_{yy}(k)\|^2_{\hat{L}^2}. \tag{143}$$

The $\hat{H}^2$ norm as a function of k is related to the H^2 norm as

$$\|f\|^2_{H^2} = \int_{-\infty}^{\infty} \|f(k)\|^2_{\hat{H}^2} dk. \tag{144}$$

A. H^2 Stability for Controlled Wave Numbers

Thanks to the backstepping transformations (78) and (79) and (88) and (89), one calculates the second order derivative of both u and V from α and its derivatives

$$\begin{aligned}\alpha_{yy} = u_{yy} &- K(k,y,y)u_y \\ &- (2K_y(k,y,y) + K_\eta(k,y,y))u \\ &- \int_0^y K_{yy}(k,y,\eta)u(t,k,\eta)d\eta\end{aligned} \tag{145}$$

$$\begin{aligned}u_{yy} = \alpha_{yy} &+ L(k,y,y)\alpha_y \\ &+ (2L_y(k,y,y) + L_\eta(k,y,y))\alpha \\ &+ \int_0^y L_{yy}(k,y,\eta)\alpha(t,k,\eta)d\eta\end{aligned} \tag{146}$$

$$\alpha_y = \frac{-1}{2\pi ki}\left(V_{yy} - K(k,y,y)V_y - \int_0^y K_y(k,y,\eta)V_y(t,k,\eta)d\eta\right) \tag{147}$$

$$V_{yy} = -2\pi ki\left(\alpha_y + L(k,y,y)\alpha + \int_0^y L_y(k,y,\eta)\alpha(t,k,\eta)d\eta\right). \tag{148}$$

It is possible then to write the following estimates, which are derived from simple estimates on α, α_y, and α_{yy} from (75)

$$\|u(t,k)\|^2_{\hat{H}^2} \le D_5 e^{-\frac{2}{5Re}t}\|u(0,k)\|^2_{\hat{H}^2} \tag{149}$$

$$\|V(t,k)\|^2_{\hat{H}^2} \le D_6 e^{-\frac{2}{5Re}t}\|V(0,k)\|^2_{\hat{H}^2}. \tag{150}$$

The positive constants D_5 and D_6 are defined as in (120) and depend only on K and L.

Using these estimates the following proposition can be stated regarding the velocity field at each k in the controlled range.

Proposition 10: For any k in the range $m < |k| < M$, the equilibrium $u(t,k,y) \equiv V(t,k,y) \equiv 0$ of the system (57)–(65) with control laws (72), (84) is exponentially stable in the H^2 sense

$$\|V(t,k)\|^2_{\hat{H}^2} + \|u(t,k)\|^2_{\hat{H}^2} \le D_7 e^{\frac{-2}{5Re}t}\left(\|V(0,k)\|^2_{\hat{H}^2} + \|u(0,k)\|^2_{\hat{H}^2}\right) \tag{151}$$

where D_7 is defined as

$$D_7 = \max\{D_5, D_6\}. \tag{152}$$

Thanks to the same argument as in Proposition 3, the following result holds for *all* wave numbers $m < |k| < M$.

Proposition 11: Consider (8)–(16) with control laws (23)–(22). Then the variables $u^*(t,x,y)$ and $V^*(t,x,y)$ defined in (93)–(94) decay exponentially in the H^2 norm

$$\|u^*(t)\|^2_{H^2} + \|V^*(t)\|^2_{H^2} \le D_8 e^{\frac{-2}{5Re}t}\left(\|u^*(0)\|^2_{H^2} + \|V^*(0)\|^2_{H^2}\right). \tag{153}$$

B. H^2 Stability for Uncontrolled Wave Numbers

For the uncontrolled wave number range, thanks to the Dirichlet boundary conditions, the $\hat{H}^2$ norm $\|u(t,k)\|_{\hat{H}^2}$ is equivalent to the norm

$$\|u(t,k)\|^2_{\hat{H}^1} + \int_0^1 |u_{yy}(t,k,y) - 4\pi^2k^2u(t,k,y)|^2 dy \tag{154}$$

i.e., to the $\hat{H}^1$ norm plus the $\hat{L}^2$ norm of the Laplacian, which we denote for short $\|\triangle_k u(k)\|^2_{\hat{L}^2}$. The proof of the norm equivalence is obtained integrating by parts

$$\begin{aligned}\|\triangle_k u(k)\|^2_{\hat{L}^2} &= \int_0^1 |-4\pi^2k^2u(y,k)+u_{yy}(y,k)|^2 dy\\ &= \int_0^1 [16\pi^4k^4|u|^2(y,k)+|u_{yy}|^2(y,k)\\ &\quad - 4\pi^2k^2(u\bar{u}_{yy}+\bar{u}u_{yy})]\,dy\\ &= 16\pi^4k^4\|u(k)\|^2_{\hat{L}^2}+\|u_{yy}(k)\|^2_{\hat{L}^2}\\ &\quad + 8\pi^2k^2\|u_y(k)\|^2_{\hat{L}^2}. \end{aligned} \tag{155}$$

The next norm equivalence property is less obvious and we state it in the following lemma.

Lemma 2: Consider u and V verifying (57)–(58). Then, for the uncontrolled wave number range, the norm $\|u\|^2_{\hat{H}^2}+\|V\|^2_{\hat{H}^2}$ is equivalent to the norm

$$\|u\|^2_{\hat{H}^1}+\|V\|^2_{\hat{H}^1}+\|u_t\|^2_{\hat{L}^2}+\|V_t\|^2_{\hat{L}^2}. \tag{156}$$

This means the Laplacian operator in norm (154) can be replaced by a time derivative, when considering the H^2 norm of u and V together.

Proof: Let us call

$$\Lambda_1 = \|u_t(t,k)\|^2_{\hat{L}^2}+\|V_t(t,k)\|^2_{\hat{L}^2} \tag{157}$$

$$\Lambda_2 = \frac{\|\triangle_k u(t,k)\|^2_{\hat{L}^2}+\|\triangle_k V(t,k)\|^2_{\hat{L}^2}}{Re^2}. \tag{158}$$

Substituting in (157) (57)–(58),

$$\Lambda_1 = \Lambda_2+\Lambda_3 \tag{159}$$

where Λ_3 contains the following terms:

$$\begin{aligned}\Lambda_3 = &-\int_0^1 \frac{-2\pi ki\triangle_k u\bar{p}+\triangle_k V\bar{p}_y}{Re}dy\\ &-2\pi ki\int_0^1 4y(1-y)\frac{\triangle_k u\bar{u}+\triangle_k V\bar{V}}{Re}dy\\ &+\int_0^1 4(1-2y)\frac{\triangle_k u\bar{V}}{Re}dy\\ &-\int_0^1 (2\pi kip\bar{u}_t+p_y\bar{V}_t)\,dy\\ &+2\pi ki\int_0^1 4y(1-y)(u\bar{u}_t+v\bar{V}_t)\,dy\\ &+\int_0^1 4(1-2y)(V\bar{u}_t)\,dy. \end{aligned} \tag{160}$$

Now one can estimate this quantity

$$|\Lambda_3| \le 48(\|u(k)\|^2_{\hat{H}^1}+\|V(k)\|^2_{\hat{H}^1})+\frac{1}{2}(\Lambda_1+\Lambda_2) \tag{161}$$

in which we have used integration by parts, Young's inequality, and Lemma 1. Therefore

$$\|u\|^2_{\hat{H}^2}+\|V\|^2_{\hat{H}^2} \le D_8\left(\|u\|^2_{\hat{H}^1}+\|V\|^2_{\hat{H}^1}+\Lambda_1\right) \tag{162}$$

and

$$\|u\|^2_{\hat{H}^1}+\|V\|^2_{\hat{H}^1}+\Lambda_1 \le D_8\left(\|u\|^2_{\hat{H}^2}+\|V\|^2_{\hat{H}^2}\right) \tag{163}$$

where $D_8 = 97\max\{Re^2, 1/Re^2\}$. ∎

From Lemma 2 one gets $\hat{H}^2$ stability for the uncontrolled wave numbers. This is obtained by considering the norm $\|u_t\|^2_{\hat{L}^2}+\|V_t\|^2_{\hat{L}^2}$ as a Lyapunov functional whose derivative can be bounded as

$$\frac{d}{dt}\frac{\|u_t\|^2_{\hat{L}^2}+\|V_t\|^2_{\hat{L}^2}}{2} \le -\frac{1}{4Re}\left(\|u_t\|^2_{\hat{L}^2}+\|V_t\|^2_{\hat{L}^2}\right) \tag{164}$$

which follows by taking the time derivative of (57)–(58) and applying the same argument as for L^2 stability. Thus

$$\begin{aligned}&\|u_t(t,k)\|^2_{\hat{L}^2}+\|V_t(t,k)\|^2_{\hat{L}^2}\\ &\quad\le \mathrm{e}^{-\frac{1}{2Re}t}\left(\|u_t(0,k)\|^2_{\hat{L}^2}+\|V_t(0,k)\|^2_{\hat{L}^2}\right). \end{aligned} \tag{165}$$

Noting that $d_1 \le 1/2Re$ and $D_3 \ge 1$, adding (165) to (139) and employing (162), (162), we obtain the following result.

Proposition 12: If $m = (1)/(32\pi Re)$ and $M = (1/\pi)\sqrt{(Re/2)}$, then for both $|k| \le m$ and $|k| \ge M$ the equilibrium $u(t,k,y) \equiv V(t,k,y) \equiv 0$ of the uncontrolled system (57)–(65) is exponentially stable in the H^2 sense

$$\begin{aligned}&\|V(t,k)\|^2_{\hat{H}^2}+\|u(t,k)\|^2_{\hat{H}^2}\\ &\quad\le D_8^2D_3\mathrm{e}^{-d_1t}\left(\|V(0,k)\|^2_{\hat{H}^2}+\|u(0,k)\|^2_{\hat{H}^2}\right). \end{aligned} \tag{166}$$

Since the decay rate in (166) is independent of k, that allows us to claim the following result for *all* uncontrolled wave numbers.

Proposition 13: The variables $\epsilon_u(t,x,y)$ and $\epsilon_V(t,x,y)$ defined as in (105)–(106) decay exponentially in the H^2 norm as

$$\begin{aligned}&\|\epsilon_u(t)\|^2_{H^2}+\|\epsilon_V(t)\|^2_{H^2}\\ &\quad\le D_8^2D_3\mathrm{e}^{-d_1t}\left(\|\epsilon_u(0)\|^2_{H^2}+\|\epsilon_V(0)\|^2_{H^2}\right). \end{aligned} \tag{167}$$

C. Analysis for All Wave Numbers

From Propositions 11 and 13, and again by the same argument as in Section V-C, the H^2 stability part of Theorem 1 is proved. One gets that

$$\begin{aligned}&\|u(t)\|^2_{H^2}+\|V(t)\|^2_{H^2}\\ &\quad\le D_9\mathrm{e}^{-d_1t}\left(\|u(0)\|^2_{H^2}+\|V(0)\|^2_{H^2}\right) \end{aligned} \tag{168}$$

where $D_9 = \max\{D_7, D_8^2D_3\}$.

VIII. Proof of Well-Posedness and Explicit Solutions

For showing well-posedness and derive the explicit solutions we decompose the system in two parts in the wave number space, using (37)–(38). The star variables represent the controlled wave numbers in physical space and are defined in (93)–(94). The epsilon variables represent the uncontrolled wave number content in physical space and are defined in (105)–(105). Consider then the initial conditions u_0 and V_0 in Fourier space. Define

$$V_0^*(x,y) = \int_{-\infty}^{\infty}\chi(k)V_0(k,y)\mathrm{e}^{2\pi ikx}dk \tag{169}$$

$$u_0^*(x,y) = \int_{-\infty}^{\infty}\chi(k)u_0(k,y)\mathrm{e}^{2\pi ikx}dk \tag{170}$$

and similarly

$$\epsilon_{V0}(x,y) = \int_{-\infty}^{\infty} (1-\chi(k))V_0(k,y)\mathrm{e}^{2\pi ikx}dk \quad (171)$$

$$\epsilon_{u0}(x,y) = \int_{-\infty}^{\infty} (1-\chi(k))u_0(k,y)\mathrm{e}^{2\pi ikx}dk. \quad (172)$$

Note that $V_0^*, u_0^*, \epsilon_{V0}, \epsilon_{u0} \in H^2(\Omega)$ and also verify the required compatibility conditions. Define the following initial-boundary value problems for the star and epsilon variables:

$$P_1 \equiv \begin{cases} (u^*, V^*) \text{ verify (8)-(9) and (17)} \\ u^*(t,x,0) = V^*(t,x,0) = 0 \\ u^*(t,x,1) = (22) \\ V^*(t,x,1) = (23) \\ u^*(0,x,y) = u_0^*(x,y), V^*(0,x,y) = V_0^*(x,y). \end{cases}$$

and

$$P_2 \equiv \begin{cases} (\epsilon_u, \epsilon_V)\text{verify(8)-(9) and (17)} \\ \epsilon_u(t,x,0) = \epsilon_V(t,x,0) = 0 \\ \epsilon_u(t,x,1) = \epsilon_V(t,x,1) = 0 \\ \epsilon_u(0,x,y) = \epsilon_{u0}(x,y), \epsilon_V(0,x,y) = \epsilon_{V0}(x,y). \end{cases}$$

By linearity and spatial invariance (implying that different wave numbers are independent of each other), the solution of the linearized Navier–Stokes equations is the sum of the solutions of P_1 and P_2. Hence, if both systems are well-posed then the original problem is well-posed too.

System P_2 is the (uncontrolled) channel flow with no-slip, no-penetration boundary conditions. See [32, Proposition 1.2, pages 265–269] for an analysis of the linear Navier–Stokes equations and their regularity that allows for unbounded domains. It is shown that with the given degree of regularity of the initial conditions, the problem is well posed in the space $L^2((0,T), H^2(\Omega))$. Combining this fact with the *a priori* bounds of Section VII, it follows that P_2 is well-posed in $L^2((0,\infty), H^2(\Omega))$. Moreover the decay rate of the epsilon variables is given in Proposition 5.

We prove now that P_1 is well-posed using a direct method, taking advantage of the possibility of writing the exact solution of the problem. This is a classical way of showing well-posedness, see, for example [14], where existence of solutions is shown by solving explicitly the problems with Fourier transform and series methods, and uniqueness is proved by energy methods.

Explicit solutions (39)–(40) are obtained in the following way. Equation (75) is a heat equation in $\alpha(t,k,y)$ and can be solved explicitly. The initial condition for this equation is

$$\alpha_0(k,y) = u_0^*(k,y) - \int_0^y K(k,y,\eta)u_0^*(k,\eta)d\eta \quad (173)$$

and, since $u_0^* \in H^2(\Omega)$, then $\alpha_0 \in H^2(\Omega)$, moreover the compatibility condition $\alpha_0(k,0) = \alpha_0(k,1) = 0$ is verified from (173) and (48). Hence, the solution of (75), a stable heat equation, is $L^2((0,\infty), H^2(\Omega))$. Using then (79) and (89), and applying the inverse Fourier transform, the solution for u^* and V^* is recovered in physical space, as given by (39)–(40). Both the inverse backstepping transformation and the inverse Fourier transform map $L^2((0,\infty), H^2(\Omega))$ back into itself; hence, the existence of a solution with the desired regularity properties follows. Explicit formulas can be written for the control laws; in particular $V_c(t)$ is well-defined as the traces $u_y(0,k)$ and $u_y(1,k)$ appearing in (25) can be computed due to the regularity of u.

Remark 8: Note that, in fact, a higher regularity can be proved for P_1, due to the smoothing properties of the heat equation. We don't pursue more than H^2 regularity in this work. Note also that regularity in x, which is determined in Fourier space by the behavior of the solution for large values of k, is guaranteed for P_1 because the solution is nonzero only for a finite subset of wave numbers. Hence, the solution of P_1 is smooth in x.

Uniqueness follows from the *a priori* bounds shown in Sections V–VII. Given two solutions in the same space, their difference verifies as well the *a priori* bounds with zero initial conditions; hence, its norm is zero for all times and both solutions must be the same.

IX. Proof of Theorem 2

Consider expressions (22)–(32).

Points i and iv are deduced trivially from the fact that (22) and (25) are defined as convolutions, and properties of the heat (24).

Point ii is verified if

$$\int_{-\infty}^{\infty} V_c(t,x)dx = 0. \quad (174)$$

From the definition of the Fourier transform of V_c

$$V_c(t,k=0) = \int_{-\infty}^{\infty} V_c(t,x)\,dx. \quad (175)$$

Therefore, as $k=0$ lies on the uncontrolled wave number range $-m < k < m$, then $V_c(t,k=0) = 0$ and the property is verified.

Point iii bounds the decay rate of kernels (26)–(28). Kernel definitions are of the form

$$Q(x-\xi,y) = \int_{-\infty}^{\infty} \chi(k)f(k,y)\mathrm{e}^{2\pi ik(x-\xi)}dk \quad (176)$$

for some f analytic in k and smooth in y. Then, integrating by parts, we find that

$$\begin{aligned} |Q(x-\xi,y)| &\leq \frac{(M-m)}{\pi|x-\xi|} \max_{m<|k|<M} \left|\frac{df}{dk}(k,y)\right| \\ &\quad + \frac{2}{\pi|x-\xi|} \max_{m<|k|<M} |f(k,y)| \\ &= \frac{C}{|x-\xi|} \end{aligned} \quad (177)$$

showing that the kernels decay at least like $1/|x-\xi|$. This bound is made explicit in Remark 2.

From the definition of the inverse Fourier transform (50), it is straightforward to show that if the real part of $f(k,y)$ is even and the imaginary part of $f(k,y)$ is odd, then the resulting $f(x,y)$ will always be real. Then, Point v can be proved showing that the functions under the integrals in (26)–(28), which are inverse Fourier transforms, have this property. This is immediate for

(27) and (28). For (26), the property must be shown for the kernel K, defined by the sequence (31)–(32). Since K is the limit of the sequence, it will have the property if all K_n share the property. This can be proved by induction. For K_0, the property is evident from its definition (31) and can be immediately verified. For K_n, if the property is assumed for K_{n-1}, then from expression (32) and taking into account that the product of even or odd functions is even and the product of an even function times an odd function is odd, then follows that K_n also shares the property. Therefore, the limit K has a real inverse transform, and kernel Q_u is real.

Point vi is deduced from the definition of the kernels (26)–(28) as truncated Fourier inverse integrals, which makes the kernels smooth in $x-\xi$. Smoothness in η is deduced from smoothness of the functions under the integrals.

For Point vii, consider expression (22) and (26). Then

$$\begin{aligned}\|U_c\|_{L^2}^2 &= \int_{-\infty}^{\infty} U_c(t,x)^2 dx \\ &= \int_{-\infty}^{\infty} |U_c|(t,k)^2 dk \\ &= \int_{-\infty}^{\infty} \chi(k) \left| \int_0^1 K(k,1,\eta) u(t,y,k) d\eta \right|^2 dk \\ &\le 2(M-m) \max_{m\le|k|\le M} \{\|K\|_\infty\} \|u(t)\|_{L^2}^2 \end{aligned} \tag{178}$$

and the result follows from Theorem 1.

On the other hand, for V_c one has to use its dynamic (24)–(25), and a Lyapunov functional consisting in half its L^2 norm. One then has, using Young's inequality

$$\begin{aligned}\frac{d}{dt}\frac{|V_c(k)|^2}{2} \le& \frac{-\pi^2 k^2}{Re}|V_c(k)|^2 \\ &+ \frac{|u_y|^2(t,k,0)+|u_y|^2(t,k,1)}{Re} \\ &+ 64\cosh(2\pi M)\|V(t,k)\|_{\hat{L}^2}^2 \end{aligned} \tag{179}$$

and supposing the control law is initialized at zero (see Remark 1), and using the H^2 norm to bound the second line of (179) one gets

$$\begin{aligned}|V_c(t,k)|^2 \le \int_0^t \mathrm{e}^{-\frac{\pi^2 m^2}{Re}(t-\tau)} \Bigg[& 10\frac{\|u(\tau,k)\|_{\hat{H}^2}^2}{Re} \\ & +64\cosh(2\pi M)\|V(\tau,k)\|_{\hat{L}^2}^2 \Bigg] d\tau. \end{aligned} \tag{180}$$

Integrating in k

$$\begin{aligned}\|V_c(t)\|_{L^2}^2 \le \int_0^t \mathrm{e}^{-\frac{\pi^2 m^2}{Re}(t-\tau)} \Bigg[& 10\frac{\|u(\tau)\|_{H^2}^2}{Re} \\ & +64\cosh(2\pi M)\|V(\tau)\|_{L^2}^2 \Bigg] d\tau \end{aligned} \tag{181}$$

and then the result follows from Theorem 1.

For Point viii, consider the jth spatial derivative of U_c and calculate its L_2 spatial norm

$$\begin{aligned}\left\| \frac{d^j}{dx^j} U_c \right\|_{L^2}^2 &= \int_{-\infty}^{\infty} \left(\frac{d^j}{dx^j} U_c(t,x) \right)^2 dx \\ &= \int_{-\infty}^{\infty} |2\pi k|^{2j} |U_c|(t,k)^2 dk \\ &\le (2\pi M)^{2j} \|U_c\|_{L^2}^2 \end{aligned} \tag{182}$$

so the result for U_c follows from Point vii. We proceed similarly for V_c, thus proving Point viii.

X. Discussion

The result was presented in 2-D for ease of notation. Since 3-D channels are spatially invariant in both streamwise and spanwise direction, it is possible to extend the design to 3-D, by applying the Fourier transform in both invariant directions and following the same steps, with some refinements which include actuation of the spanwise velocity at the wall. The result also trivially extends to periodic channel flow of arbitrary periodic box size, 2-D or 3-D; it only requires substitution of the Fourier transform by Fourier series, with all other expressions still holding.

Our control laws are presented with full state feedback. However, for parabolic PDEs, in [29], we developed an observer design methodology, which is dual to the backstepping control methodology in [28], which we extended to Navier–Stokes equations here to solve the state feedback stabilization problem for the channel flow. Extending the observer concepts in [29] to the Navier–Stokes PDEs has allowed us to also develop an observer for the channel flow, which is presented in the conference paper [33]. While the observer is of interest in its own right (one can use it to estimate turbulent flows without controlling/relaminarizing them), the state feedback controller in the present paper and the observer in [33] can be combined into an output feedback compensator, which uses measurements of $P(x,0)$ and $u_y(x,0)$ only, and the actuation of $V(x,1)$, $u(x,1)$.

Our controller requires actuation of both velocity components at the wall. An assumption made throughout the flow control literature is that the boundary values of velocity are actuated through micro-jet actuators that perform "zero-mean" blowing and suction. Effective actuation of wall velocity at angles as low as 5° relative to the wall has been demonstrated experimentally using differentially actuated pairs of jets.

Unlike in our earlier publications [1], [7], where we included DNS simulation results that demonstrated relaminarization with our controllers, we do not present simulation results in this paper. In another publication, to be submitted to a fluid mechanics journal, we will present an extension to 3-D, without the H^1, H^2 stability estimates and without the explicit closed-loop Navier–Stokes solutions (these two issues extend in a rather straightforward manner to 3-D because we deal with linearized Navier–Stokes equations), but with simulations results included. The 3-D controller will include actuation in the spanwise direction. The numerical tests will focus on turbulence-critical issues like the behavior of the controller at $k_x = 0$ for moderate-to-large k_z and other issues which come up only in 3-D.

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Rafael Vazquez (S'05) received the B.S. degrees in electrical engineering and mathematics from the University of Seville, Spain, and the M.S. and Ph.D. degrees in aerospace engineering from the University of California at San Diego, La Jolla.

He is an Assistant Professor in the Aerospace and Fluid Mechanics Department, University of Seville. He is a coauthor of the book *Control of Turbulent and Magnetohydrodynamic Channel Flows* (Birkhauser, 2007). He was a CTS Marie Curie Fellow at the Universit Paris-Sud (2005). His research interests include nonlinear control, control of distributed parameter systems, dynamical systems, and applications to flow control, orbital mechanics, air traffic control, and nanomechatronics.

Dr. Vazquez was a finalist for the Best Student Paper Award in the 2005 CDC.

Miroslav Krstic (S'92–M'95–SM'99–F'02) is the Sorenson Professor of Mechanical and Aerospace Engineering and the Director of the newly formed Center for Control Systems and Dynamics (CCSD) at the University of California at San Diego (UCSD), La Jolla. He is a coauthor of the books *Nonlinear and Adaptive Control Design* (Wiley, 1995), *Stabilization of Nonlinear Uncertain Systems* (Springer, 1998), *Flow Control by Feedback* (Springer, 2002), *Real Time Optimization by Extremum Seeking Control* (Wiley, 2003), and *Control of Turbulent and Magnetohydrodynamic Channel Flows* (Birkhauser, 2007).

Dr. Krstic received the National Science Foundation Career, ONR YI, and PECASE Awards, as well as the Axelby and the Schuck paper prizes. In 2005, he was the first engineering professor to receive the UCSD Award for Research. He is a Fellow of IEEE, a Distinguished Lecturer of the Control Systems Society, and a former CSS VP for Technical Activities. He has served as an Associate Editor for several journals and is currently an Editor for *Adaptive and Distributed Parameter Systems in Automatica*.