Available online at www.sciencedirect.com
ScienceDirect
automatica
Automatica 43 (2007) 1557 – 1564
www.elsevier.com/locate/automatica

Brief paper

# Adaptive boundary control for unstable parabolic PDEs—Part III: Output feedback examples with swapping identifiers☆

Andrey Smyshlyaev, Miroslav Krstic*

*Department of Mechanical and Aerospace Engineering, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0411, USA*

Received 30 June 2005; received in revised form 3 November 2006; accepted 15 February 2007
Available online 20 July 2007

**Abstract**

We develop output-feedback adaptive controllers for two benchmark parabolic PDEs motivated by a model of thermal instability in solid propellant rockets. Both benchmark plants are unstable, have infinite relative degree, and are controlled from the boundary. One plant has an unknown parameter in the PDE and the other in the boundary condition. In both cases the unknown parameter multiplies the measured output of the system, which is obtained with a boundary sensor located on the "opposite side" of the domain from the actuator. In comparison with the Lyapunov output-feedback adaptive controllers in Krstic and Smyshlyaev [(2005). Adaptive boundary control for unstable parabolic PDEs—Part I: Lyapunov design. *IEEE Transactions on Automatic Control*, submitted for publication], the controllers presented here employ much simpler update laws and do not require a priori knowledge about the unknown parameters. We show how our two benchmarks examples can be combined and illustrate the adaptive stabilization design by simulation.


*Keywords:* Adaptive control; Boundary control; Distributed parameter systems

## 1. Introduction

In a companion paper (Smyshlyaev & Krstic, 2007) we introduced a novel approach to *adaptive* control of PDEs where a parametrized family of boundary controllers can be combined with "swapping gradient" identifiers to yield global stability of the resulting nonlinear PDE system. Only the *state*-feedback problem was considered in Smyshlyaev and Krstic (2007). For a different, narrower, class of systems, the *output*-feedback problem is solvable by this method, which is illustrated on two benchmark examples in this paper.

We consider two parametrically uncertain, *unstable* parabolic PDE plants controlled from the *boundary*. While these benchmark plants are simple in appearance, there does not exist an adaptive control design in the literature that is applicable to them due to the fact that they have infinite relative degree. Infinite relative degree arises in applications where actuators and sensors are on the "opposite sides" of the PDE domains. The two benchmark problems in this paper are motivated by a model of thermal instability in solid propellant rockets (Boskovic & Krstic, 2003). Our control laws are adaptive versions of the explicit boundary control laws developed in Smyshlyaev and Krstic (2004, 2005). Our adaptive observers are infinite dimensional extensions of Kreisselmeier observers (Krstic, Kanellakopoulos, & Kokotovic, 1995). Our identifiers are designed using the swapping approach (Krstic et al., 1995), prevalent in adaptive control of finite dimensional systems of relative degree higher than one. These identifiers remove the need for parameter projection and low adaptation gain present in Lyapunov output-feedback designs in Krstic and Smyshlyaev (2005).

The overview of the prior literature on the subject is presented in a companion paper (Smyshlyaev & Krstic, 2007).

Although for the sake of clarity we consider two separate benchmark problems, it is possible to design an adaptive controller for a combined problem (Section 6). Another reason for a separate consideration is a slightly weaker result for the benchmark plant with the unknown parameter in the boundary

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Henri Huijberts under the direction of Editor Hassan Khalil.

* Corresponding author. Tel.: +1 858 8222406; fax: +1 858 8223107
*E-mail addresses:* asmyshly@ucsd.edu (A. Smyshlyaev), krstic@ucsd.edu (M. Krstic).

0005-1098/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2007.02.015

condition, due to an inherent difficulty observed in Bentsman and Orlov (2001) and Liu and Krstic (2001).

Throughout the paper we assume well posedness of the closed-loop systems in the interest of space and due to the parabolic character of these systems which ensures their benign behavior, as supported by numerical results that we show in this paper. For an example on how one proves well posedness, see Krstic and Smyshlyaev (2005).

**Notation.** The spatial $L_2(0,1)$ norm is denoted by $\|\cdot\|$. The temporal norms are denoted by $\mathscr{L}_\infty$ and $\mathscr{L}_2$ for $t\geqslant 0$. We denote by $l_1$ a generic function in $\mathscr{L}_\infty \cap \mathscr{L}_2$.

## 2. Benchmark plant with unknown parameter in the domain

Consider the following plant:

$$u_t(x,t) = u_{xx}(x,t) + gu(0,t), \tag{1}$$

$$u_x(0,t) = 0, \tag{2}$$

$$u(1,t) = U(t), \tag{3}$$

where $U(t)$ is a control signal. This system is inspired by a model of thermal instability in solid propellant rockets (Boskovic & Krstic, 2003). For $U(t)=0$ this system is unstable if and only if $g>2$. The plant can be written in the frequency domain as a transfer function from input $u(1)$ to output $u(0)$

$$u(0,s) = \frac{s}{(s-g)\cosh\sqrt{s}+g}u(1,s). \tag{4}$$

We can see that it has no zeros (at $s=0$ the transfer function is $2/(2-g)$) and has infinitely many poles, one of which is unstable and approximately equal to $g$ as $g\to+\infty$. So this is an infinite relative degree system.

Our main result for this problem is summarized in the following theorem.

**Theorem 1.** *Consider the system* (1)–(2) *with the controller*

$$u(1,t) = \int_0^1 \hat{k}(1,\xi)(\hat{g}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi, \tag{5}$$

$$\hat{k}(x,\xi) = \begin{cases} -\sqrt{\hat{g}}\sinh\sqrt{\hat{g}}(x-\xi), & \hat{g}\geqslant 0,\\ \sqrt{-\hat{g}}\sin\sqrt{-\hat{g}}(x-\xi), & \hat{g}<0, \end{cases} \tag{6}$$

*where an update law for* $\hat{g}$ *is*

$$\dot{\hat{g}} = \gamma\frac{(u(0,t)-\hat{g}v(0,t)-\eta(0,t))v(0,t)}{1+v^2(0,t)}, \tag{7}$$

*and the filters* $v(x,t)$, $\eta(x,t)$ *are defined as*

$$v_t(x,t) = v_{xx}(x,t) + u(0,t), \tag{8}$$

$$v_x(0,t) = 0, \tag{9}$$

$$v(1,t) = 0, \tag{10}$$

$$\eta_t(x,t) = \eta_{xx}(x,t), \tag{11}$$

$$\eta_x(0,t) = 0, \tag{12}$$

$$\eta(1,t) = u(1,t). \tag{13}$$

*If the closed loop system* (1)–(2), (5)–(13) *has a classical solution* $(u,\hat{g},v,\eta)$, *then for any* $\hat{g}(0)$ *and any initial conditions* $u_0, v_0, \eta_0 \in H_1(0,1)$, *the signals* $\hat{g}, u, v, \eta$ *are bounded and* $u$ *is regulated to zero for all* $x\in[0,1]$

$$\lim_{t\to\infty}\max_{x\in[0,1]}|u(x,t)| = 0. \tag{14}$$

Note that the control gain (6) is a smooth function of $\hat{g}$. Note also that a priori knowledge of a bound on $\hat{g}$ is not required in the swapping (5)–(13) (as opposed to Lyapunov adaptive design in Krstic & Smyshlyaev, 2005).

## 3. Proof of Theorem 1

### 3.1. Target system

Introducing the error $e=u-gv-\eta$ we get an exponentially stable system

$$e_t(x,t) = e_{xx}(x,t), \tag{15}$$

$$e_x(0,t) = 0, \tag{16}$$

$$e(1,t) = 0. \tag{17}$$

The transformation

$$\hat{w}(x,t) = \hat{g}v(x,t)+\eta(x,t) - \int_0^x \hat{k}(x,\xi)(\hat{g}v(\xi,t)+\eta(\xi,t))\,\mathrm{d}\xi \tag{18}$$

with $\hat{k}(x,\xi)$ given by (6) maps (8)–(13) into the following system (Lemma A.1):

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t) + \beta(x)\hat{e}(0,t) + \dot{\hat{g}}v(x,t) + \dot{\hat{g}}\int_0^x \alpha(x-\xi)(\hat{g}v(\xi,t)+\hat{w}(\xi,t))\,\mathrm{d}\xi, \tag{19}$$

$$\hat{w}_x(0,t) = 0, \tag{20}$$

$$\hat{w}(1,t) = 0, \tag{21}$$

where

$$\alpha(x) = -\frac{1}{\hat{g}}\hat{k}(x,0), \tag{22}$$

$$\beta(x) = \hat{k}_\xi(x,0) = \begin{cases} \hat{g}\cosh\sqrt{\hat{g}}x, & \hat{g}\geqslant 0,\\ \hat{g}\cos\sqrt{-\hat{g}}x, & \hat{g}<0. \end{cases} \tag{23}$$

If the parameter $g$ were known, we could set $\hat{g}\equiv g$ which makes (19)–(21) the plain heat equation. We can see that in the adaptive case three additional terms are introduced: one is proportional to $\hat{e}(0)$ and the other two are proportional to $\dot{\hat{g}}$. To prove closed loop stability we first show that the update law properties guarantee that $\hat{e}(0)$ and $\dot{\hat{g}}$ are "small".

### 3.2. Adaptive law properties

We take the following equation as a parametric model

$$e(0,t)=u(0,t)-gv(0,t)-\eta(0,t). \tag{24}$$

The estimation error is

$$\hat{e}(0,t)=u(0,t)-\hat{g}v(0,t)-\eta(0,t). \tag{25}$$

We use the gradient update law

$$\dot{\hat{g}}=\gamma\frac{\hat{e}(0,t)v(0,t)}{1+v^2(0,t)}. \tag{26}$$

**Lemma 2.** *The adaptive law* (26) *guarantees the following properties*:

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}\in\mathscr{L}_2\cap\mathscr{L}_\infty,\quad \tilde{g}\in\mathscr{L}_\infty,\quad \dot{\hat{g}}\in\mathscr{L}_2\cap\mathscr{L}_\infty. \tag{27}$$

**Proof.** Using a Lyapunov function

$$V=\frac{1}{2}\int_0^1 e^2\,\mathrm{d}x+\frac{1}{2\gamma}\tilde{g}^2 \tag{28}$$

we get

$$\begin{aligned}\dot{V}&=-\int_0^1 e_x^2\,\mathrm{d}x-\frac{\tilde{g}\hat{e}(0)v(0)}{1+v^2(0)}\\&\leqslant-\int_0^1 e_x^2\,\mathrm{d}x-\frac{\hat{e}^2(0)}{1+v^2(0)}+\frac{e(0)\hat{e}(0)}{1+v^2(0)}\\&\leqslant-\|e_x\|^2-\frac{\hat{e}^2(0)}{1+v^2(0)}+\frac{\|e_x\||\hat{e}(0)|}{\sqrt{1+v^2(0)}}\\&\leqslant-\frac{1}{2}\|e_x\|^2-\frac{1}{2}\frac{\hat{e}^2(0)}{1+v^2(0)}.\end{aligned} \tag{29}$$

This gives the following properties:

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}\in\mathscr{L}_2,\quad \tilde{g}\in\mathscr{L}_\infty. \tag{30}$$

Since

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}=\frac{e(0,t)}{\sqrt{1+v^2(0,t)}}+\tilde{g}\frac{v(0,t)}{\sqrt{1+v^2(0,t)}}, \tag{31}$$

$$\dot{\hat{g}}=\gamma\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}\frac{v(0,t)}{\sqrt{1+v^2(0,t)}}, \tag{32}$$

we get (27). □

The explicit bound on $\hat{g}$ in terms of initial conditions of all the signals can be obtained from (29):

$$\begin{aligned}\hat{g}^2(t)&=2g^2+2\left(\tilde{g}(0)^2+\gamma\int_0^1 e^2(x,0)\,\mathrm{d}x\right)\\&\leqslant 2g^2+2(g-\hat{g}(0))^2\\&\quad+2\gamma\int_0^1(u(x,0)-gv(x,0)-\eta(x,0))^2\,\mathrm{d}x.\end{aligned} \tag{33}$$

We denote the bound on $\hat{g}$ by $g_0$. The above properties imply that functions $\alpha$ and $\beta$ are bounded, let us denote these bounds by $\alpha_0$ and $\beta_0$.

### 3.3. Boundedness

The filter $v$ can be rewritten in the following way:

$$v_t(x,t)=v_{xx}(x,t)+\hat{w}(0,t)+\hat{e}(0,t), \tag{34}$$

$$v_x(0,t)=0, \tag{35}$$

$$v(1,t)=0. \tag{36}$$

We have two interconnected systems $\hat{w}$, $v$ driven by a signal $\hat{e}(0,t)$ with properties (27). Consider a Lyapunov function

$$V_v=\frac{1}{2}\int_0^1 v^2(x)\,\mathrm{d}x+\frac{1}{2}\int_0^1 v_x^2(x)\,\mathrm{d}x. \tag{37}$$

We include the $H_1$ norm in the Lyapunov function because the signal $\hat{e}(0)$ is normalized by $1+v^2(0)$ and $v^2(0)$ can only be bounded by $\|v_x\|^2$. Using Young's, Poincare's, and Agmon's inequalities we have[1]

$$\begin{aligned}\dot{V}_v=&-\int_0^1 v_x^2\,\mathrm{d}x+(\hat{w}(0)+\hat{e}(0))\int_0^1 v\,\mathrm{d}x\\&-\int_0^1 v_{xx}^2\,\mathrm{d}x-(\hat{w}(0)+\hat{e}(0))\int_0^1 v_{xx}\,\mathrm{d}x\\\leqslant&-\|v_x\|^2+\frac{1}{8}\|v\|^2+4\frac{\hat{e}^2(0)}{1+v^2(0)}(1+\|v_x\|^2)\\&+4\|\hat{w}_x\|^2-\|v_{xx}\|^2+\frac{1}{2}\|v_{xx}\|^2+\|\hat{w}_x\|^2\\&+\frac{\hat{e}^2(0)}{1+v^2(0)}(1+\|v_x\|^2)\\\leqslant&-\frac{1}{2}\|v_x\|^2-\frac{1}{2}\|v_{xx}\|^2+5\|\hat{w}_x\|^2+l_1\|v_x\|^2+l_1,\end{aligned} \tag{38}$$

where $l_1$ is a generic function of time in $\mathscr{L}_1\cap\mathscr{L}_\infty$. Using the following Lyapunov function for the $\hat{w}$-system:

$$V_{\hat{w}}=\frac{1}{2}\int_0^1\hat{w}^2(x)\,\mathrm{d}x \tag{39}$$

we get

$$\begin{aligned}\dot{V}_{\hat{w}}=&-\int_0^1\hat{w}_x^2\,\mathrm{d}x+\hat{e}(0)\int_0^1\beta\hat{w}\,\mathrm{d}x+\dot{\hat{g}}\int_0^1\hat{w}v\,\mathrm{d}x\\&+\dot{\hat{g}}\int_0^1\hat{w}(x)\int_0^x\alpha(x-y)(\hat{g}v(y)+\hat{w}(y))\,\mathrm{d}y\,\mathrm{d}x\\\leqslant&-\|\hat{w}_x\|^2+\frac{c_1}{2}\|\hat{w}\|^2+\frac{\beta_0^2}{2c_1}\frac{\hat{e}^2(0)}{1+v^2(0)}(1+\|v_x\|^2)\\&+\frac{|\dot{\hat{g}}|^2(1+\alpha_0g_0)^2}{2c_1}\|v\|^2+c_1\|\hat{w}\|^2+\frac{|\dot{\hat{g}}|^2\alpha_0^2}{2c_1}\|\hat{w}\|^2\\\leqslant&-(1-6c_1)\|\hat{w}_x\|^2+l_1\|\hat{w}\|^2+l_1\|v_x\|^2+l_1.\end{aligned} \tag{40}$$

[1] We drop the dependence on time in the proofs to reduce notational burden.

Choosing $c_1 = 1/24$ and using a Lyapunov function $V = V_{\hat{w}} + (1/20)V_v$, we get

$$\begin{aligned}\dot{V} \leqslant & -\tfrac{1}{2}\|\hat{w}_x\|^2 - \tfrac{1}{40}\|v_x\|^2 - \tfrac{1}{40}\|v_{xx}\|^2 \\ & + l_1\|\hat{w}\|^2 + l_1\|v_x\|^2 + l_1 \\ \leqslant & -\tfrac{1}{4}V + l_1 V + l_1 \end{aligned} \tag{41}$$

and by Lemma A.2 we obtain $\|\hat{w}\|, \|v\|, \|v_x\| \in \mathscr{L}_2 \cap \mathscr{L}_\infty$. Using these properties we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}_x\|^2 \leqslant & -\|\hat{w}_{xx}\|^2 + \beta_0|\hat{e}(0)|\|\hat{w}_{xx}\| \\ & + |\dot{\hat{g}}|\|\hat{w}_{xx}\|((1+\alpha_0 g_0)\|v\| + \alpha_0\|\hat{w}\|) \\ \leqslant & -\frac{1}{8}\|\hat{w}_x\|^2 + l_1,\end{aligned} \tag{42}$$

so that $\|\hat{w}_x\| \in \mathscr{L}_2 \cap \mathscr{L}_\infty$.

### 3.4. Regulation

Using the fact that $\|v_x\|$, $\|\hat{w}_x\|$ are bounded we get

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2 + \|\hat{w}\|^2)\right| \leqslant l_1\|\hat{w}_x\|^2 + l_1\|v_x\|^2 + l_1 < \infty. \tag{43}$$

By Barbalat's lemma $\|\hat{w}\| \to 0$, $\|v\| \to 0$. From (A.2) we have $\|\eta\| \to 0$ and $\|\eta_x\|$ is bounded. Since $u = e + qv + \eta$, we get $\|u\| \to 0$ and $\|u_x\|$ is bounded. Finally, using Agmon's inequality we get

$$\lim_{t\to\infty} \max_{x\in[0,1]} |u(x,t)| \leqslant \lim_{t\to\infty} (2\|u\|\|u_x\|)^{1/2} = 0. \tag{44}$$

## 4. Benchmark plant with unknown parameter in the boundary condition

Consider the following plant:

$$u_t(x,t) = u_{xx}(x,t), \tag{45}$$

$$u_x(0,t) = -qu(0,t), \tag{46}$$

$$u(1,t) = U(t), \tag{47}$$

where $U(t)$ is the control signal. This is an example of a system with a parametric uncertainty in the boundary condition, a hard-to-stabilize case even with full state feedback with in-domain actuation (Bentsman & Orlov, 2001). With $U(t) \equiv 0$ this PDE is unstable if and only if $q > 1$. The plant can be written in the frequency domain as a transfer function from input $u(1)$ to output $u(0)$

$$u(0,s) = \frac{\sqrt{s}}{\sqrt{s}\cosh\sqrt{s} - q\sinh\sqrt{s}}u(1,s). \tag{48}$$

Since this transfer function has infinitely many poles and no zeros (at $s = 0$ the transfer function is $1/(1-q)$), this is an infinite relative degree system. One of the poles is unstable and is approximately equal to $q^2$ as $q \to +\infty$.

For the case of known $q$ the transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,\xi)u(\xi,t)\,\mathrm{d}\xi \tag{49}$$

was used in Smyshlyaev and Krstic (2004) to map (45)–(46) into the target system

$$w_t(x,t) = w_{xx}(x,t) - cw(x,t), \tag{50}$$

$$w_x(0,t) = -qw(0,t), \tag{51}$$

$$w(1,t) = 0, \tag{52}$$

which is exponentially stable for $c \geqslant \max\{q|q|, 0\}$. However, this stability condition cannot be used when $q$ is unknown. Instead, let us use (49) to map (45)–(46) into a different target system,

$$w_t(x,t) = w_{xx}(x,t), \tag{53}$$

$$w_x(0,t) = 0, \tag{54}$$

$$w(1,t) = 0. \tag{55}$$

It can be shown that the kernel $k(x,\xi)$ must satisfy the following conditions:

$$k_{xx}(x,\xi) - k_{\xi\xi}(x,\xi) = 0, \tag{56}$$

$$k_\xi(x,0) = -qk(x,0), \tag{57}$$

$$k(x,x) = -q. \tag{58}$$

The solution to this PDE is

$$k(x,\xi) = -qe^{q(x-\xi)}. \tag{59}$$

Suppose now that we want to stabilize the plant (45)–(47) when $q$ is unknown. We have the following result.

**Theorem 3.** *Consider the system* (45)–(46) *with the control*

$$u(1,t) = -\int_0^1 \hat{q}e^{\hat{q}(1-\xi)}(\hat{q}v(\xi,t) + \eta(\xi,t))\,\mathrm{d}\xi, \tag{60}$$

*where the update law for* $\hat{q}$ *is*

$$\dot{\hat{q}} = \gamma\frac{(u(0,t) - \hat{q}v(0,t) - \eta(0,t))v(0,t)}{1 + v^2(0,t)}, \tag{61}$$

*and the filters* $v(x,t)$, $\eta(x,t)$ *are defined as*

$$v_t(x,t) = v_{xx}(x,t), \tag{62}$$

$$v_x(0,t) = -u(0,t), \tag{63}$$

$$v(1,t) = 0, \tag{64}$$

$$\eta_t(x,t) = \eta_{xx}(x,t), \tag{65}$$

$$\eta_x(0,t) = 0, \tag{66}$$

$$\eta(1,t) = u(1,t). \tag{67}$$

*If the closed loop system* (45)–(46), (60)–(67) *has a classical solution* $(u, \hat{q}, v, \eta)$, *then for any* $\hat{q}(0)$ *and any initial conditions* $u_0,\ v_0,\ \eta_0 \in H_1(0,1)$, *the signals* $\hat{q}(t)$, $\|u\|$, $\|v\|$, $\|\eta\|$ *are bounded and* $\|u\|$ *is regulated to zero*:

$$\lim_{t\to\infty} \|u\| = 0. \tag{68}$$

*In addition, u is square integrable in t for all* $x \in [0,1]$.

Although the plants considered in Sections 2 ($g$-system) and 4 ($q$-system) look quite similar, the adaptive stabilization problem for the latter is substantially harder due to uncertainty in the boundary condition. The proof becomes harder and the end result is a little weaker—$L_2$ boundedness and regulation instead of pointwise boundedness and regulation.

## 5. Proof of Theorem 3

### 5.1. Target system

Introducing the error $e = u - qv - \eta$ we get an exponentially stable system

$$e_t(x,t) = e_{xx}(x,t), \tag{69}$$

$$e_x(0,t) = 0, \tag{70}$$

$$e(1,t) = 0. \tag{71}$$

The transformation

$$\hat{w}(x,t) = \hat{q}v(x,t) + \eta(x,t) + \int_0^x \hat{q}e^{\hat{q}(x-\xi)}(\hat{q}v(\xi,t) + \eta(\xi,t))\,\mathrm{d}\xi \tag{72}$$

maps (45)–(46), (60) into the following system (Lemma A.1):

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t) + \hat{q}^2 e^{\hat{q}x}\hat{e}(0,t) + \dot{\hat{q}}v + \dot{\hat{q}}\int_0^x e^{\hat{q}(x-\xi)}(\hat{q}v(\xi,t) + \hat{w}(\xi,t))\,\mathrm{d}\xi, \tag{73}$$

$$\hat{w}_x(0,t) = -\hat{q}\hat{e}(0,t), \tag{74}$$

$$\hat{w}(1,t) = 0. \tag{75}$$

### 5.2. Adaptive law properties

This step is almost the same as in Section 3 for the $g$-system. We take the following equation as a parametric model:

$$e(0,t) = u(0,t) - qv(0,t) - \eta(0,t). \tag{76}$$

The estimation error is

$$\hat{e}(0,t) = u(0,t) - \hat{q}v(0,t) - \eta(0,t). \tag{77}$$

Using the gradient update law

$$\dot{\hat{q}} = \gamma\frac{\hat{e}(0,t)v(0,t)}{1+v^2(0,t)} \tag{78}$$

we get the following properties (as in Lemma 2)

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}} \in \mathscr{L}_2 \cap \mathscr{L}_\infty, \quad \tilde{q} \in \mathscr{L}_\infty, \quad \dot{\hat{q}} \in \mathscr{L}_2 \cap \mathscr{L}_\infty. \tag{79}$$

We denote the bound on $\hat{q}$ by $q_0$.

### 5.3. Boundedness

First we rewrite $v$-filter as

$$v_t(x,t) = v_{xx}(x,t), \tag{80}$$

$$v_x(0,t) = -\hat{w}(0,t) - \hat{e}(0,t), \tag{81}$$

$$v(1,t) = 0. \tag{82}$$

We have two interconnected systems for $\hat{w}$ and $v$ driven by the signal $\hat{e}(0,t)$ with properties (79). Consider a Lyapunov function

$$V = \frac{1}{2}\int_0^1 \hat{w}^2(x)\,\mathrm{d}x + \frac{1}{2}\int_0^1 v^2(x)\,\mathrm{d}x. \tag{83}$$

We have

$$\begin{aligned}\dot{V} = &-\hat{w}(0)\hat{w}_x(0) - \int_0^1 \hat{w}_x^2\,\mathrm{d}x + \dot{\hat{q}}\int_0^1 \hat{w}(x)v(x)\,\mathrm{d}x \\ &+ \dot{\hat{q}}\int_0^1 \hat{w}(x)\int_0^x e^{\hat{q}(x-\xi)}(\hat{q}v(\xi) + \hat{w}(\xi))\,\mathrm{d}\xi\,\mathrm{d}x \\ &+ \hat{e}(0)\int_0^1 \hat{q}^2 e^{\hat{q}x}\hat{w}(x)\,\mathrm{d}x - v(0)v_x(0) - \int_0^1 v_x^2\,\mathrm{d}x \\ \leqslant &-\|\hat{w}_x\|^2 + |\hat{e}(0)|(q_0|\hat{w}(0)| + q_0^2 e^{q_0}\|\hat{w}\|) + c_1\|\hat{w}\|^2 \\ &+ \frac{(1+q_0e^{q_0})^2|\dot{\hat{q}}|^2}{2c_1}\|v\|^2 + \frac{e^{2q_0}|\dot{\hat{q}}|^2}{2c_1}\|\hat{w}\|^2 \\ &- \|v_x\|^2 + \frac{1}{2}\|v_x\|^2 + \frac{1}{2}\|\hat{w}_x\|^2 + |v(0)||\hat{e}(0)|.\end{aligned} \tag{84}$$

Estimates of particular terms:

$$\begin{aligned}q_0^2e^{q_0}|\hat{e}(0)|\|\hat{w}\| &\leqslant q_0^2e^{q_0}\|\hat{w}\|\frac{\hat{e}(0)}{\sqrt{1+v^2(0)}}(1+|v(0)|) \\ &\leqslant c_5\|\hat{w}\|^2 + \frac{q_0^4e^{2q_0}}{4c_5}\frac{\hat{e}^2(0)}{1+v^2(0)} + c_6\|v_x\|^2 \\ &\quad + \frac{q_0^4e^{2q_0}}{4c_6}\frac{\hat{e}^2(0)}{1+v^2(0)}\|\hat{w}\|^2 \\ &\leqslant c_5\|\hat{w}\|^2 + c_6\|v_x\|^2 + l_1\|\hat{w}\|^2 + l_1,\end{aligned} \tag{85}$$

$$\begin{aligned}|v(0)||\hat{e}(0)| &\leqslant \frac{|v(0)||\hat{e}(0)|}{1+v^2(0)}(1+2\|v\|\|v_x\|) \\ &\leqslant \frac{c_7}{2}\|v_x\|^2 + \frac{1}{2c_7}\frac{\hat{e}^2(0)}{1+v^2(0)} + \frac{c_7}{2}\|v_x\|^2 \\ &\quad + \frac{2}{c_7}\left(\frac{|v(0)||\hat{e}(0)|}{1+v^2(0)}\right)^2\|v\|^2 \\ &\leqslant c_7\|v_x\|^2 + l_1\|v\|^2 + l_1,\end{aligned} \tag{86}$$

$$
\begin{aligned}
q_0|\hat{e}(0)||\hat{w}(0)| &\leqslant q_0|\hat{w}(0)|\frac{\hat{e}(0)}{\sqrt{1+v^2(0)}}(1+|v(0)|)\\
&\leqslant c_2\|\hat{w}_x\|^2+\frac{q_0^2}{4c_2}\frac{\hat{e}^2(0)}{1+v^2(0)}\\
&\quad+2q_0\sqrt{\|\hat{w}\|\|\hat{w}_x\|\|v\|\|v_x\|}\frac{|\hat{e}(0)|}{\sqrt{1+v^2(0)}}\\
&\leqslant c_2\|\hat{w}_x\|^2+l_1\\
&\quad+\frac{q_0|\hat{e}(0)|}{\sqrt{1+v^2(0)}}(\|\hat{w}\|\|\hat{w}_x\|+\|v\|\|v_x\|)\\
&\leqslant c_2\|\hat{w}_x\|^2+l_1+c_3\|\hat{w}_x\|^2+c_4\|v_x\|^2\\
&\quad+q_0^2\frac{\hat{e}^2(0)}{1+v^2(0)}\left(\frac{\|v\|^2}{4c_3}+\frac{\|\hat{w}\|^2}{4c_4}\right)\\
&\leqslant c_2\|\hat{w}_x\|^2+c_3\|\hat{w}_x\|^2+c_4\|v_x\|^2\\
&\quad+l_1\|v\|^2+l_1\|\hat{w}\|^2+l_1. \qquad (87)
\end{aligned}
$$

In the last inequality we used the fact that $\dot{\hat{q}}^2$ is an $l_1$ function. We have

$$
\begin{aligned}
\dot{V} \leqslant &-(\tfrac{1}{2}-4c_1-c_2-c_3-4c_5)\|\hat{w}_x\|^2+l_1\|\hat{w}\|^2\\
&-(\tfrac{1}{2}-c_4-c_6-c_7)\|v_x\|^2+l_1\|v\|^2+l_1. \qquad (88)
\end{aligned}
$$

Choosing $4c_1=c_2=c_3=4c_5=\frac{1}{16}$, $c_4=c_6=c_7=\frac{1}{12}$, we get

$$\dot{V} \leqslant -\tfrac{1}{8}V+l_1V+l_1 \qquad (89)$$

and by Lemma A.2 we obtain $\|\hat{w}\|,\|v\| \in \mathscr{L}_2 \cap \mathscr{L}_\infty$.

### 5.4. Regulation

It is easy to see from (89) that $\dot{V}$ is bounded from above. By using an alternative to Barbalat's lemma (Liu & Krstic, 2001, Lemma 3.1) we get $V \to 0$, that is $\|\hat{w}\| \to 0$, $\|v\| \to 0$. From (A.3) we have $\|\eta\| \to 0$. Since $u=e+qv+\eta$, we get $\|u\| \to 0$.

By integrating (88) we get $\|\hat{w}_x\|,\|v_x\| \in \mathscr{L}_2$, and from (A.3) $\|\eta_x\| \in \mathscr{L}_2$ and therefore $\|u_x\| \in \mathscr{L}_2$. Square integrability in time of $u(x,t)$ for all $x \in [0,1]$ follows from Agmon's inequality.

## 6. Plant with two unknown parameters

For the sake of clarity and due to different adaptive regulation properties that can be achieved, we considered two benchmark problems separately. It is also possible to design an output-feedback adaptive controller for the combined system

$$u_t(x,t)=u_{xx}(x,t)+gu(0,t), \qquad (90)$$

$$u_x(0,t)=-qu(0,t), \qquad (91)$$

This system is unstable if and only if $2q+g>2$. The nonadaptive control law can be designed based on the controllers for separate problems by using the method described in Smyshlyaev and Krstic (2004, Section VIII-E). We state here the stabilization result without a proof.

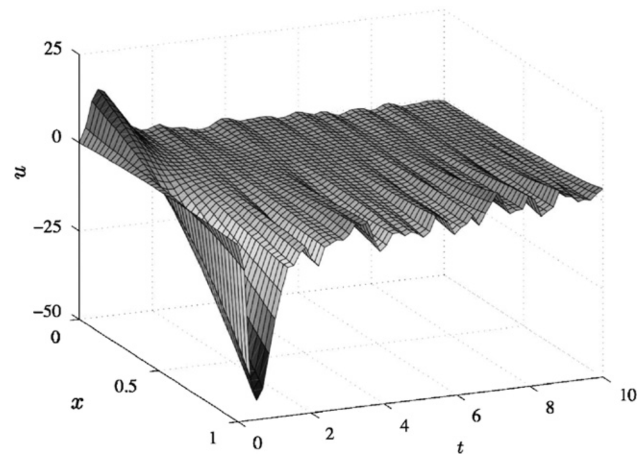



Fig. 1. The state $u(x,t)$ with the adaptive output-feedback controller (92).

**Theorem 4.** *Consider the plant* (90)–(91) *with the controller*

$$u(1)=\int_0^1 \frac{r_1^2e^{r_1(1-x)}-r_2^2e^{r_2(1-x)}}{2\sqrt{\hat{g}+\hat{q}^2/4}}(\hat{g}v+\hat{q}p+\eta)\,\mathrm{d}x, \qquad (92)$$

*where the update laws for* $\hat{g}$ *and* $\hat{q}$ *are*

$$\dot{\hat{g}}=\gamma_1\frac{\hat{e}(0)v(0)}{1+v^2(0)+p^2(0)}, \qquad \dot{\hat{q}}=\gamma_2\frac{\hat{e}(0)p(0)}{1+v^2(0)+p^2(0)}, \qquad (93)$$

*the input filter is*

$$\eta_t=\eta_{xx}, \qquad (94)$$

$$\eta_x(0)=0, \qquad (95)$$

$$\eta(1)=u(1) \qquad (96)$$

*and the output filters are*

$$
\begin{aligned}
&v_t=v_{xx}+u(0), \quad p_t=p_{xx},\\
&v_x(0)=0, \quad p_x(0)=-u(0),\\
&v(1)=0, \quad p(1)=0 \qquad (97)
\end{aligned}
$$

*with* $\hat{e}(0)=u(0)-\hat{g}v(0)-\hat{q}p(0)-\eta(0)$ *and*

$$r_{1,2}=\frac{\hat{q}}{2}\mp\sqrt{\hat{g}+\frac{\hat{q}^2}{4}} \qquad (98)$$

*If the closed loop system* (90)–(98) *has a classical solution* $(u,\hat{g},\hat{q},v,p,\eta)$, *then for any* $\hat{g}(0)$, $\hat{q}(0)$ *and any initial conditions* $u_0$, $v_0$, $p_0$, $\eta_0 \in H_1(0,1)$, *the signals* $\hat{g}(t)$, $\hat{q}(t)$, $\|u\|$, $\|v\|$, $\|p\|$, $\|\eta\|$ *are bounded and* $\|u\|$ *is regulated to zero*:

$$\lim_{t\to\infty}\|u\|=0. \qquad (99)$$

*In addition, u is square integrable in t for all* $x \in [0,1]$.

**Remark 1.** If the expression $\hat{g}+\hat{q}^2/4$ becomes negative, $r_{1,2}$ become complex. However, the control gain in (92) remains real and well defined.

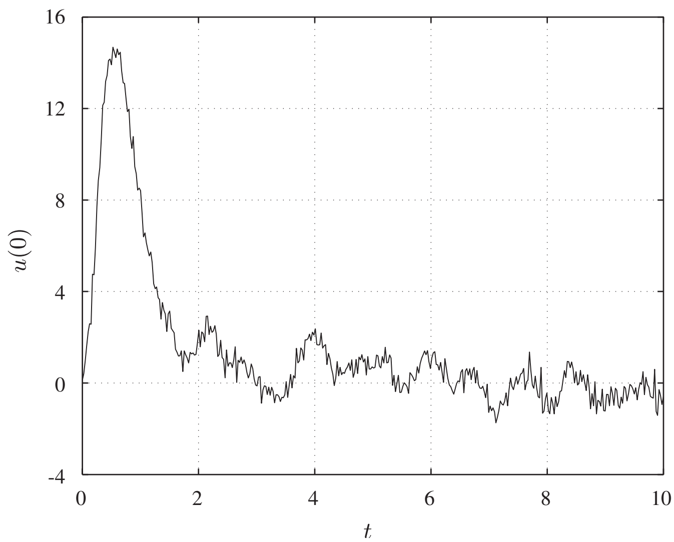



Fig. 2. The plant output $u(0,t)$.

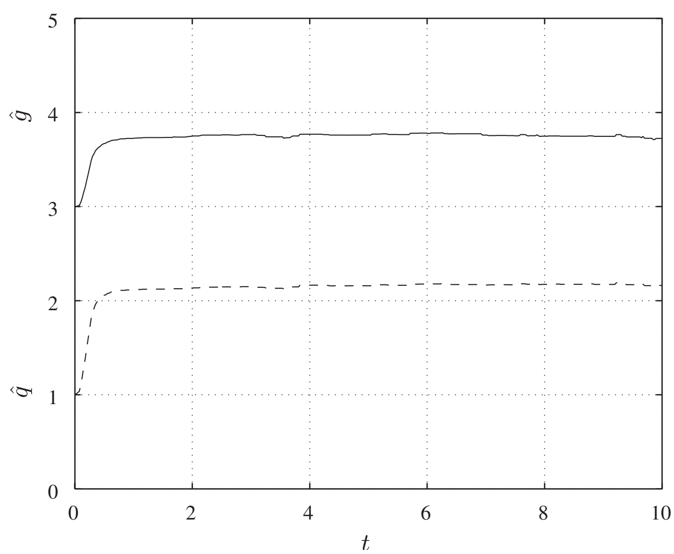



Fig. 3. The parameters $\hat{g}$ (solid) and $\hat{q}$ (dashed). The unknown parameters are set to $g=4$ and $q=2$.

## 7. Simulations

We present now the results of closed-loop simulations for the system (90)–(91). The plant parameters are set to $g=4$ and $q=2$, with these values the unstable eigenvalue $\approx 10$. For the update laws we take $\hat{g}(0)=3$, $\hat{q}(0)=1$, and $\gamma_1=\gamma_2=15$. We also assume that the measurements are noisy and there is an external disturbance (white noise both in space and time) in the plant. In Fig. 1 (state response) we can see that although the instability occurs at the $x=0$ boundary, the system is successfully regulated to zero by the control from the opposite boundary. The plant output corrupted by a sensor noise is shown in Fig. 2. In Fig. 3 the parameter estimates are shown to converge to some stabilizing values, which are slightly different from the true values (this is expected, since there is no persistency of excitation). The deadzone was used to prevent the parameter drift. In case of a larger amount of noise either the parameters start to drift or, if the deadzone is made large enough to stop the drift, the performance deteriorates.

## 8. Conclusions

It would be highly desirable to develop output-feedback versions of the state-feedback designs for reaction–advection–diffusion systems in Krstic and Smyshlyaev (2005) and Smyshlyaev and Krstic (2007). While the nonadaptive versions of such results were developed in Smyshlyaev and Krstic (2005), we have so far not been able to make them adaptive in a way that guarantees global stability. This may contradict the finite-dimensional intuition where output-feedback adaptive designs are available for a very general class of linear systems (Ioannou & Sun, 1996). However, those designs rely on transfer function representations or particular canonical state space forms—steps that do not easily translate into the PDE framework, particularly if one wants to preserve a finite parametrization.

In general, swapping-based adaptive schemes (Smyshlyaev & Krstic, 2007) have a considerably higher dynamic order than Lyapunov-based schemes (Krstic & Smyshlyaev, 2005). However, for the systems studied in this paper we have been able to use the same set of Kreisselmeier filters for both designing an observer and for achieving a static parametrization from which a gradient update law is derived. Thus, the dynamic order for the output-feedback designs in the present paper and in Krstic and Smyshlyaev (2005) is the same. The advantage of the swapping update laws in the present paper is that they are considerably simpler, whereas the advantage of the Lyapunov update laws in Krstic and Smyshlyaev (2005) is that they are derived from a complete Lyapunov function that incorporates the plant, the filters, and the update law, providing a tighter control over transient performance.

## Appendix A.

**Lemma A.1.** *The transformation* (18) *maps the system* (1)–(2), (5) *into* (19)–(21). *The transformation* (72) *maps the system* (45)–(46), (60) *into* (73)–(75).

**Proof.** It is easy to check that boundary conditions (20) and (21) are satisfied. Substituting (18) into (1) we get

$$\hat{w}_t = \hat{w}_{xx} + \dot{\hat{g}} v - \dot{\hat{g}} \int_0^x \{(\hat{k}_{\hat{g}}(x,\xi)\hat{g} + \hat{k}(x,\xi))v(\xi) + \hat{k}_{\hat{g}}(x,\xi)\eta(\xi)\}\,\mathrm{d}\xi + \hat{k}_\xi(x,0)\hat{e}(0). \tag{A.1}$$

To express the signal $\eta$ in terms of $v$ and $\hat{w}$ we use the inverse transformation to (18):

$$\hat{g}v(x,t) + \eta(x,t) = \hat{w}(x,t) - \hat{g}\int_0^x (x-\xi)\hat{w}(\xi,t)\,\mathrm{d}\xi. \tag{A.2}$$

Changing the order of integration and taking the necessary derivatives of $\hat{k}(x,\xi)$ we come to (19)–(21).

The second part of the lemma is proved in the same way. It is easy to check that (72) satisfies the boundary conditions (74) and (75). Substituting (72) into (45) we get (A.1) but with $\hat{g}$ changed to $\hat{q}$ everywhere. To express the signal $\eta$ in terms of $v$ and $\hat{w}$ we use the inverse transformation of (72):

$$\hat{q}v(x,t)+\eta(x,t)=\hat{w}(x,t)-\hat{q}\int_0^x \hat{w}(\xi,t)\,\mathrm{d}\xi. \tag{A.3}$$

Changing the order of integration and taking necessary derivatives of $k(x,\xi,\hat{q})$ we come to (73)–(75). □

**Lemma A.2** (Krstic et al., 1995, Lemma B.6). *Let $v$, $l_1$, and $l_2$ be real-valued functions defined on $R_+$, and let $c$ be a positive constant. If $l_1$ and $l_2$ are nonnegative and in $\mathscr{L}_1$ and satisfy the differential inequality*

$$\dot{v}\leqslant -cv+l_1(t)v+l_2(t), \quad v(0)\geqslant 0 \tag{A.4}$$

*then $v\in\mathscr{L}_\infty\bigcap\mathscr{L}_1$.*

## References

Bentsman, J., & Orlov, Y. (2001). Reduced spatial order model reference adaptive control of spatially varying distributed parameter systems of parabolic and hyperbolic types. *International Journal of Adaptive Control and Signal Processing*, *15*, 679–696.

Boskovic, D. M., & Krstic, M. (2003). Stabilization of a solid propellant rocket instability by state feedback. *International Journal of Robust and Nonlinear Control*, *13*, 483–495.

Ioannou, P., & Sun, J. (1996). *Robust adaptive control*. Englewood Cliffs, NJ: Prentice-Hall.

Krstic, M., Kanellakopoulos, I., & Kokotovic, P. (1995). *Nonlinear and adaptive control design*. Boston: Wiley.

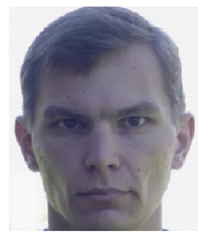
Krstic, M., & Smyshlyaev, A. (2005). Adaptive boundary control for unstable parabolic PDEs—Part I: Lyapunov design. *IEEE Transactions on Automatic Control*, accepted for publication.

Liu, W., & Krstic, M. (2001). Adaptive control of Burgers' equation with unknown viscosity. *International Journal for Adaptive Control and Signal Processing*, *15*, 745–766.

Smyshlyaev, A., & Krstic, M. (2004). Closed form boundary state feedbacks for a class of 1D partial integro-differential equations. *IEEE Transactions on Automatic Control*, *49*(12), 2185–2202.

Smyshlyaev, A., & Krstic, M. (2005). Backstepping observers for a class of parabolic PDEs. *Systems & Control Letters*, *54*, 613–625.

Smyshlyaev, A., & Krstic, M. (2007). Adaptive boundary control for unstable parabolic PDEs—Part II: Estimation-based designs. *Automatica*, in press, doi:10.1016/j.automatica.2007.02.014.



**Andrey Smyshlyaev** received his B.S. and M.S. degrees in Aerospace Engineering from the Moscow Institute of Physics and Technology in 1999 and 2001, and Ph.D. degree in Mechanical Engineering from University of California at San Diego in 2006. He is now working as a postdoctoral scholar at UCSD. His research interests include control of distributed parameter systems, adaptive control, and nonlinear control.

**Miroslav Krstic** is the Sorenson Professor of Mechanical and Aerospace Engineering and the Director of the newly formed Center for Control, Systems, and Dynamics (CCSD) at UCSD. Krstic is a coauthor of the books Nonlinear and Adaptive Control Design (1995), Stabilization of Nonlinear Uncertain Systems (1998), Flow Control by Feedback (2002), and Real Time Optimization by Extremum Seeking Control (2003). He received the NSF Career, ONR YI, and PECASE Awards, as well as the Axelby and the Schuck paper prizes. In 2005 he was the first engineering professor to receive the UCSD Award for Research. Krstic is a Fellow of IEEE, a Distinguished Lecturer of the Control Systems Society, and a former CSS VP for Technical Activities. He had served as AE for several journals and is currently Editor for Adaptive and Distributed Parameter Systems in Automatica.