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Control Lyapunov functions for adaptive nonlinear stabilization ☆

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Abstract

For the problem of stabilization of nonlinear systems linear in unknown constant parameters, we introduce the concept of an adaptive control Lyapunov function (aclf) and use Sontag's constructive proof of Artstein's theorem to design an adaptive controller. In this framework the problem of adaptive stabilization of a nonlinear system is reduced to the problem of nonadaptive stabilization of a modified system. To illustrate the construction of aclf's we give an adaptive backstepping lemma which recovers our earlier design.

Keywords: Nonlinear control; Adaptive stabilization; Adaptive control Lyapunov functions; Backstepping

1. Introduction

We consider the problem of global feedback stabilization of systems of the form

$$\dot{x} = f(x) + F(x)\theta + g(x)u, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}, \tag{1.1}$$

where θ is a constant *unknown* parameter vector which can take *any* value in $\mathbb{R}^p$, the mappings $f(x), F(x)$ and $g(x)$ are smooth, and $f(0) = 0$, $F(0) = 0$. This problem is, in general, not solvable with *static* feedback. This is obvious in the scalar case $n = p = 1$, where a control law $u = \alpha(x)$ independent of θ would have the impossible task to satisfy $x[f(x) + F(x)\theta + g(x)\alpha(x)] < 0$ for all $x \neq 0$ and all $\theta \in \mathbb{R}$. Therefore, we seek *dynamic* feedback controllers to stabilize system (1.1) for all θ.

We say that system (1.1) is *globally adaptively stabilizable* if there exists a function $\alpha(x, \hat{\theta})$ smooth on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^p$ with $\alpha(0, \hat{\theta}) \equiv 0$, a smooth function $\tau(x, \hat{\theta})$, and a positive definite symmetric $p \times p$ matrix Γ, such that the dynamic controller

$$u = \alpha(x, \hat{\theta}), \tag{1.2}$$

$$\dot{\hat{\theta}} = \Gamma\tau(x, \hat{\theta}), \tag{1.3}$$

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guarantees that the solution $(x(t), \hat{\theta}(t))$ is globally bounded, and $x(t) \to 0$ as $t \to \infty$, for any value of the unknown parameter $\theta \in \mathbb{R}^p$.

As customary in adaptive control, the state $\hat{\theta}$ plays the role of an estimate of θ.

2. Adaptive stabilization and aclf's

Our approach is to replace the problem of adaptive stabilization of (1.1) by a problem of nonadaptive stabilization of a modified system. This allows us to study adaptive stabilization in the Artstein–Sontag setting of *control Lyapunov functions* (*clf*).

Definition 2.1. A smooth function $V_a : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$, positive definite and proper in x for each θ, is called an *adaptive control Lyapunov function* (aclf) for (1.1) if there exists a positive definite symmetric matrix $\Gamma \in \mathbb{R}^{p\times p}$ such that for each $\theta \in \mathbb{R}^p$, $V_a(x,\theta)$ is a clf for the modified system

$$\dot{x} = f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}}\right) + g(x)u, \tag{2.1}$$

that is, V_a satisfies

$$\inf_{u\in\mathbb{R}}\left\{\frac{\partial V_a}{\partial x}\left[f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}}\right) + g(x)u\right]\right\} < 0. \tag{2.2}$$

We now show how to design an adaptive controller (1.2), (1.3) when an aclf is known.

Theorem 2.1. *The following two statements are equivalent*:

1. *There exists a triple* (α, V_a, Γ) *such that* $\alpha(x,\theta)$ *globally asymptotically stabilizes* (2.1) *at* $x = 0$ *for each* $\theta \in \mathbb{R}^p$ *with respect to the Lyapunov function* $V_a(x,\theta)$.
2. *There exists an* aclf $V_a(x,\theta)$ *for* (1.1).

Moreover, if an aclf $V_a(x,\theta)$ *exists, then* (1.1) *is globally adaptively stabilizable.*

Proof. ($1 \Rightarrow 2$) Obvious because 1 implies that there exists a continuous function $W : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$, positive definite in x for each θ, such that

$$\frac{\partial V_a}{\partial x}\left[f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}}\right) + g(x)\alpha(x,\theta)\right] \leqslant -W(x,\theta). \tag{2.3}$$

Thus $V_a(x,\theta)$ is a clf for (2.1) for each $\theta \in \mathbb{R}^p$, and therefore it is an aclf for (1.1).

($2 \Rightarrow 1$) The proof of this part is based on Sontag's constructive proof [9] of Artstein's theorem [1]. We assume that V_a is an aclf for (1.1), that is, a clf for (2.1). Sontag's 'universal formula' applied to (2.1) gives a control law smooth on $(\mathbb{R}^n\setminus\{0\}) \times \mathbb{R}^p$:

$$\alpha(x,\theta) = \begin{cases} -\dfrac{L_{\tilde{f}}V_a + \sqrt{(L_{\tilde{f}}V_a)^2 + (L_gV_a)^4}}{L_gV_a}, & L_gV_a(x,\theta) \neq 0, \\ 0, & L_gV_a(x,\theta) = 0, \end{cases} \tag{2.4}$$

where the Lie derivatives are taken only with respect to x, and

$$\tilde{f}(x,\theta) = f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}}\right). \tag{2.5}$$

With the choice (2.4), inequality (2.3) is satisfied with the continuous function

$$W(x,\theta) = \sqrt{(L_{\tilde f} V_a(x,\theta))^2 + (L_g V_a(x,\theta))^4}, \tag{2.6}$$

which is positive definite in x for each θ, because (2.2) implies that $L_{\tilde f} V_a(x,\theta) < 0$ whenever $L_g V_a(x,\theta) = 0$ and $x \neq 0$. We note that the control law $\alpha(x,\theta)$ will be continuous at $x = 0$ if and only if the aclf V_a satisfies the following property, called the *small control property* [9]: for each $\theta \in \mathbb{R}^p$ and for any $\varepsilon > 0$ there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $|x| \leqslant \delta$, then there is some u with $|u| \leqslant \varepsilon$ such that

$$\frac{\partial V_a}{\partial x}\left[f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}}\right) + g(x)u\right] < 0. \tag{2.7}$$

Assuming the existence of an aclf we now show that (1.1) is globally adaptively stabilizable. Since (2 $\Rightarrow$ 1), there exists a triple (α, V_a, Γ) and a function W such that (2.3) is satisfied, that is,

$$\frac{\partial V_a}{\partial x}\left[f(x) + F(x)\theta + g(x)\alpha(x,\theta)\right] + \frac{\partial V_a}{\partial \theta}\Gamma\left(\frac{\partial V_a}{\partial x}F(x)\right)^{\mathrm{T}} \leqslant -W(x,\theta). \tag{2.8}$$

Consider the Lyapunov function candidate

$$V(x,\hat\theta) = V_a(x,\hat\theta) + \tfrac{1}{2}(\theta - \hat\theta)^{\mathrm{T}}\Gamma^{-1}(\theta - \hat\theta). \tag{2.9}$$

With the help of (2.8), the derivative of V along the solutions of (1.1), (1.2), (1.3), is

$$\begin{aligned}\dot V &= \frac{\partial V_a}{\partial x}[f + F\theta + g\alpha(x,\hat\theta)] + \frac{\partial V_a}{\partial \hat\theta}\Gamma\tau(x,\hat\theta) - \tilde\theta^{\mathrm{T}}\tau(x,\hat\theta) \\ &= \frac{\partial V_a}{\partial x}[f + F\hat\theta + g\alpha(x,\hat\theta)] + \frac{\partial V_a}{\partial \hat\theta}\Gamma\tau(x,\hat\theta) + \frac{\partial V_a}{\partial x}F\tilde\theta - \tilde\theta^{\mathrm{T}}\tau(x,\hat\theta) \\ &\leqslant -W(x,\hat\theta) - \frac{\partial V_a}{\partial \hat\theta}\Gamma\left(\frac{\partial V_a}{\partial x}F\right)^{\mathrm{T}} + \frac{\partial V_a}{\partial \hat\theta}\Gamma\tau(x,\hat\theta) + \tilde\theta^{\mathrm{T}}\left(\frac{\partial V_a}{\partial x}F\right)^{\mathrm{T}} - \tilde\theta^{\mathrm{T}}\tau(x,\hat\theta).\end{aligned} \tag{2.10}$$

Choosing

$$\tau(x,\hat\theta) = \left(\frac{\partial V_a}{\partial x}(x,\hat\theta)F(x)\right)^{\mathrm{T}}, \tag{2.11}$$

we get

$$\dot V \leqslant -W(x,\hat\theta), \quad \forall\theta \in \mathbb{R}^p. \tag{2.12}$$

Thus the equilibrium $x = 0$, $\hat\theta = \theta$ of (1.1)–(1.3) is globally stable, and by LaSalle's theorem, $x(t) \to 0$, that is, (1.1) is globally adaptively stabilizable. □

The adaptive controller constructed in the proof of Theorem 2.1 consists of a control law $u = \alpha(x,\hat\theta)$ given by (2.4), and an update law $\dot{\hat\theta} = \Gamma\tau(x,\hat\theta)$ with (2.11).

It is of interest to interpret this controller as a certainty equivalence controller. The certainty equivalence approach, prevalent in the adaptive control of linear systems, is not applicable to nonlinear systems without severe restrictions on system nonlinearities. This has been a major obstacle to estimation-based designs of adaptive controllers for nonlinear systems. The control law $\alpha(x,\theta)$ given by (2.4) is stabilizing for the modified system (2.1) but may not be stabilizing for the original system (1.1). However, as the proof of Theorem 2.1 shows, its certainty equivalence form $\alpha(x,\hat\theta)$ is an adaptive globally stabilizing control law for the original system (1.1). Hence, if a certainty equivalence approach is to be applied to a nonlinear system, the system is to be modified to require a control law which anticipates the parameter estimation transients. In the proof of Theorem 2.1, as well as in the tuning functions design [5], this is achieved by incorporating the *tuning function* τ in the control law α. Indeed, the formula (2.4) for α depends on τ via

$$L_{\bar{f}}V_a(x,\theta) = L_f V_a + \tau(x,\theta)^{\mathrm{T}} \left(\theta + \Gamma \left(\frac{\partial V_a}{\partial \theta} \right)^{\mathrm{T}} \right), \tag{2.13}$$

which is obtained by combining (2.5) and (2.11). Using (2.11) to rewrite the inequality (2.8) as

$$\frac{\partial V_a}{\partial x} [f(x) + F(x)\theta + g(x)\alpha(x,\theta)] + \frac{\partial V_a}{\partial \theta} \Gamma \tau(x,\theta) \leqslant -W(x,\theta), \tag{2.14}$$

it is not difficult to see that the control law (2.4) containing (2.13) prevents τ from destroying the nonpositivity of the Lyapunov derivative.

As it is always the case in adaptive control, in the proof of Theorem 2.1 we used a Lyapunov function $V(x,\hat{\theta})$ given by (2.9), which is quadratic in the parameter error $\theta - \hat{\theta}$. The quadratic form is suggested by the linear dependence of (1.1) on θ, and the fact that θ cannot be used for feedback. We will now show that the quadratic form of (2.9) is both necessary and sufficient for the existence of an aclf.

We say that system (1.1) is *globally adaptively quadratically stabilizable* if it is globally adaptively stabilizable and, in addition, there exists a smooth function $V_a(x,\theta)$ positive definite and proper in x for each θ, and a continuous function $W(x,\theta)$ positive definite in x for each θ, such that for all $(x(0),\hat{\theta}(0)) \in \mathbb{R}^{n+p}$ and all $\theta \in \mathbb{R}^p$, the derivative of (2.9) along the solutions of (1.1)–(1.3) is given by (2.12).

Corollary 2.1. *System* (1.1) *is globally adaptively quadratically stabilizable if and only if there exists an* aclf $V_a(x,\theta)$.

Proof. The 'if' part is contained in the proof of Theorem 2.1 where the Lyapunov function $V(x,\hat{\theta})$ is in the form (2.9). To prove the 'only if' part, we start by assuming global adaptive quadratic stabilizability of (1.1), and first show that $\tau(x,\hat{\theta})$ must be given by (2.11). The derivative of V along the solutions of (1.1)–(1.3), given by (2.10), is rewritten as

$$\dot{V} = \frac{\partial V_a}{\partial x}[f + F\hat{\theta} + g\alpha(x,\hat{\theta})] + \frac{\partial V_a}{\partial \hat{\theta}} \Gamma \tau(x,\hat{\theta}) - \hat{\theta}^{\mathrm{T}} \left(\left(\frac{\partial V_a}{\partial x} F \right)^{\mathrm{T}} - \tau \right) + \theta^{\mathrm{T}} \left(\left(\frac{\partial V_a}{\partial x} F \right)^{\mathrm{T}} - \tau \right). \tag{2.15}$$

This expression has to be nonpositive to satisfy (2.12). Since it is affine in θ, it can be nonpositive for all $(x,\hat{\theta}) \in \mathbb{R}^{n+p}$ and all $\theta \in \mathbb{R}^p$ only if the last term is zero, that is, only if τ is defined as in (2.11). Then, it is straightforward to verify that

$$\begin{aligned} &\frac{\partial V_a}{\partial x} \left[f(x) + F(x) \left(\hat{\theta} + \Gamma \left(\frac{\partial V_a}{\partial \hat{\theta}} \right)^{\mathrm{T}} \right) + g(x)\alpha(x,\hat{\theta}) \right] \\ &\quad = \dot{V} + \left(\tilde{\theta}^{\mathrm{T}} - \frac{\partial V_a}{\partial \hat{\theta}} \Gamma \right) \left(\tau - \left(\frac{\partial V_a}{\partial x} F \right)^{\mathrm{T}} \right) \leqslant -W(x,\hat{\theta}) \end{aligned} \tag{2.16}$$

for all $(x,\hat{\theta}) \in \mathbb{R}^{n+p}$. By $(1 \Rightarrow 2)$ in Theorem 2.1, $V_a(x,\theta)$ is an aclf for (1.1). □

3. Adaptive backstepping

With Theorem 2.1, the problem of adaptive stabilization is reduced to the problem of finding an aclf. This would be only an esthetically pleasing result if it were not for backstepping procedures [4, 2, 5, 8] with which aclf's can actually be designed.

In this section we reinterpret the adaptive backstepping design with tuning functions [5]. Using backstepping, an aclf for a higher order system is recursively constructed starting with an aclf for a lower order system.

Lemma 3.1. *If the system*

$$\dot{x} = f(x) + F(x)\theta + g(x)u, \tag{3.1}$$

is globally adaptively quadratically stabilizable with $\alpha \in C^1$, *then the augmented system*

$$\begin{aligned} \dot{x} &= f(x) + F(x)\theta + g(x)\xi, \\ \dot{\xi} &= u, \end{aligned} \tag{3.2}$$

is also globally adaptively quadratically stabilizable.

Proof. Since system (3.1) is globally adaptively stabilizable, then by Corollary 2.1 there exists an aclf $V_a(x,\theta)$, and by Theorem 2.1 it satisfies (2.3) with a control law $u = \alpha(x,\theta)$. We will now show that

$$V_1(x,\xi,\theta) = V_a(x,\theta) + \tfrac{1}{2}(\xi - \alpha(x,\theta))^2 \tag{3.3}$$

is an aclf for the augmented system (3.2) by showing that it satisfies

$$\frac{\partial V_1}{\partial(x,\xi)}\begin{bmatrix} f + F\left(\theta + \Gamma\left(\dfrac{\partial V_1}{\partial\theta}\right)^{\mathrm{T}}\right) + g\xi \\ \alpha_1(x,\xi,\theta) \end{bmatrix} \leqslant -W - (\xi-\alpha)^2 \tag{3.4}$$

with the control law

$$u = \alpha_1(x,\xi,\theta) = -\frac{\partial V_a}{\partial x}g - (\xi - \alpha) + \frac{\partial\alpha}{\partial x}(f + F\theta + g\xi) + \frac{\partial\alpha}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial x}F\right)^{\mathrm{T}} + \frac{\partial V_a}{\partial\theta}\Gamma\left(\frac{\partial\alpha}{\partial x}F\right)^{\mathrm{T}}. \tag{3.5}$$

Let us start by introducing for brevity a new error state $z = \xi - \alpha(x,\theta)$. With (3.3) we compute

$$\begin{aligned} \frac{\partial V_1}{\partial(x,\xi)}\begin{bmatrix} f + F\theta + g\xi \\ \alpha_1(x,\xi,\theta)\end{bmatrix} &= \frac{\partial V_1}{\partial x}(f + F\theta + g\xi) + \frac{\partial V_1}{\partial\xi}\alpha_1(x,\xi,\theta) \\ &= \left(\frac{\partial V_a}{\partial x} - z\frac{\partial\alpha}{\partial x}\right)(f + F\theta + g\xi) + z\alpha_1 \\ &= \frac{\partial V_a}{\partial x}(f + F\theta + g\alpha) + \frac{\partial V_a}{\partial x}gz - z\frac{\partial\alpha}{\partial x}(f + F\theta + g\xi) + z\alpha_1 \\ &= \frac{\partial V_a}{\partial x}(f + F\theta + g\alpha) + z\left(\alpha_1 + \frac{\partial V_a}{\partial x}g - \frac{\partial\alpha}{\partial x}(f + F\theta + g\xi)\right). \end{aligned} \tag{3.6}$$

On the other hand, in view of (3.3), we have

$$\begin{aligned} \frac{\partial V_1}{\partial(x,\xi)}\begin{bmatrix} F\Gamma\left(\dfrac{\partial V_1}{\partial\theta}\right)^{\mathrm{T}} \\ 0\end{bmatrix} &= \frac{\partial V_1}{\partial x}F\Gamma\left(\frac{\partial V_1}{\partial\theta}\right)^{\mathrm{T}} \\ &= \left(\frac{\partial V_a}{\partial x} - z\frac{\partial\alpha}{\partial x}\right)F\Gamma\left(\frac{\partial V_a}{\partial\theta} - z\frac{\partial\alpha}{\partial\theta}\right)^{\mathrm{T}} \\ &= \frac{\partial V_a}{\partial x}F\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^{\mathrm{T}} - z\left(\frac{\partial\alpha}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial x}F\right)^{\mathrm{T}} + \frac{\partial V_a}{\partial\theta}\Gamma\left(\frac{\partial\alpha}{\partial x}F\right)^{\mathrm{T}}\right). \end{aligned} \tag{3.7}$$

Adding (3.6) and (3.7), with (2.3) and (3.5) we get

$$\frac{\partial V_1}{\partial(x,\xi)}\begin{bmatrix} f+F\left(\theta+\Gamma\left(\frac{\partial V_1}{\partial\theta}\right)^{\mathrm{T}}\right)+g\xi \\ \alpha_1(x,\xi,\theta)\end{bmatrix}$$

$$= \frac{\partial V_a}{\partial x}(f+F\theta+g\alpha)+\frac{\partial V_a}{\partial x}F\Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^{\mathrm{T}}$$

$$+z\left(\alpha_1+\frac{\partial V_a}{\partial x}g-\frac{\partial\alpha}{\partial x}(f+F\theta+g\xi)-\frac{\partial\alpha}{\partial\theta}\Gamma\left(\frac{\partial V_1}{\partial x}F\right)^{\mathrm{T}}-\frac{\partial V_a}{\partial\theta}\Gamma\left(\frac{\partial\alpha}{\partial x}F\right)^{\mathrm{T}}\right)$$

$$\leqslant -W(x,\theta)-z^2. \tag{3.8}$$

This proves by Theorem 2.1 that $V_1(x,\xi,\theta)$ is an aclf for system (3.2), and by Corollary 2.1 this system is globally adaptively quadratically stabilizable. □

The new tuning function for system (3.2) is determined by the new aclf V_1 and given by

$$\tau_1(x,\xi,\theta)=\left(\frac{\partial V_1}{\partial(x,\xi)}\begin{bmatrix}F\\0\end{bmatrix}\right)^{\mathrm{T}}=\left(\frac{\partial V_1}{\partial x}F\right)^{\mathrm{T}}=\left[\left(\frac{\partial V_a}{\partial x}-(\xi-\alpha)\frac{\partial\alpha}{\partial x}\right)F\right]^{\mathrm{T}}$$

$$=\tau(x,\theta)-\left(\frac{\partial\alpha}{\partial x}F\right)^{\mathrm{T}}(\xi-\alpha). \tag{3.9}$$

The control law $\alpha_1(x,\xi,\theta)$ in (3.5) is only one out of many possible control laws. Once we have shown that V_1 given by (3.3) is an aclf for (3.2), we can use, for example, the C^0 control law α_1 given by Sontag's formula (2.4) with $L_{g_1}V_1=z$ and

$$L_{\tilde f_1}V_1(x,\xi,\theta)=\frac{\partial V_1}{\partial(x,\xi)}\begin{bmatrix} f+F\left(\theta+\Gamma\left(\frac{\partial V_1}{\partial\theta}\right)^{\mathrm{T}}\right)+g\xi \\ 0\end{bmatrix}$$

$$=\frac{\partial V_1}{\partial x}(f+g\xi)+\tau_1(x,\xi,\theta)^{\mathrm{T}}\left(\theta+\Gamma\left(\frac{\partial V_1}{\partial\theta}\right)^{\mathrm{T}}\right). \tag{3.10}$$

It can be shown that the following function, used as a clf in [6], is a more general aclf than (3.3):

$$V_1(x,\xi,\theta)=V_a(x,\theta)+\int_0^{\xi-\alpha(x,\theta)}\eta(s)\,\mathrm{d}s, \tag{3.11}$$

where η is a C^0 function such that $s\eta(s)>0$ whenever $s\neq 0$, $\eta'(0)>0$, and $\eta\notin\mathscr{L}^1((-\infty,0])\cup\mathscr{L}^1([0,+\infty))$.

A repeated application of Lemma 3.1 recovers our earlier result [5]:

Corollary 3.2. *The following system is globally adaptively stabilizable*

$$\begin{aligned}\dot x_i &= x_{i+1}+\varphi_i(x_1,\ldots,x_i)^{\mathrm{T}}\theta,\quad i=1,\ldots,n-1,\\ \dot x_n &= u+\varphi_n(x_1,\ldots,x_n)^{\mathrm{T}}\theta.\end{aligned} \tag{3.12}$$

4. Conclusions

The aclf framework reduces the problem of adaptive stabilization to the problem of nonadaptive stabilization of a modified system.

The adaptive stabilization problem is difficult because the function V_a which modifies the system (2.1) has to be its Lyapunov function.

The above analysis applies also to the case where the unknown parameters enter the control vector field:

$$\dot{x} = f(x) + F(x)\theta + (g(x) + G(x)\theta)u. \tag{4.1}$$

In this case the existence of an aclf V_a is equivalent to the existence of a clf for the system

$$\dot{x} = f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}}\right) + \left[g(x) + G(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial \theta}\right)^{\mathrm{T}}\right)\right]u. \tag{4.2}$$

The extension to the multi-input case is also straightforward.

A difficult open problem is: if (1.1) is globally asymptotically stabilizable for *each* θ, is it globally adaptively stabilizable, and vice versa? In other words, does the existence of a pair (α^0, V_a^0) satisfying (2.3) for $\Gamma = 0$ imply the existence of a pair (α, V_a) satisfying (2.3) for some $\Gamma > 0$, and vice versa? Adaptive Lyapunov designs available in the literature [2–5, 7, 8, 10] are all for systems which are not only globally asymptotically stabilizable for each θ, but also globally adaptively stabilizable.

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