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Adaptive boundary control for unstable parabolic PDEs—Part II: Estimation-based designs☆

Andrey Smyshlyaev, Miroslav Krstic*

Department of Mechanical and Aerospace Engineering, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0411, USA

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Abstract

The certainty equivalence approach to adaptive control is commonly used with two types of identifiers: passivity-based identifiers and swapping identifiers. The "passive" (also known as "observer-based") approach is the prevalent identification technique in existing results on adaptive control for PDEs but has so far not been used in boundary control problems. The swapping approach, prevalent in finite-dimensional adaptive control is employed here for the first time in adaptive control of PDEs. For a class of unstable parabolic PDEs we prove a separation principle result for both the passive and swapping identifiers combined with the backstepping boundary controllers. The result is applicable in any dimension. For physical reasons we restrict our attention to dimensions no higher than three. The results of the paper are illustrated by simulations.

Keywords: Adaptive control; Boundary control; Distributed parameter systems

1. Introduction

We study the boundary control problem for a class of unstable 3D reaction–advection–diffusion PDEs with *unknown coefficients*. No solution presently exists for this problem (even in 1D) due to the absence of parametrized families of controllers for such systems. We make explicit controllers introduced in Smyshlyaev and Krstic (2004) adaptive by designing parameter identifiers and substituting the parameter estimates they generate into the control law. Adaptive controllers designed in this way are referred to as "certainty equivalence." Stability of such controllers is a highly non-trivial question because the parameter estimates make the adaptive controller nonlinear even when the PDE plant is linear. In this paper we prove the "separation principle"—the global stability of such a nonlinear closed-loop PDE system.

The parameter identifiers for use in the certainty equivalence approach to adaptive control can be split into two classes: passivity-based identifiers and swapping identifiers (Krstic, Kanellakopoulos, & Kokotovic, 1995). The "passive," a.k.a. the "observer-based" approach has so far been the prevalent identification technique in existing results on adaptive control for PDEs (Bentsman & Orlov, 2001; Bohm, Demetriou, Reich, & Rosen, 1998; Hong & Bentsman, 1994; Orlov, 2000; Orlov & Bentsman, 2000; Solo & Bamieh, 1999). This approach is appealing due to its simplicity—it employs an observer in the form of a copy of the plant, plus a stabilizing error term—however, it has so far not been used in boundary control problems. The swapping approach (often called simply the "gradient" method) is the most commonly used identification method in finite-dimensional adaptive control. In this paper we report its first use in adaptive control for PDEs. Filters of the "regressor" and of the measured part of the plant are implemented to convert a dynamic parametrization of the problem (a parametrization that involves temporal derivatives) into a static one where standard gradient and least-squares estimation techniques can be used. This method has a higher dynamic order than the passivity-based method because it uses "one-filter-per-unknown parameter" instead of just one filter.

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* Corresponding author. Tel.: +1 858 8222406; fax: +1 858 8223107.
E-mail addresses: asmyshly@ucsd.edu (A. Smyshlyaev), krstic@ucsd.edu (M. Krstic).

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On the other hand, the passivity-based approach does not allow least-squares estimation.

The same class of systems is considered in Krstic and Smyshlyaev (2005) using the Lyapunov approach. While the Lyapunov approach does not employ any filters or "observers," and as a result has the lowest on-line computational cost and typically yields the strongest performance properties (Krstic et al., 1995), it has two disadvantages: its parameter update laws are much more complex than with the estimation-based approach and it necessitates the use of parameter projection and low adaptation gain, which are not needed with the estimation-based approach (except for keeping the estimate of the diffusion coefficient positive).

The three designs (Lyapunov, passive, and swapping) have different measurement requirements. The Lyapunov design requires the measurement of the plant state, the passive design also requires the measurement of its derivatives, and the swapping design also requires the measurement of its second derivatives. The need for spatial differentiation brings up a concern about sensitivity to noise. This problem can be remedied to a certain extent by spatial "low pass" filtering, in the same way that PD control is implemented in practice.

In the class of reaction–advection–diffusion PDEs for which we design identifiers, all three classes of coefficients are allowed to be unknown—the reaction coefficients, advection coefficients, and diffusion coefficients. We prove that both the passive and swapping identifiers are stable with all the coefficients unknown and present our simulations in the case where they are all unknown. However, a fundamental obstacle exists in the estimation-based designs which makes closed-loop stability very hard to prove when the *diffusion coefficient* (the coefficient multiplying the second spatial derivatives) is unknown. The reason for this is that for closed-loop stability (with unknown diffusion) one seems to need a Sobolev bound on the "estimation error" which is one order higher than what stability analysis for the identifiers provides. Thus, we state closed-loop stability for known diffusion, though we illustrate it in simulations for unknown diffusion.

We have so far not been able to develop output feedback extensions for the class of systems in the paper. This may contradict the finite-dimensional intuition where output-feedback adaptive designs are available for a very general class of linear systems (Ioannou & Sun, 1996). However, those designs rely on transfer function representations or particular canonical state space forms—steps that do not easily translate into the PDE framework, particularly if one wants to preserve a finite parametrization. In a companion paper (Smyshlyaev & Krstic, 2007) we present examples of output-feedback swapping designs for systems where the parametric uncertainty multiplies only the measured (boundary) variable of the PDE.

Early works on adaptive control of infinite-dimensional systems were for plants stabilizable by non-identifier based high gain feedback (Logemann & Townley, 1997), under a relative degree one assumption. State-feedback model reference adaptive control (MRAC) was extended to PDEs in Hong and Bentsman (1994), Bohm et al. (1998), Solo and Bamieh (1999), Orlov (2000) and Bentsman and Orlov (2001) but not for the case of boundary control. Efforts in Demetriou and Ito (2003), Wen and Balas (1989) made use of positive realness assumptions where relative degree one is implicit, except in some examples where this restriction is cleverly overcome. Stochastic adaptive LQR with least-squares parameter estimation and state feedback was pursued in Duncan, Maslowski, and Pasik-Duncan (1994). Adaptive control of nonlinear PDEs was studied in Liu and Krstic (2001), Kobayashi (2001, 2002). Adaptive controllers for nonlinear systems on lattices were designed in Jovanovic and Bamieh (2005). An experimentally validated adaptive boundary controller for a flexible beam was presented in de Queiroz, Dawson, Agarwal, and Zhang (1999).

Throughout the paper we assume well posedness of the closed-loop systems in the interest of space and due to the parabolic character of these systems which ensures their benign behavior, as supported by numerical results that we show in this paper. An example on how one proves well posedness is given in Krstic and Smyshlyaev (2005).

The paper is organized as follows. First, we explore a simple PDE with one unknown coefficient to illustrate the methodology of control and identifier design and the proof idea. Then in Sections 3 and 4 we design and analyze a passive identifier for a PDE with several unknown parameters in a 3D setting. The adaptive design with a swapping identifier is presented in Sections 5 and 6. The results are illustrated by a 2D simulation in Section 7.

Notation: The spatial $L_2(0,1)$ norm is denoted by $\|\cdot\|$. The temporal norms are denoted by $\mathscr{L}_\infty$, $\mathscr{L}_1$, and $\mathscr{L}_2$ for $t \geqslant 0$. We denote by l_1 a generic function in $\mathscr{L}_1$. The symbols $I_1(\cdot)$, $J_1(\cdot)$ denote the corresponding Bessel functions.

2. Benchmark plant

In this section we consider a simple plant to illustrate the main ideas of our approach in a tutorial way without the extensive notation that is needed in higher dimension like 2D and 3D and with more than one physical parameter. Consider a 1D unstable heat equation

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \tag{1}$$

$$u(0,t) = 0, \tag{2}$$

$$u(1,t) = U(t), \tag{3}$$

with one unknown parameter λ. Our objective is to regulate the state of this system to zero from the boundary with Dirichlet actuation $U(t)$. For $U(t)=0$ this system can have an arbitrarily large number of unstable eigenvalues.

For the case of known λ, the following control method has been proposed in Smyshlyaev and Krstic (2004): use a transformation[1]

$$w(x) = u(x) - \int_0^x k(x,\xi)u(\xi)\,\mathrm{d}\xi, \tag{4}$$

[1] To reduce notational burden we suppress time dependence everywhere and x-dependence where it does not lead to a confusion.

$$k(x,\xi) = -\lambda\xi \frac{I_1(\sqrt{\lambda(x^2-\xi^2)})}{\sqrt{\lambda(x^2-\xi^2)}} \tag{5}$$

to map (1)–(2) into an exponentially stable system

$$w_t = w_{xx}, \tag{6}$$

$$w(0) = w(1) = 0. \tag{7}$$

The stabilizing control law is then given by

$$u(1) = -\int_0^1 \lambda\xi \frac{I_1(\sqrt{\lambda(1-\xi^2)})}{\sqrt{\lambda(1-\xi^2)}} u(\xi)\,\mathrm{d}\xi. \tag{8}$$

By certainty equivalence principle, the controller in case of unknown λ will be given by (8) with λ replaced by its estimate $\hat{\lambda}$:

$$u(1) = -\int_0^1 \hat{\lambda}\xi \frac{I_1(\sqrt{\hat{\lambda}(1-\xi^2)})}{\sqrt{\hat{\lambda}(1-\xi^2)}} u(\xi)\,\mathrm{d}\xi. \tag{9}$$

We now consider two different approaches to identifier design.

2.1. Design with passive identifier

Consider the following system:

$$\hat{u}_t = \hat{u}_{xx} + \hat{\lambda}u + \gamma^2(u-\hat{u})\int_0^1 u^2(x)\,\mathrm{d}x, \tag{10}$$

$$\hat{u}(0) = 0, \tag{11}$$

$$\hat{u}(1) = u(1). \tag{12}$$

Such systems are often called "observers" because they incorporate a copy of the plant though they are not used for state estimation. This identifier employs a copy of the PDE plant and an additional nonlinear term. The term "passive identifier" comes from the fact that an operator from the parameter estimation error $\tilde{\lambda} = \lambda - \hat{\lambda}$ to the inner product of u with $u - \hat{u}$ is strictly passive. The additional nonlinear term in (10) acts as nonlinear damping whose task is to ensure square integrability of $\dot{\hat{\lambda}}$ (i.e., in our notation, $\dot{\hat{\lambda}} \in \mathscr{L}_2$). This slows down the adaptation and serves as an alternative to update law normalization needed to achieve certainty equivalence. For parabolic PDEs with the in-domain actuation the identifiers of the type (10) were employed in Orlov and Bentsman (2000) (without the nonlinear damping), and the parameter estimates were shown to converge to the true parameters if the plant is excited by a sufficiently rich input. Here we are concerned only with stabilization and do not assume the persistency of excitation.

Consider the error signal $e = u - \hat{u}$ which satisfies the following PDE:

$$e_t = e_{xx} + \tilde{\lambda}u - \gamma^2 e\|u\|^2, \tag{13}$$

$$e(0) = e(1) = 0. \tag{14}$$

With a Lyapunov function

$$V = \frac{1}{2}\int_0^1 e^2(x)\,\mathrm{d}x + \frac{\tilde{\lambda}^2}{2\gamma} \tag{15}$$

we get

$$\dot{V} = -\|e_x\|^2 - \gamma^2\|e\|^2\|u\|^2 + \tilde{\lambda}\int_0^1 e(x)u(x)\,\mathrm{d}x - \frac{\tilde{\lambda}\dot{\hat{\lambda}}}{\gamma}. \tag{16}$$

Choosing the update law

$$\dot{\hat{\lambda}} = \gamma\int_0^1 (u(x) - \hat{u}(x))u(x)\,\mathrm{d}x, \tag{17}$$

we obtain

$$\dot{V} \leqslant -\|e_x\|^2 - \gamma^2\|e\|^2\|u\|^2, \tag{18}$$

which implies $V(t) \leqslant V(0)$ and from the definition of V we get that $\tilde{\lambda}$ and $\|e\|$ are bounded. Integrating (18) with respect to time from zero to infinity we get the properties $\|e_x\|,\ \|e\|\|u\| \in \mathscr{L}_2$. From the update law (17) we get $|\dot{\hat{\lambda}}| \leqslant \gamma\|e\|\|u\|$ and so $\dot{\hat{\lambda}} \in \mathscr{L}_2$.

For the case of unknown λ the transformation (4) is modified as follows:

$$\hat{w}(x) = \hat{u}(x) - \int_0^x \hat{k}(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi, \tag{19}$$

$$\hat{k}(x,\xi) = -\hat{\lambda}\xi \frac{I_1(\sqrt{\hat{\lambda}(x^2-\xi^2)})}{\sqrt{\hat{\lambda}(x^2-\xi^2)}}. \tag{20}$$

It maps (10)–(12) into the following target system (see Lemma 3 from Section 4):

$$\hat{w}_t = \hat{w}_{xx} + \dot{\hat{\lambda}}\int_0^x \frac{\xi}{2}\hat{w}(\xi)\,\mathrm{d}\xi + (\hat{\lambda} + \gamma^2\|u\|^2)e_1, \tag{21}$$

$$\hat{w}(0) = \hat{w}(1) = 0, \tag{22}$$

where

$$e_1(x) = e(x) - \int_0^x \hat{k}(x,\xi)e(\xi)\,\mathrm{d}\xi. \tag{23}$$

We observe that, in comparison to the non-adaptive target system (6)–(7), two additional terms appear in (21)–(22), both going to zero in some sense, since the identifier guarantees $\|e\|,\ \dot{\hat{\lambda}} \in \mathscr{L}_2$. The proof of boundedness of all the signals based on the joint analysis of e and $\hat{w}$ systems is shown next.

Let us denote a bound on $\hat{\lambda}$ by λ_0. The function $\hat{k}(x,\xi)$ is bounded and twice continuously differentiable with respect to

x and ξ, therefore there exist constants M_1, M_2, M_3 such that

$$\|e_1\| \leqslant M_1\|e\|, \quad \|u\| \leqslant \|\hat{u}\| + \|e\| \leqslant M_2\|\hat{w}\| + \|e\|, \tag{24}$$

$$\|u_x\| \leqslant \|\hat{u}_x\| + \|e_x\| \leqslant M_3\|\hat{w}_x\| + \|e_x\|. \tag{25}$$

To prove boundedness of all the signals, we estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}\|^2 = -\int_0^1 \hat{w}_x^2\,\mathrm{d}x + \dot{\hat{\lambda}}\int_0^1 \hat{w}(x)\int_0^x \frac{\xi}{2}\hat{w}(\xi)\,\mathrm{d}\xi\,\mathrm{d}x + (\hat{\lambda} + \gamma^2\|u\|^2)\int_0^1 e_1\hat{w}\,\mathrm{d}x. \tag{26}$$

Using Agmon's, Poincaré's, and Young's inequalities one can show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}\|^2 \leqslant -\frac{1}{16}\|\hat{w}\|^2 + l_1\|\hat{w}\|^2 + l_1, \tag{27}$$

where l_1 denotes a generic function in $\mathscr{L}_1$. The last inequality follows from the properties $\dot{\hat{\lambda}}$, $\|u\|\|e\|$, $\|e\| \in \mathscr{L}_2$. Using Lemma A.2 we get $\|\hat{w}\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$. From (24) we get $\|u\|, \|\hat{u}\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$, and (17) implies that $\dot{\hat{\lambda}}$ is bounded. Using these properties, one can show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \hat{w}_x^2 \leqslant -\frac{1}{8}\|\hat{w}_x\|^2 + l_1 \tag{28}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 e_x^2\,\mathrm{d}x \leqslant -\frac{1}{8}\|e_x\|^2 + l_1, \tag{29}$$

and using Lemma A.2 we get $\|\hat{w}_x\|, \|e_x\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$. Using (25) we get $\|u_x\|, \|\hat{u}_x\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$. From Agmon's inequality

$$\max_{x\in[0,1]} |u(x,t)|^2 \leqslant 2\|u\|\|u_x\| \tag{30}$$

we get the boundedness of u and $\hat{u}$ for all $x \in [0, 1]$.

To show the regulation of u to zero, we note that the time derivatives of $\|e\|^2$ and $\|\hat{w}\|^2$ are bounded and using Lemma A.1 (which is an alternative to Barbalat's lemma) we get $\|\hat{w}\| \to 0$, $\|e\| \to 0$ as $t \to \infty$. From (24) it follows that $\|\hat{u}\| \to 0$ and $\|u\| \to 0$. Using Agmon's inequality and the fact that $\|u_x\|$ is bounded, we get the regulation of u to zero for all $x \in [0, 1]$:

$$\lim_{t\to\infty} \max_{x\in[0,1]} |u(x,t)| \leqslant \lim_{t\to\infty} (2\|u\|\|u_x\|)^{1/2} = 0. \tag{31}$$

2.2. *Design with swapping identifier*

We employ two filters: the state filter

$$v_t = v_{xx} + u, \tag{32}$$

$$v(0) = v(1) = 0 \tag{33}$$

and the input filter

$$\eta_t = \eta_{xx}, \tag{34}$$

$$\eta(0) = 0, \tag{35}$$

$$\eta(1) = u(1). \tag{36}$$

The "estimation" error

$$e = u - \lambda v - \eta \tag{37}$$

is then exponentially stable:

$$e_t = e_{xx}, \tag{38}$$

$$e(0) = e(1) = 0. \tag{39}$$

Using the static relationship (37) as a parametric model, we implement a "prediction error" as

$$\hat{e} = u - \hat{\lambda}v - \eta, \quad \hat{e} = e + \tilde{\lambda}v. \tag{40}$$

We choose the gradient update law with normalization

$$\dot{\hat{\lambda}} = \gamma\frac{\int_0^1 \hat{e}(x)v(x)\,\mathrm{d}x}{1 + \|v\|^2}. \tag{41}$$

With a Lyapunov function

$$V = \frac{1}{2}\int_0^1 e^2\,\mathrm{d}x + \frac{1}{8\gamma}\tilde{\lambda}^2 \tag{42}$$

we get

$$\begin{aligned}\dot{V} &\leqslant -\int_0^1 e_x^2\,\mathrm{d}x - \frac{\int_0^1 \hat{e}^2(x)\,\mathrm{d}x}{4(1+\|v\|^2)} + \frac{\int_0^1 \hat{e}(x)e(x)\,\mathrm{d}x}{4(1+\|v\|^2)} \\ &\leqslant -\|e_x\|^2 - \frac{\|\hat{e}\|^2}{4(1+\|v\|^2)} + \frac{\|e_x\|\|\hat{e}\|}{2\sqrt{1+\|v\|^2}} \\ &\leqslant -\frac{1}{2}\|e_x\|^2 - \frac{1}{8}\frac{\|\hat{e}\|^2}{1+\|v\|^2}.\end{aligned} \tag{43}$$

This gives the following properties:

$$\frac{\|\hat{e}\|}{\sqrt{1+\|v\|^2}} \in \mathscr{L}_2 \cap \mathscr{L}_\infty, \quad \tilde{\lambda} \in \mathscr{L}_\infty, \quad \dot{\hat{\lambda}} \in \mathscr{L}_2 \cap \mathscr{L}_\infty. \tag{44}$$

In contrast with the passive identifier, the normalization in the swapping identifier is employed in the update law. This makes $\dot{\hat{\lambda}}$ not only square integrable but also bounded.

We modify the transformation (4) in the following way for the case of unknown λ:

$$\hat{w}(x) = \hat{\lambda}v(x) + \eta(x) - \int_0^x \hat{k}(x,\xi)(\hat{\lambda}v(\xi) + \eta(\xi))\,\mathrm{d}\xi \tag{45}$$

with the same $\hat{k}(x, \xi)$ as in (19). Using (32)–(36) one can get the following PDE for $\hat{w}$:

$$\hat{w}_t = \hat{w}_{xx} + \hat{\lambda}\left(\hat{e}(x) - \int_0^x \hat{k}(x,\xi)\hat{e}(\xi)\,\mathrm{d}\xi\right) + \dot{\hat{\lambda}}v(x) + \dot{\hat{\lambda}}\int_0^x \left(\frac{\xi}{2}\hat{w}(\xi) - \hat{k}(x,\xi)v(\xi)\right)\mathrm{d}\xi, \tag{46}$$

$$\hat{w}(0) = \hat{w}(1) = 0. \tag{47}$$

In order to prove boundedness of all signals we rewrite the filter (32)–(33) as follows:

$$v_t = v_{xx} + \hat{e} + \hat{w} + \int_0^x \hat{l}(x,\xi)\hat{w}(\xi)\,\mathrm{d}\xi, \tag{48}$$

$$v(0) = v(1) = 0. \tag{49}$$

We have now two interconnected systems for v and $\hat{w}$, (46)–(49), which are driven by the signals $\dot{\hat{\lambda}}$, $\hat{\lambda}$, and $\hat{e}$ with properties (44). Note that the situation here is more complicated than in the passive design where we had to analyze only the $\hat{w}$-system (21)–(22). While the signal v feeds into $\hat{w}$-system (46)–(47) through a "convergent-to-zero" signal $\dot{\hat{\lambda}}$, the signal $\hat{w}$ feeds into the v-system (48)–(49) through a bounded but possibly large gain $\hat{l}$. Therefore to prove the boundedness of $\|\hat{w}\|$ and $\|v\|$ we use a weighted Lyapunov function

$$W = A\|\hat{w}\|^2 + \|v\|^2, \tag{50}$$

where A is a large enough constant (for more details on how A is selected, see the more general case in Section 6.2). One can show then that

$$\dot{W} \leqslant -\frac{1}{4A}W + l_1 W, \tag{51}$$

and with the help of Lemma A.2 we get the boundedness of $\|\hat{w}\|$ and $\|v\|$. Using this result it can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\hat{w}_x\|^2 + \|v_x\|^2) \leqslant -\|\hat{w}_{xx}\|^2 - \|v_{xx}\|^2 + l_1, \tag{52}$$

which proves that $\|\hat{w}_x\|$ and $\|v_x\|$ are bounded. From Agmon's inequality we get that $\hat{w}$ and v are bounded pointwise in x. Using Lemma A.1 we get $\|\hat{w}\| \to 0$, $\|v\| \to 0$ as $t \to \infty$. From (45) and (37) we get the pointwise boundedness of η and u and $\|u\| \to 0$. Finally, the pointwise regulation of u to zero follows from Agmon's inequality.

3. Passive identifier for a 3D plant

We present now a passivity-based design for a plant in a 3D setting:

$$u_t = \varepsilon(u_{xx} + u_{yy} + u_{zz}) + b_1 u_x + b_2 u_y + b_3 u_z + \lambda u \tag{53}$$

for $(x, y, z) \in \Omega$, where the domain Ω is a cylinder with top and bottom of arbitrary shape Γ (Fig. 1). This configuration of the domain Ω is essential because it allows us to view the problem as many 1D problems with $0 \leqslant x \leqslant 1$ and fixed y, z. We assume Dirichlet boundary conditions on the boundary $\partial\Omega$,

$$u = 0 \quad (x, y, z) \in \partial\Omega \backslash \{x = 1\}, \tag{54}$$

except at the top of the cylinder $x = 1$ where the actuation is applied:

$$u(1, y, z) = U(t, y, z) \quad (y, z) \in \Gamma. \tag{55}$$

The parameters $\varepsilon > 0$, b_1, b_2, b_3, λ are assumed to be unknown.

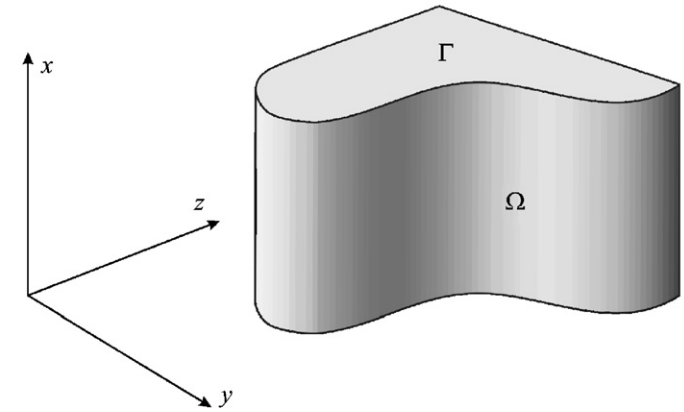

Fig. 1. The domain Ω for the plant (53).

For the notational convenience let us use the following notation later in this section:

$$\Delta u = u_{xx} + u_{yy} + u_{zz}, \quad \nabla u = (u_x, u_y, u_z)^{\mathrm{T}},$$

$$\|u\|^2 \triangleq \int\int_\Gamma \int_0^1 u^2(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \triangleq \int_\Omega u^2\,\mathrm{d}\Omega,$$

$$\|\nabla u\|^2 \triangleq \int_\Omega \nabla u \cdot \nabla u\,\mathrm{d}\Omega, \quad \mathbf{b} = (b_1, b_2, b_3)^{\mathrm{T}}. \tag{56}$$

We will employ the following "observer":

$$\hat{u}_t = \hat{\varepsilon}\Delta\hat{u} + \hat{\mathbf{b}} \cdot \nabla\hat{u} + \hat{\lambda}u + \gamma^2(u - \hat{u})\|\nabla u\|^2 \quad (x, y, z) \in \Omega, \tag{57}$$

$$\hat{u} = 0 \quad (x, y, z) \in \partial\Omega \backslash \{x = 1\}, \tag{58}$$

$$\hat{u} = u, \quad x = 1, \ (y, z) \in \Gamma. \tag{59}$$

There are two main differences compared to 1D case with one parameter in Section 2. First, since the diffusion coefficient ε is unknown we must use projection to ensure $\hat{\varepsilon} > \underline{\varepsilon} > 0$. We define the projection operator as

$$\mathrm{Proj}_{\underline{\varepsilon}}\{\tau\} = \begin{cases} 0, & \hat{\varepsilon} = \underline{\varepsilon} \text{ and } \tau < 0, \\ \tau, & \text{else.} \end{cases} \tag{60}$$

Although this operator is discontinuous it is possible to introduce a small boundary layer instead of a hard switch which will avoid dealing with Filippov solutions and noise due to frequent switching of the update law (see Krstic & Smyshlyaev, 2005 for more details). However, we use (60) here for notational clarity. Note that $\hat{\varepsilon}$ does not require the projection from above and all other parameters do not require projection at all. Second, we can see in (57) that while the diffusion and advection coefficients multiply the operators of $\hat{u}$, the reaction coefficient multiplies u in the observer. This is necessary in order to eliminate any λ-dependence in the error system so that it is stable.

The error signal $e = u - \hat{u}$ satisfies the following PDE:

$$e_t = \hat{\varepsilon}\Delta e + \hat{\mathbf{b}} \cdot \nabla e + \tilde{\varepsilon}\Delta u + \tilde{\mathbf{b}} \cdot \nabla u + \tilde{\lambda}u - \gamma^2 e\|\nabla u\|^2, \tag{61}$$

$$e = 0 \quad (x, y, z) \in \partial\Omega. \tag{62}$$

Using a Lyapunov function

$$V=\frac{1}{2}\int_{\Omega}e^2\,\mathrm{d}\Omega+\frac{\tilde{\varepsilon}^2}{2\gamma_1}+\frac{|\tilde{\mathbf{b}}|^2}{2\gamma_2}+\frac{\tilde{\lambda}^2}{2\gamma_3} \tag{63}$$

we get

$$\begin{aligned}\dot V=&-\hat{\varepsilon}\|\nabla e\|^2-\gamma^2\|e\|^2\|\nabla u\|^2\\&+\tilde{\varepsilon}\int_{\Omega}e\Delta u\,\mathrm{d}\Omega+\int_{\Omega}e(\tilde{\mathbf{b}}\cdot\nabla u)\,\mathrm{d}\Omega\\&+\tilde{\lambda}\int_{\Omega}eu\,\mathrm{d}\Omega-\frac{1}{\gamma_0}\tilde{\varepsilon}\dot{\hat{\varepsilon}}-\frac{1}{\gamma_1}\tilde{\mathbf{b}}\cdot\dot{\hat{\mathbf{b}}}-\frac{1}{\gamma_2}\tilde{\lambda}\dot{\hat{\lambda}}.\end{aligned} \tag{64}$$

With update laws

$$\dot{\hat{\varepsilon}}=-\gamma_0\mathrm{Proj}_{\underline{\varepsilon}}\left\{\int_{\Omega}\nabla u\cdot\nabla(u-\hat u)\,\mathrm{d}\Omega\right\}, \tag{65}$$

$$\dot{\hat{\mathbf{b}}}=\gamma_1\int_{\Omega}(u-\hat u)\nabla u\,\mathrm{d}\Omega,\quad \dot{\hat{\lambda}}=\gamma_2\int_{\Omega}(u-\hat u)u\,\mathrm{d}\Omega, \tag{66}$$

where $\gamma_0,\gamma_1,\gamma_2>0$ we get

$$\dot V\leqslant-\underline{\varepsilon}\|\nabla e\|^2-\gamma^2\|e\|^2\|\nabla u\|^2, \tag{67}$$

which implies $V(t)\leqslant V(0)$ so that $\tilde{\varepsilon}$, $|\tilde{\mathbf{b}}|$, $\tilde{\lambda}$, $\|e\|$ are bounded. Integrating (67) with respect to time from zero to infinity we get square integrability of $\|\nabla e\|$, $\|e\|\|\nabla u\|$, which, together with the update laws (65)–(66), gives square integrability of $|\dot{\hat{\mathbf{b}}}|$ and $\dot{\hat{\lambda}}$.

Lemma 1. *The identifier* (57)–(59) *with update laws* (66) *guarantees the following properties*:

$$\|\nabla e\|,\ \|e\|\|\nabla u\|\in\mathscr{L}_2,\quad \|e\|\in\mathscr{L}_\infty\cap\mathscr{L}_2, \tag{68}$$

$$\tilde{\varepsilon},\tilde b_1,\tilde b_2,\tilde b_3,\tilde{\lambda}\in\mathscr{L}_\infty,\quad \dot{\hat b}_1,\dot{\hat b}_2,\dot{\hat b}_3,\dot{\hat{\lambda}}\in\mathscr{L}_2. \tag{69}$$

We employ the following controller:

$$u(1,y,z)=-\int_0^1\frac{\hat{\lambda}+c}{\hat{\varepsilon}}\,\xi\mathrm{e}^{-\hat b_1(1-\xi)/2\hat{\varepsilon}}\times\frac{I_1\left(\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(1-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(1-\xi^2)}}\hat u(\xi,y,z)\,\mathrm{d}\xi \tag{70}$$

with $c\geqslant 0$, which is a straightforward generalization of the one proposed in Smyshlyaev and Krstic (2004) for the case of known parameters.

Starting with the result on stability of the identifier, we now turn to proving closed-loop stability. Unfortunately, it is very hard to prove the result in the case of unknown ε. This is because, while the identifier guarantees the properties (68) for $\|e\|$ and $\|\nabla e\|$, it does not provide any estimates for $\|\Delta e\|$ which are required in the case of unknown ε. Therefore for the closed-loop result we assume that ε is known and set $\hat{\varepsilon}=\varepsilon$ everywhere. The update law (65) nevertheless achieves closed-loop stability for unknown ε in simulations, as shown in Section 7.

Theorem 2. *Consider the plant* (53), (54) *with the controller* (70). *If the closed loop system that consists of* (53), (54), (70), *identifier* (57)–(59), *and update laws* (66) *has a classical solution* ($\hat{\mathbf{b}}$, $\hat{\lambda}$, u, $\hat u$), *then for any* $\hat{\mathbf{b}}(0)$, $\hat{\lambda}(0)$ *and any initial conditions* $u_0,\hat u_0\in H_1(\Omega)$, *the signals* $\hat{\mathbf{b}}$, $\hat{\lambda}$, u, $\hat u$ *are bounded and* u *is regulated to zero for all* $(x,y,z)\in\Omega$:

$$\lim_{t\to\infty}\max_{(x,y,z)\in\Omega}|u(x,y,z,t)|=0. \tag{71}$$

4. Proof of Theorem 2

We will use Poincaré and Agmon's inequalities (Temam, 1988):

$$\|u\|\leqslant d_1(\Gamma)\|\nabla u\|, \tag{72}$$

$$\max_{(x,y,z)\in\Omega}|u|\leqslant d_2(\Gamma)\|u\|_{H_1}^{1/2}\|u\|_{H_2}^{1/2}. \tag{73}$$

Here d_1 and d_2 are constants that depend only on Γ. The main difficulty in proving the result in 3D case compared to 1D case is that we need to show H_2 (instead of H_1) boundedness and H_1 (instead of L_2) regulation in order to have pointwise boundedness and regulation.

4.1. Target system

We use the following transformation:

$$\hat w(x,y,z)=\hat u(x,y,z)-\int_0^x\hat k(x,\xi)\hat u(\xi,y,z)\,\mathrm{d}\xi, \tag{74}$$

$$\hat k(x,\xi)=-\frac{\hat{\lambda}+c}{\varepsilon}\xi\mathrm{e}^{-\hat b_1(x-\xi)/2\varepsilon}\frac{I_1\left(\sqrt{\frac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}}, \tag{75}$$

which is a generalized version of the transformation presented in Smyshlyaev and Krstic (2004) for the case of known parameters.

Lemma 3. *The transformation* (74)–(75) *maps* (57)–(59) *into the target system*:

$$\begin{aligned}\hat w_t=&\varepsilon\Delta\hat w+\hat{\mathbf{b}}\cdot\nabla\hat w-c\hat w+\dot{\hat b}_1\Phi_1[\hat w]+\dot{\hat{\lambda}}\Phi_2[\hat w]\\&+(\hat{\lambda}+\gamma^2\|\nabla u\|^2)e_1,\end{aligned} \tag{76}$$

$$\hat w=0\quad (x,y,z)\in\partial\Omega, \tag{77}$$

where

$$\Phi_i[\hat w]=\int_0^x\varphi_i(x,\xi)\hat w(\xi,y,z)\,\mathrm{d}\xi, \tag{78}$$

$$e_1=e-\int_0^x\hat k(x,\xi)e(\xi,y,z)\,\mathrm{d}\xi \tag{79}$$

and

$$\varphi_1(x,\xi)=\frac{x-\xi}{2\varepsilon}\hat{k}(x,\xi)+\frac{1}{2\varepsilon}\int_\xi^x (x-\sigma)\hat{k}(x,\sigma)\hat{l}(\sigma,\xi)\,\mathrm{d}\sigma,$$

$$\varphi_2(x,\xi)=\frac{\xi}{2\varepsilon}\,\mathrm{e}^{-(\hat{b}_1/2\varepsilon)(x-\xi)}. \tag{80}$$

Proof. Substituting (74) into (57) we get

$$\hat{w}_t=\varepsilon\Delta\hat{w}+\hat{\mathbf{b}}\cdot\nabla\hat{w}-c\hat{w}+(\hat{\lambda}+\gamma^2\|\nabla u\|^2)e_1 - \int_0^x(\dot{\hat{b}}_1\hat{k}_{\hat{b}_1}(x,\xi)+\dot{\hat{\lambda}}\hat{k}_{\hat{\lambda}}(x,\xi))\hat{u}(\xi,y,z)\,\mathrm{d}\xi. \tag{81}$$

To replace $\hat{u}$ with $\hat{w}$ we use an inverse transformation

$$\hat{u}=\hat{w}+\int_0^x \hat{l}(x,\xi)\hat{w}(\xi,y,z)\,\mathrm{d}\xi, \tag{82}$$

$$\hat{l}(x,\xi)=-\frac{\hat{\lambda}+c}{\varepsilon}\,\xi\mathrm{e}^{-\hat{b}_1(x-\xi)/2\varepsilon}\frac{J_1\left(\sqrt{\frac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}}. \tag{83}$$

We have

$$\int_0^x \hat{k}_{\hat{\lambda}}(x,\xi)\hat{u}(\xi,y,z)\,\mathrm{d}\xi = \int_0^x\left(\hat{k}_{\hat{\lambda}}(x,\xi)+\int_\xi^x \hat{k}_{\hat{\lambda}}(x,\sigma)l(\sigma,\xi)\,\mathrm{d}\sigma\right) \times \hat{w}(\xi,y,z)\,\mathrm{d}\xi,, \tag{84}$$

and similarly for $\hat{b}_1$. Computing the inner integrals with the help of Prudnikov, Brychkov, and Marichev (1986) we get (76)–(80). □

We should mention that while the target system (76)–(77) is complicated, only the proof is affected by this complexity and not the design (which is simple).

4.2. Boundedness

Let us denote the bounds on $|\hat{\mathbf{b}}|$, $\hat{\lambda}$ by b_0, λ_0. Since $\hat{k}$ and $\hat{l}$ and their derivatives with respect to parameters are bounded functions, we have the estimates

$$\|e_1\|\leqslant M_1\|e\|,\quad \|\nabla u\|\leqslant M_2\|\nabla\hat{w}\|+\|\nabla e\|, \tag{85}$$

where M_1, M_2 are some constants. The functions φ_1, φ_2 are also bounded, let us denote these bounds by $\bar{\varphi}_1$, $\bar{\varphi}_2$.

First, we show the boundedness of the L_2-norm,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}\|^2=-\varepsilon\|\nabla\hat{w}\|^2+\dot{\hat{b}}_1\int_\Omega \hat{w}\Phi_1\,\mathrm{d}\Omega+\dot{\hat{\lambda}}\int_\Omega\hat{w}\Phi_2\,\mathrm{d}\Omega - c\|\hat{w}\|^2+(\hat{\lambda}+\gamma^2\|\nabla u\|^2)\int_\Omega e_1\hat{w}\,\mathrm{d}\Omega. \tag{86}$$

Using the estimate

$$\dot{\hat{b}}_1\int_\Omega\hat{w}\Phi_1\,\mathrm{d}\Omega\leqslant\frac{\varepsilon}{8d_1^2}\|\hat{w}\|^2+\frac{2}{\varepsilon}d_1^2|\dot{\hat{b}}_1|^2\bar{\varphi}_1^2\|\hat{w}\|^2 \leqslant\frac{\varepsilon}{8}\|\nabla\hat{w}\|^2+l_1\|\hat{w}\|^2, \tag{87}$$

and similarly for the term with $\dot{\hat{\lambda}}$, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}\|^2&\leqslant-\frac{3\varepsilon}{4}\|\nabla\hat{w}\|^2+l_1\|\hat{w}\|^2+M_1\lambda_0\|\hat{w}\|\|e\|\\&\quad+\gamma^2M_1\|\nabla u\|(M_2\|\nabla\hat{w}\|+\|\nabla e\|)\|\hat{w}\|\|e\|\\&\leqslant-\frac{3\varepsilon}{4}\|\nabla\hat{w}\|^2+l_1\|\hat{w}\|^2+\frac{d_1^2}{\varepsilon}M_1^2\lambda_0^2\|e\|^2\\&\quad+\frac{\varepsilon}{4d_1^2}\|\hat{w}\|^2+\frac{\varepsilon}{4}\|\nabla\hat{w}\|^2+\frac{\varepsilon}{4M_2^2}\|\nabla e\|^2\\&\quad+\frac{2}{\varepsilon}\gamma^4M_1^2M_2^2\|\nabla u\|^2\|e\|^2\|\hat{w}\|^2\\&\leqslant-\frac{\varepsilon}{4}\|\nabla\hat{w}\|^2+l_1\|\hat{w}\|^2+l_1.\end{aligned} \tag{88}$$

Using Lemma A.2 we get $\|\hat{w}\|\in\mathscr{L}_\infty\cap\mathscr{L}_2$. Integrating (88) with respect to time from zero to infinity we also get $\|\nabla\hat{w}\|\in\mathscr{L}_2$ and therefore $\|\nabla\hat{u}\|,\|\nabla u\|\in\mathscr{L}_2$.

Now let us show H_1 boundedness. In this case it is enough to consider e and $\hat{w}$ systems separately. First,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla e\|^2&=\int_\Omega\nabla e_t\nabla e\,\mathrm{d}\Omega=-\int_\Omega e_t\Delta e\,\mathrm{d}\Omega\\&\leqslant-\varepsilon\|\Delta e\|^2+b_0\|\Delta e\|\|\nabla e\|+|\tilde{\mathbf{b}}|\|\Delta e\|\|\nabla u\|\\&\quad+|\tilde{\lambda}|\|\Delta e\|\|u\|-\gamma^2\|\nabla e\|^2\|\nabla u\|^2\\&\leqslant-\varepsilon\|\Delta e\|^2+\frac{\varepsilon}{4}\|\Delta e\|^2+\frac{b_0^2}{\varepsilon}\|\nabla e\|^2+\frac{\varepsilon}{4}\|\Delta e\|^2\\&\quad+\frac{|\tilde{\mathbf{b}}|^2}{\varepsilon}\|\nabla u\|^2+\frac{\varepsilon}{4}\|\Delta e\|^2+\frac{|\tilde{\lambda}|^2}{\varepsilon}\|u\|^2\\&\leqslant-\frac{\varepsilon}{4}\|\Delta e\|^2+l_1.\end{aligned} \tag{89}$$

Using Lemma A.2 we get $\|\nabla e\|\in\mathscr{L}_\infty\cap\mathscr{L}_2$. Second,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\hat{w}\|^2&=-\varepsilon\|\Delta\hat{w}\|^2-c\|\nabla\hat{w}\|^2-\int_\Omega\Delta\hat{w}\,(\hat{\mathbf{b}}\cdot\nabla\hat{w})\,\mathrm{d}\Omega\\&\quad-\dot{\hat{b}}_1\int_\Omega\Delta\hat{w}\Phi_1\,d\Omega-\dot{\hat{\lambda}}\int_\Omega\Delta\hat{w}\Phi_2\,\mathrm{d}\Omega\\&\quad+(\hat{\lambda}+\gamma^2\|\nabla u\|^2)\int_\Omega e\Delta\hat{w}\,\mathrm{d}\Omega.\end{aligned} \tag{90}$$

Using the estimates (that follow from $\|\nabla\hat{w}\|\in\mathscr{L}_\infty\cap\mathscr{L}_2$)

$$\int_\Omega\Delta\hat{w}\,(\hat{\mathbf{b}}\cdot\nabla\hat{w})\,\mathrm{d}\Omega\leqslant b_0\|\Delta\hat{w}\|\|\nabla\hat{w}\|\leqslant\frac{\varepsilon}{8}\|\Delta\hat{w}\|^2+l_1,$$

$$\dot{\hat{b}}_1\int_\Omega\Delta\hat{w}\Phi_1\,\mathrm{d}\Omega\leqslant\frac{\varepsilon}{8}\|\Delta\hat{w}\|^2+l_1\|\hat{w}\|^2, \tag{91}$$

and similarly for the term with $\dot{\hat{\lambda}}$, it is easy to show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\hat{w}\|^2\leqslant-\frac{\varepsilon}{4}\|\Delta\hat{w}\|^2+l_1\|\nabla\hat{w}\|^2+l_1. \tag{92}$$

Using Lemma A.2 we get $\|\nabla\hat{w}\|\in\mathscr{L}_\infty\cap\mathscr{L}_2$ and therefore $\|\nabla\hat{u}\|,\|\nabla u\|\in\mathscr{L}_\infty\cap\mathscr{L}_2$. Integrating (89), (92) we also get

$\|\Delta e\|, \|\Delta\hat{w}\| \in \mathscr{L}_2$ and therefore $\|\Delta\hat{u}\|, \|\Delta u\| \in \mathscr{L}_2$. Note that from the above properties and (89)–(92) it follows that $(\mathrm{d}/\mathrm{d}t)\|\nabla e\|^2$ and $(\mathrm{d}/\mathrm{d}t)\|\nabla\hat{w}\|^2$ are bounded. By Lemma A.1 we get $\|\nabla e\|, \|\nabla\hat{w}\| \to 0$ and therefore $\|\nabla\hat{u}\|, \|\nabla u\| \to 0$ as $t \to \infty$.

In order to prove pointwise boundedness in 3D we need to show that the H_2 norms of the signals are bounded. It is more convenient to prove the boundedness of $\|\hat{w}_t\|$ and $\|e_t\|$ first and then use Eqs. (76), (61) to bound the H_2 norms. We start with

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_t\|^2 = & \int_\Omega e_t e_{tt}\,\mathrm{d}\Omega \\ \leqslant & -\varepsilon\|\nabla e_t\|^2 + |\dot{\hat{\mathbf{b}}}|\|e_t\|\|\nabla e\| + |\dot{\hat{\mathbf{b}}}|\|e_t\|\|\nabla u\| \\ & + |\tilde{\mathbf{b}}|\|\nabla e_t\|\|u_t\| + |\dot{\hat{\lambda}}|\|e_t\|\|u\| \\ & + |\tilde{\lambda}|\|e_t\|\|u_t\| + \gamma^2\|e\|\|e_t\|\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2\right|. \end{aligned} \quad (93)$$

We note that

$$\|u_t\|^2 \leqslant 2(\varepsilon^2\|\Delta u\|^2 + |\mathbf{b}|^2\|\nabla u\|^2 + \lambda^2\|u\|^2) \leqslant l_1 \quad (94)$$

and

$$\begin{aligned}&\|e\|\|e_t\|\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2\right| \\ &\leqslant 2\|e\|\|e_t\|(M_2^2\|\nabla\hat{w}\|\|\nabla\hat{w}_t\| + \|\nabla e\|\|\nabla e_t\|) \\ &\leqslant l_1 + \frac{\varepsilon}{8}\|\nabla e_t\|^2 + c_1\|\nabla\hat{w}_t\|^2,\end{aligned} \quad (95)$$

where c_1 is an arbitrary constant. We get the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_t\|^2 \leqslant -\frac{\varepsilon}{2}\|\nabla e_t\|^2 + l_1\|e_t\|^2 + c_1\|\nabla\hat{w}_t\|^2 + l_1. \quad (96)$$

Now we estimate the time derivative of $\|\hat{w}_t\|^2$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}_t\|^2 \leqslant & -\varepsilon\|\nabla\hat{w}_t\|^2 + |\dot{\hat{\mathbf{b}}}|\|\hat{w}_t\|\|\nabla\hat{w}\| - c\|\hat{w}_t\|^2 \\ & + |\ddot{\hat{b}}_1|\bar{\varphi}_1\|\hat{w}_t\|\|\hat{w}\| + |\dot{\hat{b}}_1|\|\hat{w}_t\dot{\Phi}_1\| \\ & + |\ddot{\hat{\lambda}}|\bar{\varphi}_2\|\hat{w}_t\|\|\hat{w}\| + |\dot{\hat{\lambda}}|\|\hat{w}_t\dot{\Phi}_2\| \\ & + (\lambda_0 + \gamma^2\|\nabla u\|^2)\|e_{1t}\|\|\hat{w}_t\| \\ & + \left(|\dot{\hat{\lambda}}| + \gamma^2\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2\right|\right)\|e\|M_1\|\hat{w}_t\|.\end{aligned} \quad (97)$$

Using the estimates

$$\|\hat{w}_t\dot{\Phi}_1\|^2 \leqslant \bar{\varphi}_1^2\|\hat{w}_t\|^2 + M_3\|\hat{w}_t\|^2\|\hat{w}\|^2,$$

$$\|\hat{w}_t\dot{\Phi}_2\|^2 \leqslant \bar{\varphi}_2^2\|\hat{w}_t\|^2 + M_4\|\hat{w}_t\|^2\|\hat{w}\|^2,$$

$$\begin{aligned}|\ddot{\hat{b}}_1|^2 &\leqslant 2\gamma_1^2\|e_t\|^2\|\nabla u\|^2 + 2\gamma_1^2\|e\|^2\|\nabla u_t\|^2 \\ &\leqslant l_1\|e_t\|^2 + M_5(\|\nabla\hat{w}_t\|^2 + \|\nabla e_t\|^2),\end{aligned}$$

$$\begin{aligned}|\ddot{\hat{\lambda}}|^2 &\leqslant 2\gamma_2^2\|e_t\|^2\|u\|^2 + 2\gamma_2^2\|e\|^2\|u_t\|^2 \\ &\leqslant l_1\|e_t\|^2 + l_1,\end{aligned}$$

$$\|e_{1t}\|^2 \leqslant 2M_1^2\|e_t\|^2 + M_6\|e\|^2, \quad (98)$$

one can show that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}_t\|^2 \leqslant & -\left(\frac{\varepsilon}{4} - c_2\right)\|\nabla\hat{w}_t\|^2 + \frac{4\lambda_0^2M_1^2d_1^2}{\varepsilon}\|e_t\|^2 + l_1 \\ & + (c_2 + c_3)\|\nabla e_t\|^2 + l_1\|\hat{w}_t\|^2 + l_1\|e_t\|^2.\end{aligned} \quad (99)$$

Combining (99) and (96) with a weighting constant A we get

$$\begin{aligned}&\frac{A}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}_t\|^2 \\ &\leqslant -\left(\frac{\varepsilon}{4} - c_2 - c_1A\right)\|\nabla\hat{w}_t\|^2 \\ &\quad - \left(\frac{\varepsilon}{2}A - \frac{4\lambda_0^2M_1^2d_1^4}{\varepsilon} - c_2 - c_3\right)\|\nabla e_t\|^2 \\ &\quad + l_1\|\hat{w}_t\|^2 + \|e_t\|^2 + l_1.\end{aligned} \quad (100)$$

Choosing $A = 1 + 8\lambda_0^2M_1^2d_1^4\varepsilon^{-2}$, $c_1 = \varepsilon/(16A)$, $c_2 = c_3 = \varepsilon/8$, we get

$$\begin{aligned}\frac{A}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}_t\|^2 \leqslant & -\frac{\varepsilon}{16}\|\nabla\hat{w}_t\|^2 - \frac{\varepsilon}{4}\|\nabla e_t\|^2 \\ & + l_1\|\hat{w}_t\|^2 + \|e_t\|^2 + l_1.\end{aligned} \quad (101)$$

By Lemma A.2 $\|\hat{w}_t\|, \|e_t\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$ and therefore $\|\hat{u}_t\|, \|u_t\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$. From (57) and (53) we get $\|\Delta\hat{u}\|, \|\Delta u\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$. Using now Agmon's inequality (73) we get the regulation result:

$$\lim_{t\to\infty}\max_{(x,y,z)\in\Omega}|u| \leqslant d_2\lim_{t\to\infty}\|u\|_{H_1}^{1/2}\|u\|_{H_2}^{1/2} = 0. \quad (102)$$

5. Swapping identifier for a reaction–advection–diffusion plant

Let us consider now a swapping-based approach for the plant

$$u_t = \varepsilon u_{xx} + bu_x + \lambda u, \quad (103)$$

$$u(0) = 0, \quad (104)$$

$$u(1) = U(t) \quad (105)$$

with three unknown parameters ε, b, λ. We restrict our attention to the 1D case here. The result can be readily extended to the 3D plant (53) in a similar fashion as in Section 3 for passive identifiers.

We need to employ four (the number of uncertain parameters plus one) filters. Let us first write the "estimation error" in the form

$$e = u - \varepsilon\psi - bp - \lambda v - \eta, \quad (106)$$

where v, p, ψ are filters for u, u_x, and u_{xx}, respectively,

$$v_t = \hat{\varepsilon}v_{xx} + \hat{b}v_x + u, \quad (107)$$

$$v(0) = v(1) = 0, \quad (108)$$

$$p_t = \hat{\varepsilon}p_{xx} + \hat{b}p_x + u_x, \quad (109)$$

$$p(0) = p(1) = 0, \quad (110)$$

$$\psi_t = \hat{\varepsilon}\psi_{xx} + \hat{b}\psi_x + u_{xx}, \quad (111)$$

$$\psi(0)=\psi(1)=0, \tag{112}$$

and η is the following filter:

$$\eta_t=\hat{\varepsilon}\eta_{xx}+\hat{b}\eta_x-\hat{b}u_x-\hat{\varepsilon}u_{xx}, \tag{113}$$

$$\eta(0)=0, \tag{114}$$

$$\eta(1)=u(1). \tag{115}$$

Note that in the case of known ε or b, the filter η is modified by dropping the terms $\hat{\varepsilon}u_{xx}$ or $\hat{b}u_x$ in (113), so that there is no need to measure u_{xx} or u_x.

With the filters (107)–(115) the estimation error (106) satisfies the following exponentially stable equation:

$$e_t=\hat{\varepsilon}e_{xx}+\hat{b}e_x, \tag{116}$$

$$e(0)=e(1)=0. \tag{117}$$

We implement a "prediction error" as

$$\hat{e}=u-\hat{\varepsilon}\psi-\hat{b}p-\hat{\lambda}v-\eta, \tag{118}$$

which is related to the estimation error by

$$\hat{e}=e+\tilde{\varepsilon}\psi+\tilde{b}p+\tilde{\lambda}v. \tag{119}$$

One important difference with respect to the benchmark plant (1)–(3) is that the diffusion coefficient ε is now unknown and we must use projection to ensure $\hat{\varepsilon}>\underline{\varepsilon}>0$ to keep parabolic character of the systems involved in the adaptive scheme. The projection operator is defined by (60).

We choose gradient update laws with normalization

$$\dot{\hat{\varepsilon}}=\gamma_1\mathrm{Proj}_{\underline{\varepsilon}}\left\{\frac{\int_0^1\hat{e}(x)\psi(x)\,\mathrm{d}x}{1+\|\psi\|^2+\|p\|^2+\|v\|^2}\right\}, \tag{120}$$

$$\dot{\hat{b}}=\gamma_2\frac{\int_0^1\hat{e}(x)p(x)\,\mathrm{d}x}{1+\|\psi\|^2+\|p\|^2+\|v\|^2}, \tag{121}$$

$$\dot{\hat{\lambda}}=\gamma_3\frac{\int_0^1\hat{e}(x)v(x)\,\mathrm{d}x}{1+\|\psi\|^2+\|p\|^2+\|v\|^2}, \tag{122}$$

where $\gamma_1,\gamma_2,\gamma_3>0$.

Lemma 4. *The update laws* (120)–(122) *guarantee the following properties*:

$$\tilde{\varepsilon},\tilde{b},\tilde{\lambda}\in\mathscr{L}_\infty,\quad \dot{\hat{\varepsilon}},\dot{\hat{b}},\dot{\hat{\lambda}}\in\mathscr{L}_2\cap\mathscr{L}_\infty, \tag{123}$$

$$\frac{\|\hat{e}\|}{\sqrt{1+\|\psi\|^2+\|p\|^2+\|v\|^2}}\in\mathscr{L}_2\cap\mathscr{L}_\infty. \tag{124}$$

Proof. With a Lyapunov function

$$V=\frac{1}{2}\int_0^1 e^2\,\mathrm{d}x+\frac{1}{8\gamma_1}\tilde{\varepsilon}^2+\frac{1}{8\gamma_2}\tilde{b}^2+\frac{1}{8\gamma_3}\tilde{\lambda}^2 \tag{125}$$

we get

$$\begin{aligned}\dot{V}&\leqslant-\int_0^1 e_x^2\,\mathrm{d}x-\frac{\int_0^1\hat{e}(\tilde{\varepsilon}\psi+\tilde{b}p+\tilde{\lambda}v)\,\mathrm{d}x}{4(1+\|\psi\|^2+\|p\|^2+\|v\|^2)}\\&\leqslant-\|e_x\|^2-\frac{\int_0^1\hat{e}^2(x)\,\mathrm{d}x+\int_0^1\hat{e}(x)e(x)\,\mathrm{d}x}{4(1+\|\psi\|^2+\|p\|^2+\|v\|^2)}\\&\leqslant-\|e_x\|^2-\frac{\|\hat{e}\|^2}{4(1+\|\psi\|^2+\|p\|^2+\|v\|^2)}\\&\quad+\frac{\|e_x\|\|\hat{e}\|}{2\sqrt{1+\|\psi\|^2+\|p\|^2+\|v\|^2}}\\&\leqslant-\frac{1}{2}\|e_x\|^2-\frac{1}{8}\frac{\|\hat{e}\|^2}{1+\|\psi\|^2+\|p\|^2+\|v\|^2}.\end{aligned} \tag{126}$$

This gives

$$\frac{\|\hat{e}\|}{\sqrt{1+\|\psi\|^2+\|p\|^2+\|v\|^2}}\in\mathscr{L}_2 \tag{127}$$

and the boundedness of $\tilde{\varepsilon}$, $\tilde{b}$, $\tilde{\lambda}$. From (119) we get

$$\frac{\|\hat{e}\|}{\sqrt{1+\|\psi\|^2+\|p\|^2+\|v\|^2}}\in\mathscr{L}_\infty \tag{128}$$

and from the update laws (120)–(122) the boundedness and square integrability of $\dot{\hat{\varepsilon}}$, $\dot{\hat{b}}$, and $\dot{\hat{\lambda}}$ follows. □

We use the controller

$$\begin{aligned}u(1)=&-\int_0^1\frac{\hat{\lambda}+c}{\hat{\varepsilon}}\,\xi\mathrm{e}^{-\hat{b}(1-\xi)/2\hat{\varepsilon}}\frac{I_1\left(\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(1-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(1-\xi^2)}}\\&\times(\hat{\varepsilon}\psi(\xi)+\hat{b}p(\xi)+\hat{\lambda}v(\xi)+\eta(\xi))\,\mathrm{d}\xi\end{aligned} \tag{129}$$

with $c\geqslant 0$. The properties of the closed-loop system with this control law will be established in the next section.

As in the case of passivity-based design, it is very hard to prove the closed-loop stability of the swapping-based scheme in the case of unknown ε. The reason for this is that while we have the properties (124) for $\|\hat{e}\|$, we cannot obtain any a priori estimates for $\|\hat{e}_x\|$ which are needed in the proof for a plant with unknown ε. However, the update law (120) is successful in simulations, as shown in Section 7.

In the next section we are going to prove the following result for a plant with known ε.

Theorem 5. *Consider the plant* (103), (104) *with the controller* (129). *If the closed-loop system that consists of* (103)–(104), (129), *the filters* (107)–(109), (115) *and update laws* (121)–(122) *has a classical solution* $(\hat{b},\hat{\lambda},v,p,\eta,u)$, *then for any* $\hat{b}(0)$, $\hat{\lambda}(0)$ *and any initial conditions* $v_0,p_0,\eta_0,u_0\in H_1(0,1)$, *the signals* $\hat{b}$, $\hat{\lambda}$, v, p, η, u *are bounded and* u *is regulated to zero for all* $x\in[0,1]$:

$$\lim_{t\to\infty}\max_{x\in[0,1]}|u(x,t)|=0. \tag{130}$$

6. Proof of Theorem 5

6.1. Target system

We use the following transformation:

$$\hat{w}(x)=\hat{b}p(x)+\hat{\lambda}v(x)+\eta(x) - \int_0^x \hat{k}(x,\xi)(\hat{b}p(\xi)+\hat{\lambda}v(\xi)+\eta(\xi))\,\mathrm{d}\xi \tag{131}$$

with the inverse

$$\hat{b}p(x)+\hat{\lambda}v(x)+\eta(x)=\hat{w}(x)+\int_0^x \hat{l}(x,\xi)\hat{w}(\xi)\,\mathrm{d}\xi, \tag{132}$$

where $\hat{k}(x,\xi)$ and $\hat{l}(x,\xi)$ are given by

$$\hat{k}(x,\xi)=-\frac{\hat{\lambda}+c}{\varepsilon}\,\xi\mathrm{e}^{-\hat{b}(x-\xi)/2\varepsilon}\frac{I_1\left(\sqrt{\dfrac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}\right)}{\sqrt{\dfrac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}}, \tag{133}$$

$$\hat{l}(x,\xi)=-\frac{\hat{\lambda}+c}{\varepsilon}\,\xi\mathrm{e}^{-\hat{b}(x-\xi)/2\varepsilon}\frac{J_1\left(\sqrt{\dfrac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}\right)}{\sqrt{\dfrac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}}. \tag{134}$$

Lemma 6. *The transformation* (131)–(133) *produces the following target system*:

$$\hat{w}_t=\varepsilon\hat{w}_{xx}+\hat{b}\hat{w}_x-c\hat{w}+K[\dot{\hat{b}}p+\dot{\hat{\lambda}}v]+\hat{\lambda}K[\hat{e}] + \int_0^x(\dot{\hat{b}}\varphi_1(x,\xi)+\dot{\hat{\lambda}}\varphi_2(x,\xi))\hat{w}(\xi)\,\mathrm{d}\xi, \tag{135}$$

$$\hat{w}(0)=\hat{w}(1)=0, \tag{136}$$

where

$$K[v]=v(x)-\int_0^x \hat{k}(x,\xi)v(\xi)\,\mathrm{d}\xi, \tag{137}$$

$\varphi_1(x,\xi)$ is given by (80), *and*

$$\varphi_2(x,\xi)=\frac{\xi}{2\varepsilon}\,\mathrm{e}^{-(\hat{b}/2\varepsilon)(x-\xi)}. \tag{138}$$

Proof. Substituting (131) into (103) we get

$$\hat{w}_t=\varepsilon\hat{w}_{xx}+\hat{b}\hat{w}_x-c\hat{w}+K[\dot{\hat{b}}p+\dot{\hat{\lambda}}v]+\hat{\lambda}K[\hat{e}] - \int_0^x(\dot{\hat{b}}\hat{k}_{\hat{b}}(x,\xi)+\dot{\hat{\lambda}}\hat{k}_{\hat{\lambda}}(x,\xi))(\hat{b}p+\hat{\lambda}v+\eta)\,\mathrm{d}\xi. \tag{139}$$

Using the inverse transformation (132) we replace $(\hat{b}p+\hat{\lambda}v+\eta)$ in (139) by $\hat{w}$. Changing the order of the integration and computing the inner integral we get (135). □

We point out that, similarly to the case of the passive identifier design, the target system (135)–(136) is complex while the design itself is simple.

6.2. Boundedness

Let us use (118) and (132) to write the state u in filters (107)–(110) as

$$u(x)=\hat{e}(x)+\hat{w}(x)+\int_0^x \hat{l}(x,\xi)\hat{w}(\xi)\,\mathrm{d}\xi. \tag{140}$$

We have now three interconnected systems for $\hat{w}$, v, and p with external signals $\hat{e}$, $\dot{\hat{b}}$, $\dot{\hat{\lambda}}$ which go to zero in some sense due to the identifier properties (123)–(124).

The identifier properties imply that $\hat{k}$ and $\hat{l}$ are bounded and thus φ_1, φ_2 are bounded. We denote these bounds by $\bar{\varphi}_1$, $\bar{\varphi}_2$. The bounds on $\hat{b}$, $\hat{\lambda}$ are denoted by b_0, λ_0, respectively. We have the following estimates:

$$\int_0^1 \hat{w}(x)\int_0^x \varphi_i(x,\xi)\hat{w}(\xi)\,\mathrm{d}\xi\,\mathrm{d}x \leqslant \bar{\varphi}_i\|\hat{w}\|^2, \tag{141}$$

$$\int_0^1 \hat{w}(x)K[\hat{e}]\,\mathrm{d}x \leqslant M_1\|\hat{w}\|\|\hat{e}\|, \tag{142}$$

$$\|u\|\leqslant\|\hat{e}\|+M_2\|\hat{w}\|, \tag{143}$$

where M_1 and M_2 are some constants that depend on the bounds b_0 and λ_0.

We are now going to perform an L_2 Lyapunov analysis of the $(\hat{w}, v, p)$ system. We start with

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}\|^2 &\leqslant -\varepsilon\|\hat{w}_x\|^2+\lambda_0M_1\|\hat{w}\|\|\hat{e}\|+M_1\|\hat{w}\||\dot{\hat{b}}|\|p\| \\ &\quad + M_1\|\hat{w}\||\dot{\hat{\lambda}}|\|v\|+(|\dot{\hat{b}}|\bar{\varphi}_1+|\dot{\hat{\lambda}}|\bar{\varphi}_2)\|\hat{w}\|^2 \\ &\leqslant -\varepsilon\|\hat{w}_x\|^2+\frac{\varepsilon}{16}\|\hat{w}\|^2+\frac{4\lambda_0^2M_1^2}{\varepsilon}\|\hat{e}\|^2 \\ &\quad + c_1(\|p\|^2+\|v\|^2)+\frac{M_1^2}{4c_1}(|\dot{\hat{b}}|^2+|\dot{\hat{\lambda}}|^2)\|\hat{w}\|^2 \\ &\quad + \frac{\varepsilon}{16}\|\hat{w}\|^2+\frac{8}{\varepsilon}(|\dot{\hat{b}}|^2\bar{\varphi}_1^2+|\dot{\hat{\lambda}}|^2\bar{\varphi}_2^2)\|\hat{w}\|^2.\end{aligned} \tag{144}$$

Here by c_1 we denoted an arbitrary constant that will be defined later. Note that in the estimates we do not use the gain $c\geqslant 0$ to help stabilize the system.

Using properties (124) we have

$$\|\hat{e}\|^2\leqslant l_1\|p\|^2+l_1\|v\|^2+l_1 \tag{145}$$

so (144) can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}\|^2\leqslant-\frac{\varepsilon}{2}\|\hat{w}_x\|^2+c_1(\|p\|^2+\|v\|^2) + l_1(\|\hat{w}\|^2+\|p\|^2+\|v\|^2)+l_1. \tag{146}$$

We do a Lyapunov analysis for the filter v now:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2\leqslant-\varepsilon\|v_x\|^2+\int_0^1 vu\,\mathrm{d}x. \tag{147}$$

Using (140) we have the estimate

$$\int_0^1 vu\,\mathrm{d}x \leqslant M_2\|v\|\|\hat{w}\| + \|v\|\|\hat{e}\| \leqslant \frac{\varepsilon}{16}\|v\|^2 + \frac{4M_2^2}{\varepsilon}\|\hat{w}\|^2 + \frac{\varepsilon}{16}\|v\|^2 + l_1\|p\|^2 + l_1\|v\|^2 + l_1. \tag{148}$$

With this estimate (147) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 \leqslant -\frac{\varepsilon}{2}\|v_x\|^2 + \frac{4M_2^2}{\varepsilon}\|\hat{w}\|^2 + l_1\|p\|^2 + l_1\|v\|^2 + l_1. \tag{149}$$

In a similar way one can show for the filter p

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p\|^2 \leqslant -\frac{\varepsilon}{2}\|p_x\|^2 + \frac{M_2^2}{\varepsilon}\|\hat{w}\|^2 + l_1\|p\|^2 + l_1\|v\|^2 + l_1. \tag{150}$$

With a composite Lyapunov function

$$V = \frac{A}{2}\|\hat{w}\|^2 + \frac{1}{2}\|v\|^2 + \frac{1}{2}\|p\|^2, \tag{151}$$

where A is a constant yet to be defined, we get

$$\dot{V} \leqslant -\left(\frac{\varepsilon}{2}A - \frac{20M_2^2}{\varepsilon}\right)\|\hat{w}_x\|^2 - \left(\frac{\varepsilon}{2} - 4c_1A\right)(\|v_x\|^2 + \|p_x\|^2) + l_1V. \tag{152}$$

Choosing $A = 1 + 40M_2^2\varepsilon^{-2}$ and $c_1 = \varepsilon/(16A)$ we get

$$\dot{V} \leqslant -\frac{\varepsilon}{4A}V + l_1V. \tag{153}$$

Using Lemma A.2 we get $V \in \mathscr{L}_\infty \cap \mathscr{L}_1$. Note that V depends on A, which depends on M_2, which depends on b_0 and λ_0, which in turn depend on the initial conditions of the system. However, $A \geqslant 1$, which implies that $\|\hat{w}\|^2, \|v\|^2, \|p\|^2 \leqslant 2V$, and hence $\|\hat{w}\|, \|v\|, \|p\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$. Integrating (152) we also get $\|\hat{w}_x\|, \|v_x\|, \|p_x\| \in \mathscr{L}_2$.

We proceed now to H_1 analysis (it is needed to establish pointwise boundedness). We start with

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}_x\|^2 &= \int_0^1 \hat{w}_x\hat{w}_{xt}\,\mathrm{d}x = -\int_0^1 \hat{w}_{xx}\hat{w}\,\mathrm{d}x \\ &\leqslant -\varepsilon\|\hat{w}_{xx}\|^2 + \|\hat{w}_{xx}\|(b_0\|\hat{w}_x\| + \lambda_0M_1\|\hat{e}\|) \\ &\quad + M_1\|\hat{w}_{xx}\|\left(|\dot{\hat{b}}|\|p\| + |\dot{\hat{\lambda}}|\|v\|\right) \\ &\quad + (|\dot{\hat{b}}|\bar{\varphi}_1 + |\dot{\hat{\lambda}}|\bar{\varphi}_2)\|\hat{w}_{xx}\|\|\hat{w}\| \\ &\leqslant -\varepsilon\|\hat{w}_{xx}\|^2 + \frac{\varepsilon}{4}\|\hat{w}_{xx}\|^2 + \frac{2\lambda_0^2M_1^2}{\varepsilon}\|\hat{e}\|^2 \\ &\quad + \frac{2b_0^2}{\varepsilon}\|\hat{w}_x\|^2 + \frac{4M_1^2}{\varepsilon}\left(|\dot{\hat{b}}|^2\|p\|^2 + |\dot{\hat{\lambda}}|^2\|v\|^2\right) \\ &\quad + \frac{\varepsilon}{4}\|\hat{w}_{xx}\|^2 + \frac{4}{\varepsilon}\left(|\dot{\hat{b}}|^2\bar{\varphi}_1^2 + |\dot{\hat{\lambda}}|^2\bar{\varphi}_2^2\right)\|\hat{w}\|^2 \\ &\leqslant -\frac{\varepsilon}{2}\|\hat{w}_{xx}\|^2 + l_1.\end{aligned} \tag{154}$$

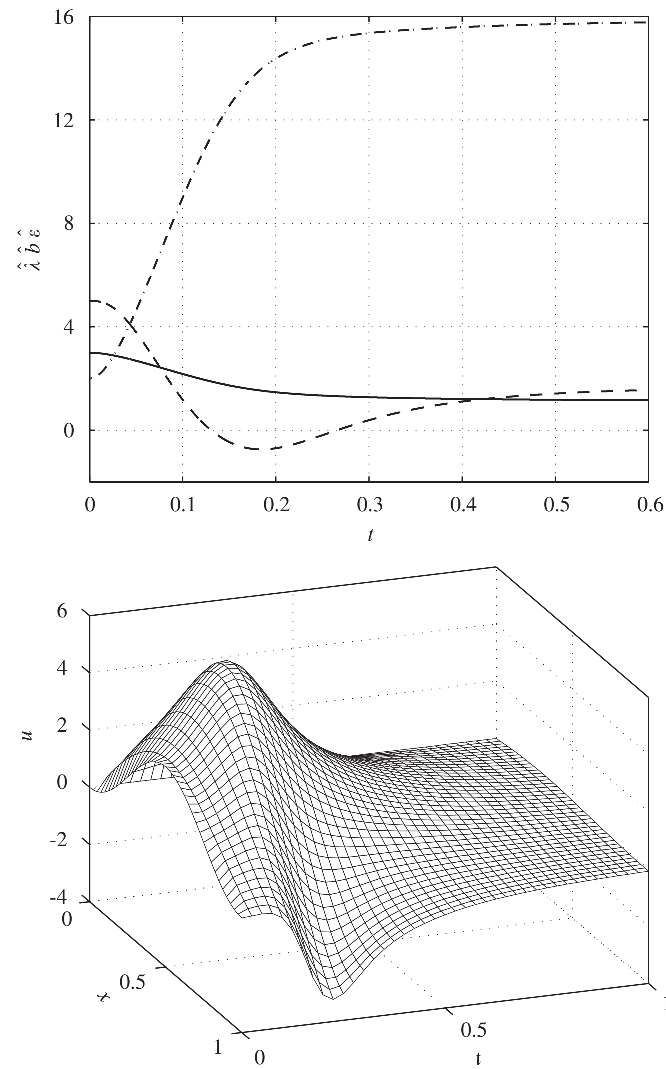

Fig. 2. The parameter estimates and the closed-loop state for the plant (103)–(105) with adaptive controller based on swapping identifier (solid, $\hat{\varepsilon}$; dashed, $\hat{b}$; dash-dotted, $\hat{\lambda}$).

By Lemma A.2 we get $\|\hat{w}_x\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$. For the filter v we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_x\|^2 &\leqslant -\varepsilon\|v_{xx}\|^2 + b_0\|v_x\|\|v_{xx}\| + \|v_{xx}\|\|u\| \\ &\leqslant -\varepsilon\|v_{xx}\|^2 + \frac{\varepsilon}{2}\|v_{xx}\|^2 + \frac{b_0^2}{\varepsilon}\|v_x\|^2 + \frac{1}{\varepsilon}\|u\|^2 \\ &\leqslant -\frac{\varepsilon}{2}\|v_{xx}\|^2 + l_1.\end{aligned} \tag{155}$$

By Lemma A.2 $\|v_x\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$. For the filter p we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_x\|^2 &\leqslant -\varepsilon\|p_{xx}\|^2 + b_0\|p_x\|\|p_{xx}\| + \|p_{xx}\|\|u_x\| \\ &\leqslant -\frac{\varepsilon}{2}\|p_{xx}\|^2 + \frac{b_0^2}{\varepsilon}\|p_x\|^2 + \frac{1}{\varepsilon}\|u_x\|^2.\end{aligned} \tag{156}$$

Since

$$\begin{aligned}\|u_x\|^2 &\leqslant 2\|\hat{e}_x\|^2 + 2M_3\|\hat{w}_x\|^2 \\ &\leqslant 4\|e_x\|^2 + 4|\tilde{b}|^2\|p_x\|^2 + 4|\tilde{\lambda}|^2\|v_x\|^2 \leqslant l_1,\end{aligned} \tag{157}$$

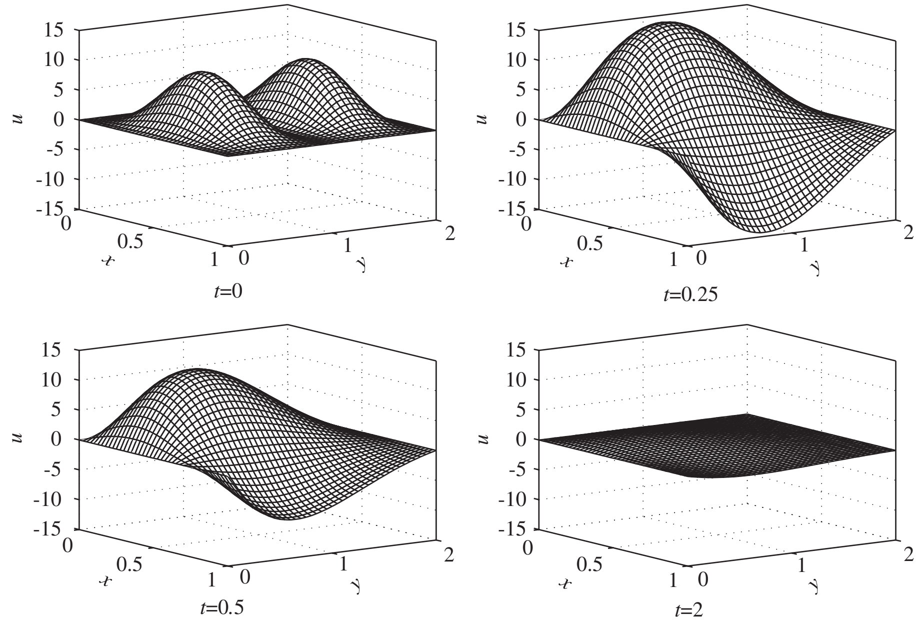

Fig. 3. The closed-loop state for the plant (161) at different times.

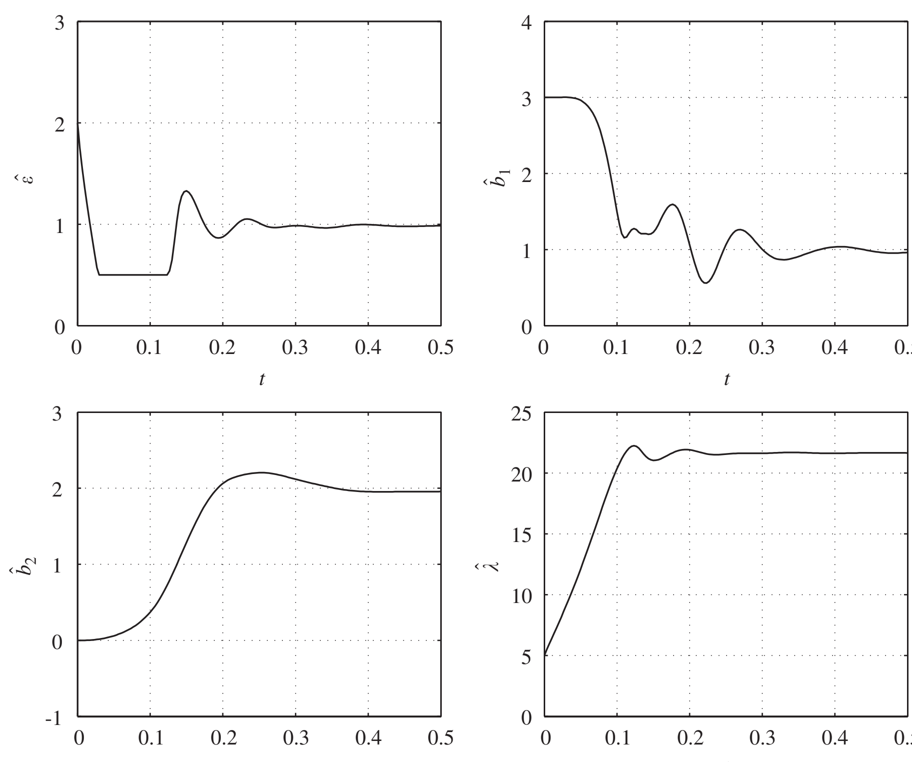

Fig. 4. The parameter estimates for the plant (161) with adaptive controller based on passive identifier.

we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_x\|^2 \leqslant -\frac{\varepsilon}{2}\|p_{xx}\|^2 + l_1. \tag{158}$$

and by Lemma A.2 $\|p_x\| \in \mathscr{L}_\infty \cap \mathscr{L}_2$. By Agmon's inequality we get the pointwise boundedness of signals $\hat{w}$, v, and p. From (132) we get the boundedness of η. Since $u = e + bp + \lambda v + \eta$, the state u is also bounded.

In order to prove regulation we notice from (153) that

$$|\dot{V}| \leqslant \frac{\varepsilon}{4A}|V| + |l_1 V| < \infty, \tag{159}$$

where we used the fact that l_1 is a bounded function in this case. By Lemma A.1 we get $V \to 0$ and thus $\hat{w}$, v, $p \to 0$. From (132) we get $\eta \to 0$ and therefore (106) implies $u \to 0$ as $t \to \infty$. Using the boundedness of $\|u_x\|$ by Agmon's inequality we get

$$\lim_{t\to\infty} \max_{x\in[0,1]} |u(x,t)| \leqslant \lim_{t\to\infty} 2\|u\|^{1/2}\|u_x\|^{1/2} = 0. \tag{160}$$

7. Simulations

We first demonstrate the design with a swapping identifier on a 1D plant (103)–(105) with parameters $\varepsilon = 1$, $b = 2$, $\lambda = 15$. The plant has one unstable eigenvalue at 4.1. Initial estimates are set to $\hat{\varepsilon}(0) = 3$, $\hat{b}(0) = 5$, $\hat{\lambda}(0) = 2$. We implemented an implicit BTCS finite-difference scheme in MATLAB, with 100-step discretization in space. The results of the simulation are presented in Fig. 2. Even though only the identifier properties (and not the closed-loop stabilization result) were proved in the case of an unknown diffusion coefficient, the adaptive controller successfully stabilizes the system. As expected for an adaptive regulation problem, the parameter estimates converge close to, but not exactly to the true parameter values.

For the demonstration of the design with passive identifier we consider a 2D plant (which is easier to vizualize than a 3D system) with four unknown parameters ε, b_1, b_2, and λ:

$$u_t = \varepsilon(u_{xx} + u_{yy}) + b_1 u_x + b_2 u_y + \lambda u \tag{161}$$

on the rectangle $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant L$ with actuation applied on the side with $x = 1$ and Dirichlet boundary conditions on the other three sides. Even though for 2D case the actuation needs to be distributed, in practice only a limited number of actuators would be used, enough for the controller to perform well. The adaptive laws (65)–(66) are modified in a straightforward way from the 3D to the 2D setting. We set the simulation parameters to $\varepsilon = 1$, $b_1 = 1$, $b_2 = 2$, $\lambda = 22$, $L = 2$. With this choice the plant has two unstable eigenvalues at 8.4 and 1. Initial estimates are set to $\hat{\varepsilon}(0) = 2$, $\hat{b}_1(0) = 3$, $\hat{b}_2(0) = 0$, $\hat{\lambda}(0) = 5$ and the bound on $\hat{\varepsilon}$ from below is $\underline{\varepsilon} = 0.5$. The initial conditions for the plant and the observer are $u(x, y, 0) = 10\sin^2(\pi x)\sin^2(\pi y)$ and $\hat{u}(x, y, 0) \equiv 0$. We implemented an implicit Euler ADI finite-difference scheme in MATLAB, with 100-by-100 spatial grid. The results of the simulation are presented in Figs. 3 (several snapshots of the state) and 4 (estimates of the unknown parameters). One can see that projection keeps $\hat{\varepsilon} \geqslant \underline{\varepsilon} = 0.5$. All estimates come close to the true values at approximately $t = 0.5$ and after that the controller stabilizes the system.

8. Conclusion

Even though we considered only Dirichlet boundary conditions, the approach can be easily extended to the Neumann case. If the boundary condition at the uncontrolled end is mixed and contains a parametric uncertainty, even the output feedback extension is possible (Smyshlyaev & Krstic, 2007). However, so far we have not obtained an output-feedback result for the class of PDEs considered in this paper (boundary observers for the case of known parameters were developed in Smyshlyaev & Krstic, 2005).

Appendix A.

Lemma A.1 (*Liu & Krstic, 2001, Lemma 3.1*). *Suppose that the function $f(t)$ defined on $[0, \infty)$ satisfies the following conditions*:

(i) *$f(t) \geqslant 0$ for all $t \in [0, \infty)$,*
(ii) *$f(t)$ is differentiable on $[0, \infty)$ and there exists a constant M such that $f'(t) \leqslant M$, $\forall t \geqslant 0$*
(iii) $\int_0^\infty f(t)\,\mathrm{d}t < \infty$.

Then we have

$$\lim_{t\to\infty} f(t) = 0. \tag{A.1}$$

Lemma A.2 (*Krstic et al., 1995, Lemma B.6*). *Let v, l_1, and l_2 be real-valued functions defined on R_+, and let c be a positive constant. If l_1 and l_2 are nonnegative and in $\mathscr{L}_1$ and satisfy the differential inequality*

$$\dot{v} \leqslant -cv + l_1(t)v + l_2(t), \quad v(0) \geqslant 0 \tag{A.2}$$

then $v \in \mathscr{L}_\infty \cap \mathscr{L}_1$.

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Andrey Smyshlyaev received his B.S. and M.S. degrees in Aerospace Engineering from the Moscow Institute of Physics and Technology in 1999 and 2001, and the Ph.D. degree in Mechanical Engineering from the University of California at San Diego in 2006. He is now working as a postdoctoral scholar at UCSD. His research interests include control of distributed parameter systems, adaptive control, and nonlinear control.

Miroslav Krstic is the Sorenson Professor of Mechanical and Aerospace Engineering and the Director of the newly formed Center for Control, Systems, and Dynamics (CCSD) at UCSD. Krstic is a coauthor of the books Nonlinear and Adaptive Control Design (1995), Stabilization of Nonlinear Uncertain Systems (1998), Flow Control by Feedback (2002), and Real Time Optimization by Extremum Seeking Control (2003). He received the NSF Career, ONR YI, and PECASE Awards, as well as the Axelby and the Schuck paper prizes. In 2005 he was the first engineering professor to receive the UCSD Award for Research. Krstic is a Fellow of IEEE, a Distinguished Lecturer of the Control Systems Society, and a former CSS VP for Technical Activities. He has served as AE for several journals and is currently Editor for Adaptive and Distributed Parameter Systems in Automatica.