Technical Notes and Correspondence

Nonovershooting Control of Strict-Feedback Nonlinear Systems

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***Abstract*—A means of obtaining a nonovershooting output tracking response for single-input–single-output (SISO) strict-feedback nonlinear systems is introduced. With the proposed method, arbitrary reference trajectories can be tracked "from below" for arbitrary initial conditions (as long as the initial value of the plant output is strictly below the initial value of the reference trajectory). In addition, a design is presented for "approximately" nonovershooting control in the presence of disturbances, where the amount of overshoot can be made arbitrarily small by appropriately choosing the control gains. Finally, an output-feedback example shows the ability of our approach to ensure arbitrarily small overshoot, where the overshoot is caused by the initial condition of the unmeasured part of the state.**

***Index Terms*—Backstepping, nonovershooting control.**

I. Introduction

The problem of tracking a known reference without overshooting is of great practical importance in a number of applications. For example, in many manufacturing and machining processes, overshoot could ruin precise tolerances. The analysis of linear time invariant systems exhibiting a nonovershooting step response (or a nonnegative impulse response) has been well studied (see [1]–[3]). The design of controllers to achieve a nonovershooting response for a variety of reference inputs has also been investigated for linear systems (see [4]–[8]). However, nonlinear systems have received almost no attention. The work presented in this note employs a modified backstepping method to guarantee a nonovershooting response for strict-feedback nonlinear systems in which the output is the first state. A companion paper [9] addresses a somewhat smaller class of nonlinear systems, but a much larger class of output maps.

We consider the class of systems

$$\dot{x}_i = x_{i+1} + \varphi_i(\underline{x}_i), \qquad i = 1, \ldots, n-1 \tag{1}$$

$$\dot{x}_n = u + \varphi_n(\underline{x}_n) \tag{2}$$

$$y = x_1 \tag{3}$$

where $\underline{x}_i = [x_1, x_2, \ldots, x_i]^{\mathrm{T}}$, u is control, y is the output, and the nonlinearities $\varphi_i(\cdot)$ are $n-1$ times differentiable. The objective is to design a controller that forces the output $y(t)$ to asymptotically track a given trajectory $r(t)$, while keeping all of the states bounded, and to ensure that

$$y(t) \le r(t) \qquad \forall t \ge 0. \tag{4}$$

We show that this objective can be satisfied with an appropriate selection of control gains as long as $x_1(0) > r(0)$.

The control design based on a modified version of the backstepping method in [10] is

$$z_i = x_i - \alpha_{i-1}(\underline{x}_{i-1}, t) - r^{(i-1)}(t) \tag{5}$$

$$\alpha_0 = 0 \tag{6}$$

$$\alpha_1(x_1, t) = -c_1 z_1 - \varphi_1 \tag{7}$$

$$\alpha_i(\underline{x}_i, t) = -c_i z_i - \varphi_i + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(x_{j+1} + \varphi_j) \tag{8}$$

$$u = \alpha_n \tag{9}$$

where $c_1, \ldots, c_n$ are positive design parameters. The coordinate change $x \mapsto z$ is smoothly invertible. It can be verified that in the z coordinates the closed-loop system is

$$\dot{z}_i = -c_i z_i + z_{i+1}, \qquad i = 1, \ldots, n-1 \tag{10}$$

$$\dot{z}_n = -c_n z_n. \tag{11}$$

The idea behind our selection of the form of the z-system is simple: Due to the cascade structureof the system (10)–(11), to ensure nonovershooting response (a response such that $z_1(t) \le 0$ for all positive time), it is sufficient to ensure that $z_i(0) > 0, \forall i = 1, \ldots, n$.

In addition to achieving nonovershooting response, namely making sure that $y(t)$ approaches $r(t)$ "from below," the designs we present actually are capable of achieving arbitrary speed of convergence, which can be systematically influenced by increasing the gains c_i. We do not dwell on this point however, as it is a well known property of backstepping design, extensively covered in [10].

Remark 1: Since the system (1)–(3) is feedback linearizable, one might envision approaching the problem of control design for nonovershooting response in the following way.

1) Convert the system to a chain of integrators using a coordinate change and feedback.
2) Apply some "standard linear technique" for nonovershooting control.

Step 1) of this approach is certainly possible and it is implicit in our approach. However, step 2) is actually not trivial. The existing linear techniques for nonovershooting control assume zero initial conditions. The class of strict-feedback systems in this note includes plants whose initial condition would change from zero to a nonzero value when converting them to a chain of integrators. For example, the plant $\dot{x}_1 = x_2 + \cos(x_1), \dot{x}_2 = u$, which is converted to the chain of integrators $\dot{\chi}_1 = \chi_2, \dot{\chi}_2 = v$ using an invertible change ofvariable $\chi_1 = x_1, \chi_2 = x_2 + \cos(x_1)$ and feedback $u = \sin(x_1)(x_2 + \cos(x_1)) + v$, would have its zero initial condition $x_1(0) = x_2(0) = 0$ changed to a nonzero value $\chi_1(0) = 0, \chi_2(0) = 1$ in the χ-coordinates. The main novelty of our work is in our choice of the form of the closed-loop system (10)–(11). Due to its cascade structure, this form yields a nonovershooting response as long as all of the initial values $z_i(0)$ are negative. One should not necessarily expect that these values could be made negative with an arbitrary coordinate change. However, as we reveal, this

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Color version of Fig. 1 available online at http://ieeexplore.ieee.org.

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is possible with the coordinate change employed in the backstepping method.

Remark 2: The standard backtepping method [10] converts the strict-feedback systems into a slightly different form, $\dot{z}_1 = -c_1 z_1 + z_2, \dot{z}_i = -z_{i-1} - c_i z_i + z_{i+1}, \dot{z}_n = -z_{n-1} - c_n z_n$. The additional terms $-z_{i-1}$, though not necessary, are helpful in the adaptive control designs [10] for strict-feedback systems. For nonovershooting response however, their absence is essential.

Contributions of This Note: We present several results in this note. First, we consider the general case where the nonlinearities are possibly nonvanishing at zero, the initial conditions are possibly nonzero, and the reference trajectory is an arbitrary ($n-1$ times differentiable) function of time. In this case the gains c_i are selected on the basis of the initial conditions and the initial values of the derivatives of the reference trajectory to achieve nonovershooting tracking response. Second, we consider the case of zero initial conditions and nonlinearities that vanish at zero and derive conditions on the c_i's which depend only on *a priori* bounds for the derivatives of the reference trajectory. Third, we specialize the result to constant set points r, in which case we obtain a design that would follow from converting the system to a chain of integrators and applying model following with a reference model of order n, relative degree n, and all of its poles real, which would be a standard "linear" way of achieving nonovershooting response. Fourth, we present a design for "approximately" nonovershooting control in the presence of disturbances, where the amount of overshoot can be made arbitrarily small by appropriately choosing the control gains. Finally, we give an output-feedback example which shows the ability of our approach to ensure arbitrarily small overshoot, where the overshoot is caused by the initial condition of the unmeasured part of the state.

II. Gain Selection for Nonvanishing Nonlinearities and Nonzero Initial Conditions

In this section, we consider the case where $x_i(0)$ and $\varphi_i(\underline{x}_i(0))$ are possibly nonzero.

Theorem 1: The controller (5)–(9) applied to the plant (1)–(3) ensures that the states are globally bounded and $\lim_{t\to\infty}(y(t)-r(t)) = 0$. Furthermore, if $y(0) < r(0)$, the choice

$$c_i > \max\{\underline{c}_i, 0\} \tag{12}$$

for $i = 1, \ldots, n-1$, where

$$\begin{aligned}\underline{c}_i = {}& [\alpha_{i-1}(\underline{x}_{i-1}(0), 0) + r^{(i-1)}(0) - x_i(0)]^{-1} \\ & \times \Big[x_{i+1}(0) + \varphi_i(\underline{x}_i(0)) \\ & - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}(\underline{x}_{i-1}(0), 0)}{\partial x_j} \\ & \times (x_{j+1}(0) + \varphi_j(\underline{x}_j(0))) - r^{(i)}(0)\Big]\end{aligned} \tag{13}$$

and $c_n > 0$ guarantees that (4) is satisfied.

Proof: Global boundedness and tracking are immediate from the global asymptotic stability of the system (10)–(11) and from the invertibility of the transformation $x \mapsto z$. The proof of (4) proceeds by noting that $z_1(0) = -r(0) < 0$

$$\begin{aligned}z_{i+1}(0) = {}& c_i z_i(0) + x_{i+1}(0) + \varphi_i(\underline{x}_i(0)) \\ & - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}(\underline{x}_{i-1}(0), 0)}{\partial x_j} \\ & \times (x_{j+1}(0) + \varphi_j(\underline{x}_j(0))) - r^{(i)}(0)\end{aligned} \tag{14}$$

and, by induction, that $z_i(0) < 0, \forall i = 2, \ldots, n$.

Example 1: Consider the system

$$\dot{x}_1 = x_2 + \varphi(x_1) \tag{15}$$
$$\dot{x}_2 = x_3 \tag{16}$$
$$\dot{x}_3 = u \tag{17}$$
$$y = x_1. \tag{18}$$

A nonovershooting tracking response for the output reference signal $r(t)$ will be achieved with the gains

$$c_1 > \max\{\underline{c}_1, 0\} \tag{19}$$
$$c_2 > \max\{\underline{c}_2, 0\} \tag{20}$$

where

$$\underline{c}_1 = \frac{x_2(0) + \varphi(x_1(0)) - \dot{r}(0)}{r(0) - x_1(0)} \tag{21}$$

$$\begin{aligned}\underline{c}_2 = {}& [c_1(r(0) - x_1(0)) - \varphi(x_1(0)) \\ & + \dot{r}(0) - x_2(0)]^{-1} \\ & \times [x_3(0) + (c_1 + \varphi'(x_1(0)))(x_2(0) + \varphi(x_1(0))) \\ & - \ddot{r}(0)].\end{aligned} \tag{22}$$

Example 2: Consider the system

$$\dot{x}_1 = x_2 + x_1^2 \tag{23}$$
$$\dot{x}_2 = u \tag{24}$$
$$y = x_1. \tag{25}$$

The backstepping controller for this system is

$$u = -c_1 c_2 x_1 - (c_1 + c_2 + 2x_1)\left(x_2 + x_1^2\right) + \ddot{r} + c_2 \dot{r} + c_1 c_2 r. \tag{26}$$

The gain condition for this system is (19) with $\varphi(x_1) = x_1^2$. Let us take $r(t) = 1 - \sin(t)$ and initial conditions $x_1(0) = 0, x_2(0) = 1$. This choice of initial conditions is chosen as "hard" for the given trajectory because $\dot{x}_1(0) = x_2(0) + x_1(0)^2 = 1 > 0$, whereas $\dot{r}(0) = -1 > 0$, i.e., the initial data favor "overshooting" response. For the given data, the gain condition is $c_1 > 2$. We take $c_1 = 3$ and also $c_2 = c_1 = 3$. It is easy to see that the resulting response in the z-coordinates is $z_1(t) = -(1+t)\mathrm{e}^{-3t}, z_2(t) = -\mathrm{e}^{-3t}$. The response in the x-coordinates and the control u that produces it are given in Fig. 1. As evident from the figure on the top, the output tracks the reference trajectory without ever exceeding it.

III. Gain Selection for Vanishing Nonlinearities and Zero Initial Conditions

The formula (12) requires that the gains c_i be recomputed for each new trajectory $r(t)$—based on its initial value and the initial values of its derivatives—and for each new initial condition $x(0)$. In this section, we assume that $x(0) = 0$ and $\varphi_i(0) = 0, \forall i = 1, \ldots, n$ and that for all the trajectories considered the initial reference properties are known to satisfy

$$r(0) \geq \quad \delta_0 > 0 \tag{27}$$
$$\left|r^{(i)}(0)\right| \leq \quad \delta_i, \qquad i = 1, \ldots, n. \tag{28}$$

For *a priori* known values of $\delta_0, \delta_1, \ldots, \delta_n$, the gain selection is given by the following theorem.

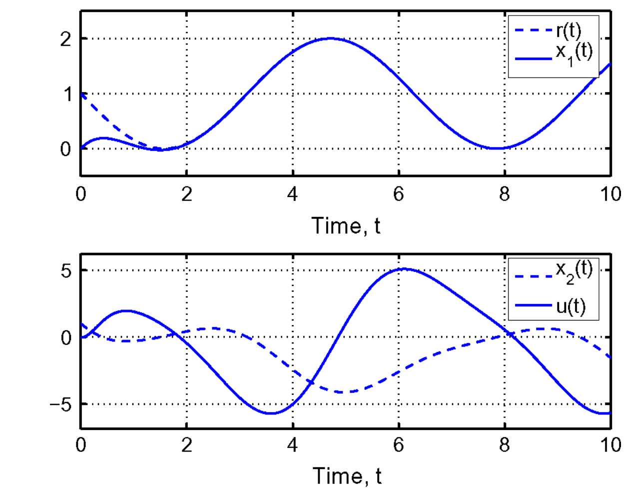

Fig. 1. Nonovershooting response for the system in Example 2.

Theorem 2: The controller (5)–(9) applied to the plant (1)–(3) ensures that the states are globally bounded and $\lim_{t\to\infty}(y(t)-r(t)) = 0$. Furthermore, the choice $c_1 > \delta_1/\delta_0$

$$c_i > \frac{\delta_i}{\delta_0 \prod_{l=1}^{i-1} c_l - \sum_{j=2}^{i-1} \delta_{j-1} \prod_{l=j}^{i-1} c_l - \delta_{i-1}} \geq 0 \tag{29}$$

for $i = 2, \ldots, n-1$, and $c_n < 0$ guarantees that (4) is satisfied.

Proof: We start by setting $x_{i+1}(0) + \varphi_i(\underline{x}_i(0)) = 0$ and $x_{j+1}(0) + \varphi_j(\underline{x}_j(0)t) = 0$ in (14), which yields

$$z_{i+1}(0) = c_i z_i(0) - r^{(i)}(0). \tag{30}$$

By induction it can be shown that

$$z_{i+1}(0) = -\left(\sum_{j=1}^{i} r^{(j-1)}(0) \prod_{l=j}^{i} c_l + r^{(i)}(0)\right) \tag{31}$$

and then that

$$\begin{aligned} z_{i+1}(0) &= -r(0)\prod_{l=1}^{i} c_l - \left(\sum_{j=2}^{i} r^{(j-1)}(0) \prod_{l=j}^{i} c_l + r^{(i)}(0)\right) \\ &\leq -\delta_0 \prod_{l=1}^{i} c_l + \sum_{j=2}^{i} \delta_{j-1} \prod_{l=j}^{i} c_l + \delta_i \\ &= -c_i\left(\delta_0 \prod_{l=1}^{i-1} c_l - \sum_{j=2}^{i-1} \delta_{j-1} \prod_{l=j}^{i-1} c_l - \delta_{i-1}\right) \\ &\quad + \delta_i. \end{aligned} \tag{32}$$

By substituting (29) we get $z_{i+1}(0) < 0$, which implies (4).

Example 3: Consider the system

$$\dot{x}_1 = x_2 + x_1^2 \tag{33}$$
$$\dot{x}_2 = x_3 \tag{34}$$
$$\dot{x}_3 = x_4 \tag{35}$$
$$\dot{x}_4 = u \tag{36}$$
$$y = x_1 \tag{37}$$

which contains a vanishing nonlinearity x_1^2. We do not illustrate here the control design (5)–(9)as it is straightforward nor do we illustrate nonovershooting closed-loop solutions for $y(t)$ as they are also straightforward and can be even calculated analytically from the linear system (10)–(11). Instead, we illustrate the condition (29). We consider a reference trajectory

$$r(t) = \cos(\omega t) \tag{38}$$

for which we have $\delta_0 = 1, \delta_1 = 0, \delta_2 = \omega^2, \delta_3 = 0$. The condition (29) is given by

$$c_1 > \frac{\delta_1}{\delta_0} = 0 \tag{39}$$
$$c_2 > \frac{\delta_2}{c_1\delta_0 - \delta_1} = \frac{\omega^2}{c_1} \tag{40}$$
$$c_3 > \frac{\delta_2}{c_1 c_2 \delta_0 - c_2\delta_1 - \delta_2} = 0. \tag{41}$$

The design parameters chosen in this fashion, along with an arbitrary positive value for c_4, guarantee that the controller (5)–(9) would yield nonovershooting convergence to (38).

For $r(t) = \text{const} > 0$ we get the following corollary of Theorem 2, which is established by plugging $\delta_1 = \cdots = \delta_n = 0$ into (29), keeping in mind that $\delta_0 > 0$.

Corollary 1: The controller (5)–(9) applied to the plant (1)–(3) ensures that the states are globally bounded and $\lim_{t\to\infty}(y(t)-r(t)) = 0$. Furthermore, for a constant reference $r(t) = r = \text{const} > 0$, the choice $c_1, \ldots, c_n > 0$ guarantees that (4) is satisfied.

Remark 3: We revisit Example 2 with $r(t) = \text{const} > 0$. The resulting response of $x_1(t)$ can be shown to be the same as theresponse of the transfer function

$$W_m(s) = \frac{c_1 c_2}{(s+c_1)(s+c_2)} \tag{42}$$

to a step input of intensity r. This means that, in the case where the nonlinearities $\varphi_i(\underline{x}_i)$ are zero at zero, and when the initial condition $x(0)$ is zero, our design is the same as performing a feedback transformation into a chain of integrators and then applying a linear model-reference controller with a reference model which is of the same order and relative degree as the plant, and which is overdamped (nonovershooting, with two real poles).

IV. Approximate Tracking for Strict-Feedback Systems With Disturbances

We consider the class of systems

$$\dot{x}_i = x_{i+1} + \varphi_i(\underline{x}_i)d(t), \qquad i = 1, \ldots, n-1 \tag{43}$$
$$\dot{x}_n = u + \varphi_n(\underline{x}_n)d(t) \tag{44}$$
$$y = x_1 \tag{45}$$

where $d(t)$ is an unknown disturbance signal such that

$$|d(t)| \leq \bar{d} \qquad \forall t \geq 0. \tag{46}$$

The bound $\bar{d}$ is assumed to be known to the designer.

The control design

$$z_i = x_i - \alpha_{i-1}(\underline{x}_{i-1}, t) - r^{(i-1)}(t) \tag{47}$$
$$\alpha_0 = 0 \tag{48}$$

$$\alpha_1(x_1,t) = -\sqrt{c_1^2+\kappa_1^2\varphi_1^2}z_1 \quad (49)$$

$$\alpha_i(\underline{x}_i,t) = -\sqrt{c_i^2+\kappa_i^2 w_i^2}z_i + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} \quad (50)$$

$$w_i = \varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j \quad (51)$$

$$u = \alpha_n \quad (52)$$

where c_i, κ_i are positive gains, results in the closed-loop system

$$\dot{z}_i = -\sqrt{c_i^2+\kappa_i^2 w_i^2}z_i + w_i d + z_{i+1}, \quad i = 1,\ldots,n-1 \quad (53)$$

$$\dot{z}_n = -\sqrt{c_n^2+\kappa_n^2 w_n^2}z_n + w_n d. \quad (54)$$

Theorem 3: The controller (47)–(52) applied to the plant (43)–(45) ensures that the states are globally bounded and

$$\limsup_{t\to\infty}|y(t)-r(t)| \le \bar{D} \quad (55)$$

where

$$\bar{D} = \bar{d}\left(\frac{1}{\kappa_1}+\frac{1}{c_1\kappa_2}+\frac{1}{c_1c_2\kappa_3}+\cdots+\frac{1}{c_1c_2\cdots c_{n-1}\kappa_n}\right) \quad (56)$$

$$= \bar{d}\sum_{i=1}^{n}\frac{1}{\kappa_i\prod_{j=1}^{i-1}}c_j \quad (57)$$

$$\le \frac{\bar{d}}{\underline{\kappa}}\left(1+\frac{1}{\underline{c}}+\cdots+\frac{1}{\underline{c}^{n-1}}\right) \quad (58)$$

and $\underline{\kappa} = \min\{\kappa_i\}, \underline{c} = \min\{c_i\}$. Furthermore, if $y(0) < r(0)$, the choice

$$c_i > \max\{\underline{c}_i, 0\} \quad (59)$$

for $i = 1,\ldots,n-1$, where

$$\underline{c}_i = \left[\alpha_{i-1}(\underline{x}_{i-1}(0),0)+r^{(i-1)}(0)-x_i(0)\right]^{-1} \times\left[x_{i+1}(0)-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}\left(\underline{x}_{i-1}(0),0\right)}{\partial x_j}x_{j+1}(0)-r^{(i)}(0)\right] \quad (60)$$

and $c_n > 0$ guarantees that

$$y(t) \le r(t)+\bar{D} \qquad \forall t \ge 0. \quad (61)$$

Proof: The approximate tracking result (55)is proved by repeated application of the cascade-ISS lemma [10, Lemma C.4]. We omit the details and concentrate on explaining the result (61),(56) on approximate nonovershooting. Let us start by denoting $s_i(t) = \sqrt{c_i^2+\kappa_i^2 w_i(\underline{x}_i(t),t)^2}$ and note that $s_i \ge c_i$ and $s_i \ge \kappa_i|w_i|$. That all of the $z_i(0)$'s are negative is shown by observing that

$$z_{i+1}(0) = s_i z_i(0) + x_{i+1}(0) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}(\underline{x}_{i-1}(0),0)}{\partial x_j}x_{j+1}(0) - r^{(i)}(0) \quad (62)$$

and by using (59) and an induction argument. Next, by applying the variation of constants formula to (53) we get

$$z_i(t) = z_i(0)e^{-\int_0^t s_i(\sigma)d\sigma} + \int_0^t e^{-\int_\tau^t s_i(\sigma)d\sigma}w_i(\tau)d(\tau)d\tau + \int_0^t e^{-\int_\tau^t s_i(\sigma)d\sigma}z_{i+1}(\tau)d\tau. \quad (63)$$

For the second term, we have

$$\begin{aligned}
&\int_0^t e^{-\int_\tau^t s_i(\sigma)d\sigma}w_i(\tau)d(\tau)d\tau \\
&\le \int_0^t e^{-\int_\tau^t s_i(\sigma)d\sigma}|w_i(\tau)|\bar{d}d\tau \\
&\le \bar{d}\int_0^t e^{-\int_\tau^t s_i(\sigma)d\sigma}\frac{1}{\kappa_i}s_i(\tau)d\tau \\
&= \frac{\bar{d}}{\kappa_i}e^{-\int_0^t s_i(\sigma)d\sigma}\int_0^t e^{\int_0^\tau s_i(\sigma)d\sigma}s_i(\tau)d\tau \\
&= \frac{\bar{d}}{\kappa_i}e^{-\int_0^t s_i(\sigma)d\sigma}\int_0^t e^{\int_0^\tau s_i(\sigma)d\sigma}d\left(\int_0^\tau s_i(\sigma)d\sigma\right) \\
&= \frac{\bar{d}}{\kappa_i}e^{-\int_0^t s_i(\sigma)d\sigma}\left(e^{\int_0^t s_i(\sigma)d\sigma}-1\right) \\
&\le \frac{\bar{d}}{\kappa_i}.
\end{aligned} \quad (64)$$

Substituting this expression into (63), we get

$$z_i(t) \le z_i(0)e^{-\int_0^t s_i(\sigma)d\sigma} + \frac{\bar{d}}{\kappa_i} + \int_0^t e^{-\int_\tau^t s_i(\sigma)d\sigma}z_{i+1}(\tau)d\tau \quad (65)$$

$$z_n(t) \le z_n(0)e^{-\int_0^t s_n(\sigma)d\sigma} + \frac{\bar{d}}{\kappa_n}. \quad (66)$$

(The second line is obtained by noting that, by definition, $z_{n+1} = 0$.) Since $s_i(t) \ge c_i$, we have that

$$\int_0^t e^{-\int_\tau^t s_{n-1}(\sigma)d\sigma}d\tau \le \frac{1}{c_{n-1}}. \quad (67)$$

With this inequality, we readily obtain

$$z_{n-1}(t) \quad \le \quad g_{n-1}(t) + \frac{\bar{d}}{\kappa_{n-1}} + \frac{\bar{d}}{c_{n-1}\kappa_n} \quad (68)$$

where

$$g_{n-1}(t) = z_{n-1}(0)e^{-\int_0^t s_{n-1}(\sigma)d\sigma} + z_n(0)\int_0^t e^{-\int_\tau^t s_{n-1}(\sigma)d\sigma-\int_0^\tau s_n(\sigma)d\sigma}d\tau \quad (69)$$

is a nonpositive function which converges to zero exponentially. Continuing in the same fashion (or, by using induction), we get

$$z_1(t) \le g_1(t) + \bar{D} \quad (70)$$

where $g_1(t)$ is a nonpositive function which converges to zero exponentially. This completes theproof.

It is clear from (61),(56)that the overshoot $\bar{D}$ can be made as small as possible by appropriately choosing $\kappa_1, \ldots, \kappa_n$ for a given value of disturbance intensity $\bar{d}$.

Our result can be readily modified to allow the presence of additional terms $\psi_i(\underline{x}_i)$, not multiplied by the disturbance $d(t)$ on the right hand side of (43)–(44).

The nonlinear damping terms $-\sqrt{c_i^2 + \kappa_i^2 w_i^2} z_i$ contain a square root for a specific reason. The same overshooting result would be achieved with simpler nonlinear damping terms $-(c_i + \kappa_i|w_i|)z_i$. However, the absolute value function is not differentiable, which would cause problems in our recursive design procedure which involves partial derivatives with respect to the state variables. For this reason, we employ the damping terms with a square root,which "softens" the absolute value.

Example: We consider a simple linear chain-of-integrators example with an additive disturbance

$$\dot{x}_1 = x_2 + d(t) \tag{71}$$

$$\dot{x}_2 = u. \tag{72}$$

We assume $x_1(0) = 0, x_2(0) = 1$ and $r(t) = 1$, $d(t) = \sin(\omega t)$. Clearly, the nonzero initial condition $x_2(0) = 1$ and the disturbance $d(t)$ are the factors that make nonovershooting control nontrivial. Following the procedure in this section, we design a controller $u = (c + \kappa)^2 z_1 - (2c + \kappa + c\kappa + \kappa^2)z_2$, where $z_1 = x_1 - 1$ and $z_2 = x_2 + (c+\kappa)z_1$. (Evidently, we have chosen $c_1 = c_2 = c$ and $\kappa_1 = \kappa_2 = \kappa$, as well as replaced the square root terms of the form $\sqrt{c_i^2 + \kappa_i^2 w_i^2}$ by linear terms of the form $c_i + \kappa_i w_i$, which is possible in this case due to linearity of the plant and due to the fact that the design procedure in this case results in w_i which are constant and positive.) To prevent overshoot due to initial condition $x_2(0)$ we chose the gains c, κ positive and such that $c + \kappa > 1$. With a lengthy calculation one can show that

$$x_1(t) = 1 + g_1(t) + A\sin(\omega t + \phi) \tag{73}$$

where ϕ is some constant, $g_1(t)$ is a nonpositive function that exponentially converges to zero and such that $g_1(0) = -A\sin(\phi - 1$, and

$$A = \sqrt{\frac{\omega^2 + (2c + \kappa + c\kappa + \kappa^2)^2}{(\omega^2 + (c+\kappa)^2)(\omega^2 + (c + c\kappa + \kappa^2)^2)}}. \tag{74}$$

From this expression it is clear that, as κ grows, the overshoot—which occurs exclusively due to the disturbance $d(t)$—is approximately $1/\kappa$.

V. Output-Feedback Case

The results in the previous sections have all used full state feedback. When only the output y is used for measurement, accompanied by an observer, achieving nonovershooting tracking becomes considerably harder. First the class of systems needs to be restricted from strict-feedback systems (which have a lower-triangular dependence on the state variables) to the systems in the *output-feedback canonical form*[10] where the nonlinearities depend only on $y = x_1$. This restriction is necessary because many of the systems in the strict-feedback class are not globally stabilizable.

An additional issue is that, while perfect tracking is achievable for systems in the output-feedback form, nonovershooting response is harder to achieve than in the state-feedback case because of the state estimation error, which acts as an unknown disturbance (even in the case of exponentially convergent state estimation, the unmeasured initial condition of the plant results in an unknown initial condition of the observer error system).

Rather than considering the general class of output-feedback systems, in this section we present an example which shows how to design a controller that achieves an approximately nonovershooting response for a system with only partial state measurement. We consider the plant

$$\dot{x}_1 = x_2 + \varphi(x_1) \tag{75}$$

$$\dot{x}_2 = u \tag{76}$$

$$y = x_1 \tag{77}$$

where x_1 is measured while x_2 is not. Since x_1 is measured, we employ a partial state observer. We use the filter

$$\dot{\xi} = -k\xi - k^2 x_1 - k\varphi(x_1) + u \tag{78}$$

where k is a positive observer gain. It can be shown that the variable $\varepsilon = x_2 - \xi - kx_1$ satisfies the differential equation $\dot{\varepsilon} = -k\varepsilon$, which means that $\varepsilon(t) = \varepsilon(0)\mathrm{e}^{-kt}$. We perform our control design on the second-order system consisting of the measured state x_1 and the filter state ξ

$$\dot{x}_1 = \xi + kx_1 + \varphi(x_1) + \varepsilon(t) \tag{79}$$

$$\dot{\xi} = u - k^2 x_1 - k\varphi(x_1) - k\xi \tag{80}$$

which is in strict-feedback form and has an unmeasured disturbance $\varepsilon(t)$. The backstepping design for this system results in the control law

$$u = -\sqrt{c_2^2 + \kappa_2^2 w^2} z_2 + k\beta - w\beta + \ddot{r} + (c_1 + \kappa_1)\dot{r} \tag{81}$$

where

$$z_1 = x_1 - r(t) \tag{82}$$

$$z_2 = \xi + kx_1 + \varphi(x_1) - \dot{r}(t) + (c_1 + \kappa_1)z_1 \tag{83}$$

$$\beta = kx_1 + \varphi(x_1) + \xi \tag{84}$$

$$w = c_1 + \kappa_1 + k + \varphi'(x_1) \tag{85}$$

which yields the closed-loop system

$$\dot{z}_1 = -(c_1 + \kappa_1)z_1 + \varepsilon(t) + z_1 \tag{86}$$

$$\dot{z}_2 = -\sqrt{c_2^2 + \kappa_2^2 w^2} z_2 + w\varepsilon(t). \tag{87}$$

Due to the exponential stability of the ε-subsystem, one can prove exponential stability of the equilibrium $z_1 = z_2 = \varepsilon = 0$, which implies asymptotic tracking for any initial conditions and any values of $c_1, c_2, \kappa_1, \kappa_2$. Moreover, using analysis as in the previous section of the note, one can arrive at the following result on approximate nonovershooting control.

Proposition 1: The output feedback controller (81),(78) applied to the plant (75)–(76) ensures that the states are globally bounded and

$$\limsup_{t\to\infty} |y(t) - r(t)| \leq \left(\frac{1}{\kappa_1} + \frac{1}{c_1\kappa_2}\right)|x_2(0) - \xi(0) - kx_1(0)|. \tag{88}$$

Furthermore, if $y(0) < r(0)$, the choice

$$c_1 > \max\left\{\frac{\xi(0) + kx_1(0) + \varphi(x_1(0)) - \dot{r}(0)}{r(0) - x_1(0)}, 0\right\} \tag{89}$$

guarantees that

$$y(t) \leq r(t) + \left(\frac{1}{\kappa_1} + \frac{1}{c_1\kappa_2}\right)|x_2(0) - \xi(0) - kx_1(0)| \quad (90)$$

for all $t \geq 0$.

From this proposition, it is clear that the amount of overshoot caused by the unknown $x_2(0)$ can be made arbitrarily small by appropriately increasing κ_1 and κ_2.

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Robust Overlapping Guaranteed Cost Control of Uncertain State-Delay Discrete-Time Systems

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***Abstract*—This note presents a new extension of the inclusion principle to cope with the problem of designing robust overlapping controllers for state-delayed discrete-time systems with norm bounded uncertainties using the concept of guaranteed cost control. Expansion-contraction relations for systems and contractibility conditions for output guaranteed cost memoryless controllers are proved, including conditions on the equality of guaranteed performance bounds. The controllers are designed in the expanded space using a linear matrix inequality (LMI) delay independent procedure specifically adapted to this class of problems. The designed controllers are then contracted and implemented into the original system. The results are specialized for the overlapping decentralized control design. The method enables an effective construction of block tridiagonal controllers. A numerical illustrative example is supplied.**

***Index Terms*—Decentralized control, delay systems, inclusion principle, large-scale systems, robust control.**

I. INTRODUCTION

Information structure constraints in feedback control systems may be classified according to the structure of the gain matrix. In fact, three different important forms of the gain matrices are usually considered: block diagonal, block tridiagonal, and double block bordered. A systematic way of the controller design with a block tridiagonal gain matrix leads to the concept of overlapping decompositions. A general mathematical framework for this approach has been called the Inclusion Principle. It usually consider two steps: first, a given dynamic system is expanded into another system with higher dimension and which includes all the information about the initial system. Second, the controller is designed for the expanded system and contracted into the original system. The main advantage of this procedure is that the expanded system appears without shared parts, which allows the design of decentralized controllers using well-known methods.

A. Relevant References

A system theoretic formulation of the Inclusion Principle has been originated in [1]–[6] and further extended to various problems such as for instance in [7], [8].

A guaranteed cost control problem for a class of uncertain state-delayed systems with quadratic performance index has been solved using LMI for the state feedback controller design in [9]–[19].

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