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Decentralized and adaptive control of nonlinear fluid flow networks

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In this paper we solve a *decentralized* nonlinear control problem where actuator valves and flow rate sensors are collocated in individual branches and do not exchange information. This is in contrast to our previous paper where a centralized controller required measurements from all the branches of the network. We solve both regulation (constant references) and tracking (time varying reference signals) problems. To eliminate conservativeness in choosing the gains of the controllers, we employ adaptation. We illustrate the results with an example reminiscent of a blood flow network.

1. Introduction

Fluid flow networks are ubiquitous as control systems – petroleum pipelines, gas and water distribution, mine ventilation systems, blood flow, etc. The objective is normally to drive flow rates through some or all the branches of the network to desired values. A compressor, a pump, or a heart, drives the flow in such a network. In order to achieve the objective one needs to actuate resistances in the network branches by valves. The problem of controlling fluid flow networks for mine ventilation received considerable attention in 1970s and 1980s (Mahdi and McPherson 1971, McPherson *et al*. 1972, Aldridge *et al*. 1976, Lee and Nutter 1980, Bogdanov and Kneller 1983, Meriluoto 1983).

In Hu *et al*. (2003) we developed nonlinear controllers for the regulation of fluid flow networks. The controllers were centralized. Each actuator needed flow measurement information from all the branches. In this paper, we solve a *decentralized* problem where valves and flow rate sensors are collocated in individual branches and do not exchange information. We solve both regulation (constant references) and tracking (time varying reference signals) problems. To eliminate conservativeness in choosing the gains of the controllers, we employ adaptation.

The paper is organized as follows. In § 2, we introduce the constitutive equations and develop separately the non-minimal and the minimal representation of the system. In § 3, we design decentralized controllers for the case when resistances in the tree branches and reference flow quantities in the links are functions of time and when they are constants, while in § 4, we design adaptive decentralized controllers for the two cases. In § 5, we illustrate the results with an example reminiscent of a blood flow network.

2. Model of a fluid flow network system

It is clear that a fluid flow network is a multivariable control problem where acting in one branch can affect

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the flow in other branches in an undesirable way. For this reason, the network needs to be approached in a model-based fashion (in much the same way as one would model an electric circuit) and as a multivariable control problem.

The control model of a fluid flow network consists of Kirchhoff's current and voltage laws (algebraic equations) and fluid dynamical equations of individual branches (differential equations). The branches are modeled using a widely accepted lumped parameter approximations of the fluid flow equations that take a form whose electric equivalent is an RL characteristic with a nonlinear resistance. To be precise, the pressure drop over a branch is approximated to be proportional to the square of the flow rate (nonlinear resistive term) and to the flow acceleration (linear inductive term).

It is clear that, due to mass conservation at the branching points (nodes) of the network, flows in many of the branches will be interdependent. Hence, the minimal system representation, written using Kirchhoff's algebraic equations and the branch characteristic differential equation will be of lower order than the number of branches.

This intuition becomes systematic when one employs graph theoretic concepts from circuit theory Desoer and Kuh (1969). Each network can be divided into a set of branches called a *tree* (they connect all the nodes of the graph without creating any loops) and the complement of the tree, called a co-tree, whose branches are referred to as the *links*. The minimal system representation of the dynamics of the network is given by the flow through the links.

2.1 *Pipe flow dynamics and Kirchhoff's laws for fluid flow networks*

In order to develop the model of a fluid flow network, we first establish the dynamical equation of one branch. For simplicity, we make the following assumptions: A1. the fluid is incompressible; A2. the temperatures in all branches are identical. Under assumptions A1 and A2, one branch of the fluid flow network is described with the following equations Kocić (1979) and Word-Smith (1980):

$$\frac{dQ_j}{dt} + K_j R_j \left|Q_j\right| Q_j = K_j H_j, \tag{1}$$

where Q_j is the flow through a branch j, R_j are resistances of the branches, H_j are pressure drops of the branches, K_j are inertia coefficients, $j = 1, \ldots, n$ and n is the number of network branches.

Like an electrical network, a fluid flow network must satisfy Kirchhoff's current law, i.e. the flow out of any node is equal to the flow into that node. Mathematically, Kirchhoff's current law for fluid flow networks can be expressed as

$$\sum_{j=1}^{n} E_{Qij} Q_j = 0, \quad i = 2, \ldots, n_c - 1, \tag{2}$$

or

$$E_Q Q = 0, \tag{3}$$

where n_c is the number of nodes in the network, Q is a vector of flow quantities, $E_Q = [E_{Qij}]$ is a full rank matrix of order $(n_c - 2) \times n$ and the values of E_{Qij} are defined as follows: $E_{Qij} = 1$ if branch j is connected to node i and the fluid flow goes away from node i, $E_{Qij} = -1$ if it goes into node i, $E_{Qij} = 0$ if branch j is not connected to node i.

Let us assume that the fluid flow network employs one main generator that is connected with the ambient outside of the network. Then the flow in the fan branch can be expressed as

$$e_{Q_m} Q = \sum_{j=1}^{n} e_{Qmj} Q_j, \tag{4}$$

or

$$e_{Q_m} Q = Q_m, \tag{5}$$

where Q_m is the flow quantity through fan (*m*ain) branch, $e_{Qm} = [e_{Qm1}, \ldots, e_{Qmn}]$ is $1 \times n$ matrix, includes the values of e_{Qmj}, that are defined as those of matrix E_Q.

Similarly, the fluid flow network also satisfies Kirchhoff's voltage law, i.e. the sum of the pressure drops around any loop in the network must be equal to zero, or mathematically,

$$\sum_{j=1}^{n} E_{Hij} H_j = 0, \quad i = 1, \ldots, l - k, \tag{6}$$

or

$$E_H H = 0, \tag{7}$$

where l is a number of the links in the network, $l = N - n_c + 1$, $N = n + 1$ is the number of branches with fan branch; H is a vector of pressure drops, $E_H = [E_{Hij}]$ is $(l - k) \times n$ mesh matrix, in which each mesh is formed by a link and a unique chain in the tree connecting two endpoints of the link, k is number of meshes, containing fan branch, it is equal to the number of links, connected to the fan branch at its end.

The elements of E_{Hij} are defined as follows: $E_{Hij}=1$ if branch j is contained in mesh i and has the same direction, $E_{Hij}=-1$ if branch j is contained in mesh i and has the opposite direction, $E_{Hij}=0$ if branch j is not contained in mesh i.

Considering meshes, containing the fan branch, express the pressure drop in it as

$$\sum_{j=1}^{n} e_{Hmij}H_j = -H_m, \quad i=1,\ldots,k, \tag{8}$$

or

$$e_{H_m}H = -H_m, \tag{9}$$

where H_m is the pressure drop of the fan branch, e_{Hm} is $k\times n$ matrix, includes the values of e_{Hmij} which is defined as E_H.

The dynamics of the fan branch can be expressed as

$$H_m = d - R_m Q_m, \tag{10}$$

where d denotes the equivalent pressure drop generated by fan, and R_m is the resistance coefficient in the fan branch.

2.2 *Minimal model of the network*

In order to establish the state equation, one has to find independent variables as states of the system. By virtue of the concepts of a tree and a link, they can easily be found. So the first step is to describe the tree of the fluid flow network such that the fan branch is contained in it, and take the flow quantities of link branches as state variables. For convenience of analysis, we label the link branches from 1 to l. Define

$$Q=\begin{bmatrix} Q_c \\ Q_a \end{bmatrix}, \quad H=\begin{bmatrix} H_c \\ H_a \end{bmatrix}, \tag{11}$$

so that Q_c and H_c matrices describe states in the links, and Q_a and H_a matrices describe them in the tree branches, excluding the fan branch.

The matrices E_H, E_Q, e_{H_m} and e_{Q_m} can be represented in the form

$$E_H = [E_{Hc}\ E_{Ha}], \quad E_Q = [E_{Qc}\ E_{Qa}], \tag{12}$$

$$e_{H_m} = [e_{H_{mc}} e_{H_{ma}}], \quad e_{Q_m} = [e_{Q_{mc}} e_{Q_{ma}}], \tag{13}$$

where

$$\begin{bmatrix} E_{H_c} \\ e_{H_{mc}} \end{bmatrix} = I_{l\times l}, \quad e_{Q_{mc}} = [0\ \ 0\ \ \cdots\ \ 1], \tag{14}$$

$$\begin{bmatrix} E_{Q_a} & 0 \\ -e_{Q_{ma}} & 1 \end{bmatrix} = I_{(N-l)\times(N-l)},$$
$$E_{Q_a} = I_{(N-l-1)\times(N-l-1)}, \quad e_{Q_{ma}} = 0. \tag{15}$$

Let us now express the tree states through link states. From (2), (11), (12) and (15), we have

$$[E_{Qc}E_{Qa}]\begin{bmatrix} Q_c \\ Q_a \end{bmatrix} = 0. \tag{16}$$

So

$$Q_a = -E_{Q_a}^{-1}E_{Q_c}Q_c = -E_{Q_c}Q_c. \tag{17}$$

With the notations

$$Q_D^2 = \mathrm{diag}\{Q_j|Q_j|\}, \quad K=\mathrm{diag}\{K_j\} = \begin{bmatrix} K_c & 0 \\ 0 & K_a \end{bmatrix}, \tag{18}$$

(1) can be rewritten as

$$\dot{Q} = -KQ_D^2R + KH. \tag{19}$$

Define also

$$R = \left[R_c^T\ \ R_a^T\right]^T, \tag{20}$$
$$Q_{cD}^2 = \mathrm{diag}\left\{Q_1|Q_1|,\ldots,Q_l|Q_l|\right\}, \tag{21}$$
$$Q_{aD}^2 = \mathrm{diag}\left\{Q_{l+1}(Q_c)\left|Q_{l+1}(Q_c)\right|,\ldots,Q_n(Q_c)\left|Q_n(Q_c)\right|\right\}, \tag{22}$$

where the dependence on Q_c in (22) should be understood in the sense of (17).

Proposition 2.1 (Hu *et al.* 2003): *The minimal model of fluid flow network system can be expressed as*

$$\dot{Q}_c = A_cQ_{cD}^2R_c + A_{ca}Q_{aD}^2R_a + B_cQ_c + C_cd, \tag{23}$$

where Q_c is the state, R_c, R_a and d are the inputs, and

$$A_c = -K_c + K_cS_{Ha}\zeta_{RQ_c}, \quad A_{ca} = K_cS_{Ha}\zeta_{RQ_a}, \tag{24}$$
$$B_c = K_c\left[S_{Ha}(\zeta_{Q_c} - \zeta_{Q_a}E_{Q_c}) + R_m(S_{Q_c} - S_{Q_a}E_{Q_c})\right], \tag{25}$$
$$C_c = K_c(S_d + S_{Ha}\zeta_d), \tag{26}$$

where

$$\zeta_{RQc} = (E_{Qc}K_cS_{Ha} + K_a)^{-1}E_{Qc}K_c, \quad (27)$$
$$\zeta_{RQa} = (E_{Qc}K_cS_{Ha} + K_a)^{-1}E_{Qa}K_a, \quad (28)$$
$$\zeta_{Qc} = -(E_{Qc}K_cS_{Ha} + K_a)^{-1}E_{Qc}K_cR_mS_{Qc}, \quad (29)$$
$$\zeta_{Qa} = -(E_{Qc}K_cS_{Ha} + K_a)^{-1}E_{Qc}K_cR_mS_{Qa}, \quad (30)$$
$$\zeta_d = -(E_{Qc}K_cS_{Ha} + K_a)^{-1}E_{Qc}K_cS_d, \quad (31)$$

and

$$S_{Ha} = -\begin{bmatrix} E_{Ha} \\ e_{H_{ma}} \end{bmatrix}, \quad (32)$$
$$S_d = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (33)$$
$$S_Q = \begin{bmatrix} 0 \\ e_{Q_m} \end{bmatrix} = [S_{Q_c} \;\; S_{Q_a}]. \quad (34)$$

One should mention that the matrix $E_{Qc}K_cS_{Ha} + K_a$ in (27)–(31), is nonsingular, which was proved in (Hu *et al.* 2003).

3. A decentralized controller

In the model (23) resistances R_c, R_a and d are considered as inputs. However, the inputs R_a and d, referred to as auxiliary inputs Kocić (1979) (thus a subscript 'a'), are not necessary, i.e. the system can be controlled successfully with R_c alone. In this section we let R_a and d be fixed at their reference values,

$$R_a = R_{ar}, \quad (35)$$
$$d = d_r, \quad (36)$$

and design a feedback law $R_c(Q_c)$. Furthermore, this control is designed as decentralized, i.e. $R_{ci} = R_{ci}(Q_{ci})$.

To prepare for stability analysis under a decentralized controller, we first need the following lemma.

Lemma 3.1: $A_c = A_c^T < 0$.

Proof: We show this using fact that $S_{Ha} = E_{Qc}^T$ and $\zeta_{RQc} = (E_{Qc}K_cS_{Ha} + K_a)^{-1}E_{Qc}K_c$:

$$\begin{aligned} A_c^{-1} &= -(I - S_{Ha}\zeta_{RQc})^{-1}K_c^{-1} \\ &= -\Big(I - E_{Qc}^T(E_{Qc}K_cS_{Ha} + K_a)^{-1}E_{Qc}K_c\Big)^{-1}K_c^{-1}. \end{aligned} \quad (37)$$

By the matrix inversion lemma Horn and Johnson (1985)

$$\begin{aligned} A_c^{-1} &= -\Big(I - E_{Qc}^T(E_{Qc}K_cS_{Ha} + K_a)^{-1}E_{Qc}K_c\Big)^{-1}K_c^{-1} \\ &= -\Big(I + E_{Qc}^TK_a^{-1}E_{Qc}K_c\Big)K_c^{-1} \\ &= -K_c^{-1} - E_{Qc}^TK_a^{-1}E_{Qc}, \end{aligned} \quad (38)$$

where matrices K_c^{-1} and K_a^{-1} exist. So, $(-K_c^{-1} - E_{Qc}^TK_a^{-1}E_{Qc})$, the inverse of A_c, also exists. From symmetry and positive definiteness of matrices K_c and K_a we conclude that A_c^{-1} is symmetric and negative definite. □

Now we can establish stability properties of the decentralized controller. In practice, we can encounter two cases: when R_{ar} and Q_{cr} are functions of time and when they are time invariant. First, consider the case when R_{ar} and Q_{cr} are functions of time and denote

$$R_{ar}^{\max} = \max_i R_{ar_i}(t). \quad (39)$$

Theorem 3.1: *Suppose* $Q_{cri}(t) \neq 0, \quad \forall i, \forall t \geq 0$. *Then there exist sufficiently large positive constants* M_1 *and* M_2, *which are independent of* $R_{ar}(t)$ *and* $d_r(t)$, *such that solution* $Q_c(t) = Q_{cr}(t)$ *of the system* (23), *under the control law*

$$R_{ci}(t, Q_{ci}) = \left\{\frac{Q_{cri}(t)|Q_{cri}(t)|}{Q_{ci}|Q_{ci}|}\right\}(R_{cri}(t) + R_{cEi}), \quad (40)$$

where

$$R_{cr}(t) = (Q_{crD}^2)^{-1}A_c^{-1}(\dot{Q}_{cr} - A_{ca}Q_{arD}^2R_{ar} - B_cQ_{cr} - C_cd_r), \quad (41)$$

$$\begin{aligned} R_{cE_i} = \frac{1}{Q_{cri}(t)|Q_{cri}(t)|}\Big(\lambda + M_1 + M_2R_{ar}^{\max}(2|Q_{cr}(t)| \\ + \sqrt{l}|Q_{ce_i}|)\Big)Q_{ce_i}, \quad \lambda > 0 \end{aligned} \quad (42)$$

is exponentially stable.

Proof: We first rewrite the minimal model (23) as

$$\dot{Q}_c = A_cQ_{crD}^2R_{cr} + A_cQ_{crD}^2R_{cE} + A_{ca}Q_{aD}^2R_{ar} + B_cQ_c + C_cd_r. \quad (43)$$

The reference model is described by the equation

$$\dot{Q}_{cr} = A_cQ_{crD}^2R_{cr} + A_{ca}Q_{arD}^2R_{ar} + B_cQ_{cr} + C_cd_r. \quad (44)$$

So, the error model takes the form

$$\dot{Q}_{ce} = \dot{Q}_c - \dot{Q}_{cr} = A_c Q_{crD}^2 R_{cE} + A_{ca}(Q_{aD}^2 - Q_{arD}^2) R_{ar} + B_c Q_{ce}. \quad (45)$$

Denoting

$$Q_{aD}^2 - Q_{arD}^2 = \varphi(Q_{ce}), \quad (46)$$

we can rewrite (45) as

$$\dot{Q}_{ce} = A_c Q_{crD}^2 R_{cE} + A_{ca}\varphi(Q_{ce}) R_{ar} + B_c Q_{ce}. \quad (47)$$

In order to establish controller properties, take a Lyapunov function as

$$V = Q_{ce}^T(-A_c^{-1})Q_{ce}, \quad (48)$$

where the inverse of A_c exists and $(-A_c^{-1})$ is positive definite by Lemma 3.1.

The derivative of the Lyapunov function (48) is

$$\begin{aligned}\dot{V} &= -Q_{ce}^T A_c^{-1}\left[A_c Q_{crD}^2 R_{cE} + A_{ca}\varphi(Q_{ce})R_{ar} + B_c Q_{ce}\right] \\ &\quad + \left[A_c Q_{crD}^2 R_{cE} + A_{ca}\varphi(Q_{ce})R_{ar} + B_c Q_{ce}\right]^T(-A_c^{-1})Q_{ce} \\ &= 2Q_{ce}^T\left[-Q_{crD}^2 R_{cE} - A_c^{-1}(A_{ca}\varphi(Q_{ce})R_{ar} + B_c Q_{ce})\right] \\ &= 2Q_{ce}^T\left[-Q_{crD}^2 R_{cE} + \Psi(Q_{ce})Q_{ce}\right], \end{aligned} \quad (49)$$

where

$$\Psi(Q_{ce})Q_{ce} = -A_c^{-1}(A_{ca}\varphi(Q_{ce})R_{ar} + B_c Q_{ce}). \quad (50)$$

We are looking for a controller in a form such that

$$Q_{ce}^T \Psi(Q_{ce}) Q_{ce} \le Q_{ce}^T \Pi(Q_{ce}) Q_{ce}, \quad (51)$$

where Π is a diagonal matrix. First, let us express $\varphi(Q_{ce})$ as a function of links' states:

$$\varphi(Q_{ce}) = \begin{bmatrix} Q_{a1}|Q_{a1}| - Q_{ar1}|Q_{ar1}| & & 0 \\ & \ddots & \\ 0 & & Q_{a,N-l-1}|Q_{a,N-l-1}| - Q_{ar,N-l-1}|Q_{ar,N-l-1}| \end{bmatrix}, \quad (52)$$

or

$$\begin{aligned}\varphi_i(Q_{ce}) &= Q_{ai}|Q_{ai}| - Q_{ari}|Q_{ari}| \\ &= Q_{aei}\begin{Bmatrix} |Q_{ai}| + |Q_{ari}|, & Q_{ai}Q_{ari} > 0 \\ \dfrac{Q_{ai}^2 + Q_{ari}^2}{|Q_{ai}| + |Q_{ari}|}, & Q_{ai}Q_{ari} < 0 \end{Bmatrix} \\ &= -E_{Qc}^i Q_{ce} \\ &\quad \times \begin{Bmatrix} |E_{Qc}^i Q_c| + |E_{Qc}^i Q_{cr}|, & E_{Qc}^i Q_c E_{Qc}^i Q_{cr} > 0 \\ \dfrac{(E_{Qc}^i Q_c)^2 + (E_{Qc}^i Q_{cr})^2}{|E_{Qc}^i Q_c| + |E_{Qc}^i Q_{cr}|}, & E_{Qc}^i Q_c E_{Qc}^i Q_{cr} < 0 \end{Bmatrix}, \end{aligned} \quad (53)$$

where $i = 1, \ldots, N-l-1$, and E_{Qc}^i is i-th row of matrix E_{Qc}. It is easy to see that

$$\begin{aligned}|\varphi_i(Q_{ce})| &= \Big|Q_{ai}|Q_{ai}| - Q_{ari}|Q_{ari}|\Big| \le |Q_{aei}|(|Q_{ai}| + |Q_{ari}|) \\ &= |E_{Qc}^i Q_{ce}|(|E_{Qc}^i Q_c| + |E_{Qc}^i Q_{cr}|). \end{aligned} \quad (54)$$

In order to find matrix Π we need to evaluate $Q_{ce}^T \Psi(Q_{ce}) Q_{ce}$. For simplicity of notation let us denote $\|\cdot\| = |\cdot|$. With the notation, from (39) we get

$$\begin{aligned} Q_{ce}^T\Psi(Q_{ce})Q_{ce} &= -Q_{ce}^T A_c^{-1}(A_{ca}\varphi(Q_{ce})R_{ar} + B_c Q_{ce}) \\ &\le |A_c^{-1}B_c||Q_{ce}|^2 + |A_c^{-1}A_{ca}||\mathrm{diag}\{R_{ar}\}|| \\ &\quad \times \mathrm{col}(\varphi_i(Q_{ce}))|_2 |Q_{ce}| \\ &\le |A_c^{-1}B_c||Q_{ce}|^2 + |A_c^{-1}A_{ca}|R_{ar}^{\max}| \\ &\quad \times \mathrm{col}(\varphi_i(Q_{ce}))|_1 |Q_{ce}| \\ &\le |A_c^{-1}B_c||Q_{ce}|^2 + |A_c^{-1}A_{ca}|R_{ar}^{\max} \\ &\quad \times \left[\sum_{i=1}^{l}(|E_{Qc}^i Q_c| + |E_{Qc}^i Q_{cr}|)|E_{Qc}^i Q_{ce}|\right]|Q_{ce}| \\ &= |A_c^{-1}B_c||Q_{ce}|^2 + |A_c^{-1}A_{ca}|R_{ar}^{\max} \\ &\quad \times \left[\sum_{i=1}^{l}(|E_{Qc}^i Q_{ce}| + 2|E_{Qc}^i Q_{cr}|) \right. \\ &\quad \left. \times |E_{Qc}^i Q_{ce}|\right]|Q_{ce}| \end{aligned}$$

$$
\begin{aligned}
&= |A_c^{-1}B_c||Q_{ce}|^2 + |A_c^{-1}A_{ca}|R_{ar}^{\max} \\
&\quad \times \left[\sum_{i=1}^{l} |E_{Qc}^i Q_{ce}|^2 + 2\sum_{i=1}^{l} |E_{Qc}^i Q_{cr}| \right.\\
&\quad \left. \times |E_{Qc}^i Q_{ce}| \right] |Q_{ce}| \\
&\le |A_c^{-1}B_c||Q_{ce}|^2 + |A_c^{-1}A_{ca}|R_{ar}^{\max} \\
&\quad \times \left[|Q_{ce}|^2 \sum_{i=1}^{l} |E_{Qc}^i|^2 + 2|Q_{ce}||Q_{cr}| \sum_{i=1}^{l} |E_{Qc}^i|^2 \right] |Q_{ce}| \\
&= |A_c^{-1}B_c||Q_{ce}|^2 + |A_c^{-1}A_{ca}|R_{ar}^{\max} \sum_{i=1}^{l} |E_{Qc}^i|^2 \\
&\quad \times (|Q_{ce}| + 2|Q_{cr}|)|Q_{ce}|^2 \\
&= (|A_c^{-1}B_c| + |A_c^{-1}A_{ca}||E_{Qc}|_F^2 R_{ar}^{\max} \\
&\quad \times (|Q_{ce}| + 2|Q_{cr}|))|Q_{ce}|^2,
\end{aligned}
$$

where $|E_{Qc}|_F^2 = \sum_{i=1}^{l} |E_{Qc}^i|^2$. Let us denote

$$
|A_c^{-1}B_c| = M_1, \quad |A_c^{-1}A_{ca}||E_{Qc}|_F^2 = M_2. \tag{56}
$$

Then we can rewrite (55) as

$$
\begin{aligned}
Q_{ce}^T \Psi(Q_{ce}) Q_{ce} \le{} & (M_1 + 2M_2 R_{ar}^{\max}|Q_{cr}|)|Q_{ce}|^2 \\
& + M_2 R_{ar}^{\max}|Q_{ce}|^3.
\end{aligned} \tag{57}
$$

Using expressions

$$
|Q_{ce}|^2 = \sum_{i=1}^{l} Q_{ce_i}^2, \qquad |Q_{ce}|^3 = \left(\sum_{i=1}^{l} Q_{ce_i}^2 \right)^{3/2} \le \sqrt{l} \sum_{i=1}^{l} |Q_{ce_i}|^3, \tag{58}
$$

we have

$$
\begin{aligned}
Q_{ce}^T \Psi(Q_{ce}) Q_{ce} \le{} & (M_1 + 2M_2 R_{ar}^{\max}|Q_{cr}|) \sum_{i=1}^{l} Q_{ce_i}^2 \\
& + \sqrt{l} M_2 R_{ar}^{\max} \sum_{i=1}^{l} |Q_{ce_i}|^3.
\end{aligned} \tag{59}
$$

So we get the following matrix Π:

$$
\Pi(Q_{ce}) = \left(M_1 + 2M_2 R_{ar}^{\max}|Q_{cr}|\right) I + \sqrt{l} M_2 R_{ar}^{\max} \operatorname{diag}\{|Q_{ce_i}|\}. \tag{60}
$$

Choosing the control law in the form (42), we get the following derivative of Lyapunov function (49)

$$
\begin{aligned}
\dot{V} &= 2Q_{ce}^T\left[-Q_{crD}^2 R_{cE} + \Psi(Q_{ce})Q_{ce}\right] \\
&\le 2Q_{ce}^T\left[-Q_{crD}^2 R_{cE} + \Pi(Q_{ce})Q_{ce}\right] \\
&= 2Q_{ce}^T\Big[-Q_{crD}^2 R_{cE} + \operatorname{diag}\Big\{M_1 + 2M_2 R_{ar}^{\max}|Q_{cr}| \\
&\qquad + \sqrt{l} M_2 R_{ar}^{\max}|Q_{ce_i}|\Big\} Q_{ce}\Big] \\
&\le -2\lambda |Q_{ce}|^2.
\end{aligned} \tag{61}
$$

From (61) we conclude that the system (23) is exponentially stable. □

Let us now consider the case when d_r, R_{ar} and Q_{cr} are constants.

Theorem 3.2: *Suppose $Q_{cri} \neq 0$, $\forall i$, $\forall t \ge 0$. There exist sufficiently large positive constants M_3 and M_4, such that the system* (23), *under the control law* (40) *with* (41) *and*

$$
R_{cE_i} = \frac{1}{Q_{cri}|Q_{cri}|}(\lambda + M_3 + M_4|Q_{ce_i}|)Q_{ce_i}, \quad \lambda > 0 \tag{62}
$$

is exponentially stable at $Q_c = Q_{cr}$.

Proof: Using notations

$$
\begin{aligned}
& |A_c^{-1}B_c| + 2|A_c^{-1}A_{ca}|R_{ar}^{\max}|E_{Qc}|_F^2|Q_{cr}| = M_3, \\
& \sqrt{l}|A_c^{-1}A_{ca}|R_{ar}^{\max}|E_{Qc}|_F^2 = M_4,
\end{aligned} \tag{63}
$$

we can rewrite (55) as

$$
\begin{aligned}
Q_{ce}^T \Psi(Q_{ce}) Q_{ce} &\le M_3|Q_{ce}|^2 + \frac{M_4}{\sqrt{l}}|Q_{ce}|^3 \\
&\le M_3 \sum_{i=1}^{l} Q_{ce_i}^2 + M_4 \sum_{i=1}^{l} |Q_{ce_i}|^3.
\end{aligned} \tag{64}
$$

Choosing control law in the form (62), we get the following derivative of Lyapunov function (49)

$$
\begin{aligned}
\dot{V} &= 2Q_{ce}^T\left[-Q_{crD}^2 R_{cE} + \Psi(Q_{ce})Q_{ce}\right] \\
&\le 2Q_{ce}^T\left[-Q_{crD}^2 R_{cE} + \operatorname{diag}\{M_3 + M_4|Q_{ce_i}|\}Q_{ce}\right] \\
&\le -2\lambda |Q_{ce}|^2.
\end{aligned} \tag{65}
$$

From (65) we conclude that the system (23) is exponentially stable. □

4. An adaptive decentralized controller

As in the previous chapter, we first consider the case when d_r, R_{ar} and Q_{cr} are functions of time.

Theorem 4.1: *Suppose* $Q_{cri}(t) \neq 0$, $\forall i$, $\forall t \geq 0$. *Then the adaptive controller*, *consisting of* (40), (41), *and*

$$R_{cE_i} = \frac{1}{Q_{cri}(t)|Q_{cri}(t)|}\Big(\lambda + \hat{\theta}_1 + \hat{\theta}_2 R_{ar}^{\max}(2|Q_{cr}(t)| + \sqrt{l}|Q_{ce_i}|)\Big)Q_{ce_i}, \tag{66}$$

$$\dot{\hat{\theta}}_1 = \sum_{i=1}^{l} Q_{ce_i}^2, \tag{67}$$

$$\dot{\hat{\theta}}_2 = R_{ar}^{\max}(t)\left(2|Q_{cr}(t)|\sum_{i=1}^{l} Q_{ce_i}^2 + \sqrt{l}\sum_{i=1}^{l}|Q_{ce_i}|^3\right) \tag{68}$$

guarantees global stability of the solution $Q_c(t) = Q_{cr}(t)$, $\hat{\theta}_1(t) = M_1$, $\hat{\theta}_2(t) = M_2$ *and asymptotic tracking* $\lim_{t\to\infty}(Q_c(t) - Q_{cr}(t)) = 0$. *The global stability of the given equilibrium means*, *in particular*, *that* $Q_c(t), \hat{\theta}_i(t)$ *are globally bounded.*

Proof: Choose the following Lyapunov function

$$V = Q_{ce}^T(-A_c^{-1})Q_{ce} + \tilde{\theta}_1^2 + \tilde{\theta}_2^2, \tag{69}$$

where $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$, $\theta_i = M_i$, $i = 1, 2$ are estimation errors. Taking into account that $\dot{\tilde{\theta}}_i = -\dot{\hat{\theta}}_i$, we get the following derivative of (69)

$$\begin{aligned}\dot{V} &= 2Q_{ce}^T(-A_c^{-1})\dot{Q}_{ce} + 2\tilde{\theta}_1\dot{\tilde{\theta}}_1 + 2\tilde{\theta}_2\dot{\tilde{\theta}}_2 \\ &= 2Q_{ce}^T\big[-Q_{crD}^2 R_{cE} + \Psi(Q_{ce})Q_{ce}\big] - 2\tilde{\theta}_1\dot{\hat{\theta}}_1 - 2\tilde{\theta}_2\dot{\hat{\theta}}_2 \\ &\leq 2Q_{ce}^T\Big[-Q_{crD}^2 R_{cE} + ((\theta_1 + 2\theta_2 R_{ar}^{\max}|Q_{cr}|)I \\ &\quad + \sqrt{l}\theta_2 R_{ar}^{\max}\mathrm{diag}\{|Q_{ce_i}|\})Q_{ce}\Big] - 2\tilde{\theta}_1\dot{\hat{\theta}}_1 - 2\tilde{\theta}_2\dot{\hat{\theta}}_2. \end{aligned} \tag{70}$$

Taking R_{cEi} in the form (66), we have

$$\begin{aligned}\dot{V} &\leq -2\lambda|Q_{ce}|^2 + 2Q_{ce}^T\tilde{\theta}_1 Q_{ce} + 4Q_{ce}^T\tilde{\theta}_2 R_{ar}^{\max}|Q_{cr}|Q_{ce} \\ &\quad + 2\tilde{\theta}_2\sqrt{l}R_{ar}^{\max}\sum_{i=1}^{l}|Q_{ce_i}|^3 - 2\tilde{\theta}_1\dot{\hat{\theta}}_1 - 2\tilde{\theta}_2\dot{\hat{\theta}}_2 \\ &= -2\lambda|Q_{ce}|^2 + 2\tilde{\theta}_1(Q_{ce}^T Q_{ce} - \dot{\hat{\theta}}_1) \\ &\quad + 2\tilde{\theta}_2\left(2R_{ar}^{\max}|Q_{cr}|\sum_{i=1}^{l}Q_{ce_i}^2 + \sqrt{l}R_{ar}^{\max}\sum_{i=1}^{l}|Q_{ce_i}|^3 - \dot{\hat{\theta}}_2\right).\end{aligned} \tag{71}$$

Now, choosing $\dot{\hat{\theta}}_1$ and $\dot{\hat{\theta}}_2$ as in equations (67) and (68) respectively, rewrite (71) as

$$\dot{V} \leq -2\lambda|Q_{ce}|^2. \tag{72}$$

From (72), by La Salle's theorem, the result follows. The boundedness of $V(t)$ guarantees the global boundedness of $Q_c(t), \hat{\theta}_i(t)$. □

Consider now the case when $d_r = \text{const}$, $R_{ar} = \text{const}$ and $Q_{cr} = \text{const}$.

Theorem 4.2: *Suppose* $Q_{cri} \neq 0$, $\forall i$, $\forall t \geq 0$. *Then the adaptive controller*, *consisting of* (40), (41), *and*

$$R_{cE_i} = \frac{1}{Q_{cri}|Q_{cri}|}\Big(\lambda + \hat{\theta}_3 + \hat{\theta}_4|Q_{ce_i}|\Big)Q_{ce_i}, \tag{73}$$

$$\dot{\hat{\theta}}_3 = \sum_{i=1}^{l} Q_{ce_i}^2, \tag{74}$$

$$\dot{\hat{\theta}}_4 = \sum_{i=1}^{l} |Q_{ce_i}|^3, \tag{75}$$

guarantees global stability of the equilibrium $Q_c = Q_{cr}$, $\hat{\theta}_3 = M_3$, $\hat{\theta}_4 = M_4$ *and regulation of* $Q_c(t)$ *to* Q_{cr}. *The global stability of the given equilibrium means*, *in particular*, *that* $Q_c(t), \hat{\theta}_i(t)$ *are globally bounded.*

Proof: Choose Lyapunov function

$$V = Q_{ce}^T(-A_c^{-1})Q_{ce} + \tilde{\theta}_3^2 + \tilde{\theta}_4^2. \tag{76}$$

Its derivative is

$$\begin{aligned}\dot{V} &= 2Q_{ce}^T\big[-Q_{crD}^2 R_{cE} + \Psi(Q_{ce})Q_{ce}\big] - 2\tilde{\theta}_3\dot{\hat{\theta}}_3 - 2\tilde{\theta}_4\dot{\hat{\theta}}_4 \\ &\leq Q_{ce}^T\big[-Q_{crD}^2 R_{cE} + (\theta_3 I + \theta_4\mathrm{diag}\{|Q_{ce_i}|\})Q_{ce}\big] \\ &\quad - 2\tilde{\theta}_3\dot{\hat{\theta}}_3 - 2\tilde{\theta}_4\dot{\hat{\theta}}_4.\end{aligned} \tag{77}$$

Taking R_{cEi} in the form (73), we have

$$\begin{aligned}\dot{V} &\leq -2\lambda|Q_{ce}|^2 + 2Q_{ce}^T\tilde{\theta}_3 Q_{ce} + 2Q_{ce}^T\tilde{\theta}_4\mathrm{diag}\{|Q_{ce_i}|\}Q_{ce} \\ &\quad - 2\tilde{\theta}_3\dot{\hat{\theta}}_3 - 2\tilde{\theta}_4\dot{\hat{\theta}}_4.\end{aligned} \tag{78}$$

Now, choosing $\dot{\hat{\theta}}_3$ and $\dot{\hat{\theta}}_4$ as in equations (74) and (75) respectively, we get

$$\dot{V} \leq -2\lambda|Q_{ce}|^2, \tag{79}$$

which, by La Salle's theorem, proves the result. □

5. Example

We consider an example reminiscent of a blood circulation system. Let the network be forced with a periodic generator (like the heart). Without control, the flow rates would tend to be periodic through all of the branches. We set as an objective to keep the flow rate in one of the branches (for example, branch 3 in figure 1) as constant, while allowing the other branches to have periodic flow. As we shall see, the solution amounts to more than just making the resistance periodic in branch 3 – all of the co-tree branches participate in solving the tracking problem.

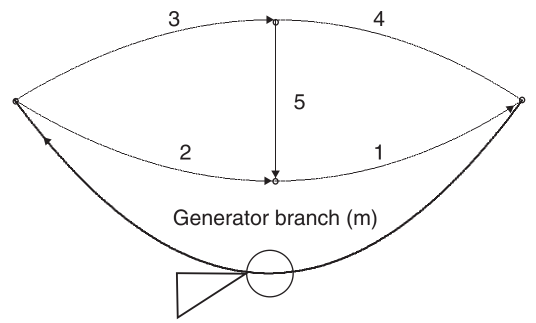

Figure 1. Fluid flow network system with 5 branches and generator branch.

The network in figure 1 consists of 4 nodes, 5 branches and 1 main generator branch, whose pressure forcing is

$$d_r(t) = 10 + 5\sin t. \tag{80}$$

Let branches 4, 5 and m (the generator branch) be the tree of the network, and let the reference flow rates in the co-tree be $Q_{1_r}(t) = 2 + 1.5\sin t$, $Q_{2_r}(t) = 2.2 + \sin t$, and

$$Q_{3_r}(t) = 1.5 = \text{const.} \tag{81}$$

Let the resistances in the uncontrolled branches be constant, $R_{4_r} = 1.5$, $R_{5_r} = 0.1$. The objective is to find decentralized feedback laws $R_1(Q_1)$, $R_2(Q_2)$ and $R_3(Q_3)$.

Define

$$Q_c = [Q_1 \ \ Q_2 \ \ Q_3]^T, \quad Q_a = [Q_4 \ \ Q_5]^T,$$
$$H_c = [H_1 \ \ H_2 \ \ H_3]^T, \quad H_a = [H_4 \ \ H_5]^T. \tag{82}$$

Choose the parameters of the system as $R_m = 1$, $K_i = 1$, $i = 1, \dots, 5$ and the initial conditions as $Q_c(0) = [5, \ 1.5, \ 4]^T$. We apply the feedback law given in Theorem 4.1.

In figure 2 the responses of the system are shown for $t = 12$ s. One can see that in controlled branches the

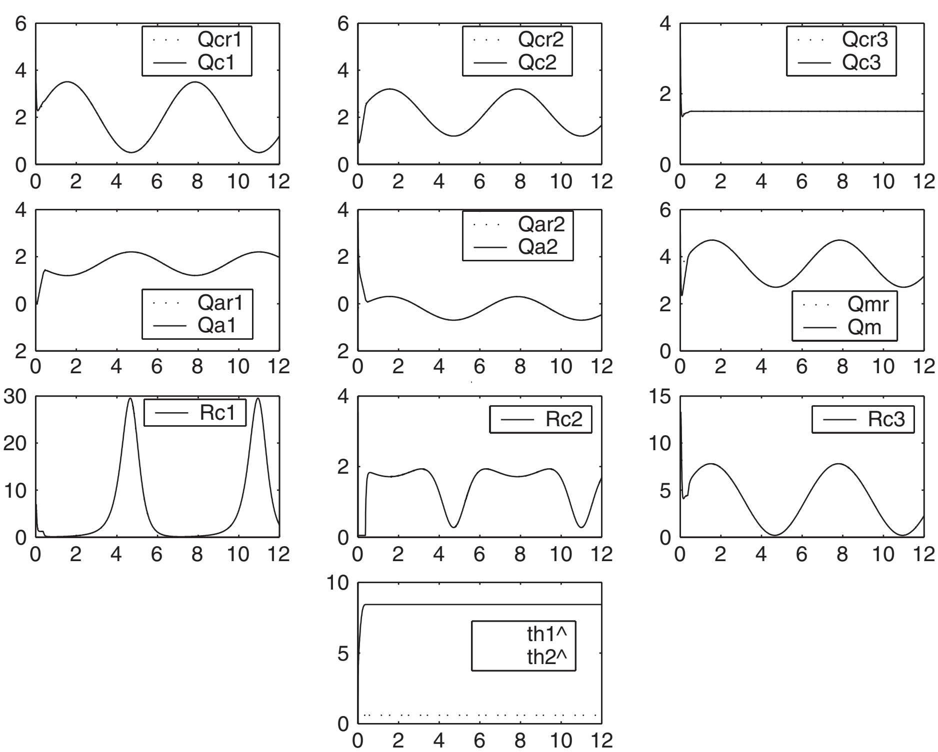

Figure 2. Adaptive controller from Theorem 4.1 applied to the example network. Note that Q_{c_3} is regulated to a constant, while the flows in the other branches remain periodic.

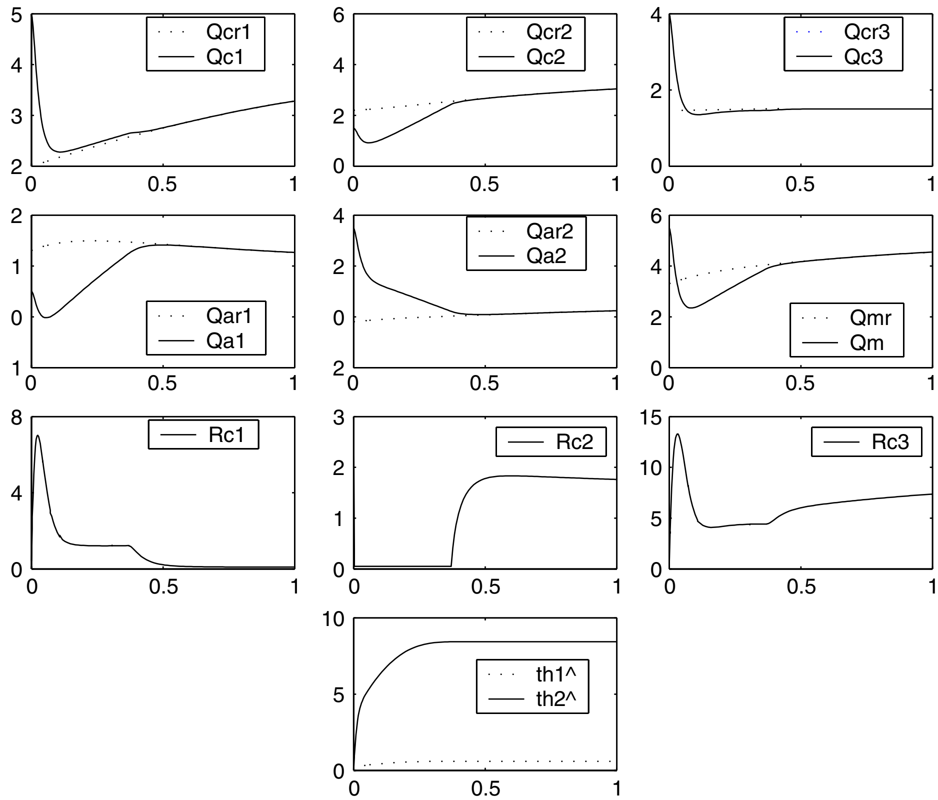

Figure 3. Zooming in to the transient from 0 to 1 s. Tracking is achieved in about 0.4 s. Note the saturation in R_{c_2}, to which the feedback system is shown to be robust.

desired flows are obtained. In other branches, where the flows cannot be controlled independently, the flow remains periodic. In figure 3 transients are shown for $t = 1$ s. Control inputs are not allowed to be less than $R_{\min} = 0.1$, so after controls R_{c_1} and R_{c_2} reach lower bound, they stay constant until flow rates decrease in the first branch or increase in the second branch. After approximately 0.4 s, flow rates achieve their references and no more saturation is observed. The controller gains in this case are $\theta_1 = 2$, $\theta_2 = 10.68$. The estimates achieve lesser values, as θ_1 and θ_2 are conservative.

6. Conclusions

In this paper, we developed nonlinear control algorithms that are implementable using collocated flow sensors and actuators. Further research should explore the applicability of this methodology to blood flow problems.

Some of the costliest vascular diseases arise in the venous system. In human body the veins carry about two-thirds of the blood volume. The venous blood above the heart – in upper limbs and the head – flows to the heart due to gravity. In lower limbs it flows towards the heart due to the *vis a tergo*, the calf pump and the *vis a fronte* factors. The *valves* control the direction of the flow to prevent the retrograde wave of the flow. If a valve is incompetent or destroyed, reflux arises, which leads to and aggrevates venous insufficiency Browse *et al.* (1999). After the function of venous valves was discovered, the interest in artificial valves arose. At present several types of artificial venous valves exist (Dimatteo and Marshall 2002, Duerig and Melzer 2003, Duerig and Stockel 2000, Jayaraman 2001, Pavcnik *et al.* 1999, Pavcnik *et al.* 2001, Pavcnik *et al.* 2002, Pavcnik 2002a, Pavcnik 2002b, Shaolian and von Hoffmann 2001, Thorpe *et al.* 2000, Uflacker 1993). All of them are *passive*, or uncontrolled valves. In this work, we developed control laws for *active* valves. While there could be many natural valves, or passive artificial valves in the part of the vein between two junctions (a *branch*), it is enough to have one *actively controlled* valve per branch. While the example in § 5 was not exactly motivated by venous disease, our related efforts on vascular networks Koroleva and Krstić (2005) concentrate on this problem more closely.

Acknowledgement

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References

M.D. Aldridge, R.E. Swartwout, N.S. Smith Jr, R.S. Nutter and J.L. Boyles, ‘‘Electronic monitoring and control of mine ventilation’’, in *Proceedings of the 3rd WVU Conference on Coal Mine Electrotechnology*, WV, USA: Morgantown, 1976.

V.O. Bogdanov and D.V. Kneller, ‘‘Identification-based algorithm of mine ventilation control’’, in *Proceedings of the 4th IFAC Symposium on Automation in Mining, Mineral and Metal Processing*, Finland: Helsinki, 1983.

N.L. Browse, K.G. Burnand, A.T. Irvine and N.M. Wilson, *Diseases of the Veins*, New York, USA, Oxford University Press, 1999.

C.A. Desoer and E.S. Kuh, *Basic Circuit Theory*, New York, USA, McGraw-Hill Book Company, 1969.

K. Dimatteo and P. Marshall, Implantable prosthetic valve. U.S. Patent 6,440,164, 2002.

T.W. Duerig and A. Melzer, Stent-based venous valves. U.S. Patent 6,503,272, 2003.

T.W. Duerig and D. Stockel, Composite self expanding stent device having a restraining element. U.S. Patent 6,086,610, 2000.

R.A. Horn and C.R. Johnson, *Matrix Analysis*, New York, USA, Cambridge University Press, 1985.

Yu. Hu, O.I. Koroleva and M. Krstić, ‘‘Nonlinear control of mine ventilation networks’’, *Systems and Control Letters*, 49, pp. 239–254, 2003.

S. Jayaraman, Stent, stent graft and stent valve. U.S. Patent 6,245,102, 2001.

D.D. Kocić , On the autonomy of local systems in mine ventilation control, 2nd Mine Ventilation Congress, Reno, USA, 1979.

O.I. Koroleva and M. Krstić, ‘‘Averaging analysis of periodically forced fluid networks’’, *Automatica*, 41, pp. 129–135, 2005.

K.Y. Lee and R.S. Nutter, A mine-wide algorithm for ventilation control in coal mines. Ias Annual Meeting, Cincinnati, OH, USA, 1980.

A.A. Mahdi and M.J. McPherson, ‘‘An introduction to automatic control of mine ventilation systems’’, *Mining Technology*, 53, 1971.

M.J. McPherson, A.A. Mahdi and D. Goh, *The Automatic Control of Mine Ventilation*, London, UK: Colloquium on Measurement and Control in Coal Mining, 1972.

T. Meriluoto, ‘‘A modular mine ventilation control system. Automation in Mining, Mineral and Metal Processing’’, in *Proceedings of the 4th IFAC Symposium*, Finland: Helsinki, 1983.

D. Pavcnik, B.T. Uchida, F.S. Keller, C. Corless and J. Rösch, ‘‘Retrievable IVC square stent filter: experimental study’’, *Cardiovasc Intervent Radiol*, 22, pp. 239–245, 1999.

D. Pavcnik, B.T. Uchida, H.A. Timmermans, F.S. Keller and J. Rösch, ‘‘Square stent: a new self-expandable endoluminal device and its applications’’, *Cardiovasc. Intervent. Radiol.*, 24, pp. 207–217, 2001.

D. Pavcnik, B.T. Uchida, H.A. Timmermans, C.L. Corless, M. O'hara, N. Toyota, G.L. Moneta, F.S. Keller and J. Rösch, ‘‘Percutaneous bioprosthetic venous valve: a long term study in sheep’’, *J. Vasc. Surg.*, 35, pp. 598–602, 2002.

D. Pavcnik, ‘‘Chronic venous insufficiency and bioprosthetic bicuspid square stent based venous valve for transcatheter placement’’, *Acta Clinica Croatica*, 41, pp. 93–97, 2002a.

D. Pavcnik, The Portland valve. Fourth Pacific Vascular Symposium of Venous Disease, Kohala Coast, Big Island of Hawaii, USA, 2002b.

S.M. Shaolian and Von G. Hoffmann, Transluminally implantable venous valve. U.S. Patent 6,299,637, 2001.

P.E. Thorpe, F.J. Osse and L.O. Correa, The valve stent: development of a percutaneous prosthesis for treatment of valvular unsufficiency. The 12th Annual Meeting of the American Venous Forum, Phoenix, AZ, USA, 2000.

R. Uflacker, Percutaneously introduced artificial venous valve: experimental use in pigs, Annual Meeting of the Western Angiographic and Interventional Society, Portland, OR, USA, 1993.

A.J. Ward-Smith, *Internal Fluid Flow – The Fluid Dynamics of Flow in Pipes and Ducts*, Oxford: Clarendon press, 1980.