INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2006; **16**:801–818
Published online 29 August 2006 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc.1098

Systematization of approaches to adaptive boundary stabilization of PDEs

Miroslav Krstic*,†

Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, U.S.A.

SUMMARY

While adaptive control of finite dimensional systems is an advanced field that has produced adaptive control methods for a very general class of LTI systems, adaptive control techniques have been developed for only a few of the classes of PDEs for which non-adaptive controllers exist. We present a catalog of approaches for the design of adaptive controllers for PDEs controlled from a boundary and containing unknown destabilizing parameters affecting the interior of the domain. We differentiate between two major classes of schemes: Lyapunov schemes and certainty equivalence schemes. Within the certainty equivalence class, two types of identifier designs are pursued: passivity-based and swapping designs. Each of those designs is applicable to two types of parametrizations: the plant model in its original form (which we refer to as the 'u-model') and a transformed model to which a *backstepping* transformation has been applied (which we refer to as the 'w-model'). Hence, a large number of control algorithms result from combining different design tools—Lyapunov schemes, w-passive schemes, u-swapping schemes, etc.

Our method builds upon the explicitly parametrized control formulae that we introduced in our earlier work on non-adaptive backstepping control for PDEs. These formulae allow us to develop tunable controllers that avoid solving Riccati or Bezout equations at each time step.

This paper is primarily a tutorial. Its purpose is to provide structure that helps the future reader of five other papers currently under review which contain the detailed proofs for the designs presented here. Additionally, the paper can serve as an entry point for a non-expert reader interested in an introduction to adaptive boundary control of PDEs. For this reason, our presentation proceeds through a series of examples, which are generalized in the companion papers. Copyright © 2006 John Wiley & Sons, Ltd.

Received 20 December 2005; Revised 25 March 2006; Accepted 2 April 2006

KEY WORDS: backstepping; Lyapunov stabilization; boundary control; distributed parameter systems; adaptive control

---

*Correspondence to: Miroslav Krstic, Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, U.S.A.
† E-mail: krstic@ucsd.edu

Contract/grant sponsor: NSF; contract/grant number: CMS-0329662

1. INTRODUCTION

While adaptive control of finite dimensional systems is a mature area that has produced adaptive control methods for most LTI systems of interest [1], adaptive control techniques have been developed for only a few of the classes of PDE for which non-adaptive controllers exist. In this paper, we present a systematization of several new approaches to adaptive stabilization for parabolic PDEs controlled from a boundary and containing unknown destabilizing parameters affecting the interior of the domain.

1.1. Literature overview

The early efforts on adaptive control of distributed parameter systems were using tuning of a scalar gain to a high level to stabilize some classes of (relatively degree one) infinite dimensional plants (see the survey by Logemann and Townley [2] for an exhaustive list of references). Model reference (MRAC) type schemes were designed by Hong and Bentsman [3], Bohm *et al.* [4], and Bentsman and Orlov [5]. While the focus in these papers is on functional, spatially dependent parametric uncertainty and the proofs of identifiability, the control is distributed in the PDE domain, allowing access to all the uncertain terms, akin to an infinite set of parallel first-order systems. Positive realness has played an important role in the work of Demetriou and Ito [6]. Adaptive linear quadratic control with least-squares estimation was pursued by Duncan *et al.* [7] for stochastic evolution equations with unbounded input operators and stable uncontrolled dynamics (assumption A3). Nonlinear PDEs have also received some attention. Liu and Krstic [8] and Kobayashi [9] considered a Burgers equation with various parametric uncertainties; Kobayashi [10] also considered the Kuramoto–Sivashinsky equation. Jovanovic and Bamieh [11] designed adaptive controllers for nonlinear systems on lattices, which include applications like infinite vehicular platoons or infinite arrays of microcantilevers. An experimentally validated adaptive boundary controller got a flexible beam was presented by de Queiroz *et al.* [12].

1.2. The backstepping approach

In this paper we present results for several open-loop unstable parabolic PDE systems controlled by boundary control. We assume that physical parameters in those systems like reaction, diffusion, or advection coefficients are unknown. We design explicit adaptive control laws to stabilize these systems despite parametric uncertainty.

Problems like the ones considered here frequently arise in applications that incorporate thermal-fluid or chemically reacting dynamics. No solution exists in the previous literature for adaptive *boundary*—control of such problems because of the absence of parametrized families of controllers for parabolic PDE systems. In a recent paper [13], Smyshlyaev and Krstic developed explicit formulae for boundary control of a class of parabolic PDEs that includes the systems considered here. Those formulae are not only explicit functions of the spatial co-ordinates of the PDE, but also depend explicitly on the physical parameters of the plant. This is a quality not shared by standard methods like LQR methods for PDEs because parametrized solutions to Riccati equations cannot be obtained. While an adaptive version of an LQR approach would require a solution to a high-dimensional Riccati equation at each time step, our approach only requires that parameter updates be plugged into the control formula.

DOI: 10.1002/rnc

1.3. The categories of designs

We differentiate between two major classes of schemes: Lyapunov schemes and certainty equivalence schemes. Within the certainty equivalence class, two types of identifier designs are pursued: passivity-based and swapping designs. Each of those designs is applicable to two types of parametrizations: the plant model in its original form (which we refer to as the 'u-model') and a transformed model to which a *backstepping* transformation has been applied (which we refer to as the 'w-model'). Hence, a large number of control algorithms result from combining different design tools—Lyapunov schemes, w-passive schemes, u-swapping schemes, etc.

1.4. The objective of this paper

This paper is primarily a tutorial. Its purpose is to help the future reader of the papers [14–18], which are currently under review and contain the detailed proofs for the designs presented here, to understand a broader context for the individual approaches and the tradeoffs between the them. Additionally, the paper is meant to serve as an entry point for a non-expert reader interested in an introduction to adaptive boundary control of PDEs. For this reason, our presentation proceeds through a series of benchmark examples, which are generalized in the companion papers. While the bulk of the designs presented in this paper are from References [14, 15, 17], the results in Sections 6.2, 6.4, 7, and 8 are presented here for the first time.

2. CATEGORIZATION OF ADAPTIVE CONTROLLERS AND IDENTIFIERS

Stability is the central issue in adaptive control because one often starts with an unstable plant and no knowledge of its parameters. Approaches to adaptive control can be divided on the basis of how closed-loop stability is achieved into two groups:

- Lyapunov approach,
- certainty equivalence approach.

The *Lyapunov approach* directly addresses the issue of closed loop stability and results in controllers and identifiers designed jointly, with all the states of the closed loop system (plant, parameter estimator, state estimator) incorporated into a single Lyapunov function. Lyapunov adaptive controllers possess the best transient performance properties, however they are often more complex and this approach is not applicable as broadly as the certainty equivalence approach.

The term *certainty equivalence* (CE) approach refers to a broad group of methods where the controller and the identifier are designed separately. The controller is designed, in a form parametrized by the unknown parameters[‡] as if they were known. The parameter identifier is designed separately, without taking closed-loop stability into account, but only with an objective that the parameter estimation error be bounded and the output estimation error and the derivative of the parameter estimate be square integrable in time. That the controller/

[‡] In this text we consider only indirect approaches as the direct approaches do not naturally extend from ODEs to PDEs.

DOI: 10.1002/rnc

identifier pair would guarantee closed-loop stability is highly non-obvious and typically difficult to prove, resulting in transient performance inferior to the Lyapunov design. However, the CE approaches have an advantage in implementation because they combine easier-to-design controller and identifier modules.

Parameter identifiers for use in the certainty equivalence approach can be split into two classes:

- passivity-based identifiers,
- swapping identifiers.

The *passivity-based* method uses a copy of the plant, with the unknown parameter replaced by its estimate, to generate a parametric model which is passive from the parameter estimation error to the error between the plant state and the state of its copy. Sometimes this method is referred to as 'observer-based' method because it uses a copy of the plant. We avoid this name because it is misleading—the 'observer' does not serve the purpose of state estimation—in fact, this method is seldom used in output-feedback adaptive control problems.

The *swapping* method is perhaps the most common method of parameter estimation in adaptive control. Filters of the 'regressor' and of the measured part of the plant are implemented to convert a dynamic parametrization of the problem (given by the plant's dynamic model) into a static parametrization where standard gradient and least squares estimation techniques can be used. Because of the prevalence of this method, it is often (incorrectly) referred to in the literature as simply the 'gradient' or 'least-squares' method, even though such terms only describe the form of the update law and not the approach used to eliminate the dynamics from the parametrization of the problem. The swapping method uses the highest order of dynamics of all identifier approaches. Lyapunov is the lowest in this respect as it only incorporates the dynamics of the parameter update, and passivity-based is better than swapping because it uses only one filter, as opposed to 'one-filter-per-unknown-parameter' in the case of the swapping approach. Despite its high dynamic order, the swapping approach is popular because it is the most transparent (its stability proof is the simplest due to the static parametrization) and it is the only method that allows least-squares estimation.

Both the passivity-based approach and the swapping approach can be applied to plant models that are linear (affine, to be precise) in the unknown parameter. However, these two methods are also applicable to models that arise in stability analysis of the controlled PDE systems. Such models are linear in parameter estimation errors. While more complicated than the basic plant models, and thus leading to somewhat more complicated identifiers, they result in easier closed-loop analysis because the 'error systems' corresponding to the control problem and that corresponding to the identification problem are the same. In our presentation the plant state will be denoted by $u(t,X)$, where t is time and X is the spatial co-ordinate (which is a scalar in 1D and a vector in 2D and 3D). The stability under feedback will be done in a different, transformed variable denoted by $w(t,X)$. Both the u-model and the w-model will be valid parametric models for which passivity-based and swapping identifiers can be designed, each with its own advantages.

We will therefore be developing four categories of identifiers for the certainty equivalence approach: u-passive, w-passive, u-swapping, and w-swapping. In summary, taking

DOI: 10.1002/rnc

into account also the Lyapunov approach, adaptive controllers will be developed in the following categories:

Lyapunov	Certainty equivalence			
	Passive		Swapping	
	u-passive	w-passive	u-swapping	w-swapping

3. BENCHMARK SYSTEMS

We present adaptive designs for three benchmark plants that capture issues that recur in most PDE problems. Our benchmark systems are parabolic PDEs in 1D, all three unstable, with parametric uncertainties appearing in various ways in the domain and in the boundary condition:

λ-system:

$$u_t = u_{xx} + \lambda u \tag{1}$$

$$u(0) = 0 \tag{2}$$

g-system:

$$u_t = u_{xx} + gu(0) \tag{3}$$

$$u_x(0) = 0 \tag{4}$$

q-system:

$$u_t = u_{xx} \tag{5}$$

$$u_x(0) = -qu(0) \tag{6}$$

The parameters λ, g, q in (1), (3), (5), respectively, are assumed to be unknown. The variable u depends on time t and space x. The arguments of the function $u(t,x)$ will be suppressed whenever possible, to reduce notation. Symbols $u(0)$ and $u_x(0)$ refer to the boundary conditions at $x=0$, where the dependence on time is suppressed in the notation.

The systems will be controlled through the boundary input $u(1)$. In the absence of control, when $\lambda > \pi^2$, $g > 2$, $q > 1$, the corresponding systems are unstable.

4. CONTROLLERS

For λ, g, q known, explicit control formulae were derived in Reference [13]. With the estimates for λ, g, q, the controllers become:[§]

λ-controller:

$$u(1) = -\hat{\lambda} \int_0^1 x \frac{I_1\left(\sqrt{\hat{\lambda}(1-x^2)}\right)}{\sqrt{\hat{\lambda}(1-x^2)}} u(x)\, \mathrm{d}x \tag{7}$$

[§]The symbols $I_1(\cdot), J_1(\cdot), I_2(\cdot)$, etc., denote Bessel functions of the appropriate kinds.

DOI: 10.1002/rnc

g-controller:

$$u(1) = -\int_0^1 \sqrt{\hat{g}}\sinh(\sqrt{\hat{g}}(1-x))u(x)\,\mathrm{d}x \tag{8}$$

q-controller:

$$u(1) = -\int_0^1 \hat{q}\,\mathrm{e}^{\hat{q}(1-x)}u(x)\,\mathrm{d}x \tag{9}$$

These controllers were motivated by 'backstepping' (spatially causal) changes of variable which transform the closed-loop system into the heat equation $w_t = w_{xx}$, which is exponentially stable. The three respective transformations and their inverses are given by
λ-transformation:

$$w(x) = u(x) + \int_0^x \hat{\lambda}\xi \frac{I_1\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)}{\sqrt{\hat{\lambda}(x^2-\xi^2)}}u(\xi)\,\mathrm{d}\xi \tag{10}$$

$$u(x) = w(x) - \int_0^x \hat{\lambda}\xi \frac{J_1\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)}{\sqrt{\hat{\lambda}(x^2-\xi^2)}}w(\xi)\,\mathrm{d}\xi \tag{11}$$

g-transformation:

$$w(x) = u(x) + \int_0^x \sqrt{\hat{g}}\sinh(\sqrt{\hat{g}}(x-\xi))u(\xi)\,\mathrm{d}\xi \tag{12}$$

$$u(x) = w(x) + \hat{g}\int_0^x (x-\xi)w(\xi)\,\mathrm{d}\xi \tag{13}$$

q-transformation:

$$w(x) = u(x) + \int_0^x \hat{q}\,\mathrm{e}^{\hat{q}(x-\xi)}u(\xi)\,\mathrm{d}\xi \tag{14}$$

$$u(x) = w(x) + \hat{q}\int_0^x w(\xi)\,\mathrm{d}\xi \tag{15}$$

While for $\hat{\lambda} = \lambda$, $\hat{g} = g$, $\hat{q} = q$ the transformed variables are governed by the heat equation $w_t = w_{xx}$, when the estimates are imperfect and, moreover, time variable, the transformed systems are much more complicated:
λ-target system:

$$w_t = w_{xx} + \dot{\hat{\lambda}}\int_0^x \frac{\xi}{2}w(\xi)\,\mathrm{d}\xi + \tilde{\lambda}w \tag{16}$$

$$w(0) = 0 \tag{17}$$

$$w(1) = 0 \tag{18}$$

g-target system:

$$w_t = w_{xx} + \dot{\hat{g}} \int_0^x \frac{\sinh(\sqrt{\hat{g}}(x-\xi))}{\sqrt{\hat{g}}} w(\xi)\,\mathrm{d}\xi + \tilde{g} w(0) \cosh(\sqrt{\hat{g}}x) \tag{19}$$

$$w_x(0) = 0 \tag{20}$$

$$w(1) = 0 \tag{21}$$

q-target system:

$$w_t = w_{xx} + \dot{\hat{q}} \int_0^x \mathrm{e}^{\hat{q}(x-\xi)} w(\xi)\,\mathrm{d}\xi \tag{22}$$

$$w_x(0) = -\tilde{q} w(0) \tag{23}$$

$$w(1) = 0 \tag{24}$$

where $\tilde{\lambda} = \lambda - \hat{\lambda}$, $\tilde{g} = g - \hat{g}$, $\tilde{q} = q - \hat{q}$ are the parameter estimation errors and $\dot{\hat{\lambda}}, \dot{\hat{g}}, \dot{\hat{q}}$ are the derivatives of the parameter estimates, which will be defined by the parameter update laws, yet to be designed.

5. LYAPUNOV DESIGN

Even for linear finite dimensional systems, quadratic Lyapunov functions work for adaptive stabilization only for a very restrictive class of systems of relative degree one. For systems with higher relative degree Lyapunov-based adaptive controllers become nonlinear even when the plants are linear, and the corresponding Lyapunov functions are highly nonlinear [19].

Inspired by Praly's old idea for adaptive nonlinear control in the presence of growth conditions [20], we have identified a class of Lyapunov functions suitable for boundary control problems for linear PDEs. This Lyapunov function is

$$V = \frac{1}{2}\log(1 + \|w\|^2) + \frac{1}{2\gamma}\tilde{\theta}^2 \tag{25}$$

where γ is a positive adaptation gain, $\|w\|$ denotes the spatial L_2 norm of $w(x,t)$, and $\tilde{\theta}$ denotes a generic parameter estimation error, i.e. $\tilde{\theta} = \tilde{\lambda}$, $\tilde{g}$, or $\tilde{q}$. The logarithm in (25) is crucial. Due to this term one can tolerate the potentially destabilizing effect of the derivatives $\dot{\hat{\lambda}}, \dot{\hat{g}}, \dot{\hat{q}}$ in (16), (19), and (22).

The update laws designed with the Lyapunov function (25) are [14]:

λ-update:

$$\dot{\hat{\lambda}} = \gamma \frac{\|w\|^2}{1 + \|w\|^2}, \quad 0 < \gamma < 1 \tag{26}$$

g-update:

$$\dot{\hat{g}} = \frac{\gamma}{1 + \|w\|^2} \mathrm{Proj}_{[\underline{g}, \bar{g}]}\left\{ w(0) \int_0^1 w(x)\cosh(\sqrt{\hat{g}}x)\,\mathrm{d}x \right\}, \quad \gamma < \frac{1}{2}\mathrm{e}^{-2\sqrt{\bar{g}}} \tag{27}$$

DOI: 10.1002/rnc

q-update:

$$\dot{\hat{q}} = \frac{\gamma}{1 + \|w\|^2} \mathrm{Proj}_{[\underline{q},\bar{q}]}\{w(0)^2\}, \quad \gamma < \frac{\sqrt{2}}{2} \mathrm{e}^{-\bar{q}} \tag{28}$$

Except for the λ-update law (26), the Lyapunov update laws (28) and (28) employ parameter projection defined as

$$\mathrm{Proj}_{[\underline{\theta},\bar{\theta}]}\{a\} = \begin{cases} 0, & \hat{\theta} = \underline{\theta} \text{ and } a < 0 \\ 0, & \hat{\theta} = \bar{\theta} \text{ and } a > 0 \\ a & \text{else} \end{cases} \tag{29}$$

It is assumed that bounds $\bar{g} > \underline{g} \geqslant 0$ and $\bar{q} > \underline{q} \geqslant 0$ are *a priori* known for g and q. In addition, the update laws require restrictions on the size of the adaptation gain γ. The upper bounds on γ that guarantee stability are known to the designer.

In addition to the limit on the adaptation gain, the Lyapunov update laws incorporate normalization by $1 + \|w\|^2$. This normalization slows down the adaptation to prevent the harmful effect of the derivatives $\dot{\hat{\lambda}}, \dot{\hat{g}}, \dot{\hat{q}}$ in (16), (19), and (22). While normalization is common in adaptive laws of swapping type, it is seldom possible to incorporate it into Lyapunov schemes where it is much more common that the effect of fast adaptation is compensated by additional terms in the control law.

Closed-loop adaptive systems are nonlinear even when the plants are linear. For example, in the simplest case (the λ-plant), the closed-loop is given by

$$w_t = w_{xx} + \frac{\gamma}{2} \frac{\|w\|^2}{1 + \|w\|^2} \int_0^x \xi w(\xi)\, \mathrm{d}\xi + \tilde{\lambda} w \tag{30}$$

$$w(0) = 0 \tag{31}$$

$$w(1) = 0 \tag{32}$$

$$\dot{\hat{\lambda}} = -\gamma \frac{\|w\|^2}{1 + \|w\|^2} \tag{33}$$

Besides the quadratic nonlinearities in the update law, the system has a product nonlinearity $\tilde{\lambda} w$ and the nonlinearity that has arisen from $\dot{\hat{\lambda}}$ on the right-hand side of w_t. Despite these nonlinearities, boundedness and regulation to zero are achieved globally (for arbitrarily large initial conditions of the plant $u(x,t)$). By boundedness and regulation we mean not just the properties that hold for the spatial L_2 norm $\sqrt{\int_0^1 u(x,t)^2\, \mathrm{d}x}$ but also pointwise in x. This property requires H_1 stability analysis, which goes beyond the Lyapunov function (25). It is shown in Reference [14].

6. CERTAINTY EQUIVALENCE DESIGN

6.1. *u*-passive identifier

The u-passive identifiers are designed on the basis of the parametric models (1)–(6). While the Lyapunov identifiers are finite dimensional for finite dimensional unknown parameters (in fact,

DOI: 10.1002/rnc

they are one dimensional in our examples), the u-passive identifiers each employ a copy of the PDE plant, which makes them infinite dimensional even when the unknown parameter is a scalar. The 'observers' and the update laws are stated next [15]:
λ-system:

$$\hat{u}_t = \hat{u}_{xx} + \hat{\lambda} u + \gamma \|u\|^2 (u - \hat{u}) \tag{34}$$

$$\hat{u}(0) = 0 \tag{35}$$

$$\hat{u}(1) = u(1) \tag{36}$$

$$\dot{\hat{\lambda}} = \gamma \int_0^1 (u(x) - \hat{u}(x)) u(x)\, \mathrm{d}x \tag{37}$$

g-system:

$$\hat{u}_t = \hat{u}_{xx} + \hat{g} u(0) + \gamma u(0)^2 (u - \hat{u}) \tag{38}$$

$$\hat{u}_x(0) = 0 \tag{39}$$

$$\hat{u}(1) = u(1) \tag{40}$$

$$\dot{\hat{g}} = \gamma u(0) \int_0^1 (u(x) - \hat{u}(x))\, \mathrm{d}x \tag{41}$$

q-system:

$$\hat{u}_t = \hat{u}_{xx} \tag{42}$$

$$\hat{u}_x(0) = -\hat{q} u(0) - \gamma^2 u(0)^2 (u(0) - \hat{u}(0)) \tag{43}$$

$$\hat{u}(1) = u(1) \tag{44}$$

$$\dot{\hat{q}} = \gamma u(0)(u(0) - \hat{u}(0)) \tag{45}$$

The properties of these identifiers are established with the Lyapunov function

$$V = \frac{1}{2} \|u - \hat{u}\|^2 + \frac{1}{2\gamma} \tilde{\theta}^2 \tag{46}$$

where $\tilde{\theta}$ denotes $\tilde{\lambda}$, $\tilde{g}$, or $\tilde{q}$. It can be shown that

$$\dot{V} \leqslant - \|(u - \hat{u})_x\|^2 - \dot{\hat{\theta}}^2 \tag{47}$$

This establishes that $\|u - \hat{u}\|$ and $\tilde{\theta}$ are bounded and $\|(u - \hat{u})_x\|$ and $\dot{\hat{\theta}}$ are square integrable over infinite time. These properties are essential for proving boundedness and regulation of $u(t, x)$.

The term 'passive identifier' comes from the fact that the nonlinear operator from, say, $\tilde{\lambda}$ to $\int_0^1 (u(x) - \hat{u}(x)) u(x)\, \mathrm{d}x$ is strictly passive. This property is achieved by adding the observer $\hat{u}$.

DOI: 10.1002/rnc

The terms like $+\gamma\|u\|^2(u-\hat{u})$ in (34) act as nonlinear damping terms whose task is to ensure square integrability of $\dot{\hat{\lambda}}$. They slow down the adaptation and act as an alternative to update law normalization.

6.2. *w-passive identifier*

Consider the 'target systems' (16)–(24). These systems incorporate the unknown parameters through the parameter errors $\tilde{\lambda}, \tilde{g}, \tilde{q}$ and thus are valid parametric models. It can be noted that, for example, (16) would be strictly passive from $\tilde{\lambda}$ to $\|w\|^2$ if it weren't for the perturbation $\dot{\hat{\lambda}}\int_0^x (\xi/2)w(\xi)\,\mathrm{d}\xi$. The observers in w-passive identifiers serve the purpose of eliminating those perturbations. The identifiers are defined as follows:¶

λ-system:

$$\hat{w}_t = \hat{w}_{xx} + \dot{\hat{\lambda}}\int_0^x \frac{\xi}{2} w(\xi)\,\mathrm{d}\xi + \gamma^2\|w\|^2(w-\hat{w}) \tag{48}$$

$$\hat{w}(0) = 0 \tag{49}$$

$$\hat{w}(1) = 0 \tag{50}$$

$$\dot{\hat{\lambda}} = \gamma\int_0^1 (w(x)-\hat{w}(x))w(x)\,\mathrm{d}x \tag{51}$$

g-system:

$$\hat{w}_t = \hat{w}_{xx} + \dot{\hat{g}}\int_0^x \frac{\sinh\left(\sqrt{\hat{g}}(x-\xi)\right)}{\sqrt{\hat{g}}} w(\xi)\,\mathrm{d}\xi + \gamma^2 w(0)^2(w-\hat{w}) \tag{52}$$

$$\hat{w}_x(0) = 0 \tag{53}$$

$$\hat{w}(1) = 0 \tag{54}$$

$$\dot{\hat{g}} = \gamma w(0)\int_0^1 (w(x)-\hat{w}(x))\cosh(\sqrt{\hat{g}}x)\,\mathrm{d}x \tag{55}$$

q-system:

$$\hat{w}_t = \hat{w}_{xx} + \dot{\hat{q}}\int_0^x \mathrm{e}^{\hat{q}(x-\xi)} w(\xi)\,\mathrm{d}\xi \tag{56}$$

$$\hat{w}_x(0) = -\gamma^2 w(0)^2(w(0)-\hat{w}(0)) \tag{57}$$

$$\hat{w}(1) = 0 \tag{58}$$

$$\dot{\hat{q}} = \gamma w(0)(w(0)-\hat{w}(0)) \tag{59}$$

¶These designs are present only in this paper. They are not contained in Reference [15] or the other companion papers.

DOI: 10.1002/rnc

The properties of this identifier are established with the Lyapunov function

$$V = \frac{1}{2}\|w - \hat{w}\|^2 + \frac{1}{2\gamma}\tilde{\theta}^2 \tag{60}$$

where $\tilde{\theta}$ denotes $\tilde{\lambda}$, $\tilde{g}$, or $\tilde{q}$. It can be shown that

$$\dot{V} \leqslant -\|(w - \hat{w})_x\|^2 - \dot{\hat{\theta}}^2 \tag{61}$$

which implies boundedness of $\|w - \hat{w}\|$ and $\tilde{\theta}$ and square integrability of $\|(w - \hat{w})_x\|$ and $\dot{\hat{\theta}}$.

It is evident in the target system 'observers' (48), (52), (56) that the explicit form of the backstepping transformations is the key to designing the w-passive identifiers.

6.3. *u-swapping identifier*

This class of identifiers would be the most readily recognizable for a reader with lay knowledge of identification. Filters are employed which convert the dynamic models (1)–(6) into static parametric models. For all three problems the update law is chosen as the normalized gradient scheme [15],

$$\dot{\hat{\theta}} = \gamma \frac{\int_0^1 (u(x) - \hat{\theta} v(x) - \eta(x)) v(x)\,\mathrm{d}x}{1 + \|v\|^2} \tag{62}$$

where $\hat{\theta}$ denotes $\hat{\lambda}$, $\hat{g}$, or $\hat{q}$. Of the two filters η and v, one is common to all three systems,[‖]

$$\eta_t = \eta_{xx} \tag{63}$$

$$\eta_x(0) = 0 \tag{64}$$

$$\eta(1) = u(1) \tag{65}$$

and the other is given as:

λ-system:

$$v_t = v_{xx} + u \tag{66}$$

$$v(0) = 0 \tag{67}$$

$$v(1) = 0 \tag{68}$$

g-system:

$$v_t = v_{xx} + u(0) \tag{69}$$

$$v_x(0) = 0 \tag{70}$$

$$v(1) = 0 \tag{71}$$

q-system:

$$v_t = v_{xx} \tag{72}$$

$$v_x(0) = -u(0) \tag{73}$$

[‖] For the λ-system, $\eta(0) = 0$ rather than $\eta_x(0) = 0$.

DOI: 10.1002/rnc

$$v(1) = 0 \tag{74}$$

For the Lyapunov function

$$V = \frac{1}{2}\|u - \theta v - \eta\|^2 + \frac{1}{2\gamma}\tilde{\theta}^2 \tag{75}$$

it can be proved that

$$\dot{V} \leqslant -\frac{1}{2}\|(u - \theta v - \eta)_x\|^2 - \frac{1}{2\gamma^2}\dot{\hat{\theta}}^2 \tag{76}$$

The above identifiers look extremely simple, however, they employ the highest dynamic order and the proof of stability for u-swapping scheme is the most complicated of all the schemes because the regressor in the output estimation error $v\tilde{\theta}$ is not closely related to the regressor in the target system. For instance, in the g-system, the former regressor is v (which is a filtered version of $u(0)$), whereas the latter regressor is $u(0)\cosh(\sqrt{\hat{g}}x)$.

6.4. *w-swapping identifier*

The w-swapping identifiers use the 'target systems' (16)–(24) as the parametric models. They all employ the update law**

$$\dot{\hat{\theta}} = \gamma \frac{\int_0^1 (w(x) - \hat{\theta}p(x) - \psi(x))p(x)\,\mathrm{d}x}{1 + \|p\|^2} \tag{77}$$

and two separate filters p and ψ defined as

λ-system:

$$p_t = p_{xx} + w \tag{78}$$

$$p(0) = 0 \tag{79}$$

$$p(1) = 0 \tag{80}$$

$$\psi_t = \psi_{xx} + \dot{\hat{\lambda}} \int_0^x \frac{\xi}{2} w(\xi)\,\mathrm{d}\xi - w\hat{\lambda} \tag{81}$$

$$\psi(0) = 0 \tag{82}$$

$$\psi(1) = 0 \tag{83}$$

g-system:

$$p_t = p_{xx} + w(0)\cosh(\sqrt{\hat{g}}x) \tag{84}$$

$$p_x(0) = 0 \tag{85}$$

$$p(1) = 0 \tag{86}$$

** These designs are present only in this paper. They are not contained in Reference [15] or the other companion papers.

DOI: 10.1002/rnc

$$\psi_t = \psi_{xx} + \dot{\hat{g}} \int_0^x \frac{\sinh\left(\sqrt{\hat{g}}(x-\xi)\right)}{\sqrt{\hat{g}}} w(\xi)\,\mathrm{d}\xi - \hat{g}w(0)\cosh(\sqrt{\hat{g}}x) \tag{87}$$

$$\psi_x(0) = 0 \tag{88}$$

$$\psi(1) = 0 \tag{89}$$

q-system:

$$p_t = p_{xx} \tag{90}$$

$$p_x(0) = -w(0) \tag{91}$$

$$p(1) = 0 \tag{92}$$

$$\psi_t = \psi_{xx} + \dot{\hat{q}} \int_0^x \mathrm{e}^{\hat{q}(x-\xi)} w(\xi)\,\mathrm{d}\xi \tag{93}$$

$$\psi(0) = \hat{q}w(0) \tag{94}$$

$$\psi(1) = 0 \tag{95}$$

For the Lyapunov function

$$V = \frac{1}{2}\|w - \theta p - \psi\|^2 + \frac{1}{2\gamma}\tilde{\theta}^2 \tag{96}$$

it can be proved that

$$\dot{V} \leqslant -\frac{1}{2}\|(w - \theta p - \psi)_x\|^2 - \frac{1}{2\gamma^2}\dot{\hat{\theta}}^2 \tag{97}$$

7. TRADEOFFS BETWEEN THE DESIGNS

With five designs per benchmark problem, the reader will probably wonder why so many different designs are needed. As we will explain here, each design has some advantage over the others, so it is important to be aware of all the five design options.

We make a comparison for the case of the g-plant. The Lyapunov identifier is given by (27), the u-passive identifier by (38)–(41), the w-passive identifier by (52)–(55), the u-swapping identifier by (62), (63)–(65), (69)–(71), and the w-swapping identifier by (77), (84)–(89).

The Lyapunov identifier (27) clearly has the lowest dynamic order. It employs only one scalar differential equation, whereas the other designs in addition incorporate PDEs. However, the Lyapunov identifier is functionally more complex than the u-passive and the u-swapping identifiers because the Lyapunov identifier incorporates the change of variable $u(x) \mapsto w(x)$, which is non-dynamic but nevertheless high dimensional (integration in x).

DOI: 10.1002/rnc

Between the identifiers based on the u-model, the u-passive identifier (38)–(41) has a lower dynamic order than the u-swapping identifier (62), (63)–(65),(69)–(71) because, while the former incorporates only one PDE (the 'observer' $\hat{u}$), the latter incorporates two PDEs (the input filter η and the output filter v). However, the u-swapping identifier is able to employ the standard gradient update law, with simple normalization, whereas the u-passive identifier utilizes an unusual form of nonlinear damping in the 'observer.'

The w-passive (52)–(55) and w-swapping (77), (84)–(89) identifiers may seem a little harder to justify because they have both the high dynamic order of the u-passive and u-swapping identifiers, as well as the functional complexity of the Lyapunov identifier. However, their advantage is in the fact that they are based on the w-system as the parametric model. This quality endows them with transient performance properties that are easier to quantify, as demonstrated in Reference [19] for finite dimensional nonlinear systems (note that the x-models in Reference [19] correspond to u-models here and z-models in Reference [19] correspond to w-models here).

8. STABILITY

For each of the three benchmark systems we have presented five adaptive schemes, for a total of 15 schemes. While stability of the identifiers (taken separately from the plant stability) is quite immediate for some of the schemes, the *closed-loop* stability of the complete dynamics consisting of the plant, controller, parameter update law, and the filters, is far from immediate. Even in the simplest among the cases the stability analysis is quite involved, as in the case of classical adaptive control for linear ODEs [1]. In this section we give a stability proof for one of the 15 schemes presented earlier to give an idea of what is involved in such analysis. This proof is not contained in Reference [15] or the other companion papers (the particular scheme we analyse is presented only here).

Consider the w-passive identifier (48)–(51) for the λ-system given in its error form by (16)–(18). From (60), (61) we get boundedness of $\|w - \hat{w}\|$ and $\tilde{\theta}$ and square integrability of $\|(w - \hat{w})_x\|$ and $\dot{\hat{\lambda}}$. However, it is not the boundedness of $w - \hat{w}$ that we need but the boundedness of both w and $\hat{w}$. Consider the Lyapunov function $U = \frac{1}{2}\int_0^1 \hat{w}^2\,\mathrm{d}x$. With some calculations that involve integration by parts and Cauchy–Schwartz and triangle inequalities, we get

$$\dot{U} \leqslant -\|\hat{w}_x\|^2 + \frac{|\dot{\hat{\lambda}}|}{2}\|\hat{w}\|(\|\hat{w}\| + \|e\|) + \gamma^2\|w\|(\|\hat{w}\| + \|e\|)\|\hat{w}\|\|e\| \tag{98}$$

where $e = w - \hat{w}$. Denoting $l = (\frac{1}{2}|\dot{\hat{\lambda}}| + \gamma^2\|w\|\|e\|)^2$, using Poincare's and Young's inequalities, we get

$$\dot{U} \leqslant -\tfrac{1}{4}U + 6l(t)U + \tfrac{1}{4}\|e(t)\|^2 \tag{99}$$

Since the functions $l(t)$ and $\|e(t)\|^2$ are integrable over infinite time,†† using Reference [19, Lemma B.6], we get from (99) that $\|\hat{w}(t)\|$ is bounded and square integrable, which together with boundedness and square integrability of $\|e(t)\|$, implies the same properties for $\|w(t)\|$ and $\dot{\hat{\lambda}}$.

One of the difficulties in working with PDEs is that the boundedness of $\|w\|$ does not imply boundedness of $w(t,x)$ pointwise in x. To show the boundedness of $\max_{x\in[0,1]}|w(t,x)|$ we can

†† It can be shown independently of (61) that $\|w\|^2\|e\|^2$ in l is integrable.

DOI: 10.1002/rnc

show the boundedness of $\|w_x(t)\|$ and use Agmon's inequality. Towards this end, we first derive the bound

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_x(t)\|^2 \leqslant \left(\tilde{\lambda}^2 + \frac{|\dot{\hat{\lambda}}|^2}{4}\right)\|w(t)\|^2 \tag{100}$$

Since $\tilde{\lambda}, \dot{\hat{\lambda}}$ are bounded and $\|w(t)\|$ is integrable, integrating (100) with respect to time we get boundedness of $\|w_x(t)\|$.

Next, we set out to prove regulation. First we note from (16)–(18) that

$$\frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2\right| \leqslant \|w_x\|^2 + |\dot{\hat{\lambda}}|\|w\|^2 + |\tilde{\lambda}|\|w\|^2 < \infty \tag{101}$$

So, $\|w(t)\|^2, (\mathrm{d}/\mathrm{d}t)\|w(t)\|^2$ are bounded and $\|w(t)\|^2$ is integrable. By Barbalat's lemma $\|w(t)\| \to 0$ as $t \to \infty$. To show the convergence to zero for all $x \in [0,1]$, we use Agmon's inequality and the fact that $\|w_x(t)\|$ is bounded:

$$\lim_{t\to\infty} \max_{x\in[0,1]} |w(x,t)| \leqslant \lim_{t\to\infty} (2\,\|w(t)\|\|w_x(t)\|)^{1/2} = 0 \tag{102}$$

The remaining step is to show that the properties established for w also hold for u. This is done by proving from (11) that $\|u\| \leqslant C\|w\|$ and $\max_{x\in[0,1]}|u(x,t)| \leqslant C \max_{x\in[0,1]}|w(x,t)|$ for some finite C, which yields boundedness and regulation of $u(x,t)$ pointwise in $x \in [0,1]$.

9. DESIGN FOR SYSTEMS WITH UNKNOWN DIFFUSION AND ADVECTION COEFFICIENTS

In this section, we show how one can also incorporate adaptation for unknown diffusion and advection coefficients—in addition to unknown reaction coefficients. Consider the system

$$u_t = \varepsilon u_{xx} + b u_x + \lambda u \tag{103}$$

$$u(0) = 0 \tag{104}$$

where ε, b, λ are unknown constants. The control law for this system is

$$u(1) = -\int_0^1 \frac{\hat{\lambda}}{\hat{\varepsilon}} x\, \mathrm{e}^{-(\hat{b}/2\hat{\varepsilon})(1-x)} \frac{I_1\left(\sqrt{(\hat{\lambda}/\hat{\varepsilon})(1-x^2)}\right)}{\sqrt{(\hat{\lambda}/\hat{\varepsilon})(1-x^2)}} u(x)\,\mathrm{d}x \tag{105}$$

where $\hat{\varepsilon}, \hat{b}, \hat{\lambda}$ are the estimates of ε, b, λ. The Lyapunov approach results in the update laws [14]

$$\dot{\hat{\lambda}} = \gamma \frac{\|w\|^2}{1 + \|w\|^2} \tag{106}$$

$$\dot{\hat{b}} = \gamma \frac{\int_0^1 w(x) \int_0^x \varphi(x,\xi) w(\xi)\,\mathrm{d}\xi\,\mathrm{d}x}{1 + \|w\|^2} \tag{107}$$

$$\dot{\hat{\varepsilon}} = -\frac{\hat{\lambda}\dot{\hat{\lambda}} + \hat{b}\dot{\hat{b}}}{\hat{\varepsilon}} \tag{108}$$

DOI: 10.1002/rnc

where

$$w(x) = u(x) - \int_0^x k(x,\xi)u(\xi)\,\mathrm{d}\xi \tag{109}$$

$$k(x,\xi) = -\frac{\hat{\lambda}}{\hat{\varepsilon}}\xi\,\mathrm{e}^{-(\hat{b}/2\hat{\varepsilon})(x-\xi)}\frac{I_1\left(\sqrt{(\hat{\lambda}/\hat{\varepsilon})(x^2-\xi^2)}\right)}{\sqrt{(\hat{\lambda}/\hat{\varepsilon})(x^2-\xi^2)}} \tag{110}$$

$$\varphi(x,\xi) = \mathrm{div}\,k(x,\xi) + \int_{\xi^x}(\mathrm{div}\,k(x,\sigma))l(\sigma,\xi)\,\mathrm{d}\sigma \tag{111}$$

$$\mathrm{div}\,k(x,\xi) = \frac{1}{\xi}k(x,\xi) + \frac{\hat{\lambda}}{\hat{\varepsilon}}\mathrm{e}^{-(\hat{b}/2\hat{\varepsilon})(x-\xi)}\frac{\xi}{x+\xi}I_2\left(\sqrt{\frac{\hat{\lambda}}{\hat{\varepsilon}}(x^2-\xi^2)}\right) \tag{112}$$

and projection is used (though we do not explicitly include it in the definition of the update laws) to keep the parameter estimates within *a priori* bounds $[\underline{\lambda},\bar{\lambda}]$, $[\underline{b},\bar{b}]$, and $[\underline{\varepsilon},\bar{\varepsilon}]$, where $\underline{\varepsilon}>0$. As in the previous Lyapunov designs, γ is limited by an upper bound which can be *a priori* computed.

10. ADAPTIVE CONTROL IN THE PRESENCE OF FUNCTIONAL PARAMETRIC UNCERTAINTIES

In the previous sections we considered only unknown parameters of scalar (λ, g, q) or vector $([\varepsilon, b, \lambda])$ type. In heterogeneous media (or in non-Cartesian co-ordinate systems) physical parameters like diffusion, viscosity, reaction, convection, etc., can be non-constant. Our method is capable of handling such problems. We will illustrate this on the simplest example that fits the space limit:

$$u_t = u_{xx} + \lambda(x)u \tag{113}$$

with uncontrolled boundary condition $u(0)=0$ and a boundary controller

$$u(1) = \int_0^1 k(1,\xi,\hat{\lambda})u(\xi)\,\mathrm{d}\xi \tag{114}$$

where $\hat{\lambda}(t,x)$ is the online functional estimate of $\lambda(x)$, and the gain kernel $k(1,\xi,\hat{\lambda})$ is obtained by solving the integral equation

$$k(x,\xi,\hat{\lambda}) = -\frac{1}{4}\int_{x-\xi}^{x+\xi}\hat{\lambda}\left(\frac{\zeta}{2}\right)\mathrm{d}\zeta + \frac{1}{4}\int_{x-\xi}^{x+\xi}\int_0^{x-\xi}\hat{\lambda}\left(\frac{\zeta-\sigma}{2}\right)k\left(\frac{\zeta+\sigma}{2},\frac{\zeta-\sigma}{2},\hat{\lambda}\right)\mathrm{d}\sigma\,\mathrm{d}\zeta \tag{115}$$

for each new update of $\hat{\lambda}(t,x)$. Well posedness of this integral equation for arbitrary continuous functions $\hat{\lambda}(x)$ was proved in Reference [13] and several methods for solving it (symbolically or numerically) were proposed and illustrated. It was observed that the computational expense of solving this equation is at least an order of magnitude lower than solving a Riccati equation.

DOI: 10.1002/rnc

The adaptive controller (114)–(115) is made stabilizing with the Lyapunov update law [17]

$$\hat{\lambda}_t(t,x) = \gamma \frac{u(t,x)(w(t,x) - \int_x^1 k(\xi,x,\hat{\lambda}(t))w(t,\xi)\,d\xi)}{1+\|w(t)\|^2} \tag{116}$$

for γ sufficiently small and with w defined as $w(x) = u(x) - \int_0^1 k(x,\xi,\hat{\lambda})u(\xi)\,d\xi$.

11. FUTURE WORK AND OPEN PROBLEMS

All of the designs presented here are for the state feedback problem. Output feedback designs for the g- and q-benchmarks are presented in Reference [16]. These extensions are relatively straightforward because the plants are already in a form which is a PDE analog of the 'observer canonical form'. Output feedback designs for the λ- and $\lambda(x)$-benchmarks are much more complex because they first require a transformation of the plant into the 'PDE observer canonical form'. These designs are presented in Reference [18].

While the extension from parabolic to hyperbolic PDEs is the obvious next problem of interest, the most exciting opportunities lie in developing adaptive tracking controllers for PDEs where the system output, for example $u(0,t)$, is being forced to track a prescribed reference signal $u_r(0,t)$ using control $u(1,t)$ at the other boundary. To solve such a problem one first needs to solve the motion planning problem, i.e. finding the function $u_r(x,t)$ that corresponds to the output reference $u_r(0,t)$. Even in the non-adaptive case this is a very challenging problem. We have recently solved the motion planning problem in closed form for reference trajectories that are sinusoidal, exponential, and polynomial functions of time. The remaining work is to develop adaptive versions of these results, with stabilization around the reference trajectories.

These advancements will also create exciting prospects for the analysis of persistence of excitation of the new adaptive boundary control schemes. Since the PE has to be enforced by scalar actuation from the boundary, achieving parameter convergence will be a challenging problem.

ACKNOWLEDGEMENT

This work was supported by NSF grant number CMS-0329662.

REFERENCES

1. Ioannou P, Sun J. *Robust Adaptive Control*. Prentice-Hall: Englewood Cliffs, NJ, 1996.
2. Logemann H, Townley S. Adaptive stabilization without identification for distributed parameter systems: an overview. *IMA Journal of Mathematical Control and Information* 1997; **14**:175–206.
3. Hong KS, Bentsman J. Direct adaptive control of parabolic systems: algorithm synthesis, and convergence, and stability analysis. *IEEE Transactions on Automatic Control* 1994; **39**:2018–2033.
4. Bohm M, Demetriou MA, Reich S, Rosen IG. Model reference adaptive control of distributed parameter systems. *SIAM Journal on Control and Optimization* 1998; **36**(1):33–81.
5. Bentsman J, Orlov Y. Reduced spatial order model reference adaptive control of spatially varying distributed parameter systems of parabolic and hyperbolic types. *International Journal of Adaptive Control and Signal Processing* 2001; **15**:679–696.
6. Demetriou MA, Ito K. Optimal on-line parameter estimation for a class of infinite dimensional systems using Kalman filters. *Proceedings of the American Control Conference*, Denver, Colorado, U.S.A., 2003.

DOI: 10.1002/rnc

7. Duncan TE, Maslowski B, Pasik-Duncan B. Adaptive boundary and point control of linear stochastic distributed parameter systems. *SIAM Journal on Control and Optimization* 1994; **32**(3):648–672.
8. Liu W, Krstic M. Adaptive control of Burgers' equation with unknown viscosity. *International Journal of Adaptive Control and Signal Processing* 2001; **15**:745–766.
9. Kobayashi T. Adaptive regulator design of a viscous Burgers' system by boundary control. *IMA Journal of Mathematical Control and Information* 2001; **18**:427–437.
10. Kobayashi T. Adaptive stabilization of the Kuramoto–Sivashinsky equation. *International Journal of Systems Science* 2002; **33**:175–180.
11. Jovanovic M, Bamieh B. Lyapunov-based distributed control of systems on lattices. *IEEE Transactions on Automatic Control* 2005; **50**:422–433.
12. Queiroz de MS, Dawson DM, Agarwal M, Zhang F. Adaptive nonlinear boundary control of a flexible link robot arm. *IEEE Transactions on Robotics and Automation* 1999; **15**(4):779–787.
13. Smyshlyaev A, Krstic M. Closed form boundary state feedbacks for a class of 1-D partial integro-differential equations. *IEEE Transactions on Automatic Control* 2004; **49**:2185–2202.
14. Krstic M. Adaptive boundary control for unstable parabolic PDEs—Part I: Lyapunov design. *IEEE Transactions on Automatic Control*, submitted.
15. Smyshlyaev A, Krstic M. Adaptive boundary control for unstable parabolic PDEs—Part II: Estimation-based designs. *Automatica*, submitted.
16. Smyshlyaev A, Krstic M. Adaptive boundary control for unstable parabolic PDEs—Part III: Output-feedback examples with swapping identifiers. *Automatica*, submitted.
17. Smyshlyaev A, Krstic M. Adaptive boundary control of reaction-diffusion-advection PDEs with spatially varying parameters. *2006 American Control Conference*, Minneapolis, Minnesota, U.S.A., 2006.
18. Smyshlyaev A, Krstic M. Output-feedback adaptive control for parabolic PDEs. *2006 Conference on Decision and Control*, San Diego, California, U.S.A., 2006.
19. Krstic M, Kanellakopoulos I, Kokotovic P. *Nonlinear and Adaptive Control Design*. Wiley: New York, 1995.
20. Praly L. Adaptive regulation: Lyapunov design with a growth condition. *International Journal of Adaptive Control and Signal Processing* 1992; **6**:329–351.

DOI: 10.1002/rnc