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Brief paper

On control design for PDEs with space-dependent diffusivity or time-dependent reactivity☆

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Abstract

In this paper the recently introduced backstepping method for boundary control of linear partial differential equations (PDEs) is extended to plants with non-constant diffusivity/thermal conductivity and time-varying coefficients. The boundary stabilization problem is converted to a problem of solving a specific Klein–Gordon-type linear hyperbolic PDE. This PDE is then solved for a family of system parameters resulting in closed-form boundary controllers. The results of a numerical simulation are presented for the case when an explicit solution is not available.

Keywords: Distributed parameter systems; Boundary control; Backstepping

1. Introduction

Methods for boundary control of linear parabolic PDEs are well established (see e.g., Curtain & Zwart, 1995; Lasiecka & Triggiani, 2000). However, even in simple cases the existing results are not explicit and require numerical solution, e.g., solving an operator Riccati equation in the case of the LQR method. In recent papers (Liu, 2003; Smyshlyaev & Krstic, 2004, 2005), a new method based on an infinite-dimensional version of the backstepping technique (Krstic, Kanellakopoulos, & Kokotovic, 1995) was introduced. By exploiting the structure, this method allows easier solution to the boundary stabilization problem and in many cases leads to closed form results.

In this paper, we further extend this approach to the parabolic 1D PDEs with space-dependent thermal conductivity/diffusivity and time-varying coefficients. The time-varying problem is novel compared to Smyshlyaev and Krstic (2004) and with respect to PDE control literature in general. The extension to non-constant diffusion makes the ideas in Smyshlyaev and Krstic (2004) applicable to inhomogeneous media.

Although only the state-feedback results are presented in this paper, it was shown by Smyshlyaev and Krstic (2005) that dual output-feedback results can be obtained. This means that every closed-form controller can be used to get a closed-form observer, and thus a closed form output feedback compensator. All the controllers in the paper can also be modified to be inverse optimal (Smyshlyaev & Krstic, 2004), i.e., minimize a cost functional that puts penalty on both state and control giving stability margins and the reduced control effort.

2. Plant with non-constant diffusion coefficient

2.1. Problem statement

Consider the following plant:

$$u_t(x,t) = \varepsilon(x)u_{xx}(x,t) + b(x)u_x(x) + \lambda(x)u(x,t), \quad (1)$$

$$u_x(0,t) = qu(0,t). \quad (2)$$

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We assume that $\varepsilon(x)>0, \forall x \in [0,1]$ and $b, \lambda \in C^1[0,1]$, $\varepsilon \in C^3[0,1]$, q is an arbitrary constant ($q=+\infty$ handles the Dirichlet case). The PDE (1)–(2) describes a wide variety of thermal/fluid systems including, but not limited to, heat conduction in non-homogeneous materials (Carslaw & Jaeger, 1959) and chemical tubular reactor (Boskovic & Krstic, 2002). The open-loop system ($u(1,t)=0$) is unstable with arbitrarily many unstable eigenvalues even in the case of the constant coefficients. The objective is to stabilize the zero solution $u\equiv 0$ by using $u(1,t)$ (Dirichlet actuation) or $u_x(1,t)$ (Neumann actuation) as a control input. We will consider only the Dirichlet actuation in this paper since the extension to the Neumann case is straightforward (Smyshlyaev & Krstic, 2004).

Without loss of generality, we assume $b(x)\equiv 0$ since it can be eliminated from the equation with the transformation

$$u(x,t)\mapsto u(x,t)\mathrm{e}^{-\int_0^x (b(\tau)/2\varepsilon(\tau))\,\mathrm{d}\tau}, \tag{3}$$

and the appropriate changes of the parameters

$$\lambda \mapsto \lambda + \frac{b'}{2} + \frac{b^2}{4\varepsilon} - \frac{b\varepsilon'}{2\varepsilon}, \quad q \mapsto q - \frac{b(0)}{2\varepsilon(0)}. \tag{4}$$

We do not consider other (integral and local) terms from Smyshlyaev and Krstic (2004) for clarity, they can be easily included and do not affect the analysis.

The main idea of our method is to use a coordinate transformation

$$w(x,t)=u(x,t)-\int_0^x k(x,y)u(y,t)\,\mathrm{d}y \tag{5}$$

to map (1) and (2) into the stable target system:

$$w_t(x,t)=\varepsilon(x)w_{xx}(x,t)-cw(x,t), \tag{6}$$

$$w_x(0,t)=qw(0,t), \tag{7}$$

$$w(1,t)=0, \tag{8}$$

where a free parameter c can be used to set the desired rate of stability. Once we find transformation (5) (namely $k(x,y)$), the boundary condition (8) gives the feedback controller in the form

$$u(1,t)=\int_0^1 k(1,y)u(y,t)\,\mathrm{d}y. \tag{9}$$

From (3) and (5), it follows that for the case of non-zero $b(x)$ the kernel is transformed in the following way:

$$k(x,y)\mapsto k(x,y)\mathrm{e}^{-\int_y^x (b(\tau)/2\varepsilon(\tau))\,\mathrm{d}\tau}. \tag{10}$$

We face two problems now: establish a stability condition for the target system (6)–(8) and find the equation for the transformation kernel $k(x,y)$ and possibly solve it.

2.2. Stability analysis

Lemma 1. *System* (6)–(8) *is exponentially stable under the condition*

$$c > \frac{\varepsilon''_{\max}}{2} + \frac{\bar{q}^2}{\varepsilon_{\min}}, \tag{11}$$

where

$$\varepsilon''_{\max} = \max_{x\in[0,1]}\{\varepsilon''(x)\}, \quad \varepsilon_{\min} = \min_{x\in[0,1]}\{\varepsilon(x)\},$$

$$\bar{q} = \max\left\{0, \frac{\varepsilon'(0)}{2} - q\varepsilon(0)\right\}. \tag{12}$$

Proof. Consider a Lyapunov function

$$V=\frac{1}{2}\int_0^1 w^2(x,t)\,\mathrm{d}x. \tag{13}$$

Using Poincare's and Agmon's inequalities, we get

$$\begin{aligned}\dot{V} &= -w_x(0,t)w(0,t)\varepsilon(0) - \int_0^1 \varepsilon'(x)ww_x\,\mathrm{d}x \\ &\quad - \int_0^1 \varepsilon(x)w_x^2\,\mathrm{d}x - c\int_0^1 w^2\,\mathrm{d}x \\ &\leqslant \bar{q}w^2(0,t) - \int_0^1 \varepsilon(x)w_x^2\,\mathrm{d}x \\ &\quad - \int_0^1 \left(c - \frac{\varepsilon''(x)}{2}\right) w^2\,\mathrm{d}x \\ &\leqslant -\int_0^1 \left(c - \frac{\varepsilon''_{\max}}{2} - \frac{\bar{q}^2}{\varepsilon_{\min}}\right) w^2\,\mathrm{d}x,\end{aligned} \tag{14}$$

which, due to (11), gives the stability result. □

Estimate (11) is rather conservative, it can be substantially improved for specific $\varepsilon(x)$.

2.3. Kernel PDE analysis

Substitution of (5) into (6)–(8) and (1)–(2) leads to the following PDE for $k(x,y)$:

$$\begin{aligned}&\varepsilon(x)k_{xx}(x,y) - (\varepsilon(y)k(x,y))_{yy} \\ &\quad = (\lambda(y)+c)k(x,y)\end{aligned} \tag{15}$$

for $0<y<x<1$ with boundary conditions

$$k_y(x,0)=(q-\varepsilon'(0)/\varepsilon(0))k(x,0), \tag{16}$$

$$2\varepsilon(x)\frac{\mathrm{d}}{\mathrm{d}x}k(x,x) = -\varepsilon'(x)k(x,x) - \lambda(x) - c, \tag{17}$$

$$k(0,0)=0. \tag{18}$$

By solving the ODE (17) and (18) with respect to $k(x,x)$ the last two conditions can be combined into one:

$$k(x,x) = -\frac{1}{2\sqrt{\varepsilon(x)}}\int_0^x \frac{(\lambda(\tau)+c)}{\sqrt{\varepsilon(\tau)}}\,\mathrm{d}\tau. \tag{19}$$

The PDE (15), (16), (19) is more complicated than the one in Smyshlyaev and Krstic (2004) since the first derivatives of k appear and the coefficients depend on both x and y. Note, that not only ε, but ε' and ε'' are also involved. Our goal now is to manipulate (15) into the form for which the analysis from Smyshlyaev and Krstic (2004) can be applied.

First, we transform this hyperbolic PDE into the canonical form by introducing the change of variables:

$$\breve{k}(\bar{x},\bar{y}) = \varepsilon(y)k(x,y), \quad \bar{x}=\varphi(x), \quad \bar{y}=\varphi(y),$$
$$\varphi(x) = \sqrt{\varepsilon(0)}\int_0^x \frac{\mathrm{d}\tau}{\sqrt{\varepsilon(\tau)}}. \tag{20}$$

In these variables the PDE (15), (16), (19) becomes

$$\breve{k}_{\bar{x}\bar{x}} - \breve{k}_{\bar{y}\bar{y}} = \frac{\varepsilon'(x)}{2\sqrt{\varepsilon(0)\varepsilon(x)}}\breve{k}_{\bar{x}} - \frac{\varepsilon'(y)}{2\sqrt{\varepsilon(0)\varepsilon(y)}}\breve{k}_{\bar{y}} + \varepsilon^{-1}(0)(\lambda(y)+c)\breve{k}, \tag{21}$$

$$\breve{k}_{\bar{y}}(\bar{x},0) = q\breve{k}(\bar{x},0), \tag{22}$$

$$\breve{k}(\bar{x},\bar{x}) = -\frac{1}{2}\sqrt{\frac{\varepsilon(x)}{\varepsilon(0)}}\int_0^{\bar{x}}(\lambda(\varphi^{-1}(\xi))+c)\,\mathrm{d}\xi. \tag{23}$$

For clarity we leave old variables in the coefficients for a while. The second step is to further simplify the equation by eliminating the terms with the first derivatives of $\breve{k}$. It is possible in this case since the coefficients in front of these terms depend only on a single variable. We introduce

$$\bar{k}(\bar{x},\bar{y}) = (\varepsilon(x)\varepsilon(y))^{-1/4}\breve{k}(\bar{x},\bar{y}), \tag{24}$$

which now satisfies the following PDE:

$$\varepsilon(0)(\bar{k}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{k}_{\bar{y}\bar{y}}(\bar{x},\bar{y})) = \bar{\lambda}(\bar{x},\bar{y})\bar{k}(\bar{x},\bar{y}), \tag{25}$$

with boundary conditions

$$\bar{k}_{\bar{y}}(\bar{x},0) = \left(q - \frac{\varepsilon'(0)}{4\varepsilon(0)}\right)\bar{k}(\bar{x},0), \tag{26}$$

$$\bar{k}(\bar{x},\bar{x}) = -\frac{1}{2\sqrt{\varepsilon(0)}}\int_0^{\bar{x}}(\lambda(\varphi^{-1}(\xi))+c)\,\mathrm{d}\xi, \tag{27}$$

where

$$\bar{\lambda}(\bar{x},\bar{y}) = \frac{3}{16}\left(\frac{\varepsilon'^2(x)}{\varepsilon(x)} - \frac{\varepsilon'^2(y)}{\varepsilon(y)}\right) + \frac{1}{4}(\varepsilon''(y)-\varepsilon''(x)) + \lambda(y) + c. \tag{28}$$

and x, y are given in (20).

We can see now from (25) to (28) that when $\varepsilon(x)$ is not a constant, there is only one qualitative change to the PDE, namely the coefficient $\bar{\lambda}(\bar{x},\bar{y})$ depends on both $\bar{x}$ and $\bar{y}$ (it depends only on $\bar{y}$ when $\varepsilon(x)=\text{const}$). In the proof of well-posedness of the PDE (25)–(27) only the bound on this coefficient is used by Smyshlyaev and Krstic (2004) and thus the same proof applies here. The closed-loop stability follows from the stability of the target system (6)–(8) (Lemma 1) along with the invertibility of transformation (5) (because of the smooth kernel $k(x,y)$, see Liu, 2003, for details). The results can be summarized in the following theorem.

Theorem 2. *The PDE* (25)–(27) *has a unique* $C^2(0<\bar{y}<\bar{x}<\varphi(1))$ *solution. For any initial condition* $u_0 \in L_2(0,1)$ *system*(1), (2), (9) *with k given by* (20), (24), (25)–(27) *has a unique classical solution* $u \in C^{2,1}((0,1)\times(0,\infty))$ *and is exponentially stable at the origin,* $u\equiv 0$, *in the* $L_2(0,1)$ *and* $H_1(0,1)$ *norms.*

3. Closed-form controllers

By extending the results of Smyshlyaev and Krstic (2004) to the case of space-dependent $\varepsilon(x)$ we open many new opportunities to find families of closed-form controllers for some classes of $\varepsilon(x)$. We consider two cases.

3.1. Plant with constant λ

Consider the following plant:

$$u_t(x,t) = \varepsilon(x)u_{xx}(x,t) + \lambda u(x,t), \tag{29}$$
$$u(0,t) = 0. \tag{30}$$

Here $\lambda=\text{const}$. The boundary condition at zero end can be Neumann or mixed as well (see Remark 1 after Theorem 3). The PDE (25)–(27) takes the form

$$\varepsilon(0)(\bar{k}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{k}_{\bar{y}\bar{y}}(\bar{x},\bar{y})) = \bar{\lambda}(\bar{x},\bar{y})\bar{k}(\bar{x},\bar{y}), \tag{31}$$
$$\bar{k}(\bar{x},0) = 0, \tag{32}$$
$$\bar{k}(\bar{x},\bar{x}) = -\frac{\lambda+c}{2\sqrt{\varepsilon(0)}}\bar{x}. \tag{33}$$

Suppose that for some constant C, we have

$$\frac{3}{16}\frac{\varepsilon'^2(x)}{\varepsilon(x)} - \frac{1}{4}\varepsilon''(x) = C. \tag{34}$$

As one can see from (28), in this case $\bar{\lambda}(\bar{x},\bar{y}) = \lambda + c = \text{const}$. The PDE (31)–(33) can be solved now in closed form (Smyshlyaev & Krstic, 2004):

$$\bar{k}(\bar{x},\bar{y}) = -\bar{y}\frac{\lambda+c}{\sqrt{\varepsilon(0)}}\frac{I_1\left(\sqrt{((\lambda+c)/\varepsilon(0))(\bar{x}^2-\bar{y}^2)}\right)}{\sqrt{((\lambda+c)/\varepsilon(0))(\bar{x}^2-\bar{y}^2)}}, \tag{35}$$

where I_1 is a modified Bessel function of order one. There are two solutions to the ODE (34). The first solution is

$$\varepsilon(x) = \varepsilon_0(x-x_0)^2, \tag{36}$$

where ε_0 and x_0 are arbitrary (not violating the condition $\varepsilon(x)>0$) constant parameters and $C=\varepsilon_0/4$.

The other solution is three-parametric and thus is more interesting:

$$\varepsilon(x) = \varepsilon_0(1+\theta_0(x-x_0)^2)^2, \tag{37}$$

where ε_0, θ_0, x_0 are arbitrary constants and $C=-\varepsilon_0\theta_0$. This solution can give a very good approximation on $x\in[0,1]$ to many functions, including (36). So, we will focus our attention on solution (37).

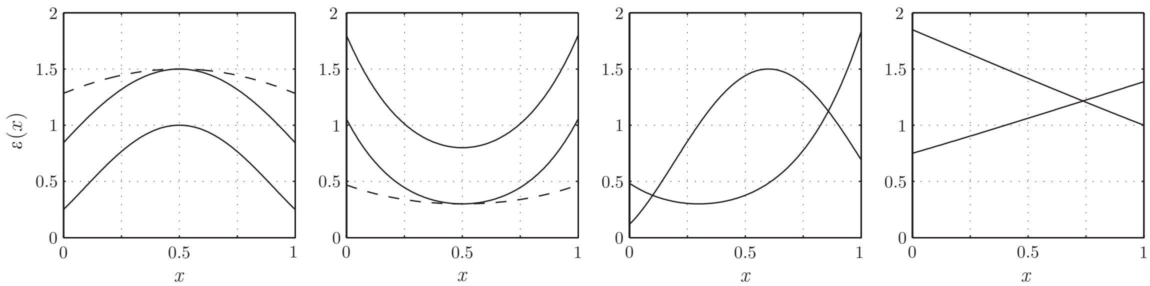

Fig. 1. The function $\varepsilon(x)$ for different values of ε_0, θ_0, and x_0. From left to right: peak value and flatness are arbitrary; extremum can be set to max or min; location of extremum is arbitrary; linear functions matched.

Function (37) always has one maximum or one minimum (for the range of the parameters that do not violate the condition $\varepsilon(x) > 0$). The value and the location of the maximum (minimum) can be arbitrarily set by ε_0 and x_0, correspondingly. The sign of θ_0 determines if it is a maximum or minimum and the value of θ_0 can set arbitrary "sharpness" of the extremum (Fig. 1 a–c). By selecting the extremum outside of the region [0,1] and changing its value and sharpness we can almost perfectly match any linear function as well (Fig. 1d).

Theorem 3. *Controller* (9) *with*

$$k(x,y) = -\bar{y}\frac{\lambda+c}{\sqrt{\varepsilon(0)}}\frac{\varepsilon^{1/4}(x)}{\varepsilon^{3/4}(y)} \times \frac{I_1\left(\sqrt{((\lambda+c)/\varepsilon(0))(\bar{x}^2-\bar{y}^2)}\right)}{\sqrt{((\lambda+c)/\varepsilon(0))(\bar{x}^2-\bar{y}^2)}}, \tag{38}$$

where $\bar{x} = \varphi(x)$, $\bar{y} = \varphi(y)$,

$$\varphi(\xi) = \frac{1+\theta_0 x_0^2}{\sqrt{\theta_0}} \times \left(\operatorname{atan}\left(\sqrt{\theta_0}(\xi - x_0)\right) + \operatorname{atan}\left(\sqrt{\theta_0}x_0\right)\right) \tag{39}$$

exponentially stabilizes the zero solution of system (29) and (30) *with* $\varepsilon(x)$ *given by* (37).

Remark 1. If the boundary condition (30) is changed to $u_x(0,t)=0$, the only change in the control gain (38) would be the leading factor $\bar{x}$ instead of $\bar{y}$. For the mixed boundary condition $u_x(0,t)=qu(0,t)$ the closed form solution is also possible and can be inferred from Smyshlyaev and Krstic (2004).

3.2. *Unstable heat equation with non-constant thermal conductivity*

Many problems (e.g., heat conduction in non-homogeneous materials (Carslaw & Jaeger, 1959)) have a structure different from that of (29). The heat equation with space-dependent thermal conductivity is usually written as

$$u_t(x,t) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\varepsilon(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x,t)\right) + \lambda u(x,t), \tag{40}$$

$$u(0,t) = 0. \tag{41}$$

With a change of variables $u = \sqrt{\varepsilon(x)}v$, we have

$$v_t = \varepsilon(x)v_{xx} + \left(\lambda + \frac{\varepsilon'^2(x)}{4\varepsilon(x)} - \frac{\varepsilon''(x)}{2}\right)v, \tag{42}$$

$$v(0,t) = 0. \tag{43}$$

One can see now that if the expression in the brackets in (42) is constant, then we can apply the results of Section 3.1. By direct substitution of solutions (36) and (37) we find that only (36) makes this expression constant (equal to λ). Using (35), (36) with (20), (24) we get the following result.

Theorem 4. *Controller* (9) *with*

$$k(x,y) = -\bar{y}\frac{\lambda+c}{\sqrt{\varepsilon(0)}}\frac{\varepsilon^{3/4}(x)}{\varepsilon^{5/4}(y)} \times \frac{I_1\left(\sqrt{((\lambda+c)/\varepsilon(0))(\bar{x}^2-\bar{y}^2)}\right)}{\sqrt{((\lambda+c)/\varepsilon(0))(\bar{x}^2-\bar{y}^2)}}, \tag{44}$$

where

$$\bar{x} = -x_0\log(1 - x/x_0), \quad \bar{y} = -x_0\log(1-y/x_0) \tag{45}$$

exponentially stabilizes the zero solution of system (40) and (41) *with* $\varepsilon(x)$ *given by* (36).

Note, that Remark 1 holds here as well. Since the minimum of function (36) is always a zero, x_0 should be chosen outside of the region [0,1] to keep $\varepsilon(x) > 0$ for $x \in [0,1]$. This means that $\varepsilon(x)$ given by (36) can approximate linear functions on [0,1] very well. In Fig. 2, the function $\varepsilon(x)$ and the corresponding control gains are shown for different parameter values.

4. Plant with time-dependent coefficients

The next natural extension of the method is including time-dependence in the equation coefficients. Consider the

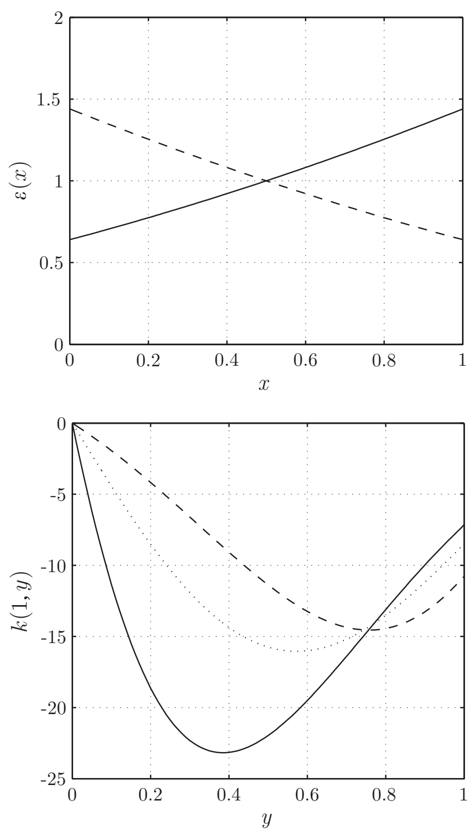

Fig. 2. The function $\varepsilon(x)$ from (36) and the corresponding kernel $k(1, y)$ for different parameter values. Dotted line shows the kernel for $\varepsilon(x) \equiv 1$.

following plant:

$$u_t(x,t) = \varepsilon_0 u_{xx}(x,t) + \lambda(x,t)u(x,t), \tag{46}$$

$$u_x(0,t) = qu(0,t), \tag{47}$$

where λ is now the continuous function of time.

The PDEs of type (46) and (47) can arise for example in the trajectory tracking problems for nonlinear distributed parameter systems. Following our approach we search for the transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y,t)u(y,t)\,\mathrm{d}y \tag{48}$$

that maps (46) and (47) into the exponentially stable target system (6)–(8) with $\varepsilon(x) = \varepsilon_0$. The difference with time-invariant case is that the transformation kernel now depends on the third variable — time. After substitution of (48) into (46) and (47), (6)–(8) we get the following kernel PDE for $k = k(x, y, t)$:

$$k_t = \varepsilon_0(k_{xx} - k_{yy}) - (\lambda(y,t) + c)k \tag{49}$$

for $0 < y < x < 1$ with boundary conditions

$$k_y(x,0,t) = qk(x,0,t), \tag{50}$$

$$k(x,x,t) = -\frac{1}{2\varepsilon_0}\int_0^x (\lambda(\xi,t) + c)\,\mathrm{d}\xi. \tag{51}$$

The stabilization problem is now converted to the problem of solvability of (49)–(51). PDEs of this type have been studied by Colton (1977) who proved that they are well-posed on a finite time interval. For the general $\lambda(x,t)$ PDE (49)–(51) needs to be solved numerically.

5. Closed-form controllers for $\lambda = \lambda(t)$

We present here explicit controllers that can stabilize the following plant with smooth $\lambda(t)$:

$$u_t(x,t) = \varepsilon_0 u_{xx}(x,t) + \lambda(t)u(x,t), \tag{52}$$

$$u(0,t) = 0. \tag{53}$$

For this system the PDE (49)–(51) takes the form

$$k_t(x,y,t) = k_{xx}(x,y,t) - k_{yy}(x,y,t) - \lambda(t)k(x,y,t), \tag{54}$$

$$k(x,0,t) = 0, \tag{55}$$

$$k(x,x,t) = -\frac{x}{2}\lambda(t). \tag{56}$$

Without loss of generality, we have set $\varepsilon_0 = 1$, $c = 0$ here. Let us make the following change of variables:

$$k(x,y,t) = -\frac{y}{2}\mathrm{e}^{-\int_0^t \lambda(\tau)\,\mathrm{d}\tau} f(z,t), \quad z = \sqrt{x^2 - y^2}. \tag{57}$$

We get the following PDE in one spatial variable for the function $f(z,t)$:

$$f_t(z,t) = f_{zz}(z,t) + 3z^{-1}f_z(z,t) \tag{58}$$

with boundary conditions

$$f_z(0,t) = 0, \quad f(0,t) = \lambda(t)\mathrm{e}^{\int_0^t \lambda(\tau)\,\mathrm{d}\tau} :=F(t). \tag{59}$$

The $C^{2,1}_{z,t}$ solution to this problem is (Polianin, 2002)

$$f(z,t) = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}\left(\frac{z}{2}\right)^{2n} F^{(n)}(t). \tag{60}$$

This solution is rather explicit. Since $z \leqslant 1$ and squared factorial increases very fast with n, one can obtain very accurate approximations to $f(z,t)$ using just several terms of the sum.

It can be shown that operator (48) from u to w, as functions of x, is bounded invertible in both $L_2(0, 1)$ and $H_1(0, 1)$, uniformly in time.

Theorem 5. *The controller*

$$u(1,t) = -\int_0^1 y\,\frac{u(y,t)}{2\mathrm{e}^{\int_0^t \lambda(\tau)\,\mathrm{d}\tau}} \times \sum_{n=0}^{\infty} \frac{(1-y^2)^n F^{(n)}(t)}{4^n n!(n+1)!}\,\mathrm{d}y \tag{61}$$

exponentially stabilizes the system (52) *and* (53).

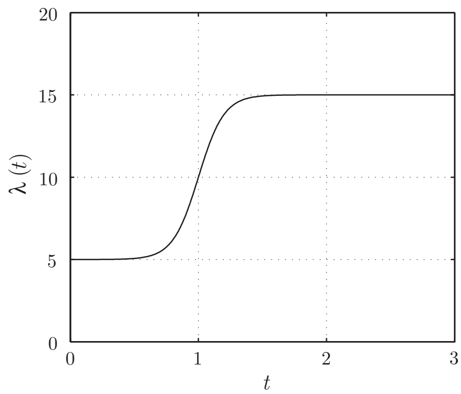

Fig. 3. $\lambda(t)$ from (63) for $\lambda_0=10$, $\omega_0=5$, and $t_0=1$.

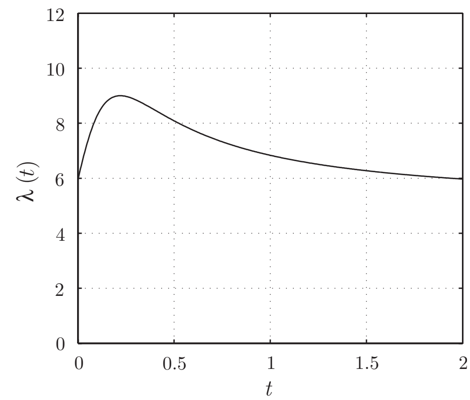

Fig. 4. $\lambda(t)$ from (66) for $\lambda_0=5$, $a=0.03$, and $b=0.25$.

Remark 2. If the boundary condition (53) is changed to $u_x(0,t)=0$, the only difference in the controller (61) would be the leading factor $(1/2)$ instead of $(y/2)$.

There are two cases when it is easy to compute the series (60) in closed form: when $F(t)$ is a combination of exponentials (since it is easy to compute the nth derivative of $F(t)$ in this case) or a polynomial (since the series is finite). Let us consider two examples.

Example 1 (*A rapid transition between two levels*). Let $F(t)$ be

$$F(t)=\mathrm{e}^{\lambda_0 t}\{\lambda_0\cosh\omega_0(t-t_0)+\sinh\omega_0(t-t_0)\}, \tag{62}$$

where λ_0, ω_0 and t_0 are arbitrary constants. This $F(t)$ corresponds to the following $\lambda(t)$:

$$\lambda(t)=\lambda_0+\omega_0\tanh(\omega_0(t-t_0)). \tag{63}$$

This $\lambda(t)$ approximates a rapid change from a constant level $\lambda_0-\omega_0$ to a constant level $\lambda_0+\omega_0$ at a time $t=t_0$ (Fig. 3). Substituting (62) into (60) and computing the sum we get the following control gain:

$$\begin{aligned} k(x,y,t) &= -\frac{y}{2\sqrt{x^2-y^2}\cosh(\omega_0(t-t_0))} \\ &\quad\times\left\{\sqrt{\lambda_0+\omega_0}\,I_1\left(\sqrt{(\lambda_0+\omega_0)(x^2-y^2)}\right)\mathrm{e}^{-\omega_0(t-t_0)}\right. \\ &\quad+\sqrt{\lambda_0-\omega_0}\,I_1\left(\sqrt{(\lambda_0-\omega_0)(x^2-y^2)}\right) \\ &\quad\left.\times\ \mathrm{e}^{\omega_0(t-t_0)}\right\}. \end{aligned} \tag{64}$$

Example 2 (*One-peak*). Let $F(t)$ be

$$F(t)=\mathrm{e}^{\lambda_0 t}(\lambda_0((t+a)^2+b^2)+2(t+a)), \tag{65}$$

where λ_0, a and $b\neq 0$ are arbitrary constants. This $F(t)$ corresponds to the following $\lambda(t)$:

$$\lambda(t)=\lambda_0+\frac{2(t+a)}{(t+a)^2+b^2}. \tag{66}$$

This $\lambda(t)$ can approximate some "one-peak" functions (Fig. 4). Substituting (65) into (60) and computing the sum we get the following control gain:

$$\begin{aligned} k(x,y,t) &= -\lambda_0 y\frac{I_1\left(\sqrt{\lambda_0}z\right)}{\sqrt{\lambda_0}z}-y\frac{t+a}{(t+a)^2+b^2}I_0\left(\sqrt{\lambda_0}z\right) \\ &\quad-\frac{y}{4\sqrt{\lambda_0}}\frac{zI_1\left(\sqrt{\lambda_0}z\right)}{(t+a)^2+b^2}, \\ z &= \sqrt{x^2-y^2}, \end{aligned} \tag{67}$$

where I_0 and I_1 are modified Bessel functions.

We should mention that there is a simpler solution to the problem of stabilization of (52) and (53) which is obtained by converting it by a change of variables $u(x,t)=v(x,t)\exp\int_0^t\lambda(\tau)\,\mathrm{d}\tau$ into a PDE with constant coefficients $v_t=\varepsilon_0 v_{xx}$, $v(0,t)=0$. This problem can be then stabilized using the results of Smyshlyaev and Krstic (2004) with the controller

$$u(1,t)=-\int_0^1\frac{c}{\varepsilon_0}y\frac{I_1\left(\sqrt{(c/\varepsilon_0)(1-y^2)}\right)}{\sqrt{(c/\varepsilon_0)(1-y^2)}}u(y,t)\,\mathrm{d}y. \tag{68}$$

The decay rate of the closed-loop v-system is equal to the decay rate of the target system, i.e., $\mathrm{e}^{-(c+\varepsilon_0\pi^2)t}$. So the closed-loop stability of u-system is guaranteed by satisfying the condition (Khalil, 1996, p. 226)

$$c>\limsup_{t\to\infty}\lambda(t)-\varepsilon_0\pi^2, \tag{69}$$

or $c>-\varepsilon_0\pi^2$ if $\lambda\in L_1(0,\infty)\cup L_2(0,\infty)$.

Although controller (68) stabilizes (52) and (53) for any $\lambda(t)$, it is most suitable for the cases when minimum and maximum values of $\lambda(t)$ are close, for example when it is a

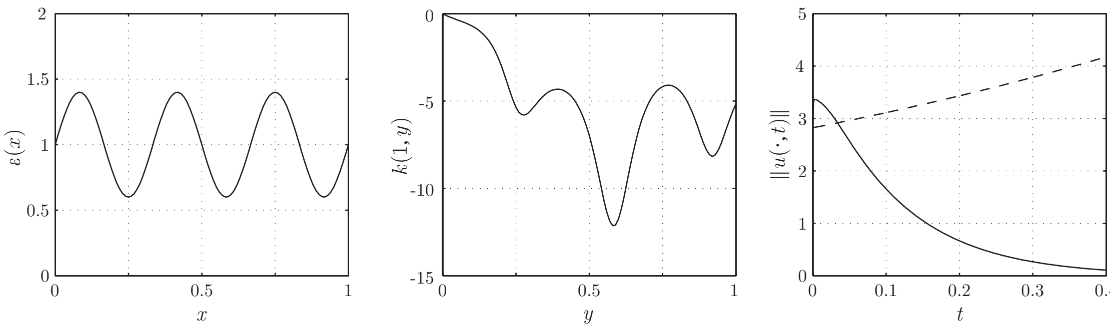

Fig. 5. The simulation results for (1) and (2) with controller (9). From left to right: $\varepsilon(x)$; the gain kernel; the L_2-norm of open loop (dashed) and closed loop (solid) response.

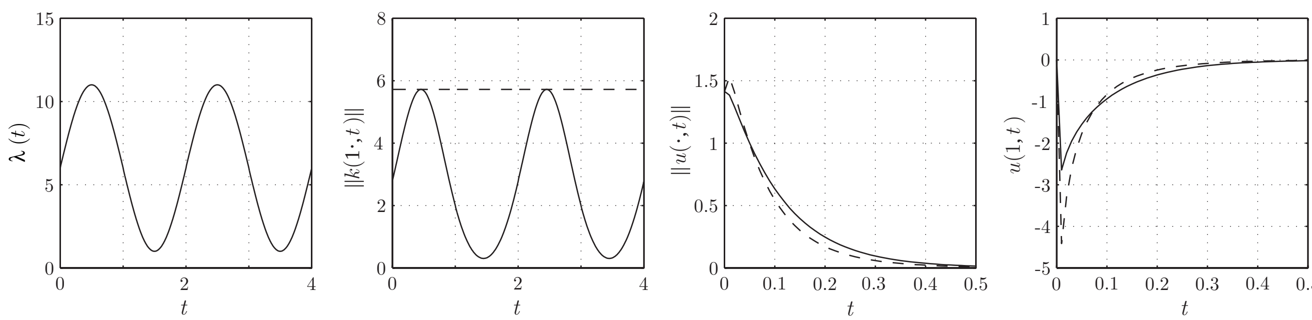

Fig. 6. The simulation results for (46) and (47) with controllers (61) (solid) and (68) (dashed). From left to right: $\lambda(t)$; L_2-norm of the gain kernel; L_2-norm of the state; the control effort.

constant plus sinusoid with small amplitude. When $\lambda(t)$ has significant drops and rises, this method will use unnecessarily large initial control effort (Example 1) or result in poor initial performance (Example 2) (see next section). For such cases design (61) is advantageous.

6. Simulations

We presented several closed-form boundary controllers for stabilization of parabolic PDEs with time- and spatial-dependent parameters. When the plant coefficients cannot be accurately approximated by families of functions for which explicit solutions can be found, the kernel PDE (15)–(19) or (49)–(51) should be solved numerically. As shown by Smyshlyaev and Krstic (2004), it is an order of magnitude easier computational problem than the problem of solving a Riccati equation. For the case of non-constant diffusivity $\varepsilon(x)=1+0.4\sin(6\pi x)$ the results of simulation are presented in Fig. 5. The controller was implemented on a coarse grid with only 6 points. For the time-dependent case of $\lambda(t) = 6+5\sin(\pi t)$ the results are shown in Fig. 6. We can see the advantage of controller (61) over (68) for this type of $\lambda(t)$—lower control effort in the initial transient.

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