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Systems & Control Letters 54 (2005) 613–625
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Backstepping observers for a class of parabolic PDEs☆

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Received 15 April 2004; received in revised form 26 October 2004; accepted 8 November 2004
Available online 29 December 2004

Abstract

In this paper we design exponentially convergent observers for a class of parabolic partial integro-differential equations (P(I)DEs) with only boundary sensing available. The problem is posed as a problem of designing an invertible coordinate transformation of the observer error system into an exponentially stable target system. Observer gain (output injection function) is shown to satisfy a well-posed hyperbolic PDE that is closely related to the hyperbolic PDE governing backstepping control gain for the state-feedback problem. For several physically relevant problems the observer gains are obtained in closed form. The observer gains are then used for an output-feedback design in both collocated and anti-collocated setting of sensor and actuator. The order of the resulting compensator can be substantially lowered without affecting stability. Explicit solutions of a closed loop system are found in particular cases.

Keywords: Parabolic PDE; Observer; Output feedback; Backstepping

1. Introduction

In this paper we propose backstepping-based infinite dimensional observers for a class of linear parabolic partial integro-differential equations with sensing restricted to the boundary.

To solve this problem we draw inspiration from a recent paper of Krener and Kang [9] in which a finite dimensional backstepping observer is proposed for nonlinear ODEs. They discover and exploit a triangular structure dual to that for the backstepping controller design [10]. The complexities present due to nonlinearities in finite dimension make the Krener–Kang observer non-global. This limitation is not an issue in our problem, as the class of parabolic PDEs we consider is linear. Our observers, due to the infinite dimension, take a form in which they are almost unrecognizable as Krener–Kang observers, however their structure is exactly that of Krener and Kang, where duality with backstepping control is exploited.

Our observer design for linear parabolic PDEs involves a linear Volterra transformation of the observer error system into a heat equation, with the aid of output injection. The transformation kernel satisfies a linear hyperbolic (Klein–Gordon type) PDE dual to the PDE studied for the state-feedback problem by

☆ This work was supported by NSF.
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doi:10.1016/j.sysconle.2004.11.001

Liu [12]. While observers are of interest on their own merit as state estimators/forecasters, putting them together with our earlier boundary stabilizers [15] yields output feedback compensators for a class of parabolic PDEs.

Past efforts in linear observer design for PDEs include the infinite dimensional Luenberger approach [7,11]. A unified treatment of both interior and boundary observations/control generalized to semilinear problems can be found in [1]. Fuji [8] and Nambu [14] developed auxiliary functional observers to stabilize diffusion equations using boundary observation and feedback. For the general Pritchard–Salamon class of state-space systems a number of frequency-domain results has been established on stabilization during the last decade (see, e.g. [6,13] for surveys). Christofides [5] developed nonlinear output-feedback controllers for parabolic PDE systems for which the eigenspectrum can be separated into a finite-dimensional slow part and an infinite-dimensional stable fast part.

While our method is certainly not the first solution to the problems of boundary observer design or output-feedback boundary control, it has several distinguishing features. First of all, it takes advantage of the structure of the system, resulting in a problem of solving a linear hyperbolic PDE for the gain kernel, an object much easier, both conceptually and computationally, than operator Riccati equations arising in LQG approaches to boundary control. Second, the problem is solved essentially by calculus making a design procedure clear and constructive and the analysis easy in contrast to standard abstract approaches (semigroups, etc.). Last but not the least, for a number of physically relevant problems we are able to find the observer/controller kernels in closed form, i.e., as explicit functions of the spatial variable. This, in turn, allows to even find closed-loop solutions explicitly.

The paper is organized as follows. In Section 2 we introduce a class of parabolic PDEs and formulate the problem. The observers for anti-collocated and collocated sensor/actuator pairs are designed in Sections 3 and 4, respectively. In Section 5 the observers are combined with backstepping controllers to obtain a solution to the output-feedback problem. For certain classes of PDEs the observers and compensators are constructed explicitly in Section 6.

2. Problem statement

We consider the following class of parabolic PDEs:

$$\begin{aligned} u_t(x,t) = \varepsilon u_{xx}(x,t) + b(x)u_x(x,t) \\ + \lambda(x)u(x,t) + g(x)u(0,t) \\ + \int_0^x f(x,y)u(y,t)\,\mathrm{d}y, \end{aligned} \tag{1}$$

for $x \in (0,1)$, $t>0$, with boundary conditions:[1]

$$u_x(0,t) = qu(0,t), \tag{2}$$

$$u(1,t) = U(t) \quad \text{or} \quad u_x(1,t) = U(t) \tag{3}$$

and under the assumption

$$\begin{aligned} &\varepsilon > 0, \quad q \in \mathbb{R}, \quad \lambda, g \in C^1[0,1], \\ &f \in C^1([0,1]\times[0,1]). \end{aligned} \tag{4}$$

Without loss of generality we can set $b(x) \equiv 0$, since it can be eliminated from the equation with the transformation

$$u(x,t) \mapsto u(x,t)\mathrm{e}^{-(1/2\varepsilon)\int_0^x b(\tau)\,\mathrm{d}\tau} \tag{5}$$

and the appropriate changes of parameters q, $\lambda(x)$, $g(x)$, and $f(x,y)$.

The PDE (1)–(2) is actuated at $x=1$ (using either Dirichlet or Neumann actuation) by a boundary input $U(t)$ that can be any function of time or a feedback law.

The problem is to design an exponentially convergent observer for the plant with only boundary measurements available. The observer design depends on the type (Dirichlet/Neumann) and the location of measurement and actuation. We consider two setups: the anti-collocated setup, when sensor and actuator are placed at the opposite ends, and the collocated case, when sensor and actuator are placed at the same end. There is not much technical difference between the cases of Dirichlet and Neumann actuation, so we pick one (Neumann) for anti-collocated case and the other (Dirichlet) for collocated case. We use the backstepping state-feedback results of [15], therefore we put them into the appendix for easy reference.

[1] The case of Dirichlet boundary condition at the zero end can be handled by setting $q=+\infty$.

3. Observer design for anti-collocated setup

Suppose the only available measurement of our system is at $x=0$, the opposite end to actuation. We propose the following observer for system (1)–(3) with Dirichlet actuation:

$$\hat{u}_t(x,t) = \varepsilon \hat{u}_{xx}(x,t) + \lambda(x)\hat{u}(x,t) + g(x)u(0,t) + \int_0^x f(x,y)\hat{u}(y,t)\,\mathrm{d}y + p_1(x)[u(0,t) - \hat{u}(0,t)], \tag{6}$$

$$\hat{u}_x(0,t) = qu(0,t) + p_{10}[u(0,t) - \hat{u}(0,t)], \tag{7}$$

$$\hat{u}(1,t) = U(t). \tag{8}$$

Here $p_1(x)$ and p_{10} are output injection functions (p_{10} is a constant) *to be designed*. Note that we introduce output injection not only in Eq. (6) but also at the boundary where measurement is available. We also implicitly use the additional output injection here in a form $q(u(0,t) - \hat{u}(0,t))$ that cancels the dependency on q in the error dynamics.

Observer (6)–(8) is in the standard form of "copy of the system plus injection of the output estimation error," i.e., it mimics the finite-dimensional case where observers of the form $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$ are used for plants $\dot{x} = Ax + Bu,\ \ y = Cx$. This standard form allows us to pursue duality between the observer and the controller design—to find the observer gain function using the solution to the stabilization problem we found in [15], similar to the way duality is used to find the gains of a Luenberger observer based on the pole placement control algorithm, or to the way duality is used to construct a Kalman filter based on the LQR design.

The observer error $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$ satisfies the following PDE:

$$\tilde{u}_t(x,t) = \varepsilon \tilde{u}_{xx}(x,t) + \lambda(x)\tilde{u}(x,t) + \int_0^x f(x,y)\tilde{u}(y,t)\,\mathrm{d}y - p_1(x)\tilde{u}(0,t), \tag{9}$$

$$\tilde{u}_x(0,t) = -p_{10}\tilde{u}(0,t), \tag{10}$$

$$\tilde{u}(1,t) = 0. \tag{11}$$

Observer gains $p_1(x)$ and p_{10} should be now chosen to stabilize system (9)–(11). We solve the problem of stabilization of (9)–(11) by the same integral transformation approach as the (state feedback) boundary *control* problem reviewed in Appendix A. We look for a backstepping-like coordinate transformation

$$\tilde{u}(x,t) = \tilde{w}(x,t) - \int_0^x p(x,y)\tilde{w}(y,t)\,\mathrm{d}y \tag{12}$$

that transforms system (9)–(11) into the exponentially stable (for $\tilde{c} \geqslant 0$) system

$$\tilde{w}_t(x,t) = \varepsilon \tilde{w}_{xx}(x,t) - \tilde{c}\tilde{w}(x,t), \quad x \in (0,1), \tag{13}$$

$$\tilde{w}_x(0,t) = 0, \tag{14}$$

$$\tilde{w}(1,t) = 0. \tag{15}$$

The free parameter $\tilde{c}$ can be used to set the desired observer convergence speed. It is in general different from the analogous coefficient c in control design since one usually wants the estimator to be faster than the state feedback closed-loop dynamics.

By substituting (12) into (9)–(11) we obtain a set of conditions on the kernel $p(x,y)$ in the form of the hyperbolic PDE

$$\varepsilon p_{yy}(x,y) - \varepsilon p_{xx}(x,y) = (\lambda(x) + c)p(x,y) - f(x,y) + \int_y^x p(\xi,y)f(x,\xi)\,\mathrm{d}\xi, \tag{16}$$

for $(x,y) \in \mathscr{T} = \{x, y : 0 < y < x < 1\}$, with the boundary conditions

$$\frac{\mathrm{d}}{\mathrm{d}x} p(x,x) = \frac{1}{2\varepsilon}(\lambda(x) + c), \tag{17}$$

$$p(1,y) = 0 \tag{18}$$

that yield

$$\tilde{w}_t(x,t) = \varepsilon \tilde{w}_{xx}(x,t) - \tilde{c}\tilde{w}(x,t) - \varepsilon p(x,0)\tilde{w}_x(0,t) + (\varepsilon p_y(x,0) - p_1(x))\tilde{w}(0,t), \tag{19}$$

$$\tilde{w}_x(0,t) = (p(0,0) - p_{10})\tilde{w}(0,t), \tag{20}$$

$$\tilde{w}(1,t) = 0. \tag{21}$$

Comparing this with (13)–(15), it follows that the observer gains should be chosen as

$$p_1(x) = \varepsilon p_y(x,0), \quad p_{10} = p(0,0). \tag{22}$$

The problem is first to prove that PDE (16)–(18) is well-posed. Once the solution $p(x,y)$ to the problem (16)–(18) is found, the observer gains can be obtained from (22).

Let us make a change of variables

$$\breve{x}=1-y, \quad \breve{y}=1-x, \quad \breve{\lambda}(\breve{y})=\lambda(x),$$
$$\breve{f}(\breve{x},\breve{y})=f(x,y), \quad \breve{p}(\breve{x},\breve{y})=p(x,y). \tag{23}$$

In these new variables problem (16)–(18) becomes

$$\varepsilon\breve{p}_{\breve{x}\breve{x}}(\breve{x},\breve{y})-\varepsilon\breve{p}_{\breve{y}\breve{y}}(\breve{x},\breve{y}) = (\breve{\lambda}(\breve{y})+\tilde{c})\breve{p}(\breve{x},\breve{y})-\breve{f}(\breve{x},\breve{y}) + \int_{\breve{y}}^{\breve{x}} \breve{p}(\breve{x},\xi)\breve{f}(\xi,\breve{y})\,\mathrm{d}\xi, \quad (\breve{x},\breve{y})\in\mathscr{T}, \tag{24}$$

$$\breve{p}(\breve{x},0)=0, \tag{25}$$

$$\breve{p}(\breve{x},\breve{x})=-\frac{1}{2\varepsilon}\int_0^{\breve{x}} (\breve{\lambda}(\xi)+\tilde{c})\,\mathrm{d}\xi. \tag{26}$$

This PDE is in class (A.7)–(A.9) from Appendix A (with $q=\infty$, $g(x)=0$, c replaced by $\tilde{c}$, λ replaced by $\breve{\lambda}$, and f replaced by $\breve{f}$). Hence, using Theorem A.1 we obtain the following result.

Theorem 1. *Eq.* (16) *with boundary conditions* (17)–(18) *has a unique $C^2(\mathscr{T})$ solution. The kernel $r(x,y)$ of the inverse transformation*

$$\tilde{w}(x,t)=\tilde{u}(x,t)+\int_0^x r(x,y)\tilde{u}(y,t)\,\mathrm{d}y \tag{27}$$

is also a unique $C^2(\mathscr{T})$ function.

The fact that the observer gain in transposed and switched variables satisfies the same class of PDEs as control gain is reminiscent of the duality property of state-feedback and observer design problems for linear finite-dimensional systems. The difference between the equations for observer and control gains is due to the fact that the observer error system does not contain terms with $g(x)$ and q because $u(0,t)$ is measured.

The observer gains in the new coordinates are given by

$$p_1(x)=-\varepsilon\breve{p}_{\breve{x}}(1,1-x), \quad p_{10}=\breve{p}(1,1). \tag{28}$$

The exponential stability of the target system (13)–(15) and invertibility of transformation (12) (established in Theorem 1) imply the exponential stability of (9)–(11) both in L_2 and H_1 (see [2,12] for details and references). The result is formulated in the following theorem.

Theorem 2. *Let $p(x,y)$ be the solution of system* (16)–(18). *Then for any $\tilde{u}_0(x)\in L_2(0,1)$ system* (9)–(11) *with $p_1(x)$ and p_{10} given by Eq.* (22) *has a unique classical solution $\tilde{u}(x,t)\in C^{2,1}((0,1)\times(0,\infty))$. Additionally, the origin $\tilde{u}(x,t)\equiv 0$ is exponentially stable in the $L_2(0,1)$ and $H_1(0,1)$ norms.*

This result can be readily extended to the Neumann type of actuation as well. All the computational issues related to solving (16)–(18) numerically (for cases when closed form solutions can be obtained see Section 6) are addressed in [15].

4. Observer design for collocated setup

Suppose now that the only available measurement is at the same end with actuation ($x=1$). We will concentrate on the case with $u(1,t)$ measured and $u_x(1,t)$ actuated which is the usual setting for thermal/chemical problems (temperature/concentration is available and the gradients are used for actuation). It is quite straightforward to adapt the design to the opposite setting which usually occurs in fluid problems (shear stress is measured and velocity is a control variable).

We solve this problem with a restriction on class (1)–(2) by setting $f(x,y)\equiv 0$, $g(x)\equiv 0$. This restriction is necessary because the observer problem in the collocated case is "upper-triangular," thus the "lower-triangular" terms with $g(x)$ and $f(x,y)$ are not allowed.

Consider the following observer:

$$\hat{u}_t(x,t)=\varepsilon\hat{u}_{xx}(x,t)+\lambda(x)\hat{u}(x,t) + p_1(x)[u(1,t)-\hat{u}(1,t)], \tag{29}$$

$$\hat{u}_x(0,t)=q\hat{u}(0,t), \tag{30}$$

$$\hat{u}_x(1,t)=-p_{10}[u(1,t)-\hat{u}(1,t)]+U(t). \tag{31}$$

Here $p_1(x)$ and p_{10} are output injection functions *to be designed*. The difference with the anti-collocated case (apart from injecting $u(1,t)$ instead of $u(0,t)$) is that the gain p_{10} is introduced in the other boundary condition.

The observer error $\tilde{u}(x)$ satisfies the equation

$$\tilde{u}_t(x,t) = \varepsilon\tilde{u}_{xx}(x,t) + \lambda(x)\tilde{u}(x,t) - p_1(x)\tilde{u}(1,t), \tag{32}$$

$$\tilde{u}_x(0,t) = q\tilde{u}(0,t), \tag{33}$$

$$\tilde{u}_x(1,t) = p_{10}\tilde{u}(1,t). \tag{34}$$

We are looking for the transformation:

$$\tilde{u}(x,t) = \tilde{w}(x,t) - \int_x^1 p(x,y)\tilde{w}(y,t)\,\mathrm{d}y \tag{35}$$

that transforms (32)–(34) into the exponentially stable (for $\tilde{c} \geqslant \varepsilon/2 + \max\{0, -\varepsilon q|q|\}$) target system

$$\tilde{w}_t(x,t) = \varepsilon\tilde{w}_{xx}(x,t) - \tilde{c}\tilde{w}(x,t), \quad x \in (0,1), \tag{36}$$

$$\tilde{w}_x(0,t) = q\tilde{w}(0,t), \tag{37}$$

$$\tilde{w}_x(1,t) = 0. \tag{38}$$

Note, that transformation (35) is in upper-triangular form. By substituting (35) into (32)–(34) we get the set of conditions on the kernel $p(x,y)$ in the form of hyperbolic PDE

$$\varepsilon p_{yy}(x,y) - \varepsilon p_{xx}(x,y) = (\lambda(x) + \tilde{c})p(x,y) \tag{39}$$

with the boundary conditions

$$p_x(0,y) = qp(0,y), \tag{40}$$

$$p(x,x) = -\frac{1}{2\varepsilon}\int_0^x (\lambda(\xi) + \tilde{c})\,\mathrm{d}\xi \tag{41}$$

that yield

$$\begin{aligned}\tilde{w}_t(x,t) = {} & \varepsilon\tilde{w}_{xx}(x,t) - \tilde{c}\tilde{w}(x,t) \\ & + \varepsilon p(x,1)\tilde{w}_x(1,t) \\ & - (\varepsilon p_y(x,1) + p_1(x))\tilde{w}(1,t),\end{aligned} \tag{42}$$

$$\tilde{w}_x(0,t) = q\tilde{w}(0,t), \tag{43}$$

$$\tilde{w}_x(1,t) = (p_{10} - p(1,1))\tilde{w}(1,t). \tag{44}$$

Comparing this with (36)–(38), it follows that the observer gains should be chosen as

$$p_1(x) = -\varepsilon p_y(x,1), \quad p_{10} = p(1,1). \tag{45}$$

Once the solution $p(x,y)$ to problem (39)–(41) is found, the observer gains can be obtained from (45). Similar to the anti-collocated case we introduce new variables

$$\breve{x} = y, \quad \breve{y} = x, \quad \breve{p}(\breve{x},\breve{y}) = p(x,y), \tag{46}$$

in which (39)–(41) becomes

$$\varepsilon\breve{p}_{\breve{x}\breve{x}}(\breve{x},\breve{y}) - \varepsilon\breve{p}_{\breve{y}\breve{y}}(\breve{x},\breve{y}) = (\lambda(\breve{y}) + \tilde{c})\breve{p}(\breve{x},\breve{y}), \quad (\breve{x},\breve{y}) \in \mathscr{T} \tag{47}$$

$$\breve{p}_{\breve{y}}(\breve{x},0) = q\breve{p}(\breve{x},0), \tag{48}$$

$$\breve{p}(\breve{x},\breve{x}) = -\frac{1}{2\varepsilon}\int_0^{\breve{x}} (\lambda(\xi) + \tilde{c})\,\mathrm{d}\xi, \tag{49}$$

This is exactly the same PDE as (A.7)–(A.9) for $k(\breve{x},\breve{y})$ (with c replaced by $\tilde{c}$) and therefore the existence and uniqueness of the solution of (39)–(41) and invertibility of transformation (35) immediately follow. The duality between the observer and control design is even more evident here than in the anti-collocated case: the kernel of the observer transformation (35) is equal to the kernel of the control transformation (A.1) with switched variables, $p(x,y) = k(y,x)$ (for the same rate of convergence, i.e., $\tilde{c} = c$). The observer gains in the new coordinates are given by

$$p_1(x) = -\varepsilon\tilde{p}_x(1,x), \quad p_{10} = \tilde{p}(1,1). \tag{50}$$

For $\tilde{c} = c$ these gains are equal (up to a constant factor $-\varepsilon$) to the control gains.

A kernel well posedness result similar to Theorem 1 holds here. By similar argument to one for the anti-collocated case we obtain the following result.

Theorem 3. *Let $p(x,y)$ be the solution of system* (39)–(41). *Then for any $\tilde{u}_0(x) \in L_2(0,1)$ system* (32)–(34) *with $p_1(x)$ and p_{10} given by* (45) *has a unique classical solution $\tilde{u}(x,t) \in C^{2,1}((0,1) \times (0,\infty))$. Additionally, the origin $\tilde{u}(x,t) \equiv 0$ is exponentially stable in the $L_2(0,1)$ and $H_1(0,1)$ norms.*

5. Output feedback control laws

The exponentially convergent observers developed in previous sections are independent of the control input and can be used with any controller. In this section we combine these observers with their natural dual controllers—backstepping controllers—to solve the output-feedback problem fully by backstepping.

5.1. Anti-collocated setup

Theorem 4. *Let* $k_1(x)$ *be the solution of* (A.5), (A.7)–(A.9), $p_1(x)$, p_{10} *be the solutions of* (16)–(18), (22) *and let the assumptions* (4), $\tilde{c}\geqslant 0$, *and* $c\geqslant \max\{0,-\varepsilon q|q|\}$ *hold. Then for any* u_0, $\hat{u}_0\in L_2(0,1)$ *the system consisting of plant* (1)–(2), *the controller*

$$u(1,t)=\int_0^1 k_1(y)\hat{u}(y,t)\,\mathrm{d}y \tag{51}$$

and the observer

$$\hat{u}_t(x,t)=\varepsilon\hat{u}_{xx}(x,t)+\lambda(x)\hat{u}(x,t)+g(x)u(0,t)+\int_0^x f(x,y)\hat{u}(y,t)\,\mathrm{d}y+p_1(x)[u(0,t)-\hat{u}(0,t)], \tag{52}$$

$$\hat{u}_x(0,t)=qu(0,t)+p_{10}[u(0,t)-\hat{u}(0,t)], \tag{53}$$

$$\hat{u}(1,t)=\int_0^1 k_1(y)\hat{u}(y,t)\,\mathrm{d}y \tag{54}$$

has a unique classical solution $u(x,t)$, $\hat{u}(x,t)\in C^{2,1}((0,1)\times(0,\infty))$ *and is exponentially stable at the origin,* $u(x,t)\equiv 0$, $\hat{u}(x,t)\equiv 0$, *in the* $L_2(0,1)$ *and* $H_1(0,1)$ *norms.*

Proof. The coordinate transformation

$$\hat{w}(x,t)=\hat{u}(x,t)-\int_0^x k(x,y)\hat{u}(y,t)\,\mathrm{d}y \tag{55}$$

maps (52)–(54) into the system

$$\hat{w}_t(x,t)=\varepsilon\hat{w}_{xx}(x,t)-c\hat{w}(x,t)+\left\{p_1(x)+g(x)-\int_0^x k(x,y)(p_1(y)+g(y))\,\mathrm{d}y\right\}\tilde{w}(0,t), \tag{56}$$

$$\hat{w}_x(0,t)=q\hat{w}(0,t)+(p_{10}+q)\tilde{w}(0,t), \tag{57}$$

$$\hat{w}(1,t)=0. \tag{58}$$

The $\tilde{w}$-system (13)–(15) and the homogeneous part of the $\hat{w}$-system (56)–(58) (without $\tilde{w}(0,t)$, where $\tilde{w}(0,t)$ is driving the $\hat{w}$-system (56)–(57) through a C^1 function of x) are exponentially stable heat equations. The interconnection of the two heat equations $(\hat{w},\tilde{w})$ is a cascade, and therefore the combined $(\hat{w},\tilde{w})$ system is exponentially stable in L^2 and H^1. Hence, the system $(\hat{u},\tilde{u})$ is also exponentially stable since it is related to $(\hat{w},\tilde{w})$ by the invertible coordinate transformation (12) and (55).This directly implies the closed-loop stability of $(u,\hat{u})$. □

5.2. Collocated setup

Theorem 5. *Let* $k_1(x)$, $k_2(x)$ *be the solutions of* (A.5)–(A.9), $p_1(x)$, p_{10} *be the solutions of* (39)–(41), (45) *and let the assumptions* (4) *and* $\tilde{c}$, $c\geqslant\varepsilon/2+\max\{0,-\varepsilon q|q|\}$ *hold. Then for any* u_0, $\hat{u}_0\in L_2(0,1)$ *the system consisting of plant* (1)–(2) ($g(x)\equiv 0$, $f(x,y)\equiv 0$), *the controller*

$$u_x(1,t)=k_1(1)u(1,t)+\int_0^1 k_2(y)\hat{u}(y,t)\,\mathrm{d}y \tag{59}$$

and the observer

$$\hat{u}_t(x,t)=\varepsilon\hat{u}_{xx}(x,t)+\lambda(x)\hat{u}(x,t)+p_1(x)[u(1,t)-\hat{u}(1,t)], \tag{60}$$

$$\hat{u}_x(0,t)=q\hat{u}(0,t), \tag{61}$$

$$\hat{u}_x(1,t)=k_1(1)u(1,t)+\int_0^1 k_2(y)\hat{u}(y,t)\,\mathrm{d}y, \tag{62}$$

has a unique classical solution $u(x,t)$, $\hat{u}(x,t)\in C^{2,1}((0,1)\times(0,\infty))$ *and is exponentially stable at the origin,* $u(x,t)\equiv 0$, $\hat{u}(x,t)\equiv 0$, *in the* $L_2(0,1)$ *and* $H_1(0,1)$ *norms.*

Proof. Very similar to the proof of Theorem 4. □

6. Explicit construction

For some classes of systems our approach gives explicit observers and output feedbacks which is not the case with existing methods. In this section we present several important cases.[2]

[2] For the sake of notational simplicity we set $\tilde{c}=c$ in this section.

6.1. Unstable heat equation

6.1.1. Observer design

Consider the unstable heat equation with boundary actuation and sensing:[3]

$$u_t = \varepsilon u_{xx} + \lambda_0 u, \tag{63}$$

$$u_x(0) = 0, \tag{64}$$

$$u(1) = U(t). \tag{65}$$

The open-loop system (63)–(64) (with $U = 0$) is unstable with arbitrarily many unstable eigenvalues.

Let us consider the anti-collocated setup. Eqs. (24)–(26) for the observer gain takes the form

$$\breve{p}_{\breve{x}\breve{x}}(\breve{x}, \breve{y}) - \breve{p}_{\breve{y}\breve{y}}(\breve{x}, \breve{y}) = \lambda \breve{p}(\breve{x}, \breve{y}), \tag{66}$$

$$\breve{p}(\breve{x}, 0) = 0, \tag{67}$$

$$\breve{p}(\breve{x}, \breve{x}) = -\lambda \frac{\breve{x}}{2}, \tag{68}$$

where $\lambda = (\lambda_0 + c)/\varepsilon$. The solution to (66)–(68) is [15]

$$\breve{p}(\breve{x}, \breve{y}) = -\lambda \breve{y} \frac{I_1(\sqrt{\lambda(\breve{x}^2 - \breve{y}^2)})}{\sqrt{\lambda(\breve{x}^2 - \breve{y}^2)}}. \tag{69}$$

I_1 is the modified Bessel function of the first order. Using (28) we obtain the observer gains

$$p_1(x) = \varepsilon \frac{\lambda(1-x)}{x(2-x)} I_2\left(\sqrt{\lambda x(2-x)}\right),$$
$$p_{10} = -\lambda/2. \tag{70}$$

In Fig. 1 the observer gain $p_1(x)$ is shown for different values of the parameter λ. The exponential convergence of the observer for $\lambda = 5$ is illustrated in Fig. 2. We can see that observer converges to the plant even though the plant is unstable.

6.1.2. Output feedback compensator

We can now write the explicit solution to the output-feedback problem. The gain kernel for the state-feedback problem has been found in [15] by solving (A.7)–(A.9) analytically:

$$k(x, y) = -\lambda x \frac{I_1(\sqrt{\lambda(x^2 - y^2)})}{\sqrt{\lambda(x^2 - y^2)}}. \tag{71}$$

[3] Throughout this section we drop (x, t)-dependence for clarity wherever it is possible, so $u(0, t) = u(0)$, $u(x, t) = u$, etc.

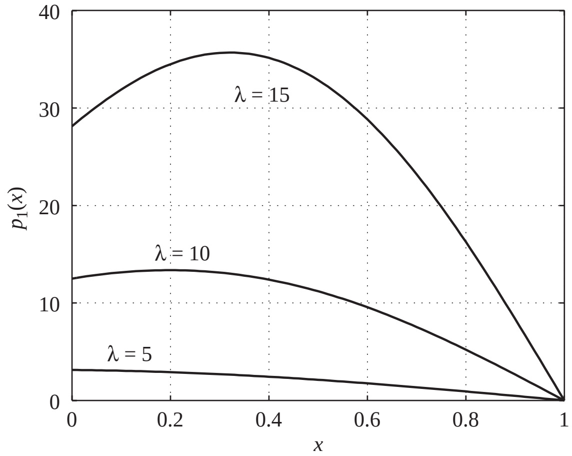

Fig. 1. Observer gain for the unstable heat equation.

Using (71), (70), and Theorem 4 we get the following result.

Theorem 6. *The controller*

$$u(1) = -\int_0^1 \lambda \frac{I_1(\sqrt{\lambda(1-y^2)})}{\sqrt{\lambda(1-y^2)}} \hat{u}(y)\, dy \tag{72}$$

with the observer

$$\hat{u}_t = \varepsilon \hat{u}_{xx} + \lambda_0 \hat{u} + \varepsilon \frac{\lambda(1-x)}{x(2-x)} I_2\left(\sqrt{\lambda x(2-x)}\right) \times [u(0) - \hat{u}(0)], \tag{73}$$

$$\hat{u}_x(0) = -\frac{\lambda}{2}[u(0) - \hat{u}(0)], \tag{74}$$

$$\hat{u}(1) = -\int_0^1 \lambda \frac{I_1(\sqrt{\lambda(1-y^2)})}{\sqrt{\lambda(1-y^2)}} \hat{u}(y)\, dy \tag{75}$$

stabilizes the zero solution of system (63)–(64).

The above result can be easily extended for Neumann type of actuation.

The closed-loop system has been simulated with $\varepsilon = 1$, $\lambda_0 = 10$, $c = 5$, $u(x, 0) = 2e^{-2x} \sin(\pi x)$. With this choice of parameters the open-loop system has two unstable eigenvalues. The plant and the observer are discretized using a finite difference method. Since designs exist where, in principle, the order of the observer can be as low as the number of unstable eigenvalues, we design the low-order compensator by

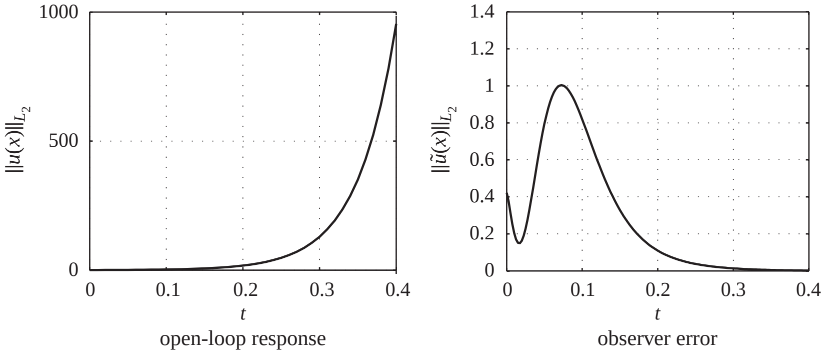

Fig. 2. Exponential convergence of the observer for the unstable heat equation.

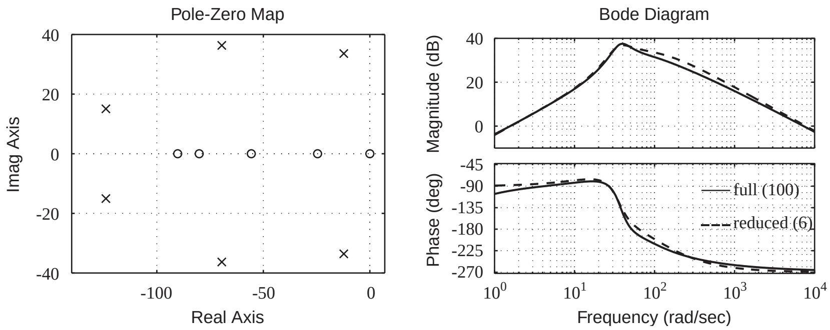

Fig. 3. Pole-zero map and Bode plot of the compensator for unstable heat equation.

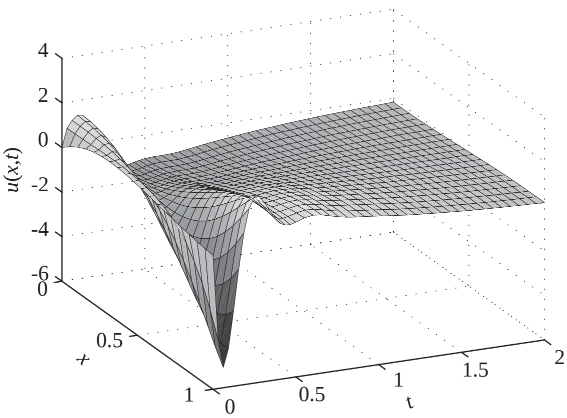

Fig. 4. Closed-loop response with a low-order compensator.

taking a coarse 6-point grid (keeping the fine discretization of the plant, 100 points in our case, for simulation). In Fig. 3, the pole-zero map and Bode plots of the low-order compensator are shown. The reduced order compensator is able to stabilize the system (Fig. 4).

6.1.3. Closed-loop solution

With every part of our design being explicit we can even write the closed loop solution of system (63)–(64) together with compensator (72)–(75) explicitly, in terms of the initial conditions $u_0(x)$, $\hat{u}_0(x)$.

Theorem 7. *The solution to the closed loop system* (63)–(64), (72)–(75) *is*

$$u(x,t) = 2\sum_{n=0}^{\infty} \mathrm{e}^{-(c+\mu_n^2)t} \cos\sqrt{\lambda+\mu_n^2}\,x \times \left\{ \int_0^1 \psi_n(\xi)u_0(\xi)\,\mathrm{d}\xi + \mu_n(-1)^n \times \left(C_n t + \sum_{m\neq n}^{\infty} C_m \frac{1-\mathrm{e}^{(\mu_n^2-\mu_m^2)t}}{\mu_n^2-\mu_m^2} \right) \right\}, \quad (76)$$

where $\mu_n = \pi(n + 1/2)$,

$$C_n = 2\left(\int_0^1 \lambda \frac{I_1(\sqrt{\lambda\xi(2-\xi)})}{\sqrt{\lambda\xi(2-\xi)}}\,\psi_n'(\xi)\,\mathrm{d}\xi\right) \times \left(\int_0^1 \frac{\sin\sqrt{\lambda+\mu_n^2}(1-\xi)}{\sqrt{\lambda+\mu_n^2}}(u_0(\xi) - \hat{u}_0(\xi))\,\mathrm{d}\xi\right), \tag{77}$$

$$\psi_n(x) = \cos(\mu_n x) + \int_0^x \lambda\xi \frac{I_1(\sqrt{\lambda(x^2-\xi^2)})}{\sqrt{\lambda(x^2-\xi^2)}} \times \cos(\mu_n\xi)\,\mathrm{d}\xi. \tag{78}$$

Proof. We set $\varepsilon = 1$ for simplicity. We start by solving the damped heat equation (13)–(15):

$$\tilde{w}(x,t) = 2\sum_{n=0}^{\infty} \mathrm{e}^{-(c+\mu_n^2)t}\cos(\mu_n x) \times \int_0^1 \tilde{w}_0(\xi)\cos(\mu_n\xi)\,\mathrm{d}\xi,$$
$$\mu_n = \pi\left(n + \frac{1}{2}\right). \tag{79}$$

The initial condition $\tilde{w}_0$ can be calculated explicitly from $\tilde{u}_0$ via (27). Substituting the result into (12), changing the order of integration, and calculating some of the integrals we obtain

$$\tilde{u}(x,t) = 2\sum_{n=0}^{\infty} \mathrm{e}^{-(c+\mu_n^2)t}\frac{\mu_n}{\lambda+\mu_n^2}\sin\sqrt{\lambda+\mu_n^2}(1-x) \times \int_0^1\left(\sin\mu_n\xi + \int_\xi^1 \lambda\xi\frac{I_1(\sqrt{y^2-\xi^2})}{\sqrt{y^2-\xi^2}} \times \sin\mu_n y\,\mathrm{d}y\right)\tilde{u}_0(1-\xi)\,\mathrm{d}\xi. \tag{80}$$

Now we solve the controller target system

$$w_t(x,t) = w_{xx}(x,t) - cw(x,t), \tag{81}$$

$$w_x(0,t) = 0,$$
$$w(1,t) = -\int_0^1 k(1,y)\tilde{u}(y)\,\mathrm{d}y \equiv \mathrm{d}(t), \tag{82}$$

where the boundary condition $w(1,t)$ appears due to output-feedback instead of state-feedback:

$$w(x,t) = 2\sum_{n=0}^{\infty}\mathrm{e}^{-(c+\mu_n^2)t}\cos(\mu_n x) \times \left(\int_0^1 w_0(\xi)\cos(\mu_n\xi)\,\mathrm{d}\xi + (-1)^n\mu_n\int_0^t \mathrm{e}^{(c+\mu_n^2)\tau}\,\mathrm{d}(\tau)\,\mathrm{d}\tau\right). \tag{83}$$

The initial condition w_0 can be calculated explicitly from u_0 via (A.1). Substituting (83) into the inverse transformation (A.10) and calculating the integrals we obtain (76)–(78). □

6.2. Chemical tubular reactor

Another case in which we can find the explicit gains is the heat equation with a non-constant coefficient:

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \lambda_{\alpha\beta}(x)u(x,t),$$
$$x \in (0,1), \tag{84}$$

$$u(0,t) = 0, \tag{85}$$

where

$$\lambda_{\alpha\beta}(x) = \frac{2\varepsilon\alpha^2}{\cosh^2(\alpha x - \beta)}. \tag{86}$$

The coefficient $\lambda_{\alpha\beta}(x)$ parameterizes a family of "one-peak" functions. The free parameters α and β are chosen so that the maximum of $\lambda_{\alpha\beta}(x)$ is $2\alpha^2$ and is achieved at $x = \beta/\alpha$. Examples of $\lambda_{\alpha\beta}(x)$ for different values of α and β are shown in Fig. 5. Equations of the form (84)–(85) often describe the heat/mass transfer systems with heat generation or volumetric chemical reactions, for example chemical tubular reactor (see [3] and references therein) . The open-loop system (84)–(85) (with $u(1) = 0$) is unstable for all three cases shown in Fig. 5.

Since the plant is in the diagonal form (there are no terms with $g(x)$ and $f(x,y)$), we choose to collocate the sensor and the actuator at $x = 1$. Following our approach we easily get the following result.

Theorem 8. *The controller*

$$u_x(1,t) = -\alpha(\tanh\beta - \tanh(\beta-\alpha))u(1) + \int_0^1 k_2(y)\hat{u}(y,t)\,\mathrm{d}y \tag{87}$$

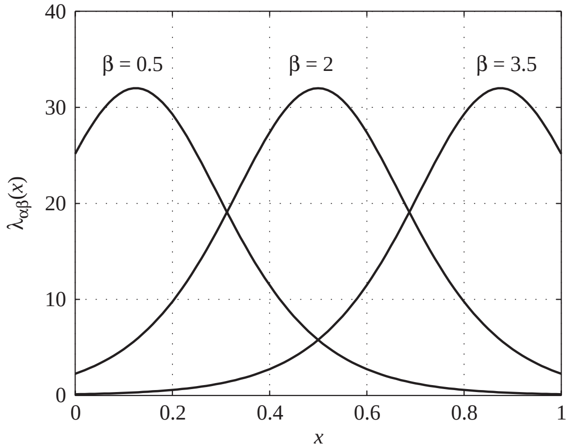

Fig. 5. “One-peak” $\lambda_{\alpha\beta}(x)$ for $\alpha=4$ and $\varepsilon=1$.

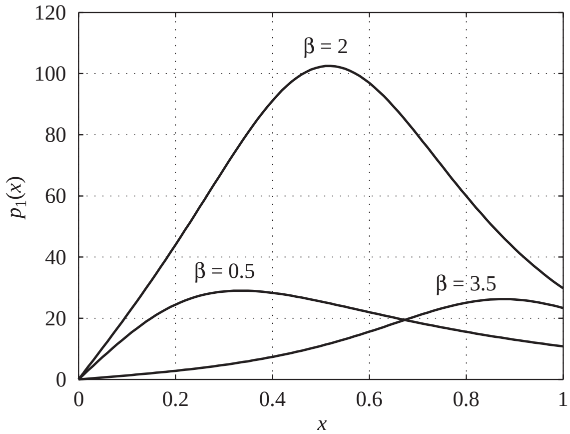

Fig. 6. Observer gain for $\alpha=4$ and $\varepsilon=1$.

with the observer

$$\hat{u}_t(x,t)=\varepsilon\hat{u}_{xx}(x,t)+\lambda_{\alpha\beta}(x)\hat{u}(x,t) + p_1(x)[u(1)-\hat{u}(1)], \quad (88)$$

$$\hat{u}(0,t)=0, \quad (89)$$

$$\hat{u}_x(1,t)= -\alpha(\tanh\beta-\tanh(\beta-\alpha))\hat{u}(1) + \int_0^1 k_2(y)\hat{u}(y,t)\,\mathrm{d}y, \quad (90)$$

where

$$k_2(x)=p_1(x)=\varepsilon\alpha^2\tanh(\beta)\mathrm{e}^{(1-x)\alpha\tanh\beta} \times(\tanh\beta-\tanh(\beta-\alpha x)) \quad (91)$$

stabilizes the zero solution of system (84)–(85).

Proof. The stabilizing kernel $k_2(x)$ for (84)–(85) was obtained in [15]. Using (50) we get the observer gain (91). The stability of the closed-loop system is ensured by Theorem 5. □

In Fig. 6 the observer gains corresponding to $\lambda_{\alpha\beta}(x)$ from Fig. 5 are shown.

6.3. Combining solutions

In Sections 6.1 and 6.2 we considered two interesting examples of solving an output-feedback problem explicitly, in a closed form. One can actually combine these two solutions to get a solution for a heat equation with $\lambda(x)=\lambda_0+\lambda_{\alpha\beta}(x)$. It can be done in two steps. First, transform the error system into the target system (36)–(38) with $c=-\lambda_0$. It will give a PDE for $p(x,y)$ with $\lambda(x)=\lambda_{\alpha\beta}(x)$ whose solution we know. The target system will not be stable, but it will have constant coefficients. Second, stabilize this target system with $p_1(x)$ corresponding to a constant λ_0. The resulting gain will be expressed in quadratures in terms of gains for λ_0 and $\lambda_{\alpha\beta}(x)$. Denote by $p^{\alpha\beta}(x,y)$ and $p^{\lambda}(x,y)$ the observer gains for the heat equation with $\lambda(x)=\lambda_{\alpha\beta}(x)$ and $\lambda(x)=\lambda_0$ $(a=(\lambda_0+c)/\varepsilon)$, respectively. Then following the procedure described above we obtain the observer gain for the heat equation with $\lambda(x)=\lambda_0+\lambda_{\alpha\beta}(x)$:

$$p_1(x)=p_1^{\lambda}(x)+p_1^{\alpha\beta}(x)+\varepsilon p_{10}^{\lambda}p^{\alpha\beta}(x,1) - \int_x^1 p^{\alpha\beta}(x,\xi)p_1^{\lambda}(\xi)\,\mathrm{d}\xi, \quad (92)$$

$$p_{10}=p_{10}^{\lambda}+p_{10}^{\alpha\beta}. \quad (93)$$

For example for $\beta=0$ one can get the closed-form solution

$$p_1(x)=\frac{\varepsilon\lambda}{1-x^2}I_2\left(\sqrt{\lambda(1-x^2)}\right) + \varepsilon\lambda\alpha\tanh(\alpha x)\frac{I_1\left(\sqrt{\lambda(1-x^2)}\right)}{\sqrt{\lambda(1-x^2)}}. \quad (94)$$

The control gain kernel can be obtained from (92) to (93) using (50). Thus, we can obtain the explicit solution to an output-feedback problem for a heat equation with non-constant coefficients and arbitrary level of instability.

The explicit control gains (and thus the observer gains) for even more complicated plants are available in [15].

6.4. Frequency domain compensator

The solutions obtained in previous sections can be used to get explicit compensator transfer functions (treating $u(0,t)$ or $u(1,t)$ as an input and $u(1,t)$ or $u_x(1,t)$ as an output). We illustrate this point with the following system inspired by a solid propellant rocket model [4]:

$$u_t(x,t) = u_{xx}(x,t) + gu(0,t), \tag{95}$$

$$u_x(0,t) = 0, \tag{96}$$

$$u(1,t) = U(t). \tag{97}$$

The observer

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + gu(0,t), \tag{98}$$

$$\hat{u}_x(0,t) = 0, \tag{99}$$

$$\hat{u}(1,t) = U(t) \tag{100}$$

with direct injection of the reaction term $gu(0,t)$ is exponentially convergent. The stabilizing controller, whose state-feedback version was found in [15], is

$$u(1,t) = U(t) = -\sqrt{g}\int_0^1 \sinh\left(\sqrt{g}(1-y)\right) \times \hat{u}(y,t)\,dy. \tag{101}$$

We want to find a transfer function from the input $u(0,t)$ to the output $u(1,t)$, i.e., $u(1,s) = -C(s)u(0,s)$. Taking the Laplace transform of (98)–(100), setting the initial condition to zero, $\hat{u}(x,0)=0$, we have (for simplicity of notation we denote by $\hat{u}(x,s)$ and $u(0,s)$ the Laplace transforms of $\hat{u}(x,t)$ and $u(0,t)$, respectively):

$$s\hat{u}(x,s) = \hat{u}_{xx}(x,s) + gu(0,s), \tag{102}$$

$$\hat{u}_x(0,s) = 0, \tag{103}$$

$$\hat{u}(1,s) = -\sqrt{g}\int_0^1 \sinh\left(\sqrt{g}(1-y)\right) \times \hat{u}(y,s)\,dy. \tag{104}$$

Eq. (102) with boundary conditions (103)–(104) is a second-order ODE with respect to x (we regard s as a parameter). The solution of (102) satisfying (103) is

$$\hat{u}(x,s) = \hat{u}(0,s)\cosh\left(\sqrt{s}x\right) + \frac{g}{s}\left(1-\cosh\left(\sqrt{s}x\right)\right)u(0,s). \tag{105}$$

Using boundary condition (104) we obtain $\hat{u}(0,s)$

$$\hat{u}(0,s) = \frac{\cosh(\sqrt{s}) - \cosh(\sqrt{g})}{s\cosh(\sqrt{s}) - g\cosh(\sqrt{g})}\,gu(0,s). \tag{106}$$

Substituting now (106) into (105) with $x=1$ we obtain the following result:

Theorem 9. *The transfer function of system* (98)–(101) *with $u(0,t)$ as an input and $u(1,t)$ as an output is*

$$C(s) = \frac{g}{s}\left(-1 + \frac{(s-g)\cosh(\sqrt{s})\cosh(\sqrt{g})}{s\cosh(\sqrt{s}) - g\cosh(\sqrt{g})}\right). \tag{107}$$

The validation of application of the above procedure for linear parabolic PDEs (which proves that $C(s)$ is indeed a transfer function) can be found in [7, Chapter 4]. Note that $s=0$ is not a pole:

$$C(0) = \frac{g}{2} + \frac{1}{\cosh(\sqrt{g})} - 1. \tag{108}$$

The transfer function (107) has infinitely many poles, all of them are real and negative. The Bode plots of $C(s)$ for $g=8$ are presented in Fig. 7. It is evident from the Bode plots that $C(s)$ can be approximated by a second-order, relative degree one transfer function. For example, a pretty good estimate would be

$$C(s) \approx 60\frac{s+17}{s^2+25s+320}. \tag{109}$$

The relative degree one nature of the compensator is the result of employing a full order (rather than a reduced order) observer.

Appendix A. Backstepping control design overview

The stabilization problem for class (1)–(2) was solved in [15] by finding a backstepping-style integral

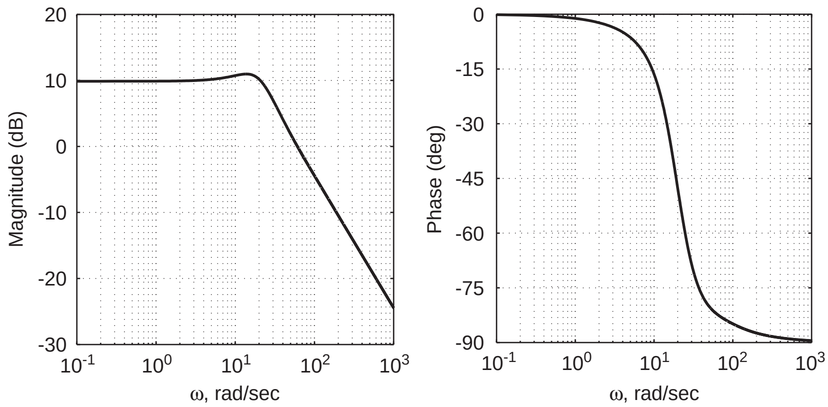

Fig. 7. Bode plot of $C(j\omega)$ for $g=8$.

transformation

$$w(x,t)=u(x,t)-\int_0^x k(x,y)u(y,t)\,\mathrm{d}y \tag{A.1}$$

that maps system (1)–(2) into the system

$$w_t(x,t)=\varepsilon w_{xx}(x,t)-cw(x,t), \quad x\in(0,1), \tag{A.2}$$

$$w_x(0,t)=qw(0,t), \tag{A.3}$$

$$w(1,t)=0 \quad \text{or } w_x(1,t)=0, \tag{A.4}$$

which is exponentially stable for $c>\varepsilon\bar{q}^2$ where $\bar{q}=\max\{0,-q\}$. Once the kernel $k(x,y)$ of the transformation (A.1) is found, the stabilizing boundary controls at $x=1$ can be obtained in the form

$$u(1,t)=\int_0^1 k_1(y)u(y,t)\,\mathrm{d}y,\qquad k_1(y)=k(1,y), \tag{A.5}$$

$$u_x(1,t)=k_1(1)u(1,t)+\int_0^1 k_2(y)u(y,t)\,\mathrm{d}y,\qquad k_2(y)=k_x(1,y). \tag{A.6}$$

It was shown in [15] that the control gain kernel $k(x,y)$ satisfies the following hyperbolic PDE:

$$\begin{aligned}\varepsilon k_{xx}(x,y)-\varepsilon k_{yy}(x,y) &=(\lambda(y)+c)k(x,y)-f(x,y)\\ &\quad+\int_y^x k(x,\xi)f(\xi,y)\,\mathrm{d}\xi,\end{aligned} \tag{A.7}$$

for $(x,y)\in\mathscr{T}=\{x,y:0<y<x<1\}$ with boundary conditions

$$\varepsilon k_y(x,0)=\varepsilon qk(x,0)+g(x)-\int_0^x k(x,y)g(y)\,\mathrm{d}y, \tag{A.8}$$

$$k(x,x)=-\frac{1}{2\varepsilon}\int_0^x(\lambda(\xi)+c)\,\mathrm{d}\xi \tag{A.9}$$

and the following theorem proved.

Theorem A.1. *Eq.* (A.7)–(A.9) *has a unique* $C^2(\mathscr{T})$ *solution. The kernel* $l(x,y)$ *of the inverse transformation*

$$u(x,t)=w(x,t)+\int_0^x l(x,y)w(y,t)\,\mathrm{d}y \tag{A.10}$$

is also a unique C^2 *function.*

This result (meaning invertibility of (A.1)) along with stability of the target system (A.2)–(A.4) allowed us to prove the closed-loop stability and well posedness of system (1)–(2), (A.5) (or (A.6)).

Theorem A.2. *For any* $u_0\in L_2(0,1)$ system (1)–(2), (A.5) (*or* (A.6)) *with assumptions* (4), $c\geqslant\varepsilon\bar{q}^2$ (*or* $c\geqslant\varepsilon\bar{q}^2+\varepsilon/2$) *and the kernel* $k_1(y)=k(1,y)$ (*or* $k_2(y)=k_x(1,y)$) *has a unique classical solution* $u(x,t)\in C^{2,1}((0,1)\times(0,\infty))$ *and is exponentially stable at the origin,* $u(x,t)\equiv 0$, *in the* $L_2(0,1)$ *and* $H_1(0,1)$ *norms.*

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