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Stability of partial difference equations governing control gains in infinite-dimensional backstepping☆

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Abstract

We examine the stability properties of a class of LTV difference equations on an infinite-dimensional state space that arise in backstepping designs for parabolic PDEs. The nominal system matrix of the difference equation has a special structure: all of its powers have entries that are −1, 0, or 1, and all of the eigenvalues of the matrix are on the unit circle. The difference equation is driven by initial conditions, additive forcing, and a system matrix perturbation, all of which depend on problem data (for example, viscosity and reactivity in the case of a reaction–diffusion equation), and all of which go to zero as the discretization step in the backstepping design goes to zero. All of these observations, combined with the fact that the equation evolves only in a number of steps equal to the dimension of its state space, combined with the discrete Gronwall inequality, establish that the difference equation has bounded solutions. This, in turn, guarantees the existence of a state-feedback gain kernel in the backstepping control law. With this approach we greatly expand, relative to our previous results, the class of parabolic PDEs to which backstepping is applicable.

Keywords: LTV difference equations; Stability; Parabolic PDEs; Boundary control; Gain kernel

1. Introduction

We consider parabolic PDEs of the form

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + b(x)u_x(x,t) + \lambda(x)u(x,t) + g(x)u(0,t) + d(x)u(\theta x,t) + \int_0^x f(x,\xi)u(\xi,t)\,\mathrm{d}\xi \quad (1.1)$$

for $x \in (0,1)$, $t > 0$, with initial condition

$$u(x,0) = u_0(x), \quad x \in [0,1], \quad (1.2)$$

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with boundary conditions [1]

$$u_x(0,t) = qu(0,t), \tag{1.3}$$

$$u(1,t) = \int_0^1 k_1(\xi)u(\xi,t)\,\mathrm{d}\xi \tag{1.4}$$

and under the assumption

$$\varepsilon > 0, \quad 0 < \theta < 1, \quad q \in \mathbb{R}, \quad b, \lambda, d, g \in L_\infty(0,1), \quad f \in L_\infty([0,1]\times[0,1]), \tag{1.5}$$

where the feedback gain kernel $k_1 \in L_\infty(0,1)$ is sought to stabilize the equilibrium $u \equiv 0$. Throughout the paper we will consider only real valued functions and the classical $L_p(0,1)$, $p \geqslant 1$ and $C^{2,1}([0,1]\times(0,\infty))$ spaces (see [11]). Relative to our previous work on the subject [1,2], which was under the assumptions $g(x)=d(x)=f(x,\xi)/q=0$ for $x, \xi \in [0,1]$, the results here are more general, and the proof technique we develop is more elegant. Our approach is to use the backstepping method for the finite difference semi-discretized approximation of (1.1) to derive an infinite-dimensional coordinate transformation that maps our system into an exponentially stable system. The coordinate transformation results in a boundary feedback control law of form (1.4). Our result is formulated in the following theorem.

Theorem 1.1. *For any $c > 0$ there exists a function $k_1 \in L_\infty(0,1)$ such that for any $u_0 \in L_\infty(0,1)$ system (1.1)–(1.4) with assumption (1.5) has a unique classical solution $u \in C^{2,1}([0,1]\times(0,\infty))$ and the trivial solution $u_{\mathrm{triv}} \equiv 0$ is exponentially stable in the $L_2(0,1)$ and maximum norms with decay rate c. More precisely, there exists a positive constant M such that for all $t > 0$*

$$\|u(t,\cdot)\|_{L_2(0,1)} \leqslant M\|u_0\|_{L_2(0,1)}\mathrm{e}^{-ct} \tag{1.6}$$

and

$$\max_{x\in[0,1]} |u(x,t)| \leqslant M \sup_{x\in[0,1]} |u_0(x)|\mathrm{e}^{-ct}. \tag{1.7}$$

The problem of boundary feedback stabilization of general parabolic equations is not new. Our papers [1,2] contain a detailed discussion of prior work, and the reader is also referred to [8,12] for extensive surveys. While these previous approaches give an existence answer to our stabilization problem, our approach offers an implementable, numerically simple solution that avoids the tasks of estimating eigenfunctions or solving operator Riccati equations, which are formidable in the case of nonconstant coefficients.

The present paper, besides generalizing the class of systems in [1,2] (the expanded class now includes, for example, the linearized model of the solid propellant rocket instability [3]), offers a very different, and remarkably more elegant proof technique. Boundedness of the coordinate transformation kernel, whose trace on the boundary is the gain kernel function k_1, is the key result. In [1,2] this result involved deriving extremely complicated formulae for the exact form of the transformation. In this paper the boundedness of the transformation is proved without solving for it. The boundedness proof is essentially stability analysis for a complicated LTV difference equation on an infinite-dimensional state space.

This paper is organized as follows. In Section 2 we lay out our strategy for the solution of the stabilization problem. We design a coordinate transformation for a semi-discretization of our system which maps it into an exponentially stable system and derive a recursive relationship for the kernel of transformation in Section 3. The recursive relationship is written in the form of a system of second-order difference equations in Section 4. Our main theorem on the stability of this system is proven in Section 5. The stability result shows the uniform boundedness of the discretized coordinate transformations as the grid is refined. This implies the existence of

[1] The case of Dirichlet boundary condition at the zero end ($q = \infty$) can be handled the same way.

the stabilizing boundary control law (1.4), as it was shown in [1,2]. For completeness we included in Section 6 a theorem on the well posedness of the controlled system (1.1)–(1.4).

2. Backstepping transformation

We look for a coordinate transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,\xi)u(\xi,t)\,d\xi, \quad x \in [0,1], \ t > 0, \tag{2.1}$$

that transforms system (1.1)–(1.4) into the exponentially stable system

$$w_t(x,t) = \varepsilon w_{xx}(x,t) + b(x)w_x(x,t) - cw(x,t), \quad x \in [0,1], \ t > 0, \tag{2.2}$$

where $c > q^2$ and the boundary conditions are

$$w_x(0,t) = qw(0,t), \quad \forall t > 0, \tag{2.3}$$

$$w(1,t) = 0, \quad \forall t > 0. \tag{2.4}$$

Once transformation (2.1) is found, it is realized through the stabilizing boundary control (1.4) with $k_1(\cdot) = k(1,\cdot)$.

Substituting (2.1) into Eq. (2.2) and using Eq. (1.1) results in the following weak formulation of a hyperbolic partial differential equation for the function k:

$$\begin{aligned} 0 = &\int_0^x (\varepsilon k_{xx}(x,\xi) - \varepsilon k_{\xi\xi}(x,\xi) + k_\xi(x,\xi)b(\xi) + k_x(x,\xi)b(x))u(\xi,t)\,d\xi \\ &- \int_0^x k(x,\xi)(\lambda(\xi) + c - b'(\xi))u(\xi,t)\,d\xi - \int_0^x k(x,\xi)(g(\xi)u(0,t) + d(\xi)u(\theta\xi,t) \\ &+ \int_0^\xi f(\xi,s)u(s,t)\,ds)\,d\xi + \varepsilon 2\left(\frac{d}{dx}k(x,x)\right)u(x,t) + \varepsilon(qk(x,0) - k_\xi(x,0))u(0,t) \\ &+ k(x,0)b(0)u(0,t) + (\lambda(x)+c)u(x,t) + g(x)u(0,t) + d(x)u(\theta x,t) + \int_0^x f(x,\xi)u(\xi,t)d\xi \end{aligned} \tag{2.5}$$

for all $x \in [0,1]$.

In order to find (2.1) in a constructive way we first discretize (1.1)–(1.4), then we develop a stabilizing coordinate transformation for the discretized system and finally we show that the discretization converges to an infinite-dimensional transformation. We define $k_{ij}^n = k((i-1)h,(j-1)h)$, $u_i^n = u((i-1)h,t)$, $b_i^n = b((i-1)h)$ for $t > 0$, $i,j = 1,\dots,n$, $n = 1,2,\dots$ where $h = 1/n$, and the finite difference discretization of the rest of the functions is defined the same way. The discretized version of coordinate transformation (2.1) now has the form

$$w^n = u^n - hK^n u^n, \quad n = 1,2,\dots \tag{2.6}$$

where

$$w^n = [w_1^n, w_2^n, \dots, w_n^n]^T, \tag{2.7}$$

$$u^n = [u_1^n, u_2^n, \dots, u_n^n]^T \tag{2.8}$$

and

$$K^n = [k_{ij}^n]_{n\times n} \tag{2.9}$$

with the convention that

$$k_{ij}^n = 0 \quad \text{for } j > i. \tag{2.10}$$

A discretization of system (1.1)–(1.4) with respect to the spatial variable x using finite differences is

$$\frac{u_1^n - u_0^n}{h} = q u_1^n, \tag{2.11}$$

$$\dot{u}_i^n = \varepsilon \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + b_i^n \frac{u_i^n - u_{i-1}^n}{h} + \lambda_i^n u_i^n + g_i^n u_1^n + d_i^n u_{[\theta i]+1}^n + h \sum_{j=1}^{i} f_{ij}^n u_j^n, \quad i = 1, \ldots, n, \tag{2.12}$$

$$u_{n+1}^n = h \sum_{j=1}^{n} k_{nj}^n u_j^n. \tag{2.13}$$

The exponentially stable transformed system (2.2)–(2.4) has the discretized form

$$\frac{w_1^n - w_0^n}{h} = q w_1^n, \tag{2.14}$$

$$\dot{w}_i^n = \varepsilon \frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{h^2} + b_i^n \frac{w_i^n - w_{i-1}^n}{h} - c w_i^n, \quad i = 1, \ldots, n, \tag{2.15}$$

$$w_{n+1}^n = 0. \tag{2.16}$$

As it was shown in [1,2], the convergence of the finite-dimensional transformation (2.6) to the infinite-dimensional one (2.1) reduces to proving the uniform boundedness of $\|K^n\|_m = \max_{i,j=1,\ldots,n} |k_{ij}^n|$ in n. Since n plays an important role, we will keep the superscript n notation throughout the paper. Any other superscript will refer to powers. We note here that $\|\cdot\|_m$ is different from the regular matrix ∞-norm.

3. Finding the gain kernel

In this section we derive a recursive relationship for the kernel $\{k_{ij}^n\}_{i,j=1,\ldots,n}$. We have from (2.14)

$$w_0^n = (1 - qh) u_1^n \tag{3.1}$$

and from (2.15)

$$\varepsilon w_{i+1}^n = (2\varepsilon + ch^2 - b_i^n h) w_i^n - (\varepsilon - b_i^n h) w_{i-1}^n + h^2 \dot{w}_i^n, \quad i = 1, \ldots, n. \tag{3.2}$$

With the help of Eqs. (2.6) and (2.12) we get

$$\begin{aligned}
\varepsilon \left(u_{i+1}^n - h \sum_{j=1}^{i} k_{ij}^n u_j^n \right) &= (2\varepsilon + ch^2 - b_i^n h) \left(u_i^n - h \sum_{j=1}^{i-1} k_{i-1,j}^n u_j^n \right) \\
&\quad - (\varepsilon - b_i^n h) \left(u_{i-1}^n - h \sum_{j=1}^{i-2} k_{i-2,j}^n u_j^n \right) + h^2 \left(\dot{u}_i^n - h \sum_{j=1}^{i-1} k_{i-1,j}^n \dot{u}_j^n \right) \\
&= \left(2\varepsilon + h^2 c - h b_i^n \right) u_i^n - (2\varepsilon + h^2 c - h b_i^n) h \sum_{j=1}^{i-1} k_{i-1,j}^n u_j^n - (\varepsilon - h b_i^n) u_{i-1}^n
\end{aligned}$$

$$+(\varepsilon - hb_i^n)h\sum_{j=1}^{i-2} k_{i-2,j}^n u_j^n + h^2\left(\varepsilon\,\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + b_i^n\,\frac{u_i^n - u_{i-1}^n}{h}\right)$$

$$+h^2\left(\lambda_i^n u_i^n + g_i^n u_1^n + d_i^n u_{[\theta i]+1}^n + h\sum_{l=1}^{i-1} f_{il}^n u_l^n - h\sum_{j=1}^{i-1} k_{i-1,j}^n \dot{u}_j^n\right)$$

$$= \varepsilon u_{i+1}^n + h^2(c+\lambda_i^n)u_i^n - (2\varepsilon + h^2 c - hb_i^n)h \times \sum_{j=1}^{i-1} k_{i-1,j}^n u_j^n + (\varepsilon - hb_i^n)h\sum_{j=1}^{i-2} k_{i-2,j}^n u_j^n$$

$$+h^2 g_i^n u_1^n + h^2 d_i^n u_{[\theta i]+1}^n + h^3\sum_{l=1}^{i-1} f_{il}^n u_l - h^3\sum_{j=1}^{i-1} k_{i-1,j}^n\left(\varepsilon\,\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + b_j^n\,\frac{u_j^n - u_{j-1}^n}{h}\right.$$

$$\left.+\lambda_j^n u_j^n + g_j^n u_1^n + d_j^n u_{[\theta j]+1}^n + h\sum_{l=1}^{j-1} f_{jl}^n u_l^n\right) \tag{3.3}$$

for $i = 1,\ldots,n$. Rearranging (3.3) and using (2.11) we obtain

$$\varepsilon\sum_{j=1}^{i} k_{ij}^n u_j^n = (2\varepsilon + ch^2 - b_i^n h)\sum_{j=1}^{i-1} k_{i-1,j}^n u_j^n$$

$$-(\varepsilon - b_i^n h)\sum_{j=1}^{i-2} k_{i-2,j}^n u_j^n - h(c+\lambda_i^n)u_i^n - hg_i^n u_1^n - hd_i^n u_{[\theta i]+1}^n$$

$$-h^2\sum_{l=1}^{i-1} f_{il}^n u_l^n + k_{i-1,1}^n(\varepsilon u_2^n - (\varepsilon(1+hq) - h^2(g_1^n + \lambda_1^n + d_1^n + b_1^n q))u_1^n)$$

$$+\sum_{j=2}^{i-1} k_{i-1,j}^n(\varepsilon u_{j+1}^n - (2\varepsilon - hb_j^n)u_j^n + (\varepsilon - hb_j^n)u_{j-1}^n + h^2 g_j^n u_1^n + h^2\lambda_j^n u_j^n + h^2 d_j^n u_{[\theta j]+1}^n)$$

$$+h^3\sum_{j=2}^{i-1}\sum_{l=1}^{j-1} k_{i-1,j}^n f_{jl}^n u_l^n. \tag{3.4}$$

In the next step we are going to use the identities

$$\sum_{j=2}^{i-1}\sum_{k=1}^{j-1} k_{i-1,j}^n f_{jk}^n u_k^n = \sum_{j=1}^{i-2}\sum_{l=j+1}^{i-1} k_{i-1,l}^n f_{lj}^n u_j^n = \sum_{j=1}^{i-2}\sum_{l=j}^{i-1} k_{i-1,l}^n f_{lj}^n u_j^n, \tag{3.5}$$

where that latter equality holds after setting $f_{lj} = 0,\ j \geqslant l$. Comparing coefficients of u_j's in (3.4) results in recursive relationships

$$k_{i1}^n = \frac{1}{\varepsilon}\Bigg[(\varepsilon - hb_i^n - \varepsilon hq + h^2(c + \lambda_1^n + b_1^n q))k_{i-1,1}^n + (\varepsilon - hb_2^n)k_{i-1,2}^n - (\varepsilon - hb_i^n)k_{i-2,1}^n$$

$$+h^2\sum_{l=1}^{i-1} k_{i-1,l}^n(g_l^n + d_l^n\delta_{[\theta l],0} + hf_{l1}^n) - hg_i^n - hd_i^n\delta_{[\theta i],0} - h(c+\lambda_1^n)\delta_{i1} - h^2 f_{i1}^n\Bigg] \tag{3.6}$$

and

$$k_{ij}^n = \frac{1}{\varepsilon}\Bigg[(h(b_j^n - b_i^n) + h^2(c+\lambda_j^n))k_{i-1,j}^n + \varepsilon k_{i-1,j-1}^n + (\varepsilon - hb_{j+1}^n)k_{i-1,j+1}^n - (\varepsilon - hb_i^n)k_{i-2,j}^n$$

$$+ h^2\sum_{l=j}^{i-1} k_{i-1,l}^n(d_l^n\delta_{[\theta l]+1,j} + hf_{lj}^n) - h^2 f_{ij}^n - h(c+\lambda_j^n)\delta_{ij} - hd_i^n\delta_{[\theta i]+1,j}\Bigg] \tag{3.7}$$

for $j = 2, \dots, i$. Our next goal is to show that the solution k_{ij}^n to these recursive relations remain uniformly bounded as $j = 1, 2, \dots, i$, $i = 1, 2, \dots, n$ and $n \to \infty$.

4. Difference equation governing the kernel

Eqs. (3.6) and (3.7) can be written in the form of a system of second-order difference equations,

$$k_{i+1}^n = \tilde{\Gamma}_i k_i^n - \left(1 - \frac{hb_{i+1}^n}{\varepsilon}\right)k_{i-1}^n + f_i^n, \quad i = 1, \dots, n-1, \tag{4.1}$$

where

$$k_i^n = [k_{i1}^n, k_{i2}^n, \dots, k_{in}^n]^{\mathrm{T}}, \tag{4.2}$$

$$(f_i^n)_1 = -\frac{h}{\varepsilon}(g_{i+1}^n + (c+\lambda_1^n)\delta_{i+1,1} + d_{i+1}^n\delta_{[\theta(i+1)],0} + hf_{i+1,1}^n), \tag{4.3}$$

$$(f_i^n)_j = -\frac{h}{\varepsilon}((c+\lambda_j^n)\delta_{i+1,j} + d_{i+1}^n\delta_{[\theta(i+1)]+1,j} + hf_{i+1,j}^n) \quad \text{for } j > 1 \tag{4.4}$$

and $\tilde{\Gamma}_i = [(\gamma_i^n)_{lj}]_{n\times n}$ with

$$(\gamma_i^n)_{lj} = \begin{cases} 0 & \text{if } j < l-1, \\ 1 & \text{if } j = l-1, \\ 1 + \frac{h}{\varepsilon}(-b_{i+1}^n - \varepsilon q + h(c + \lambda_1^n + b_1^n q + g_1^n + d_1^n)) & \text{if } j = l = 1, \\ \frac{h}{\varepsilon}(b_l^n - b_{i+1}^n + h(c+\lambda_l^n) + hd_l^n\delta_{[\theta l]+1,l}) & \text{if } j = l \neq 1, \\ 1 + \frac{h}{\varepsilon}(-b_2^n + h(g_2^n + d_2^n\delta_{[2\theta],0} + hf_{21}^n)) & \text{if } l = 1,\ j = 2, \\ 1 + \frac{h}{\varepsilon}(-b_j^n + h(d_j^n\delta_{[\theta j]+1,j-1} + hf_{jl}^n)) & \text{if } j = l+1 > 1, \\ \frac{h^2}{\varepsilon}(g_j^n + d_j^n\delta_{[\theta j],0} + hf_{j1}^n) & \text{if } l = 1,\ j > 2, \\ \frac{h^2}{\varepsilon}(d_j^n\delta_{[\theta j]+1,l} + hf_{jl}^n) & \text{if } j > l+1,\ l \neq 1. \end{cases} \tag{4.5}$$

As it is seen from (4.5), $\tilde{\Gamma}_i$ has the form

$$\tilde{\Gamma}_i = \Gamma + \Delta\Gamma_i \mathrm{O}(h), \tag{4.6}$$

where

$$\Gamma = \begin{bmatrix} 1 & 1 & & & \\ 1 & 0 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix} \tag{4.7}$$

and

$$\Delta\Gamma_i = \begin{bmatrix} 1 & 1 & & & \\ & \ddots & \ddots & \mathrm{O}(h) & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & 1 \end{bmatrix}. \tag{4.8}$$

Here and below O(h) denotes an expressions for which there exists a uniform constant $M > 0$ such that $\mathrm{O}(h) \leqslant Mh$. Definitions (4.3) and (4.4) imply that

$$(f_i^n)_j = \begin{cases} \mathrm{O}(h) & \text{if } j = 1,\ i+1, [\theta(i+1)]+1, \\ \mathrm{O}(h)^2 & \text{otherwise} \end{cases} \tag{4.9}$$

for $i = 1, \ldots, n$, and hence

$$\sup_{l \geqslant 1} \|f_l^n\|_1 \leqslant 2\,\frac{h}{\varepsilon}\left(c + \sup_{l \geqslant 1}|\lambda_l^n| + \sup_{l \geqslant 1}|d_l^n| + \sup_{l,j \geqslant 1}|f_{lj}^n| + \sup_{l \geqslant 1}|g_l^n|\right)$$
$$= \mathrm{O}(h). \tag{4.10}$$

Eq. (4.1) now produces its own initial conditions through convention (2.10), namely

$$k_0^n = 0 \tag{4.11}$$

and

$$k_1^n = \left[-\frac{h}{\varepsilon}(g_1^n + d_1^n + c + \lambda_1^n), 0, \ldots, 0\right]^{\mathrm{T}}. \tag{4.12}$$

Using notation

$$\Theta_i^n = \begin{bmatrix} k_{i-1}^n \\ k_i^n \end{bmatrix} \tag{4.13}$$

we obtain from (4.1)

$$\Theta_{i+1}^n = \widetilde{A_i^n}\Theta_i^n + \begin{bmatrix} 0 \\ f_i^n \end{bmatrix}, \tag{4.14}$$

where

$$\widetilde{A_i^n} = \begin{bmatrix} 0 & I_{n\times n} \\ -(1-b_{i+1}^n h/\varepsilon)I_{n\times n} & \tilde{\Gamma}_i \end{bmatrix} \tag{4.15}$$

and

$$F_i = \begin{bmatrix} 0 \\ f_i^n \end{bmatrix}. \tag{4.16}$$

The matrix $\widetilde{A_i^n}$ can be written as

$$\widetilde{A_i^n} = A + \Delta A_i, \tag{4.17}$$

where

$$A = \begin{bmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & \Gamma \end{bmatrix} \tag{4.18}$$

and

$$\Delta A_i = \begin{bmatrix} 0 & 0 \\ I_{n\times n} & \Delta\Gamma_i \end{bmatrix} \mathrm{O}(h). \tag{4.19}$$

Notice that $\|\Delta A_i\|_1 = \mathrm{O}(h)$. The initial condition of system (4.14) is

$$\Theta_1^n = \begin{bmatrix} k_0^n \\ k_1^n \end{bmatrix} \tag{4.20}$$

which, according to (4.11) and (4.12), has 1-norm that is $\mathrm{O}(h)$ as well. Eq. (4.14) can also be written in the following form:

$$\begin{aligned} \Theta_{i+1} &= A\Theta_i + \Delta A_i\Theta_i + F_i = A^i\Theta_1 + \sum_{j=1}^{i} A^{i-j}(\Delta A_j\Theta_j + F_j) \\ &= A^i\Theta_1 + \sum_{j=1}^{i} A^{i-j}F_j + \sum_{j=1}^{i} A^{i-j}\Delta A_j\Theta_j. \end{aligned} \tag{4.21}$$

5. Proof of the main result

With the help of the matrix difference equation (4.21) we are going to prove our main theorem on the stability of Eqs. (4.1)–(4.5). For this purpose we need to use two lemmas. The first lemma is on the boundedness and structure of powers of matrix A.

Lemma 5.1. *Assume that a matrix A is of the form*

$$A = \begin{bmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & \Gamma \end{bmatrix}, \tag{5.1}$$

where

$$\Gamma = \begin{bmatrix} 1 & 1 & & & \\ 1 & 0 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}. \tag{5.2}$$

Then powers of matrix A have the form

$$A^i = \begin{bmatrix} -P_{i-1} & P_i \\ -P_i & P_{i+1} \end{bmatrix} \quad \textit{for } i = 1, 2, \ldots, n, \tag{5.3}$$

where

$$(P_i)_{kl} = \begin{cases} 1 & \textit{if } k + l \leqslant i + 1 \textit{ or} \\ & \textit{if } k - l = i - 1, i - 3, \ldots, -i + 3, -i + 1 \textit{ and } k + l \leqslant 2n - i + 1, \\ 0 & \textit{otherwise.} \end{cases} \tag{5.4}$$

Proof. From (4.18) and (4.7) we obtain structure (5.3) of matrices A^i where the matrices P_i have to satisfy the difference equation

$$P_{i+1} = \Gamma P_i - P_{i-1}, \quad i = 1, 2, \ldots \tag{5.5}$$

with initial conditions

$$P_0 = 0 \quad \text{and} \quad P_1 = I. \tag{5.6}$$

In order to better understand the structure of P_i defined in (5.4) we provide here P_6 for $n = 10$ as an example.

$$P_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.7}$$

Substituting $(P_{i+1})_{kl}$, $(P_i)_{kl}$ and $(P_{i-1})_{kl}$ for $k, l=1,2,\ldots,n$ from (5.4) into (5.5) and using definition (4.7) of Γ we see the unique solution of (5.5) is given by (5.4). With this we obtain the statement of the lemma. □

Remark 1. Although it does not help in establishing the stability properties of our unperturbed system, it is interesting to note the special eigenvalue structure of matrix A. All the eigenvalues of A are distributed evenly along the unit circle, i.e., the eigenvalues of A are $\lambda = e^{Il\pi/(2n+1)}$, $l=1,2,\ldots,2n$, where I denotes the imaginary unit.

Proof. We first transform the characteristic equation of A to a characteristic equation that corresponds to a real symmetric matrix.

$$\begin{aligned}\det(A-\lambda I) &= \det\left(\begin{bmatrix} -\lambda & I_{n\times n} \\ -I_{n\times n} & \Gamma-\lambda \end{bmatrix}\right)\\
&= \det\left(\begin{bmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{bmatrix}\begin{bmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{bmatrix}\begin{bmatrix} -\lambda & I_{n\times n} \\ -I_{n\times n} & \Gamma-\lambda \end{bmatrix}\right)\\
&= \det\left(\begin{bmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{bmatrix}\right)\det\left(\begin{bmatrix} -I_{n\times n} & \Gamma-\lambda \\ \lambda & -I_{n\times n} \end{bmatrix}\right)\\
&= \det\left(\begin{bmatrix} -I_{n\times n} & \Gamma-\lambda \\ \lambda & -I_{n\times n} \end{bmatrix}\right)\\
&= \det(-I_{n\times n}-(\Gamma-\lambda)(-I_{n\times n})^{-1}\lambda)\det(-I_{n\times n})\\
&= (-1)^n\det(-\lambda^2-I_{n\times n}+\lambda\Gamma). \end{aligned} \tag{5.8}$$

In the fifth step above we used the identity

$$\det\begin{pmatrix} A & B \\ D & C \end{pmatrix} = \det(A-BC^{-1}D)\det(C). \tag{5.9}$$

Note that $\lambda=0$ is not a root of (5.8), hence we can factor out λ and obtain that the roots of (5.8) are identical to the roots of

$$\det(\Gamma-\chi I_{n\times n})=0, \tag{5.10}$$

where

$$\chi = \frac{\lambda^2+1}{\lambda}. \tag{5.11}$$

We now determine the eigenvalues χ of matrix Γ. Along the line of [10, Appendix II] we obtain recursive relations

$$\Delta_l(\chi)=\chi\Delta_{l-1}(\chi)-\Delta_{l-2}(\chi), \quad l=1,\ldots,n. \tag{5.12}$$

Introducing a new variable ω through the relation

$$\chi=\omega+\frac{1}{\omega} \tag{5.13}$$

we obtain by induction the general expression

$$\Delta_n(\omega) = \sum_{i=-n}^{n} \omega^i \tag{5.14}$$

that, in turn, can be written as

$$\Delta_n(\omega) = \omega^{-n}\frac{\omega^{2n+1}-1}{\omega-1}. \tag{5.15}$$

The roots of (5.15) can be easily found to be

$$\omega_l = \mathrm{e}^{Il2\pi/(2n+1)}, \quad l = 1,2,\ldots,2n. \tag{5.16}$$

Using (5.16) with (5.13) we obtain that the eigenvalues of Γ are

$$\chi_l = -2\cos\frac{l2\pi}{2n+1}, \quad l = 1,2,\ldots,n. \tag{5.17}$$

Eq. (5.11) results in the quadratic equations

$$\lambda^2 - \chi_l\lambda + 1 = 0, \quad l = 1,2,\ldots,n. \tag{5.18}$$

Using (5.17) it is easy to see that Eq. (5.18) has solutions $\lambda = \mathrm{e}^{Il\pi/(2n+1)}$, $l = 1,2,\ldots,2n$. With this we obtain the statement of the remark. □

In the proof of our main theorem we will also use the discrete time Gronwall lemma. Its proof for a more general case can be found, for example, in [5, Appendix E].

Lemma 5.2. *Assume that*

$$m_k \leqslant c + \sum_{l=0}^{k-1} m_l g_l, \tag{5.19}$$

where m and g are positive sequences. Then

$$m_k \leqslant c\exp\left\{\sum_{l=0}^{k-1} g_l\right\}. \tag{5.20}$$

We now state and prove our main theorem.

Theorem 5.1. *Solutions of the system of second-order difference equations* (4.1)–(4.5) *with initial conditions* (4.11)–(4.12) *are bounded uniformly in* n. *More precisely, there exists a constant* $C>0$, *whose size depends on the size of constants and supremum norm of functions in assumption* (1.5), *such that*

$$\sup_{n\geqslant 1}\max_{i=1,\ldots,n} \|k_i^n|_\infty \leqslant C. \tag{5.21}$$

Proof. As a result of Lemma 5.1 the powers of matrix A up to power n have entries that are equal to -1, 0 or 1. We are going to use the non-submultiplicative matrix norm

$$\|A\|_{1,\infty} = \sup_{x\neq 0}\frac{\|Ax\|\infty}{\|x\|_1} = \max_{i,j}|(A)_{ij}|$$

which in the case of our A matrix gives

$$\|A^i x\|_\infty \leqslant \|A^i\|_{1,\infty}\|x\|_1$$
$$= \|x\|_1, \quad \text{for } i = 1,2,\ldots,n. \tag{5.22}$$

Since $\|\Theta_1\|_1 = \|F_j\|_1 = \|\Delta A_j\|_1 = \mathrm{O}(h)$ for all $j = 1, 2, \ldots, n$, hence multiplication by powers of A results in $\mathrm{O}(h)$ infinity norms. We then obtain from (4.21) for $i = 1, 2, \ldots, n-1$ that

$$\begin{aligned}\|\Theta_{i+1}\|_\infty &= \left\| A^i\Theta_1 + \sum_{j=1}^{i} A^{i-j}F_j + \sum_{j=1}^{i} A^{i-j}\Delta A_j \Theta_j \right\|_\infty \\ &\leqslant \|A^i\Theta_1\|_\infty + \sum_{j=1}^{i}\|A^{i-j}F_j\|_\infty + \sum_{j=1}^{i}\|A^{i-j}\Delta A_j\|_\infty \|\Theta_j\|_\infty \\ &\leqslant \|\Theta_1\|_1 + \sum_{j=1}^{i}\|F_j\|_1 + \sum_{j=1}^{i}\|\Delta A_j\|_1\|\Theta_j\|_\infty. \end{aligned} \tag{5.23}$$

Using the discrete Gronwall lemma we obtain

$$\|\Theta_n\|_\infty \leqslant \left(\|\Theta_1\|_1 + \sum_{j=1}^{n-1}\|F_j\|_1 \right) \mathrm{e}^{\sum_{j=1}^{n}\|\Delta A_j\|_1}. \tag{5.24}$$

Since $\|\Theta_1\|_1$, $\|F_j\|_1$ and $\|\Delta A_j\|_1$ are all of order $\mathrm{O}(h)$, for $j = 1, 2, \ldots, n$,

$$\|\Theta_n\|_\infty \leqslant \mathrm{O}(1)\exp\{\mathrm{O}(1)\} = \mathrm{O}(1). \tag{5.25}$$

This proves the theorem. □

Remark 2. For the very simple case of our previous paper where $b = q = d = b = f = 0$, estimate (5.24) gives us the bound $(c + \lambda)/\varepsilon$ for the approximating kernel gain.

With this we can prove (as it was done in [1,3]) Theorem 1.1, the existence of an infinite dimensional coordinate transformation (2.1) and stabilizing boundary control (1.4).

6. Existence and uniqueness of closed-loop solutions

For completeness we establish the local in time existence and uniqueness of classical solutions to system (1.1)–(1.4) for the case of $b \in C^1(0,1)$ and continuous initial data $u_0 \in C(0,1)$ using a contraction mapping argument [9]. For less smooth initial data, namely for $u_0 \in L_\infty(0,1)$, the existence of classical solution for $t > 0$ follows from the well-known smoothing properties of the heat equation (see, e.g., [4]). Once the local in time existence obtained, the global existence follows from the stability properties.

We define

$$\bar{\lambda} = \sup_{x\in[0,1]} |\lambda(x) - b'(x)|, \quad \bar{d} = \sup_{x\in[0,1]} |d(x)|, \quad \bar{g} = \sup_{x\in[0,1]} |g(x)|, \tag{6.1}$$

$$\bar{f} = \sup_{(x,\xi)\in[0,1]\times[0,1]} |f(x,\xi)|, \quad \bar{k} = \sup_{x\in[0,1]} |k_1(x)|, \quad \bar{b} = \sup_{x\in[0,1]} |b(x)|. \tag{6.2}$$

Let

$$G(x,\xi,t,\tau) \quad = 2\sum_{n=1}^{\infty} \cos(\lambda_n x)\cos(\lambda_n \xi)\mathrm{e}^{-\varepsilon\lambda_n^2(t-\tau)} \tag{6.3}$$

denote Green's function corresponding to the heat operator

$$\mathscr{L}u = u_t - \varepsilon u_{xx}, \quad x \in (0,1), \ t > 0 \tag{6.4}$$

with boundary conditions

$$u_x(0,t) = 0, \tag{6.5}$$

$$u(1,t) = 0, \tag{6.6}$$

where

$$\lambda_n = (2n-1)\frac{\pi}{2}, \quad n = 1,2,\ldots \ . \tag{6.7}$$

The solutions to our system (1.1)–(1.4) are the fixed points of the operator

$$\begin{aligned} Fu(x,t) = {} & \int_0^1 G(x,\xi,t,\tau)u_0(\xi)\mathrm{d}\xi - \int_0^t \int_0^1 G_\xi(x,\xi,t,\tau)b(\xi)u(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\ & + \int_0^t \int_0^1 G(x,\xi,t,\tau)(g(\xi)u(0,\tau) + (\lambda(\xi) - b'(\xi))u(\xi,\tau))\,\mathrm{d}\xi\,\mathrm{d}\tau \\ & + \int_0^t \int_0^1 G(x,\xi,t,\tau)\left(d(\xi)u(\theta\xi,\tau) + \int_0^\xi f(\xi,y)u(y,\tau)\,\mathrm{d}y\right)\mathrm{d}\xi\,\mathrm{d}\tau \\ & + \int_0^t G(x,1,t,\tau)(1+b(1))\int_0^1 k_1(\xi)u(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \\ & - \int_0^t G(x,0,t,\tau)(b(0)+q)u(0,\tau)\,\mathrm{d}\tau, \quad x \in [0,1], \ t > 0. \end{aligned} \tag{6.8}$$

The local in time solvability result follows from contraction mapping argument applied to the iteration $u_{m+1} = Fu_m$ with some starting function u_1, where $Q_T = [0,1] \times [0,T]$ and

$$\max_{(x,t)\in Q_T} |u_1(x,t)| \leqslant M_0 \equiv 2 \sup_{x\in[0,L]} |u_0(x)|. \tag{6.9}$$

According to the properties of the heat equation kernel function G (see, e.g., [7]) we obtain that the function

$$\gamma(T) = \max\left\{ \max_{(x,t)\in Q_T} \int_0^t \int_0^1 |G_\xi(x,\xi,t,\tau)|\,\mathrm{d}\xi\,\mathrm{d}\tau, \max_{(x,t)\in Q_T} \int_0^t |G(x,0,t,\tau)|\,\mathrm{d}\tau, \max_{(x,t)\in Q_T} \int_0^t |G(x,1,t,\tau)|\,\mathrm{d}\tau \right\} \tag{6.10}$$

converges to zero monotonically as $T \to 0$. We now choose $T > 0$ sufficiently small such that

$$\gamma(T)(1 + \bar{g} + \bar{\lambda} + \bar{d} + \bar{f} + (1+\bar{b})\bar{k} + (\bar{b}+q)) < \tfrac{1}{2}. \tag{6.11}$$

We obtain by induction that

$$\begin{aligned} \max_{(x,t)\in Q_T} |u_{m+1}(x,t)| \leqslant {} & \sup_{x\in[0,1]} |u_0(x)| + \max_{(x,t)\in Q_T} \int_0^t \int_0^L |G_\xi(x,\xi,t,\tau)|\mathrm{d}\xi\,\mathrm{d}\tau \max_{(x,t)\in Q_T} |u_m(x,t)| \\ & + \max_{(x,t)\in Q_T} \int_0^t \int_0^L |G_\xi(x,\xi,t,\tau)|\,\mathrm{d}\xi\,\mathrm{d}\tau \left[\max_{(x,t)\in Q_T} \left(g(x)u_m(0,t) + (\lambda(x) - b'(x))u_m(x,t) \right.\right. \\ & \left.\left. + d(x)u_m(\theta x,t) + \int_0^x f(x,y)u_m(y,t)\,\mathrm{d}y\right)\right] \end{aligned}$$

$$
\begin{aligned}
&+ (1+\bar{b})\bar{k} \max_{(x,t)\in Q_T} \int_0^t |G(x,L,t,\tau)|\,\mathrm{d}\tau \max_{(x,t)\in Q_T} |u_m(x,t)| \\
&+ (\bar{b}+q) \max_{(x,t)\in Q_T} \int_0^t |G(x,0,t,\tau)|\,\mathrm{d}\tau \max_{0\leqslant t\leqslant T} |u_m(0,t)| \\
&\leqslant \tfrac{1}{2}M_0 + \gamma(T)M_0(1+\bar{g}+\bar{\lambda}+\bar{d}+\bar{f}+(1+\bar{b})\bar{k}+(\bar{b}+q)) \\
&\leqslant M_0 \qquad (6.12)
\end{aligned}
$$

for all $m = 1, 2, \ldots$. In a similar way we obtain

$$\max_{(x,t)\in Q_T} |u_{m+1}(x,t) - u_m(x,t)| \leqslant \tfrac{1}{2} \max_{(x,t)\in Q_T} |u_m(x,t) - u_{m-1}(x,t)|.$$

As a result, the sequence $\{u_m\}_{m\geqslant 1}$ converges uniformly to a unique continuous function u on Q_T for sufficiently small $T > 0$. Once the continuity of the solution is obtained, the general theory of parabolic equations (see, e.g. [6]) implies that u is a classical solution.

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