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Brief paper

Averaging analysis of periodically forced fluid networks☆

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Abstract

For a lumped parameter model of a flow network driven by a periodic generator, we apply averaging to find an approximate solution and analyze its stability. The approximate solution has three parts: mean flow due to the resistive effects of branches, a time-periodic part due to "inductive" effects, and a mean flow average correction due to the interaction of nonlinear and time varying effects. We present an example that may explain the scenario leading to venous diseases. It is shown that the widening of a branch in a venous network leads to an increase in the AC flow and a decrease in the DC flow through that branch, thus increasing the stress on venous valves, and consequently leading to a further increase in the effective width of the vein.

Keywords: Averaging analysis; Networks; Venous disease; Venous valves

1. Introduction

Fluid flow networks (ventilation, gas/water distribution, vascular) have been of interest to control engineers for three decades. The objective is to drive the flow in all branches to a desired value and several different control laws have been developed (Aldridge, Swartwout, Smith, Nutter, & Boyles, 1976; Bogdanov & Kneller, 1983; DeMoyer, 1974; Hu, Koroleva, & Krstić, 2003; Kocić, 1979; Koroleva, Krstić, & Schmid-Schonbein, 2004). We consider a different problem of finding approximate solutions to periodically forced networks without feedback control, which, to our knowledge, has not been studied.

We consider a network driven by an ideal flow generator. Combining Kirchhoff's laws for flows and pressure laws with equations for pipe flow dynamics, we get a nonlinear ODE model with a periodic right-hand side $\dot{Q}=\varepsilon f(t, Q, \varepsilon)$, where Q is the flow rate. With averaging, we find an approximate closed-form solution for Q.

Each network can be divided into a set of *tree* branches, which connect all the nodes without creating loops, and the complement of the tree, called co-tree, whose branches are called *links*. By KCL, the flows in the branches are not independent—the links are independent variables—and the flows in the tree branches depend on them.

Venous disease progression is an active area of research (Bassez, Flaud, & Chauveau, 2001; Skalak, Keller, & Secomb, 1981; Wild, Pedley, & Riley, 1977) but its mechanisms are not fully understood. Veins in lower limbs contain valves which prevent backflow. Their failure initiates the disease, followed by the vein dilation, and ultimately thrombosis, ulcer, etc. There were several attempts at mathematical analysis of blood flow (Conrad, 1969; Kresch & Noordergraaf, 1969; Lambert, 1958; Rudinger, 1966).

We start in Section 2 with a simple introductory example. In Section 3 we derive a minimal model of the network, using pipe flow dynamics equations and Kirchhoff's laws, and in Section 4 we do averaging analysis of the network. In Section 5 we consider three examples: first, a network with two branches (and a generator branch) for which we can find closed-form solution; second, a network with four branches, for which we can find closed-form solution for a constant input and then apply the main result for a periodic input;

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third, a network with five branches, for which we calculate the approximate solution numerically.

2. Introductory example

Consider a network with two parallel branches, as in Fig. 1, with $Q_{in} = Q_1 + Q_2$, $H_1 = H_2$, where Q_j are flows through the branches and H_j are pressure drops of the branches. The network is driven by a periodic generator $Q_{in}(t) = Q_0 + a\sin\omega t$. As will be explained later, the dynamic equations of the branches are $T_1\dot{Q}_1 + R_1Q_1^2 = H_1$, $T_2\dot{Q}_2 + R_2Q_2^2 = H_2$, where R_j are aero/hydrodynamic resistances and T_j are inertia coefficients. We obtain

$$T_1\dot{Q}_1 + R_1Q_1^2 = T_2\dot{Q}_{in} - T_2\dot{Q}_1 + R_2(Q_{in} - Q_1)^2. \quad (1)$$

In the case when there are no resistances, $R_1 = R_2 = 0$, we get a linear differential equation $(T_1 + T_2)\dot{Q}_1 = T_2 a\omega\cos(\omega t)$, whose solution (for zero i.c.'s) is

$$Q_1(t) = \frac{T_2}{T_1 + T_2} a\sin(\omega t). \quad (2)$$

In the static case, i.e., when $a = T_1 = T_2 = 0$, (1) becomes a quadratic equation $(R_1 - R_2)Q_1^2 + 2R_2Q_0Q_1 - R_2Q_0^2 = 0$. The solution of this equation is

$$Q_1 = \frac{\sqrt{R_2}}{\sqrt{R_1} + \sqrt{R_2}} Q_0. \quad (3)$$

Adding (2) and (3) we get an $O(a)$ approximation,

$$\hat{Q}_1(t) = \frac{\sqrt{R_2}}{\sqrt{R_1} + \sqrt{R_2}} Q_0 + \frac{T_2}{T_1 + T_2} a\sin(\omega t). \quad (4)$$

As we shall show in Section 5.2, the $O(a^2)$ approximation is

$$\overline{Q}_1(t) = \frac{\sqrt{R_2}}{\sqrt{R_1} + \sqrt{R_2}} Q_0 + \frac{T_2}{T_1 + T_2} a\sin(\omega t) + \frac{a^2}{4Q_0\sqrt{R_1R_2}} \frac{R_2T_1^2 - R_1T_2^2}{(T_1 + T_2)^2}. \quad (5)$$

The third term is the result of combined nonlinear and time-varying effects. It is obtained by the method of averaging. The rest of this paper develops this idea for general networks and discusses a possible application.

3. Pipe flow dynamics and Kirchhoff's laws

We consider a network driven by a single ideal current source with flow $Q_{in}(t)$. A branch of a fluid network is modeled as (Kocić, 1979; Hu et al., 2003; Koroleva et al., 2004; Ward-Smith, 1980)

$$T_j \frac{\mathrm{d}Q_j}{\mathrm{d}t} + R_j|Q_j|Q_j = H_j, \quad (6)$$

where Q_j is flow through a branch j, R_j are aero/hydrodynamic resistances, H_j are pressure drops of the branches,

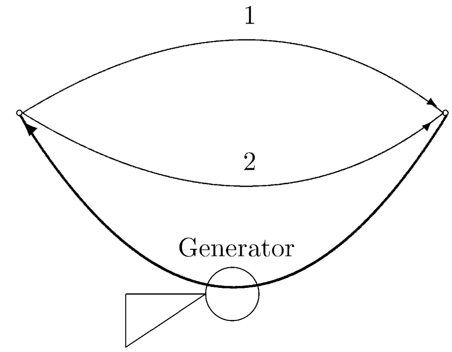

Fig. 1. Fluid network with a generator and two branches.

T_j are inertia coefficients, $j = 1, \ldots, n$ and n is the number of network branches (excluding the generator branch). For ease of notation, it is assumed throughout the paper that the equilibrium solutions for the Q_j's when $a = 0$ are positive, i.e., the reference directions, have been chosen properly and $Q_j|Q_j|$ can be replaced by Q_j^2 (except possibly during initial transients). We write this in vector form as

$$T\dot{Q} = -Q_D^2 R + H, \quad (7)$$

where $T = \mathrm{diag}\{T_j\}$, $R = \mathrm{col}\{R_j\}$ and $Q_D^2 = \mathrm{diag}\{Q_j^2\}$. Let n_c denote the number of nodes. Then $l = n - n_c + 1$ is the number of links (excluding the generator branch) and $n - l$ is the number of tree branches.

Like an electrical network, a fluid network must satisfy Kirchhoff's current law,

$$E_{Q_{in}} \begin{bmatrix} Q_{in} \\ Q \end{bmatrix} = 0,$$

or $\sum_{j=1}^{n} E_{Qij}Q_j + e_{Q_{in_i}}Q_{in} = 0$, $i = 1, \ldots, n - l$, where $n-l+1$ is the number of nodes (of which one is a "reference" node), Q is a vector of flows, $E_{Q_{in}} = [e_{Q_{in}}\ E_Q]$, and $E_Q = [E_{Qij}]$ is a full rank matrix of order $(n - l) \times n$ where $E_{Qij} = 1$ if branch j is connected to node i and the flow goes away from node i, $E_{Qij} = -1$ if it goes into node i, $E_{Qij} = 0$ if branch j is not connected to node i; e_{Qin} is an $(n - l) \times 1$ vector such that, if the generator is connected to node i and the flow goes away from node i then $e_{Qin_i} = 1$, if the flow goes into node i then $e_{Qin_i} = -1$, and $e_{Qin_i} = 0$ if the generator is not connected to node i.

Similarly, the network satisfies Kirchhoff's voltage law, $E_H H = 0$, or

$$\sum_{j=1}^{n} E_{Hij}H_j = 0, \quad i = 1, \ldots, l, \quad (8)$$

where H_j is the pressure drop of the branch j, H is a vector of pressure drops, $E_H = [E_{Hij}]$ is an $l \times n$ mesh matrix, in which each mesh (loop) is formed by a link and a unique chain in the tree connecting the two nodes of the link. The elements of E_{Hij} are defined as follows: $E_{Hij} = 1$ if branch j is contained in mesh i and has the same direction, $E_{Hij} = -1$

if branch j is contained in mesh i and has the opposite direction, $E_{Hij} = 0$ if branch j is not contained in mesh i.

We take the flows of link (co-tree) branches as state variables and include the generator branch into the set of links since its flow is given as

$$Q_{in}(t) = Q_0 + a \sin \omega t. \tag{9}$$

For convenience of analysis, we label the link branches (except the generator branch) from 1 to l. Define

$$Q = \begin{bmatrix} Q_c \\ Q_a \end{bmatrix}, \quad H = \begin{bmatrix} H_c \\ H_a \end{bmatrix}, \tag{10}$$

so that Q_c and H_c vectors describe flow and pressure drop in the links, excluding the generator branch, and Q_a and H_a in the tree branches.

The matrices E_H and $E_{Q_{in}}$ can be split into blocks $E_H = [E_{Hc}\ E_{Ha}]$, $E_{Q_{in}} = [e_{Q_{in}}\ E_{Qc}\ E_{Qa}]$, where (Desoer & Kuh, 1969; Hu et al., 2003; Koroleva et al., 2004) $E_{Qa} = I_{(n-l)\times(n-l)}$, $E_{Hc} = I_{l\times l}$, $E_{Ha} = -E_{Qc}^{\mathrm{T}}$. Hence, the structure of the network can be expressed in matrix form as

$$E = \begin{bmatrix} 0 & I & -E_{Qc}^{\mathrm{T}} \\ e_{Q_{in}} & E_{Qc} & I \end{bmatrix}.$$

Furthermore,

$$T = \begin{bmatrix} T_c & 0 \\ 0 & T_a \end{bmatrix}, \quad R = [R_c^{\mathrm{T}}\ R_a^{\mathrm{T}}]^{\mathrm{T}}.$$

4. Main result

Theorem 4.1. *Let*

$$T_0(T) = T_c + E_{Qc}^{\mathrm{T}} T_a E_{Qc}, \tag{11}$$

$$B_c(T, E) = -T_0^{-1} E_{Qc}^{\mathrm{T}} T_a e_{Qin}, \tag{12}$$

$$B_a(T, E) = -(I - E_{Qc} T_0^{-1} E_{Qc}^{\mathrm{T}} T_a) e_{Qin}, \tag{13}$$

$$U(R, T, E) = \mathrm{col}\{B_{c_i}^2 R_{c_i}\} - E_{Qc}^{\mathrm{T}} \mathrm{col}\{B_{a_i}^2 R_{a_i}\}, \tag{14}$$

$$W = \{E_{Qc_{ij}}(-E_{Qc_i} Q_{c0} - e_{Qin_i} Q_0) R_{a_i}\}_{(n-l)\times l}, \tag{15}$$

$$V(R, E, Q_0) = \mathrm{diag}\{Q_{c0_i} R_{c_i}\} + E_{Qc}^{\mathrm{T}} W, \tag{16}$$

where $Q_{c0}(R, E, Q_0)$ denotes a solution to the l-dimensional quadratic equation

$$Q_{c0D}^2 R_c - E_{Qc}^{\mathrm{T}} \mathrm{diag}\{(E_{Qc_i} Q_{c0} + e_{Qin_i} Q_0)^2\} R_a = 0 \tag{17}$$

such that V is nonsingular and $-T_0^{-1} V$ is Hurwitz. Then for a given $Q_0 > 0$, for sufficiently small a and sufficiently large ω the solutions of the system (7), (9) *locally exponentially converge to an $O(1/\omega + a^4)$ neighborhood of*

$$\overline{Q}_c(t) = Q_{c0} - \frac{a^2}{4} V^{-1} U + B_c a \sin \omega t, \tag{18}$$

$$\overline{Q}_a(t) = (-E_{Qc} Q_{c0} - e_{Qin} Q_0) + \frac{a^2}{4} E_{Qc} V^{-1} U + B_a a \sin \omega t. \tag{19}$$

Proof. With (10) the flow rates through tree branches can be expressed by flows through links:

$$Q_a = -e_{Qin} Q_{in} - E_{Qc} Q_c, \tag{20}$$

$$\dot{Q}_a = -e_{Qin} \dot{Q}_{in} - E_{Qc} \dot{Q}_c. \tag{21}$$

From branch equation (7), with (10) and (21) we get

$$\begin{bmatrix} H_c \\ H_a \end{bmatrix} = \begin{bmatrix} T_c \dot{Q}_c \\ -T_a e_{Q_{in}} \dot{Q}_{in} - T_a E_{Qc} \dot{Q}_c \end{bmatrix} + \begin{bmatrix} Q_{cD}^2 R_c \\ Q_{aD}^2 (Q_c, Q_{in}) R_a \end{bmatrix}. \tag{22}$$

Rewrite (8) as $0 = T_c \dot{Q}_c + Q_{cD}^2 R_c + E_{Qc}^{\mathrm{T}} T_a E_{Qc} \dot{Q}_c - E_{Qc}^{\mathrm{T}} Q_{aD}^2 R_a + E_{Qc}^{\mathrm{T}} T_a e_{Q_{in}} \dot{Q}_{in}$, or, rearranging this,

$$-T_0 \dot{Q}_c - E_{Qc}^{\mathrm{T}} T_a e_{Q_{in}} \dot{Q}_{in} = Q_{cD}^2 R_c - E_{Qc}^{\mathrm{T}} Q_{aD}^2 R_a, \tag{23}$$

where T_0 is invertible (Hu et al., 2003). Denote

$$\begin{aligned} \widetilde{Q}_c &= Q_c + T_0^{-1} E_{Qc}^{\mathrm{T}} T_a e_{Q_{in}} a \sin(\omega t) - Q_{c0} \\ &= Q_c - Q_{c0} - B_c a \sin(\omega t). \end{aligned} \tag{24}$$

Then (23) can be rewritten as

$$-T_0 \dot{\widetilde{Q}}_c = Q_{cD}^2 R_c - E_{Qc}^{\mathrm{T}} Q_{aD}^2 R_a. \tag{25}$$

Denote the RHS of (25) by $f(\omega t, \widetilde{Q}_c, a^2, \frac{1}{\omega}) = Q_{cD}^2 R_c - E_{Qc}^{\mathrm{T}} Q_{aD}^2 R_a = \mathrm{diag}\{2Q_{c0_i}(\widetilde{Q}_{c_i} + B_{c_i} a \sin(\omega t)) + (\widetilde{Q}_{c_i} + B_{c_i} a \sin(\omega t))^2\} R_c - E_{Qc}^{\mathrm{T}} \mathrm{diag}\{2(e_{Qin_i} Q_0 + E_{Qc_i}(\widetilde{Q}_c + Q_{c0}))(e_{Qin_i} + E_{Qc_i} B_c) a \sin \omega t\} R_a - E_{Qc}^{\mathrm{T}} \mathrm{diag}\{(e_{Qin_i} + E_{Qc_i} B_c)^2 a^2 \sin^2 \omega t\} R_a - E_{Qc}^{\mathrm{T}} \mathrm{diag}\{2E_{Qc_i} \widetilde{Q}_c (e_{Qin_i} Q_0 + E_{Qc_i} Q_{c0}) + (E_{Qc_i} \widetilde{Q}_c)^2\} R_a + f_1(Q_{c0})$, where $f_1(Q_{c0}) = Q_{c0D}^2 R_c - E_{Qc}^{\mathrm{T}} \mathrm{diag}\{(E_{Qc_i} Q_{c0} + e_{Qin_i} Q_0)^2\} R_a = 0$ according to (17). Eq. (25) becomes

$$\frac{\mathrm{d}\widetilde{Q}_c}{\mathrm{d}(\omega t)} = -\frac{1}{\omega} T_0^{-1} f\left(\omega t, \widetilde{Q}_c, a^2, \frac{1}{\omega}\right). \tag{26}$$

Next we calculate

$$\begin{aligned} f_{av}(\widetilde{Q}_c, a^2) &= \frac{1}{2\pi} \int_0^{2\pi} f(\omega t, \widetilde{Q}_c, a^2, 0)\, \mathrm{d}(\omega t) \\ &= -E_{Qc}^{\mathrm{T}} \mathrm{diag}\{(E_{Qc_i} \widetilde{Q}_c)^2 \\ &\quad + 2E_{Qc_i} \widetilde{Q}_c (e_{Qin_i} Q_0 + E_{Qc_i} Q_{c0})\} R_a \\ &\quad + \mathrm{diag}\{2Q_{co_i} \widetilde{Q}_{c_i} + \widetilde{Q}_{c_i}^2\} R_c \\ &\quad + \frac{a^2}{2} \mathrm{diag}\{B_{c_i}^2\} R_c \\ &\quad - \frac{a^2}{2} E_{Qc}^{\mathrm{T}} \mathrm{diag}\{B_{a_i}^2\} R_a. \end{aligned} \tag{27}$$

With U and V defined by (14), (16), we rewrite (27) as

$$f_{av}(\widetilde{Q}_c, a^2) = 2V \widetilde{Q}_c + \frac{a^2}{2} U + \mathrm{col}\{\widetilde{Q}_c^{\mathrm{T}} Z_i \widetilde{Q}_c\}, \tag{28}$$

where $Z_i = -\sum_{j=1}^{n-l} E_{Qc_{ji}} E^{\mathrm{T}}_{Qc_j} E_{Qc_j} R_{a_j} + \mathrm{diag}\{0, \ldots, 0, R_{c_i}, 0, \ldots, 0\}$.

So, the average system is

$$\frac{\mathrm{d}\widetilde{Q}_c}{\mathrm{d}(\omega t)} = -\frac{1}{\omega} T_0^{-1} f_{av}(\widetilde{Q}_c, a^2). \tag{29}$$

To find the equilibrium points of (29), one needs to find the solution $\widetilde{Q}_c(a^2)$ of $f_{av}(\widetilde{Q}_c, a^2) = 0$. By the implicit function theorem, since V is assumed to be invertible, there exists such a solution. It can be written as

$$\widetilde{Q}_c^{av} = -\frac{a^2}{4} V^{-1} U + O(a^4). \tag{30}$$

The Jacobian of (29) at $\widetilde{Q}_c^{av}$ is $J = -\frac{2}{\omega} T_0^{-1} V + O(a^2)$. Since $-T_0^{-1} V$ is Hurwitz, for sufficiently small a, J will also be Hurwitz. By the averaging theorem (Khalil, 2002) there exists an exponentially stable periodic solution $\widetilde{Q}_c^{2\pi/\omega}(t)$ of period $2\pi/\omega$ in the $1/\omega$-neighborhood of the average equilibrium $\widetilde{Q}_c^{av}$, that is, $\widetilde{Q}_c(t) = -\frac{a^2}{4} V^{-1} U + O(\frac{1}{\omega} + a^4) + \varepsilon^{-t}$, where ε^{-t} denotes exponentially decaying terms.[1] Rewrite (24) as $Q_c(t) = \widetilde{Q}_c(t) + Q_{c0} + B_c a \sin(\omega t)$. With (30) this becomes(18) plus $O\left(\frac{1}{\omega} + a^4\right) + \varepsilon^{-t}$, which, combined with (20), gives (19). $O(\frac{1}{\omega} + a^4) + \varepsilon^{-t}$. $\square$

Remark 4.1. The vector field (28) is a vector-valued quadratic form in $\widetilde{Q}_c$. In the scalar case the solution to the quadratic equation $f_{av}(\widetilde{Q}_c, a^2) = 0$ would be explicit. In general, only an approximate solution (for small a) can be obtained.

Remark 4.2. It is worth noting that in (18), (19) the first (respective) terms are due to the resistive part of the network (R), the third (sinusoidal) terms are due to the "inductive" part of the network (T), and the second terms are due to both resistance and inductivity.

5. Examples

5.1. Two-branch electric circuit

Consider the circuit in Fig. 2, which is driven by $I_{in}(t) = I_0 + a \sin(\omega t)$. The response of the current in the first branch is

$$I_1(s) = \frac{R_2 + L_2 s}{R_1 + R_2 + (L_1 + L_2)s} I_{in}(s),$$

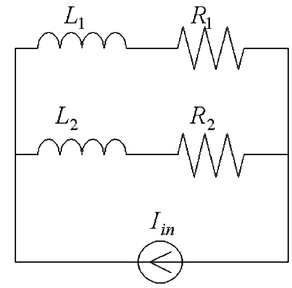

Fig. 2. Electric circuit with 2 branches and current generator.

or, in time domain

$$I_1(t) = \frac{I_0}{R_1 + R_2}\left(R_2 + \frac{L_2 R_1 - L_1 R_2}{L_1 + L_2} e^{-\frac{R_1 + R_2}{L_1 + L_2} t}\right) + a\omega \frac{L_2 R_1 - L_1 R_2}{(R_1 + R_2)^2 + \omega^2 (L_1 + L_2)^2} \times \left(-e^{-\frac{R_1 + R_2}{L_1 + L_2} t} + \cos(\omega t)\right) + \frac{(R_1 + R_2) R_2 + \omega^2 (L_1 + L_2) L_2}{(R_1 + R_2)^2 + \omega^2 (L_1 + L_2)^2} a \sin(\omega t)$$

which, for large ω and t (and for any a), becomes

$$I_1(t) \approx \frac{R_2}{R_1 + R_2} I_0 + \frac{L_2}{L_1 + L_2} a \sin(\omega t). \tag{31}$$

Thus, the DC response depends only on the resistive effects, and, for fast forcing, the AC response depends only on the inductive effects. As we shall see in the next section, the former is not the case for the fluid networks, where the resistive effect is nonlinear.

5.2. Two-branch fluid network

Let us now consider a fluid network example where we can calculate the flows in closed form. The network consists of two parallel branches, as shown in Fig.1. Branch 1 is the link and branch 2 is the tree of the network. Thus $T_c = T_1$, $T_a = T_2$, $R_c = R_1$, $R_a = R_2$, and we get $E_{Hc} = 1$, $E_{Ha} = -1$, $e_{Qin} = -1$, $E_{Qc} = 1$, $E_{Qa} = 1$. Following the procedure given in the statement of Theorem 4.1, we get $T_0 = T_1 + T_2$,

$$B_c = \frac{T_2}{T_1 + T_2}, \quad B_a = \frac{T_1}{T_1 + T_2}, \quad U = \frac{T_2^2 R_1 - T_1^2 R_2}{T_1 + T_2},$$

$V = Q_{c0} R_1 - W = Q_{c0}(R_1 - R_2) - Q_0 R_2$, $W = (Q_{c0} - Q_0) R_2$, where Q_{c0} is the solution of quadratic equation $Q_1^2 R_1 - (Q_1 - Q_0)^2 R_2 = 0$, such that $V \neq 0$ and $-T_0^{-1} V < 0$. We obtain

$$Q_{c0} = \frac{\sqrt{R_2}}{\sqrt{R_1} + \sqrt{R_2}} Q_0,$$

with $V = \sqrt{R_1 R_2} Q_0 \neq 0$, and

$$-T_0^{-1} V = -\frac{\sqrt{R_1 R_2}}{T_1 + T_2} Q_0 < 0.$$

[1] The averaging theorem requires nonlinearities $Q|Q|$ to be in C^2, which is satisfied in the neighborhood of the equilibria.

With these equations we get (5). Let us now write the average system of (1) as

$$(T_1+T_2)\dot{\tilde{Q}}_1 = f_{av}(\tilde{Q}_1, a^2) = (R_2-R_1)\tilde{Q}_1^2 - 2\sqrt{R_1R_2}Q_0\tilde{Q}_1 - \frac{a^2}{2}\frac{R_1T_2^2-R_2T_1^2}{(T_1+T_2)^2},$$

where

$$\tilde{Q}_1 = Q_1 - \frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}}Q_0 - \frac{T_2}{T_1+T_2}a\sin(\omega t).$$

While the second term in (5) represents an approximate average equilibrium, the average equilibrium in this scalar situation can be found exactly, providing a more accurate, $O(1/\omega)$ approximation

$$\check{Q}_1(t) = \frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}}Q_0 + \frac{T_2}{T_1+T_2}a\sin(\omega t) + \frac{a^2}{2Q_0\sqrt{R_1R_2}}\frac{R_2T_1^2-R_1T_2^2}{(T_1+T_2)^2} \times \frac{1}{\left(1+\sqrt{1+\frac{a^2}{2Q_0^2}\frac{R_1-R_2}{R_1R_2}\frac{R_2T_1^2-R_1T_2^2}{(T_1+T_2)^2}}\right)}, \quad (32)$$

whereas (5) is an $O(1/\omega + a^4)$ approximation.

Expression (5) is very similar to (31) for $L_i = T_i$ and $I_0 = Q_0$. One minor difference is between R_i and $\sqrt{R_i}$, where the latter appears due to the quadratic nature of resistive losses in the fluid network. The other difference is the absence of the a^2-order DC term in (31). This term appears in (5) due to the nonlinearities in the fluid network. We note that this term depends on R_i, T_i, a, Q_0—i.e., all the problem data except the frequency ω. This term becomes significant when Q_0 becomes relatively small in comparison to a but has no effect when $R_2T_1^2 \approx R_1T_2^2$, i.e., when, say, one branch is wide and smooth, while the other is narrow and rough, which follows from $T_i = \rho l_i/S_i$, $R_i = r_i l_i$, where S_i is cross-section, l_i is length, ρ is density and r_i is specific resistance.

5.3. *Peakiness and venous disease*

To examine (5) further and analyze its meaning for blood flow networks, let us introduce *peakiness*

$$P = \frac{\frac{T_2}{T_1+T_2}a}{\frac{\sqrt{R_2}}{\sqrt{R_1}+\sqrt{R_2}}Q_0 + \frac{a^2}{4Q_0\sqrt{R_1R_2}}\frac{R_2T_1^2-R_1T_2^2}{(T_1+T_2)^2}} \quad (33)$$

as the ratio of AC and DC components of the flow. Clearly, the peakiness of the input $Q_{in}(t)$ is $P_0 = a/Q_0$. Let the first branch be σ^2 times longer and μ times wider than second branch, i.e., $R_1 = rl\sigma^2$, $R_2 = rl$, $T_1 = \rho\sigma^2 l/\mu S$, $T_2 = \rho l/S$. (Taking $\sigma < 1$ or $\mu < 1$ means that the first branch is shorter, or, respectively, narrower.) Then the normalized peakiness,

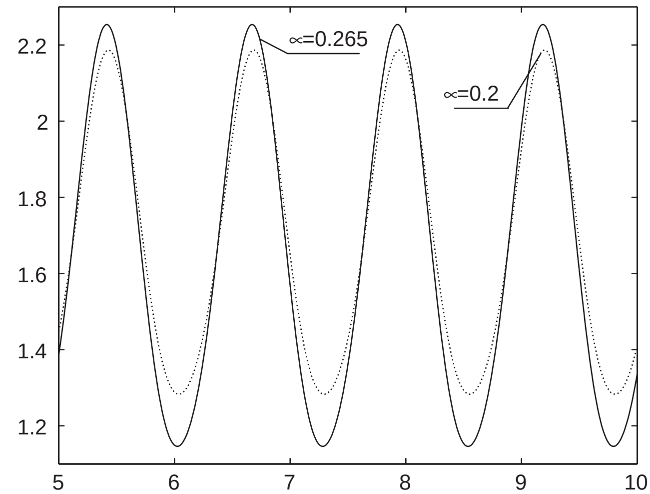

Fig. 3. Flow in the first branch with $R_1 = R_2 = 2$, $T_1 = T_2/\mu$, $T_2 = 1$ and $\mu = 0.2$ and $\mu = 0.265$.

the ratio between P and P_0, is

$$\frac{P}{P_0} = \mu\frac{(\sigma+1)(\sigma^2+\mu)}{(\sigma^2+\mu)^2 + \frac{P_0^2}{4}\sigma(\sigma+1)(\sigma^2-\mu^2)}. \quad (34)$$

The partial derivative of (34) with respect to μ is

$$\frac{\partial(P/P_0)}{\partial\mu} = \frac{1}{[(\sigma^2+\mu)^2 + \frac{P_0^2}{4}\sigma(\sigma+1)(\sigma^2-\mu^2)]^2} \times \sigma^2(\sigma+1)[(\sigma^2+\mu)^2 + \frac{P_0^2}{4}\sigma(\sigma+1)(\sigma^2+2\mu+\mu^2)]. \quad (35)$$

One can see that (35) is always positive. Thus, when a vein gets wider, the flow through it gets more “peaky”. The extra stress promotes valve failure, which further increases the vein effective width. This “positive feedback” may explain the development of venous diseases.

Since our $\overline{Q}_1(t)$ is only an estimate, we present next an exact numerical simulation illustrating the above phenomenon. Consider the case where the branch lengths are equal, $l_1 = l_2$, and the first branch is narrower than second one: $S_1 = \mu S_2$, $\mu < 1$. Let the flow in the generator branch be $Q_{in} = 3 + 2\sin 5t$, which gives $P_0 = 2/3$. Consider vein widening from $\mu = 0.2$ to $\mu = 0.265$, which corresponds to a 15% increase in diameter. Then the analytical estimates are $P/P_0(\mu=0.2) = 0.2903$, $P/P_0(\mu = 0.265) = 0.3711$, i.e., a 27% increase in peakiness. The exact responses of the system are shown in Fig. 3, with peakiness $P/P_0(\mu=0.2) = 0.455/1.735 = 0.2622$ and $P/P_0(\mu = 0.265) = 0.56/1.7 = 0.3294$. So, the actual increase in peakiness is 25%. We chose μ small because venous disease tends to develop in narrower “superficial” veins, rather than in wider “deep” veins (Browse, Burnand, Irvine, & Wilson, 1999).

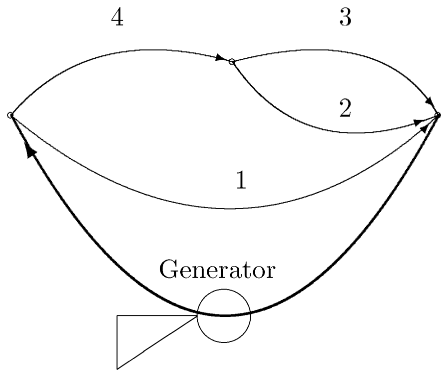

Fig. 4. Fluid network with 4 branches.

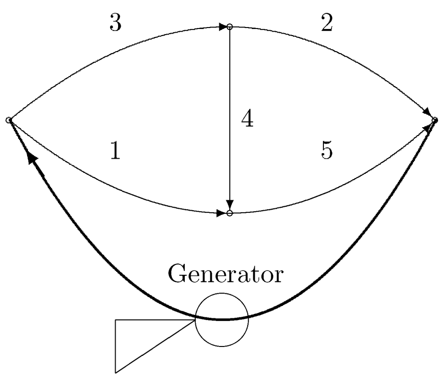

Fig. 5. Fluid network with 5 branches.

5.4. Four-branch network

In the previous example we had only one link. Now consider an example with vector equations, but with a solution in closed form. In the 4-branch network in Fig. 4, branches 3 and 4 form the tree. For simplicity, let $R_i = R$, $T_i = T$, $i = 1, \ldots, 4$. The approximations are

$$\overline{Q}_1(t) = Q_0 \frac{\sqrt{5}}{2+\sqrt{5}} - \frac{a^2}{Q_0}\frac{2}{25\sqrt{5}} + \frac{3}{5} a \sin(\omega t), \tag{36}$$

$$\overline{Q}_2(t) = Q_0 \frac{1}{2+\sqrt{5}} + \frac{a^2}{Q_0}\frac{1}{25\sqrt{5}} + \frac{1}{5} a \sin(\omega t), \tag{37}$$

$$\overline{Q}_3(t) = Q_0 \frac{1}{2+\sqrt{5}} + \frac{a^2}{Q_0}\frac{1}{25\sqrt{5}} + \frac{1}{5} a \sin(\omega t), \tag{38}$$

$$\overline{Q}_4(t) = Q_0 \frac{2}{2+\sqrt{5}} + \frac{a^2}{Q_0}\frac{2}{25\sqrt{5}} + \frac{2}{5} a \sin(\omega t). \tag{39}$$

5.5. Five-branch network

Let us now consider the network in Fig. 5, with 2 links as in the previous example, but with 3 tree branches. In this case we can find a solution only numerically. Let branches 3, 4 and 5 define the tree.

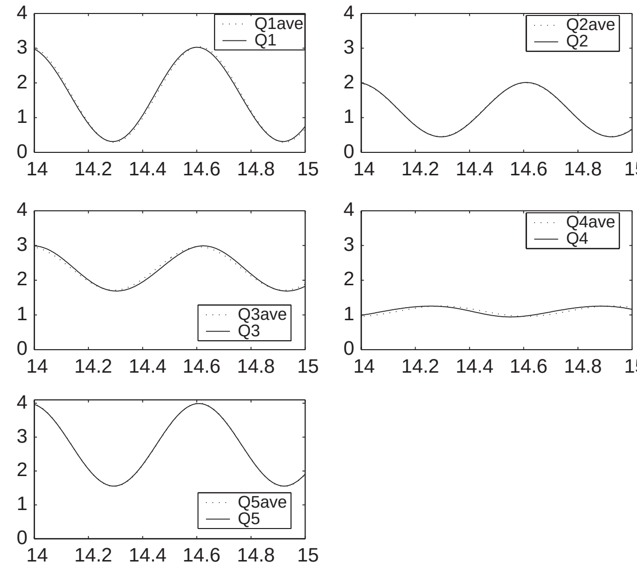

Fig. 6. Zoom in from 14 to 15 s. The flows in the five branch network.

We calculate T_0, B_c, B_a, U, W and V as in Eqs. (11)–(16). Then, Eq. (17) can be written as

$$R_1 Q_1^2 - R_3(Q_1 - Q_0)^2 - R_4(Q_1 + Q_2 - Q_0)^2 = 0, \tag{40}$$

$$R_2 Q_2^2 - R_4(Q_1 + Q_2 - Q_0)^2 - R_5(Q_2 - Q_0)^2 = 0. \tag{41}$$

To find a general closed-form solution, we need to solve a fourth-order equation analytically. This is possible in principle; however, since equations of order higher than four are not solvable analytically, we pursue a numerical solution here. Let $R_1 = 1.5$, $R_2 = 4$, $R_3 = 0.9$, $R_4 = 0.4$, $R_5 = 0.8$, $T_1 = 0.2$, $T_2 = 3.2$, $T_3 = 1.4$, $T_4 = 4$, $T_5 = 2.5$, $Q_0 = 4$, $a = 2$. By Theorem 4.1 we get

$$\overline{Q}_1(t) = 1.665 + 1.376 \sin(\omega t), \tag{42}$$

$$\overline{Q}_2(t) = 1.230 + 0.772 \sin(\omega t), \tag{43}$$

$$\overline{Q}_3(t) = 2.335 + 0.624 \sin(\omega t), \tag{44}$$

$$\overline{Q}_4(t) = 1.106 - 0.150 \sin(\omega t), \tag{45}$$

$$\overline{Q}_5(t) = 2.771 + 1.228 \sin(\omega t). \tag{46}$$

Fig. 6 shows simulations in comparison with average flows (42)–(46) for $\omega = 10$. They are almost indistinguishable. We selected the parameters to emulate the venous flow situation in a (healthy) leg. Branches 1 and 5 are deep veins, branches 2 and 3 are superficial veins and branch 4 is a “communicating” vein.

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