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Passivity and Parametric Robustness of a New Class of Adaptive Systems*†

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A recently proposed class of adaptive controllers is presented from a passivity perspective, its underlying linear nonadaptive controller is revealed, and a parametric robustness property proven.

Key Words—Adaptive control; passivity; parametric robustness; nonlinear control.

Abstract—A recently proposed recursive design of adaptive controllers for minimum phase linear systems with known but arbitrary relative degree is presented from a passivity perspective and stability is deduced from the well-known links between passivity and Lyapunov stability. With adaptation switched off, the feedback system is linear and has an additional parametric robustness property.

1. INTRODUCTION

MOST ADAPTIVE designs (Åström and Wittenmark, 1989; Goodwin and Sin, 1984; Narendra and Annaswamy, 1989; Sastry and Bodson, 1989) view multiplicative nonlinearities as 'adjustable parameters', in spite of the fact that these 'parameters' participate in highly nonlinear transients.

In a drastic departure from the existing adaptive outlook, we introduced a design of adaptive controllers as nonlinear feedback controllers (Krstić *et al.*, 1994). In this paper we reveal further useful properties of the designed adaptive systems. The most important among those are a strict passivity property of the main adaptation loop and a parametric robustness property of the underlying linear system when the adaptation is switched off. These properties belong to the list of desirable but unachieved goals of the traditional adaptive control.

Our presentation begins with preliminary developments in Section 2, where we give a detailed form of the state estimation filters. Our recursive design for plants of relative degrees one and two is presented in Section 3. The relative-degree-two scheme of this section resembles the relative-degree-two direct MRAC scheme in Narendra and Annaswamy (1989). Both schemes incorporate $\hat{\theta}$ in the control law u, and use unnormalized gradient update laws derived from strictly passive error systems. The difference is in the new form of the underlying nonadaptive controller which avoids the reparametrization required by the direct MRAC scheme and contains design coefficients which can be used to improve transients. The new scheme also avoids the need for regressor filtering.

The effectivenss of the new approach is fully expressed in the general design for relative degree ≥ 3 presented in Section 4. This section provides a new passivity perspective on the backstepping design (Kanellakopoulos *et al.*, 1991a, b; Marino and Tomei, 1991; Kanellakopoulos *et al.*, 1992; Krstić *et al.*, 1992). Similar to its earlier nonadaptive forms (Kokotović and Sussmann, 1989; Saberi *et al.*, 1990; Byrnes *et al.*, 1991; Ortega, 1991; Lozano *et al.*, 1992), this form of adaptive backstepping avoids the relative degree obstacle and leads to the desired strict passivity property for any linear minimum phase plant. Another way to avoid the relative degree obstacle, is using the high order tuners (Morse, 1992). A stability proof based on the

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links between passivity and Lyapunov stability, is outlined in Section 5.

Although the new adaptive design is nonlinear, the underlying system with the adaptation switched off is linear. This *detuned* linear system, discussed in Section 6, has a remarkable robustness property. When a bound on plant parameter uncertainty is known, design parameters can be chosen to guarantee stability without adaptation. For traditional model reference schemes, a similar parametric robust stability is achieved by introducing approximate derivatives in the control law (Sun *et al.*, 1991). The parametric robustness of the detuned linear systems indicates a separation of tasks of the nonadaptive and adaptive parts of the design. If the parameter uncertainty is small and the required values of the design parameters do not lead to high-gain feedback, then the use of adaptation can be avoided. If this is not the case, the adaptation is included to reduce the effects of parameter uncertainty. The structure of the underlying nonadaptive linear controller is given in Section 7. In Section 8, we illustrate the properties of the new adaptive systems on a relative-degree-three example.

2. PRELIMINARIES

2.1. *Problem statement*

The control objective is to asymptotically track a reference signal $y_r(t)$ with the output y of the plant

$$y(s)=\frac{B(s)}{A(s)}u(s)=\frac{b_ms^m+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}u(s). \quad (1)$$

Assumption 1. The plant is minimum phase, i.e. the polynomial $B(s)=b_ms^m+\cdots+b_1s+b_0$ is Hurwitz, and the plant order n, relative degree $\rho=n-m$, and high-frequency gain b_m are known.

Assumption 2. The reference signal $y_r(t)$ and its first ρ derivatives are known and bounded, and, in addition, $y_r^{(\rho)}(t)$ is piecewise continuous. In particular, $y_r(t)$ may be the output of a reference model of relative degree $\rho_r\geq\rho$ with a piecewise continuous input $r(t)$.

To simplify our presentation, we will consider the case where the high-frequency gain b_m is known, $b_m=1$. In a companion paper, the results have been extended to the case where only the sign of b_m is known (Krstić *et al.*, 1994).

2.2. *State estimation filters*

As in most traditional adaptive schemes, we first design state estimation filters. To this end, we represent the plant (1) as

$$\begin{aligned}\dot{x}_1&=x_2-a_{n-1}y,\\&\vdots\\\dot{x}_{\rho-1}&=x_\rho-a_{m+1}y,\\\dot{x}_\rho&=x_{\rho+1}-a_my+u,\\&\vdots\\\dot{x}_n&=-a_0y+b_0u,\\y&=x_1.\end{aligned}\quad (2)$$

In our recursive procedure, called *backstepping*, intermediate variables are successively treated as 'virtual controls'. If the parameters a_i, b_i were known and the state variables $x_2,\ldots,x_n$ were measured, backstepping would successively treat $x_2,x_3,\ldots,x_\rho$ as virtual controls. Since these variables are not measured, we must replace them by some estimates. To construct these estimates, we employ the filters

$$\begin{aligned}\dot{\xi}_n&=A_0\xi_n+ky,\\\dot{\xi}_i&=A_0\xi_i+e_{n-i}y,\quad 0\leq i\leq n-1,\\\dot{v}_i&=A_0v_i+e_{n-i}u,\quad 0\leq i\leq m,\end{aligned}\quad (3)$$

commonly used in adaptive observers (Kreisselmeier, 1977), where e_i denotes the ith coordinate vector in $\mathbb{R}^n$. The vector $k=[k_1,\ldots,k_n]^T$ is chosen so that the matrix

$$A_0=\begin{bmatrix}-k_1 & \\ \vdots & I_{n-1}\\ & 0\cdots0\\ -k_n & \end{bmatrix}$$

is Hurwitz. If the parameters a_i, b_i were known, then the vector $\xi_n-\sum_{i=0}^{n-1}a_i\xi_i+v_m+\sum_{i=0}^{m-1}b_iv_i$ would be an exponentially convergent estimate of x, because

$$\varepsilon=x-\left(\xi_n-\sum_{i=0}^{n-1}a_i\xi_i+v_m+\sum_{i=0}^{m-1}b_iv_i\right),\quad (4)$$

$$\dot{\varepsilon}=A_0\varepsilon. \quad (5)$$

From (3) it may appear that $m+n+2$ filters are needed to generate the signals ξ_i and v_i. Fortunately, this is not so. In view of $A_0^ie_n=e_{n-i}$, $0\leq i\leq n-1$ and $A_0^ne_n=-k$, employing the algebraic expressions

$$\begin{aligned}&\xi_i=A_0^i\eta,\quad 0\leq i\leq n-1,\quad \xi_n=-A_0^n\eta,\\&v_i=A_0^i\lambda,\quad 0\leq i\leq m,\end{aligned}\quad (6)$$

it is easy to prove that these signals can be obtained from only two filters, one at the output and the other at the input of the plant (1):

$$\dot{\eta} = A_0\eta + e_n y,$$
$$\dot{\lambda} = A_0\lambda + e_n u. \tag{7}$$

Let $\hat{\theta}$ be an estimate of the parameter vector

$$\theta^{\mathrm{T}} = [-a_{n-1}, \ldots, -a_0, b_{m-1}, \ldots, b_0], \tag{8}$$

and let $\tilde{\theta} = \theta - \hat{\theta}$ be the corresponding parameter estimation error. We will need a compact form of x_2. Introducing the signal row-vectors

$$\xi_{(2)} = [\xi_{n-1,2,\ldots,}\xi_{0,2}], \quad \bar{v}_{(2)} = [v_{m-1,2,\ldots,}v_{0,2}], \tag{9}$$

combining (4), (8) and (9) we obtain

$$x_2 = v_{m,2} + \xi_{n,2} + [\xi_{(2)}, \bar{v}_{(2)}]\hat{\theta} + [\xi_{(2)}, \bar{v}_{(2)}]\tilde{\theta} + \varepsilon_2. \tag{10}$$

The first three terms in this compact expression are implementable. The last two terms incorporate the estimation errors $\tilde{\theta}$ and ε_2, which are unknown. However, from (5) it is known that ε_2 is a bounded exponentially decaying signal.

The key expression (10) is the starting point of our recursive procedure.

2.3. *Backstepping with passivity*

At each step i of the recursive procedure we construct an *error system* $\mathcal{S}_i$ and a *tuning function* τ_i as its output. For each error system we design a *stabilizing function* α_i to guarantee that the operator from the parameter error input $\tilde{\theta}$ to the tuning output τ_i is *strictly passive.* At each consecutive step the order of the error system $\mathcal{S}_i$ is increased by one. The design is completed at the ρth step, where ρ is the relative degree of equation (1). A schematic representation of the recursive procedure is given in Fig. 1, where the last tuning function τ_ρ is used to close the adaptive feedback loop via the passive parameter update law $\dot{\hat{\theta}} = \Gamma\tau_\rho$ with a positive definite gain matrix Γ. In the procedure we will also use the positive constants c_i, d_i, $1 \le i \le \rho$.

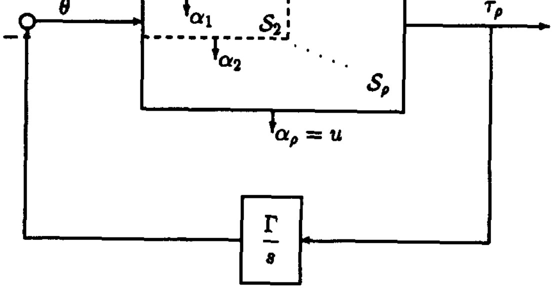

FIG. 1. The schematic representation of the design procedure.

3. DESIGN FOR RELATIVE DEGREE ONE AND TWO

For a plant of relative degree ρ, the recursive design procedure is in ρ steps. The reason the design is completed at the ρth step is that the control u appears for the first time in the ρth equation of (2). Although in this section we deal with plants of relative degree one and two, so that u should appear in the first or the second equation of (2), we will present the first two steps of the procedure in a form convenient for higher relative degree design. At each step we indicate how to complete the design if this happens to be the final step.

Step 1. We start with the equation for the tracking error $z_1 = y - y_r$, namely

$$\dot{z}_1 = x_2 - a_{n-1}y - \dot{y}_r. \tag{11}$$

Before we replace x_2 from the key expression (10), we define the *regressor* vector

$$\omega^{\mathrm{T}} = [\xi_{(2)} + e_1^{\mathrm{T}}y, \bar{v}_{(2)}], \tag{12}$$

where $e_1^{\mathrm{T}}y$ is added to the vector $[\xi_{(2)}, \bar{v}_{(2)}]$ to account for the term $a_{n-1}y$ in (11). Now the substitution of (10) and (12) into (11) results in

$$\dot{z}_1 = v_{m,2} + \xi_{n,2} + \omega^{\mathrm{T}}\hat{\theta} - \dot{y}_r + \omega^{\mathrm{T}}\tilde{\theta} + \varepsilon_2. \tag{13}$$

If the plant relative degree were $\rho = 1$, an additional term in (11) would be the actual control u for which we would design our stabilizing control law α_1. To prepare for a higher relative degree design we choose $v_{m,2}$ to be our first virtual control and design for it a stabilizing feedback law

$$\alpha_1 = -c_1 z_1 - d_1 z_1 - \xi_{n,2} + \dot{y}_r - \omega^{\mathrm{T}}\hat{\theta}. \tag{14}$$

If $v_{m,2}$ were our actual control, the substitution of $v_{m,2} = \alpha_1$ into (13) would result in

$$\dot{z}_1 = -c_1 z_1 - d_1 z_1 + \omega^{\mathrm{T}}\tilde{\theta} + \varepsilon_2, \tag{15}$$

which is an exponentially stable system perturbed by the estimation error terms $\omega^{\mathrm{T}}\tilde{\theta} + \varepsilon_2$.

This, together with (5), $\dot{\varepsilon} = A_0\varepsilon$, is our first error system $\mathcal{S}_1$ in which we treat $\tilde{\theta}$ as the input. For this input we now select our first tuning output τ_1 so that the operator from $\tilde{\theta}$ to τ_1 is strictly passive. This output can be defined using the transpose ω of the input vector ω^{T}, namely

$$\tau_1 = \omega z_1. \tag{16}$$

The strict passivity of the operator from $\tilde{\theta}$ to τ_1 follows from the fact (see Appendix A) that there exist a storage function $V_1(z_1, \varepsilon)$, and a dissipation rate $\psi_1(z_1, \varepsilon)$, such that

$$\int_0^t \tau_1^{\mathrm{T}}\tilde{\theta}\,d\sigma \ge V_1(t) - V_1(0) + \int_0^t \psi_1(\sigma)\,d\sigma. \tag{17}$$

In particular, (17) is satisfied for (15) with

$$V_1(z_1, \varepsilon) = \frac{1}{2}\left(z_1^2 + \frac{1}{d_1}\varepsilon^{\mathrm{T}}P\varepsilon\right)$$

and

$$\psi_1(z_1, \varepsilon) = c_1 z_1^2 + \frac{1}{4d_1}\varepsilon^{\mathrm{T}}\varepsilon.$$

To see this, note that from (5) and (15) we obtain

$$\begin{aligned}\dot{V}_1 &\le -c_1 z_1^2 + z_1\omega^{\mathrm{T}}\tilde{\theta} - d_1\left(z_1 - \frac{1}{2d_1}\varepsilon_2\right)^2 \\ &\quad - \frac{1}{4d_1}\varepsilon^{\mathrm{T}}\varepsilon \\ &\le -c_1 z_1^2 - \frac{1}{4d_1}\varepsilon^{\mathrm{T}}\varepsilon + z_1\omega^{\mathrm{T}}\tilde{\theta}. \qquad (18)\end{aligned}$$

For the integral of the input–output product $\tau_1^{\mathrm{T}}\tilde{\theta} = z_1\omega^{\mathrm{T}}\tilde{\theta}$ to appear, we integrate (18) over $[0, t]$ and verify that (17) is satisfied.

In view of this strict passivity property, the simplest update law for $\hat{\theta}$ would be $\dot{\hat{\theta}} = -\dot{\tilde{\theta}} = \Gamma\tau_1$, which is passive because

$$\int_0^t (-\dot{\tilde{\theta}})^{\mathrm{T}}\tau_1 d\sigma = \tfrac{1}{2}\tilde{\theta}(t)^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}(t) - \tfrac{1}{2}\tilde{\theta}(0)^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}(0). \qquad (19)$$

In the case where $\rho = 1$, this would complete our design. The true update law would be $\dot{\hat{\theta}} = \Gamma\tau_1$, and the actual feedback control law would be $u = -v_{m,2} + \alpha_1$, where α_1 is as in (14).

Step 2. In the case where $\rho > 1$, we cannot implement α_1 as a control law, and we *do not* use $\dot{\hat{\theta}} = \Gamma\tau_1$ as the update law for $\hat{\theta}$. Instead, we retain τ_1 as our first *tuning function* and α_1 as our first *stabilizing function.*

Since we cannot implement $v_{m,2} = \alpha_1$, we introduce an error variable $z_2 = v_{m,2} - \alpha_1$, which, in view of (14), satisfies the equation

$$\begin{aligned}\dot{z}_2 &= v_{m,3} - k_2 v_{m,1} - \frac{\partial\alpha_1}{\partial y}(\xi_{n,2} + v_{m,2} + \omega^{\mathrm{T}}\theta + \varepsilon_2) \\ &\quad - \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial\alpha_1}{\partial\dot{y}_r}\ddot{y}_r - \frac{\partial\alpha_1}{\partial\xi_n}(A_0\xi_n + ky) \\ &\quad - \sum_{i=0}^{n-1}\frac{\partial\alpha_1}{\partial\xi_i}(A_0\xi_i + e_{n-i}y) \\ &\quad - \sum_{i=0}^{m-1}\frac{\partial\alpha_1}{\partial v_{i,2}}(v_{i,3} - k_2 v_{i,1}) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} \\ &= v_{m,3} + \beta_2 - \frac{\partial\alpha_1}{\partial y}\omega^{\mathrm{T}}\theta - \frac{\partial\alpha_1}{\partial y}\varepsilon_2 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}, \qquad (20)\end{aligned}$$

where β_2 denotes all the known terms except for $v_{m,3}$. Combining (13), (14) and (20), we obtain

$$\begin{bmatrix}\dot{z}_1 \\ \dot{z}_2\end{bmatrix} = \begin{bmatrix} -c_1 z_1 - d_1 z_1 + z_2 + \varepsilon_2 \\ v_{m,3} + \beta_2 - \frac{\partial\alpha_1}{\partial y}\omega^{\mathrm{T}}\hat{\theta} - \frac{\partial\alpha_1}{\partial y}\varepsilon_2 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\end{bmatrix} + \begin{bmatrix}\omega^{\mathrm{T}} \\ -\frac{\partial\alpha_1}{\partial y}\omega^{\mathrm{T}}\end{bmatrix}\tilde{\theta}. \qquad (21)$$

In the case where $\rho = 2$, the actual control u would appear in the second equation of (21) and would be used to implement the feedback law α_2. When $\rho > 2$, we treat $v_{m,3}$ as a virtual control and design for it a feedback law $v_{m,3} = \alpha_2$ to achieve strict passivity of (21) from the input $\tilde{\theta}$ to a new tuning output τ_2. To define τ_2, we use the transposed input matrix as the output matrix

$$\tau_2 = \begin{bmatrix}\omega, & -\frac{\partial\alpha_1}{\partial y}\omega\end{bmatrix}\begin{bmatrix}z_1 \\ z_2\end{bmatrix} = \tau_1 - \frac{\partial\alpha_1}{\partial y}\omega z_2. \qquad (22)$$

If the relative degree were $\rho = 2$, we would choose the passive update law $\dot{\hat{\theta}} = \Gamma\tau_2$ to close the loop around the error system (21). The desired strict passivity property of (21) with $\dot{\hat{\theta}}$ replaced by $\Gamma\tau_2$ is achieved using the stabilizing function

$$\begin{aligned}\alpha_2 &= -z_1 - c_2 z_2 - d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_2 - \beta_2 \\ &\quad + \frac{\partial\alpha_1}{\partial y}\omega^{\mathrm{T}}\hat{\theta} + \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\tau_2. \qquad (23)\end{aligned}$$

Thus, our second error system $\mathscr{S}_2$ consists of $\dot{\varepsilon} = A_0\varepsilon$ and

$$\begin{aligned}\begin{bmatrix}\dot{z}_1 \\ \dot{z}_2\end{bmatrix} &= \begin{bmatrix} -c_1 - d_1 & 1 \\ -1 & -c_2 - d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2\end{bmatrix}\begin{bmatrix}z_1 \\ z_2\end{bmatrix} \\ &\quad + \begin{bmatrix}1 \\ -\frac{\partial\alpha_1}{\partial y}\end{bmatrix}\varepsilon_2 + \begin{bmatrix}\omega^{\mathrm{T}} \\ -\frac{\partial\alpha_1}{\partial y}\omega^{\mathrm{T}}\end{bmatrix}\tilde{\theta} \qquad (24) \\ \tau_2 &= \begin{bmatrix}\omega, & -\frac{\partial\alpha_1}{\partial y}\omega\end{bmatrix}\begin{bmatrix}z_1 \\ z_2\end{bmatrix}.\end{aligned}$$

To prove that in this system the operator from $\tilde{\theta}$ to τ_2 is strictly passive, we introduce

$$V_2 = V_1 + \frac{1}{2}\left(z_2^2 + \frac{1}{d_2}\varepsilon^{\mathrm{T}}P\varepsilon\right)$$

and

$$\psi_2 = \psi_1 + c_2 z_2^2 + \frac{1}{4d_2}\varepsilon^{\mathrm{T}}\varepsilon,$$

and consider

$$\begin{aligned}\dot V_2 \le\ & -c_1 z_1^2 - c_2 z_2^2 + z_1 \omega^{\mathrm T}\tilde\theta - z_2 \frac{\partial\alpha_1}{\partial y}\omega^{\mathrm T}\tilde\theta \\ & - d_1\left(z_1 - \frac{1}{2d_1}\varepsilon_2\right)^2 - d_2\left(\frac{\partial\alpha_1}{\partial y}z_2 + \frac{1}{2d_2}\varepsilon_2\right)^2 \\ & - \left(\frac{1}{4d_1}+\frac{1}{4d_2}\right)\varepsilon^{\mathrm T}\varepsilon \\ \le\ & -c_1 z_1^2 - c_2 z_2^2 - \left(\frac{1}{4d_1}+\frac{1}{4d_2}\right)\varepsilon^{\mathrm T}\varepsilon \\ & + \left(\left[\omega,\ -\frac{\partial\alpha_1}{\partial y}\omega\right]\begin{bmatrix} z_1 \\ z_2\end{bmatrix}\right)^{\mathrm T}\tilde\theta. \end{aligned} \tag{25}$$

Noting that the last term is $\tau_2^{\mathrm T}\tilde\theta$, we integrate this inequality over $[0,t]$, which proves that the operator from $\tilde\theta$ to τ_2 is strictly passive because

$$\int_0^t \tau_2^{\mathrm T}\tilde\theta \,\mathrm d\sigma \ge V_2(t) - V_2(0) + \int_0^t \psi_2(\sigma)\,\mathrm d\sigma. \tag{26}$$

This would complete our procedure for a relative-degree-two plant, in which case we would design $u = -v_{m,3} + \alpha_2$ and $\dot{\hat\theta} = \Gamma\tau_2$. In Fig. 2 we show the adaptive feedback loop for the case $\rho = 2$.

The novelty of this step is the presence in the control law α_2 of the 'nonlinear damping' term

$$-d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_2$$

and of the tuning function τ_2.

4. DESIGN FOR HIGHER RELATIVE DEGREE

The design for relative degree higher than two introduces a new tool for achieving skew-symmetry of the off-diagonal entries in the error system matrix, which is instrumental in the proof of strict passivity.

Step 3. In the case where $\rho > 2$, we cannot implement α_2 as a control law and we *do not* use $\dot{\hat\theta} = \Gamma\tau_2$ as the update law for $\hat\theta$. Instead, we retain τ_2 as our second tuning function and α_2 as our second stabilizing function.

We express the derivative of $z_3 = v_{m,3} - \alpha_2$ as

$$\dot z_3 = v_{m,4} + \beta_3 - \frac{\partial\alpha_2}{\partial y}\omega^{\mathrm T}\theta - \frac{\partial\alpha_2}{\partial y}\varepsilon_2 - \frac{\partial\alpha_2}{\partial\hat\theta}\dot{\hat\theta}, \tag{27}$$

where β_3 encompasses all the known terms for $v_{m,4}$. We now treat $v_{m,4}$ as a virtual control. Introducing the new error variable $z_4 = v_{m,4} - \alpha_3$ and using (21)–(23), we write the new error system as

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot z_3\end{bmatrix} = \begin{bmatrix} -c_1 z_1 - d_1 z_1 + z_2 + \varepsilon_2 \\ -z_1 - c_2 z_2 - d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_2 + z_3 - \frac{\partial\alpha_1}{\partial y}\varepsilon_2 - \frac{\partial\alpha_1}{\partial\hat\theta}(\dot{\hat\theta} - \Gamma\tau_2) \\ z_4 + \alpha_3 + \beta_3 - \frac{\partial\alpha_2}{\partial y}\omega^{\mathrm T}\hat\theta - \frac{\partial\alpha_2}{\partial y}\varepsilon_2 - \frac{\partial\alpha_2}{\partial\hat\theta}\dot{\hat\theta}\end{bmatrix} + \begin{bmatrix}\omega^{\mathrm T} \\ -\frac{\partial\alpha_1}{\partial y}\omega^{\mathrm T} \\ -\frac{\partial\alpha_2}{\partial y}\omega^{\mathrm T}\end{bmatrix}\tilde\theta. \tag{28}$$

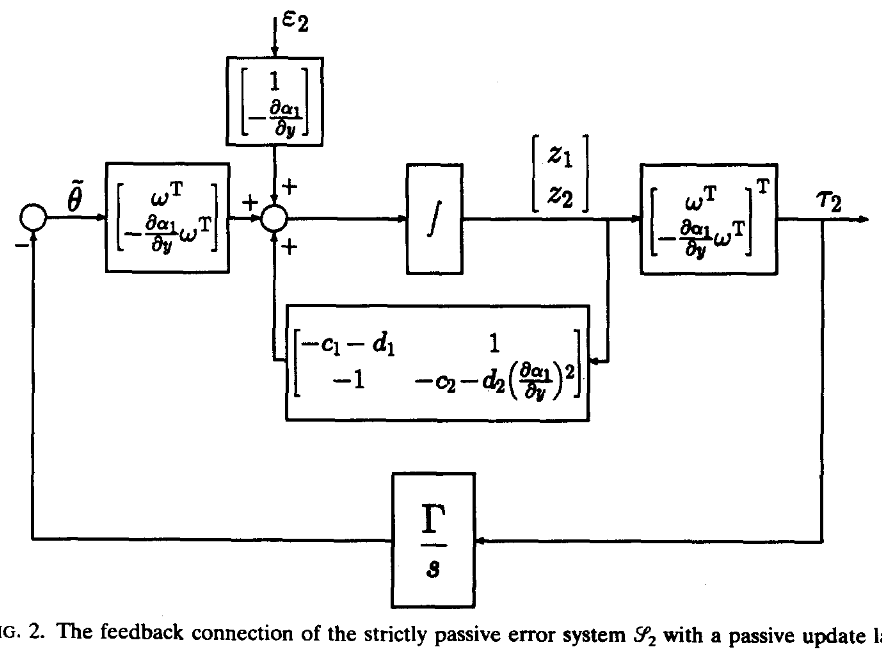

FIG. 2. The feedback connection of the strictly passive error system $\mathcal{S}_2$ with a passive update law.

The function α_3 will be chosen to achieve strict passivity of (28) from the input $\tilde{\theta}$ to a new tuning output τ_3. As in previous steps, we use the transpose of the input matrix as the output matrix, and obtain our third tuning output

$$\tau_3 = \left[\omega, \quad -\frac{\partial \alpha_1}{\partial y}\omega, \quad -\frac{\partial \alpha_2}{\partial y}\omega\right]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \tau_2 - \frac{\partial \alpha_2}{\partial y}\omega z_3. \tag{29}$$

To make the operator from $\tilde{\theta}$ to τ_3 strictly passive, we choose our third stabilizing function as

$$\alpha_3 = -z_2 - c_3 z_3 - d_3\left(\frac{\partial \alpha_2}{\partial y}\right)^2 z_3 - \beta_3 + \frac{\partial \alpha_2}{\partial y}\omega^{\mathrm{T}}\hat{\theta} + \frac{\partial \alpha_2}{\partial \hat{\theta}}\Gamma\tau_3 - \frac{\partial \alpha_1}{\partial \hat{\theta}}\frac{\partial \alpha_2}{\partial y}\Gamma\omega z_2. \tag{30}$$

The last term in (30) is our new tool introduced to counteract the effects of the term

$$-\frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma(\tau_3 - \tau_2) = \frac{\partial \alpha_1}{\partial \hat{\theta}}\frac{\partial \alpha_2}{\partial y}\Gamma\omega z_3 \triangleq \sigma_{23}\Gamma\omega z_3 \tag{31}$$

in the $\dot{z}_2$-equation. With this term we achieve the skew-symmetry of the off-diagonal entries in the error system $\mathscr{S}_3$ consisting of $\dot{\varepsilon} = A_0\varepsilon$ and

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -c_1 - d_1 & 1 & 0 \\ -1 & -c_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 & 1 + \sigma_{23}\Gamma\omega \\ 0 & -1 - \sigma_{23}\Gamma\omega & -c_3 - d_3\left(\frac{\partial \alpha_2}{\partial y}\right)^2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{\partial \alpha_1}{\partial y} \\ -\frac{\partial \alpha_2}{\partial y} \end{bmatrix}\varepsilon_2 + \begin{bmatrix} \omega^{\mathrm{T}} \\ -\frac{\partial \alpha_1}{\partial y}\omega^{\mathrm{T}} \\ -\frac{\partial \alpha_2}{\partial y}\omega^{\mathrm{T}} \end{bmatrix}\tilde{\theta}, \tag{32}$$

$$\tau_3 = \left[\omega, \quad -\frac{\partial \alpha_1}{\partial y}\omega, \quad -\frac{\partial \alpha_2}{\partial y}\omega\right]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.$$

This error system corresponds to the case $\rho = 3$ and is obtained from (28) by substituting $z_4 \equiv 0$, $\dot{\hat{\theta}} = \Gamma\tau_3$, (29) and (30).

Along with (32) we introduce

$$V_3 = V_2 + \frac{1}{2}\left(z_3^2 + \frac{1}{d_3}\varepsilon^{\mathrm{T}}P\varepsilon\right)$$

and

$$\psi_3 = \psi_2 + c_3 z_3^2 + \frac{1}{4d_3}\varepsilon^{\mathrm{T}}\varepsilon,$$

and consider

$$\begin{aligned}\dot{V}_3 &\leq -\sum_{k=1}^{3} c_k z_k^2 + \tau_3^{\mathrm{T}}\tilde{\theta} - \sum_{k=1}^{3} d_k \times\left(\frac{\partial \alpha_{k-1}}{\partial y}z_k - \frac{1}{2d_k}\varepsilon_2\right)^2 - \sum_{k=1}^{3}\frac{1}{4d_k}\varepsilon^{\mathrm{T}}\varepsilon \\ &\leq -\sum_{k=1}^{3}\left(c_k z_k^2 + \frac{1}{4d_k}\varepsilon^{\mathrm{T}}\varepsilon\right) + \tau_3^{\mathrm{T}}\tilde{\theta},\end{aligned} \tag{33}$$

where we define $\alpha_0 = -y$ for notational convenience. Integrating this inequality over $[0, t]$, we prove that the operator from $\tilde{\theta}$ to τ_3 is strictly passive because

$$\int_0^t \tau_3^{\mathrm{T}}\tilde{\theta}\,\mathrm{d}\sigma \geq V_3(t) - V_3(0) + \int_0^t \psi_3(\sigma)\,\mathrm{d}\sigma. \tag{34}$$

When $\rho > 3$, we have $z_4 \not\equiv 0$ and we do not use $\dot{\hat{\theta}} = \Gamma\tau_3$ as the update law for $\hat{\theta}$. Instead, we retain τ_3 as our third tuning function and α_3 as our third stabilizing function.

Step $i(3 < i \leq \rho)$. We express the derivative of $z_i = v_{m,i} - \alpha_{i-1}$ as

$$\dot{z}_i = v_{m,i+1} + \beta_i - \frac{\partial \alpha_{i-1}}{\partial y}\omega^{\mathrm{T}}\theta - \frac{\partial \alpha_{i-1}}{\partial y}\varepsilon_2 - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}, \tag{35}$$

where β_i encompasses all the known terms except for $v_{m,i+1}$. We now treat $v_{m,i+1}$ as a virtual control and introduce the new error variable $z_{i+1} = v_{m,i+1} - \alpha_i$. Then, denoting

$$\sigma_{kj} = \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\frac{\partial \alpha_{j-1}}{\partial y},$$

we write the ith error system as*

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \vdots \\ \dot z_j \\ \vdots \\ \dot z_i \end{bmatrix} = \begin{bmatrix} -c_1z_1 - d_1z_1 + z_2 + \varepsilon_2 \\ -z_1 - c_2z_2 - d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_2 + z_3 - \frac{\partial\alpha_1}{\partial y}\varepsilon_2 - \frac{\partial\alpha_1}{\partial\hat\theta}(\dot{\hat\theta} - \Gamma\tau_2) \\ \vdots \\ -\sum_{k=2}^{j-1}\sigma_{kj}\Gamma\omega z_k - z_{j-1} - c_jz_j - d_j\left(\frac{\partial\alpha_{j-1}}{\partial y}\right)^2 z_j + z_{j+1} - \frac{\partial\alpha_{j-1}}{\partial y}\varepsilon_2 - \frac{\partial\alpha_{j-1}}{\partial\hat\theta}(\dot{\hat\theta} - \Gamma\tau_j) \\ \vdots \\ z_{i+1} + \alpha_i + \beta_i - \frac{\partial\alpha_{i-1}}{\partial y}\omega^{\mathrm T}\hat\theta - \frac{\partial\alpha_{i-1}}{\partial y}\varepsilon_2 - \frac{\partial\alpha_{i-1}}{\partial\hat\theta}\dot{\hat\theta} \end{bmatrix} + \begin{bmatrix} \omega^{\mathrm T} \\ -\frac{\partial\alpha_1}{\partial y}\omega^{\mathrm T} \\ \vdots \\ -\frac{\partial\alpha_{j-1}}{\partial y}\omega^{\mathrm T} \\ \vdots \\ -\frac{\partial\alpha_{i-1}}{\partial y}\omega^{\mathrm T} \end{bmatrix}\tilde\theta,$$

$$3 \le j \le i-1. \qquad (36)$$

As in the previous steps, we use the transpose of the input matrix as the output matrix to obtain the ith tuning output

$$\tau_i = \left[\omega, \quad -\frac{\partial\alpha_1}{\partial y}\omega, \ldots, -\frac{\partial\alpha_{i-1}}{\partial y}\omega\right]\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \end{bmatrix}$$

$$= \tau_{i-1} - \frac{\partial\alpha_{i-1}}{\partial y}\omega z_i. \qquad (37)$$

Next, we make the operator from $\tilde\theta$ to τ_i strictly passive with the ith stabilizing function

$$\alpha_i = -z_{i-1} - c_iz_i - d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 z_i - \beta_i + \frac{\partial\alpha_{i-1}}{\partial y}\omega^{\mathrm T}\hat\theta + \frac{\partial\alpha_{i-1}}{\partial\hat\theta}\Gamma\tau_i - \sum_{k=2}^{i-1}\sigma_{ki}\Gamma\omega z_k, \qquad (38)$$

where the last term counteracts the effects of the terms

$$-\frac{\partial\alpha_{k-1}}{\partial\hat\theta}\Gamma(\tau_i - \tau_{i-1}) = \frac{\partial\alpha_{k-1}}{\partial\hat\theta}\frac{\partial\alpha_{i-1}}{\partial y}\Gamma\omega z_i = \sigma_{ki}\Gamma\omega z_i \qquad (39)$$

in the $\dot z_k$-equations, where $2 \le k \le i-1$. If $z_{i+1} \equiv 0$ and $\dot{\hat\theta} = \Gamma\tau_i$, (37) and (38) would result in the error system $\mathcal{S}_i$ consisting of $\dot\varepsilon = A_0\varepsilon$ and

$$\begin{bmatrix} \dot z_2 \\ \dot z_3 \\ \dot z_4 \\ \vdots \\ \dot z_i \end{bmatrix} = \begin{bmatrix} -c_1 - d_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 - d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 & 1 + \sigma_{23}\Gamma\omega & \cdots & \sigma_{2i}\Gamma\omega \\ 0 & -1 - \sigma_{23}\Gamma\omega & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 + \sigma_{i-1,i}\Gamma\omega \\ 0 & -\sigma_{2i}\Gamma\omega & \cdots & -1 - \sigma_{i-1,i}\Gamma\omega & -c_i - d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_i \end{bmatrix}$$

$$+ \begin{bmatrix} 1 \\ -\frac{\partial\alpha_1}{\partial y} \\ \vdots \\ -\frac{\partial\alpha_{i-1}}{\partial y} \end{bmatrix}\varepsilon_2 + \begin{bmatrix} \omega^{\mathrm T} \\ -\frac{\partial\alpha_1}{\partial y}\omega^{\mathrm T} \\ \vdots \\ -\frac{\partial\alpha_{i-1}}{\partial y}\omega^{\mathrm T} \end{bmatrix}\tilde\theta,$$

$$\tau_i = \left[\omega, -\frac{\partial\alpha_1}{\partial y}\omega, \ldots, -\frac{\partial\alpha_{i-1}}{\partial y}\omega\right]\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \end{bmatrix}. \qquad (40)$$

* For notational convenience we define $z_{p+1} \triangleq 0$.

Along with (40) we introduce

$$V_i = V_{i-1} + \frac{1}{2}\left(z_i^2 + \frac{1}{d_i}\varepsilon^{\mathrm{T}} P \varepsilon\right)$$

and

$$\psi_i = \psi_{i-1} + c_i z_i^2 + \frac{1}{4d_i}\varepsilon^{\mathrm{T}}\varepsilon,$$

and consider

$$\begin{aligned} \dot V_i &\le -\sum_{k=1}^{i} c_k z_k^2 + \tau_i^{\mathrm{T}}\tilde\theta - \sum_{k=1}^{i} d_k \\ &\quad \times \left(\frac{\partial \alpha_{k-1}}{\partial y} z_k - \frac{1}{2d_k}\varepsilon_2\right)^2 \\ &\quad - \sum_{k=1}^{i} \frac{1}{4d_k}\varepsilon^{\mathrm{T}}\varepsilon \\ &\le -\sum_{k=1}^{i}\left(c_k z_k^2 + \frac{1}{4d_k}\varepsilon^{\mathrm{T}}\varepsilon\right) + \tau_i^{\mathrm{T}}\tilde\theta. \end{aligned} \tag{41}$$

Integrating this inequality over $[0, t]$, we prove that the operator from $\tilde\theta$ to τ_i is strictly passive because

$$\int_0^t \tau_i^{\mathrm{T}}\tilde\theta \, \mathrm{d}\sigma \ge V_i(t) - V_i(0) + \int_0^t \psi_i(\sigma)\, \mathrm{d}\sigma. \tag{42}$$

Our procedure terminates at $i = \rho$, when the actual control u appears in lieu of $z_{i+1} + \alpha_i$ in (36). We complete our design with

$$u = \alpha_\rho, \tag{43}$$

$$\dot{\hat\theta} = \Gamma \tau_\rho. \tag{44}$$

5. STABILITY FROM PASSIVITY

The following global stability theorem for the closed-loop adaptive system was proven in Krstić *et al.* (1994).

Theorem 1. All the states of the closed-loop adaptive system consisting of the plant (1), filters (7), update law (44) and control law (44) are globally uniformly bounded and global asymptotic tracking is achieved: $\lim_{t\to\infty}[y(t) - y_r(t)] = 0$.

Here, we interpret it in the light of the established passivity properties.

Let us for brevity introduce the notation

$$A_z(z,t,\Gamma) = \begin{bmatrix} -c_1 - d_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 - d_2\left(\frac{\partial \alpha_1}{\partial y}\right)^2 & 1 + \sigma_{23}\Gamma\omega & \cdots & \sigma_{2\rho}\Gamma\omega \\ 0 & -1 - \sigma_{23}\Gamma\omega & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 + \sigma_{\rho-1,\rho}\Gamma\omega \\ 0 & -\sigma_{2\rho}\Gamma\omega & \cdots & -1 - \sigma_{\rho-1,\rho}\Gamma\omega & -c_\rho - d_\rho\left(\frac{\partial \alpha_{\rho-1}}{\partial y}\right)^2 \end{bmatrix}, \tag{45}$$

$$b_z(x,t,\Gamma) = \begin{bmatrix} 1 \\ -\frac{\partial \alpha_1}{\partial y} \\ \vdots \\ -\frac{\partial \alpha_{\rho-1}}{\partial y} \end{bmatrix}, \tag{46}$$

which is slightly abusive because the dependence on the states other than z is represented by t. The system

$$\mathcal{S}_\rho: \quad \begin{aligned} \dot\varepsilon &= A_0\varepsilon, \\ \dot z &= A_z(z,t,\Gamma)z + b_z(z,t,\Gamma)(\omega^{\mathrm{T}}\tilde\theta + \varepsilon_2), \\ \tau_\rho &= \omega b_z^{\mathrm{T}}(z,t,\Gamma) z, \end{aligned} \tag{47}$$

possesses a strict passivity property from $\tilde\theta$ to τ_ρ with a storage function

$$V(z,\varepsilon) = \tfrac{1}{2}(z^{\mathrm{T}} z + (1/d_0)\varepsilon^{\mathrm{T}} P \varepsilon)$$

and a dissipation rate

$$\psi(z,\varepsilon) = \textstyle\sum_{i=1}^{\rho} c_i z_i^2 + (1/4d_0)\varepsilon^{\mathrm{T}}\varepsilon,$$

where

$$d_0 \triangleq \left(\textstyle\sum_{i=1}^{\rho}(1/d_i)\right)^{-1}.$$

On the other hand,

$$-\dot{\tilde\theta} = \Gamma\tau_\rho \tag{48}$$

is passive with a storage function

$$V_\theta(\tilde\theta) = \tfrac{1}{2}\tilde\theta^{\mathrm{T}}\Gamma^{-1}\tilde\theta.$$

By Proposition A.1, the equilibrium $\varepsilon = 0$, $z = 0$, $\tilde\theta = 0$ of the dynamic system (47) and (48) is globally uniformly stable. This, together with the boundedness of $y_r, \ldots, y_r^{(\rho)}$ implies that ε, z, $\hat\theta$,

η are bounded. The proof of boundedness of λ, x, u is as in Krstić *et al.* (1994). Proposition A.1 also gives $\varepsilon(t)$, $z(t) \rightarrow 0$ as $t \rightarrow 0$.

6. ROBUSTNESS WITHOUT ADAPTATION

We will now demonstrate a robustness property of the detuned system when the adaptation is switched off, that is when $\Gamma = 0$ in (45). The detuned error system is

$$\dot{z} = A_z z + b_z(\tilde{\theta}^{\mathrm{T}}\omega + \varepsilon_2), \tag{49}$$

where $A_z = A_z(z, t, 0)$ and $b_z = b_z(z, t, 0)$, that is

$$A_z = \begin{bmatrix} -c_1 - d_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 - d_2\left(\dfrac{\partial \alpha_1}{\partial y}\right)^2 & 1 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & -1 & -c_p - d_p\left(\dfrac{\partial \alpha_{p-1}}{\partial y}\right)^2 \end{bmatrix}, \tag{50}$$

$$b_z = \begin{bmatrix} 1 \\ -\dfrac{\partial \alpha_1}{\partial y} \\ \vdots \\ -\dfrac{\partial \alpha_{p-1}}{\partial y} \end{bmatrix}. \tag{51}$$

Examining the expression (14) for α_1, then (23) for α_2, and successively through (38) for α_i, it can be established that when $\Gamma = 0$, all the derivatives $\partial\alpha_i/\partial y$ are known constants depending on c_i, d_i and $\hat{\theta}$. Hence, the matrix A_z and the vector b_z are constant. Because $\varepsilon_2(t) \rightarrow 0$ exponentially, and the system is linear, we continue with $\varepsilon_2(t) \equiv 0$. For the same reason, we let all the initial conditions be zero. Since A_z is Hurwitz (as the sum of a skew-symmetric matrix and a negative diagonal matrix), the transfer function from $\tilde{\theta}^{\mathrm{T}}\omega$ to z_1

$$\frac{\beta_z(s)}{\alpha_z(s)} = e_1^{\mathrm{T}}(sI - A_z)^{-1} b_z, \tag{52}$$

is stable, its relative degree is one, and both $\beta_z(s)$ and $\alpha_z(s)$ are monic polynomials.

The error system (49) is one of the two parts of the detuned linear feedback system. To close the feedback loop with the second part, we represent the signal $\tilde{\theta}^{\mathrm{T}}\omega$ as the output of the system $W(s)$, which is driven by y, as shown in Fig. 3. The transfer function $W(s)$ is calculated as follows:

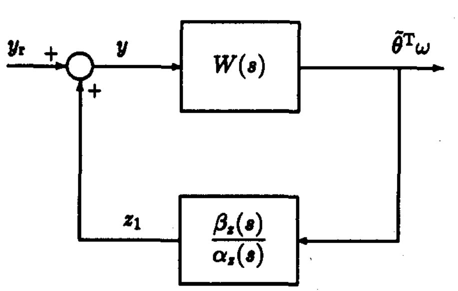

FIG. 3. The detuned linear feedback system.

$$\begin{aligned} \tilde{\theta}^{\mathrm{T}}\omega &= [-\tilde{a}_{n-1}, \ldots, -\tilde{a}_0, \tilde{b}_{m-1}, \ldots, \tilde{b}_0] \\ &\quad \times \begin{bmatrix} \xi_{(2)}^{\mathrm{T}} + e_1 y \\ \bar{v}_{(2)}^{\mathrm{T}} \end{bmatrix} \\ &= \frac{s + k_1}{s^n + k_1 s^{n-1} + \cdots + k_n} \\ &\quad \times \left([-\tilde{a}_{n-1}, \ldots, -\tilde{a}_0] \begin{bmatrix} s^{n-1} \\ \vdots \\ s \\ 1 \end{bmatrix} y \right. \\ &\quad \left. + [\tilde{b}_{m-1}, \ldots, \tilde{b}_0] \begin{bmatrix} s^{m-1} \\ \vdots \\ s \\ 1 \end{bmatrix} u \right) \\ &\triangleq \frac{s + k_1}{K(s)}(-\tilde{A}(s)y + \tilde{B}(s)u) \\ &= \frac{s + k_1}{K(s)}\left(-\tilde{A}(s)y + \tilde{B}(s)\frac{A(s)}{B(s)}y\right) \\ &= \frac{(s + k_1)(-\tilde{A}(s)B(s) + A(s)\tilde{B}(s))}{K(s)B(s)} y \\ &\triangleq W(s)y. \end{aligned} \tag{53}$$

Since $\deg \tilde{A} = n - 1$ and $\deg \tilde{B} = m - 1$, it is clear from the final expression in (53) that $W(s)$ is stable and proper. By the small gain theorem, the stability of the feedback connection of $W(s)$ with the stable strictly proper transfer function $\beta_z(s)/\alpha_z(s)$ is guaranteed if the loop operator gain is less than one. We will now show that the operator gain of $\beta_z(s)/\alpha_z(s)$ can be made arbitrarily small by a choice of the design parameters c_i, d_i, $1 \le i \le \rho$.

Differentiating $\frac{1}{2}|z|^2 \triangleq \frac{1}{2}z^{\mathrm{T}}z$ along the solutions of (49) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|z|^2\right) &= -\sum_{k=1}^{\rho} c_k z_k^2 - \sum_{k=1}^{\rho} d_k z_k^2\left(\frac{\partial\alpha_{k-1}}{\partial y}\right)^2 \\ &\quad - \sum_{k=1}^{\rho} z_k \frac{\partial\alpha_{k-1}}{\partial y}\tilde{\theta}^{\mathrm{T}}\omega \\ &= -\sum_{k=1}^{\rho} c_k z_k^2 - \sum_{k=1}^{\rho} d_k \\ &\quad \times\left[\frac{\partial\alpha_{k-1}}{\partial y} z_k + \frac{1}{2d_k}\tilde{\theta}^{\mathrm{T}}\omega\right]^2 \\ &\quad + \left(\sum_{k=1}^{\rho}\frac{1}{4d_k}\right)(\tilde{\theta}^{\mathrm{T}}\omega)^2 \\ &\le -c_0|z|^2 + \frac{1}{4d_0}(\tilde{\theta}^{\mathrm{T}}\omega)^2, \end{aligned} \tag{54}$$

where

$$c_0 = \min_{1\le k\le\rho} c_k$$

and

$$\frac{1}{d_0} = \sum_{k=1}^{\rho}\frac{1}{d_k}.$$

Upon multiplication by e^{2c_0t}, the last inequality becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(|z|^2\mathrm{e}^{2c_0t}) \le \frac{1}{2d_0}(\tilde{\theta}^{\mathrm{T}}\omega)^2\mathrm{e}^{2c_0t}. \tag{55}$$

Integrating (55) over $[0, t]$, we obtain

$$\begin{aligned}|z(t)|^2 &\le \frac{1}{2d_0}\int_0^t \mathrm{e}^{-2c_0(t-\tau)}(\tilde{\theta}^{\mathrm{T}}\omega(\tau))^2\,\mathrm{d}\tau \\ &\le \frac{1}{2d_0}\int_0^t \mathrm{e}^{-2c_0(t-\tau)}\,\mathrm{d}\tau \sup_{t\in[0,\infty)}\|\tilde{\theta}^{\mathrm{T}}\omega(t)\|^2 \\ &= \frac{1}{4c_0d_0}(1-\mathrm{e}^{-2c_0t})\,\|\tilde{\theta}^{\mathrm{T}}\omega\|_\infty^2 \\ &\le \frac{1}{4c_0d_0}\|\tilde{\theta}^{\mathrm{T}}\omega\|_\infty^2. \end{aligned} \tag{56}$$

Finally, using the triangle inequality and noting that $\|z_1\|_\infty \le \|z\|_\infty$, we arrive at

$$\|z_1\|_\infty \le \frac{1}{2\sqrt{c_0d_0}}\|\tilde{\theta}^{\mathrm{T}}\omega\|_\infty. \tag{57}$$

On the other hand, since $W(s)$, as defined in (53), is a stable proper transfer function, then

$$\|\tilde{\theta}^{\mathrm{T}}\omega\|_\infty \le \|w\|_1\,\|y\|_\infty, \tag{58}$$

where $\|w\|_1 = \int_{-\infty}^{\infty}|w(t)|\,\mathrm{d}t$, and $w(t)$ is the impulse response of $W(s)$.

To apply the small gain theorem, we note that $1/2\sqrt{c_0d_0}$ in (57) can be made arbitrarily small by a choice of c_0 and d_0. Since $\|w\|_1$ is finite and independent of c_0 and d_0, the loop gain $1/2\sqrt{c_0d_0}\,\|w\|_1$ can be made less than one. Thus, the $\mathscr{L}_\infty$-stability of the feedback system in Fig. 3 is guaranteed.

Next, we show that the $\mathscr{L}_\infty$-stability also guarantees the internal asymptotic stability of this system. Substituting (53) into

$$y - y_r = \frac{\beta_z(s)}{\alpha_z(s)}\tilde{\theta}^{\mathrm{T}}\omega \tag{59}$$

and solving for y, we obtain

$$y = \frac{\alpha_z KB}{\alpha_z KB + (s+k_1)\beta_z(\hat{B}A - \hat{A}B)}y_r. \tag{60}$$

Since $\alpha_z(s)$, $K(s)$ and $B(s)$ are all Hurwitz, if there are cancellations in the transfer function in (60), they are all in the open left half-plane, so that the denominator in (60) is also Hurwitz. We have thus shown that for sufficiently large $c_1, \ldots, c_\rho$ and $d_1, \ldots, d_\rho$ the linear system (60) is asymptotically stable.

Theorem 2. Let the norm of $W(s)$, with the fixed parameter errors $\tilde{a}_i$, $\tilde{b}_i$ in (53), be $\|w\|_1 \le \bar{w}$. Then, the linear detuned feedback system (49) and (53) is robust in the sense that the choice of design coefficients

$$c_i > \bar{w}^2\sum_{k=1}^{\rho}\frac{1}{4d_k}, \quad 1 \le i \le \rho, \tag{61}$$

guarantees its asymptotic stability.

7. THE UNDERLYING LINEAR NONADAPTIVE CONTROLLER

The block diagram in Fig. 3 does not reveal the structure of the linear controller that results in the established robustness property. We now derive the equation of this controller and include it in a more detailed block diagram.

We start with the equations representing the two blocks in Fig. 3:

$$y - y_r = \frac{\beta_z(s)}{\alpha_z(s)}\tilde{\theta}^{\mathrm{T}}\omega, \tag{62}$$

$$\tilde{\theta}^{\mathrm{T}}\omega = \frac{s+k_1}{K(s)}(-\tilde{A}(s)y + \tilde{B}(s)u). \tag{63}$$

Substituting (63) into (62) we get

$$\begin{aligned} y - y_r &= \frac{\beta_z(s)}{\alpha_z(s)}\frac{(s+k_1)}{K(s)}(-\tilde{A}(s)y + \tilde{B}(s)u) \\ &= \frac{\beta_z(s)}{\alpha_z(s)}\frac{(s+k_1)}{K(s)}(\hat{A}(s)y - \hat{B}(s)u). \end{aligned} \tag{64}$$

Collecting the terms in (64), we obtain the control law

$$\frac{(s+k_1)\beta_z(s)\hat{B}(s)}{K(s)}u = \alpha_z(s)y_r - \frac{\alpha_z(s)K(s)-(s+k_1)\beta_z(s)\hat{A}(s)}{K(s)}y. \quad (65)$$

Figure 4 shows the block diagram of the underlying linear system. The transfer function $((s+k_1)\beta_z\hat{B}-K)/K$ is strictly proper because $\beta_z(s)$, $K(s)$ and $\hat{B}(s)$ are monic. The properness of $(\alpha_z K-(s+k_1)\beta_z\hat{A})/K \triangleq C_y(s)$ is proven in Appendix B.

From the block diagram in Fig. 4 we obtain (60). The matching condition for $y(s)=y_r(s)$ is

$$\hat{A}(s)=A(s), \quad \hat{B}(s)=B(s), \quad (66)$$

under the assumption that $A(s)$ and $B(s)$ are coprime. Under this condition, the characteristic polynomial of (60) is $\alpha_z(s)K(s)B(s)$, so that the closed-loop poles consist of:

(1) the roots of $\alpha_z(s)$, i.e. the eigenvalues of the error system matrix A_z;
(2) the roots of $K(s)$, i.e. the eigenvalues of the filter matrix A_z; and
(3) the roots of $B(s)$, i.e. the zeros of the plant.

It is of interest to compare these closed-loop poles with those of the traditional MRAC whose characteristic polynomial is $A_m(s)K(s)B(s)$. Our design replaces the reference model denominator polynomial $A_m(s)$ by the error system denominator polynomial $\alpha_z(s)$. In this way, our controller allows the closed-loop poles to be placed independently of the reference model poles.

8. EXAMPLE

In this section, we illustrate the passivity and parametric robustness properties of the new design procedure on an unstable relative-degree-three plant

$$y(s)=\frac{1}{s^2(s-a)}u(s), \quad a>0 \text{ unknown}, \quad (67)$$

considered in Krstić *et al.* (1994). The control objective is to asymptotically track the output of the reference model $y_r(s)=1/(s+1)^3 r(s)$. The resulting adaptation loop described by

$$\dot{z}=\begin{bmatrix} -c_1-d_1 & 1 & 0 \\ -1 & -c_2-d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 & 1+\sigma\gamma\omega \\ 0 & -1-\sigma\gamma\omega & -c_3-d_3\left(\frac{\partial\alpha_2}{\partial y}\right)^2 \end{bmatrix} z + \begin{bmatrix} 1 \\ -\frac{\partial\alpha_1}{\partial y} \\ -\frac{\partial\alpha_2}{\partial y} \end{bmatrix}(\omega\tilde{a}+\varepsilon_2), \quad (68)$$

$$\dot{\tilde{a}} = -\gamma\left[1, -\frac{\partial\alpha_1}{\partial y}, -\frac{\partial\alpha_2}{\partial y}\right]\omega z$$

possesses the strict passivity property from $\tilde{a}$ to $-\dot{\tilde{a}}/\gamma$.

To illustrate the parametric robustness (Theorem 2), we switch off the adaptation ($\gamma=0$) at a constant estimate $\hat{a}=1$, when the parameter error $\tilde{a}=2$ is significant. With $c_1=c_2=c_3=3$, $d_1=d_2=d_3=0.1$, the resulting detuned linear system is unstable. With an

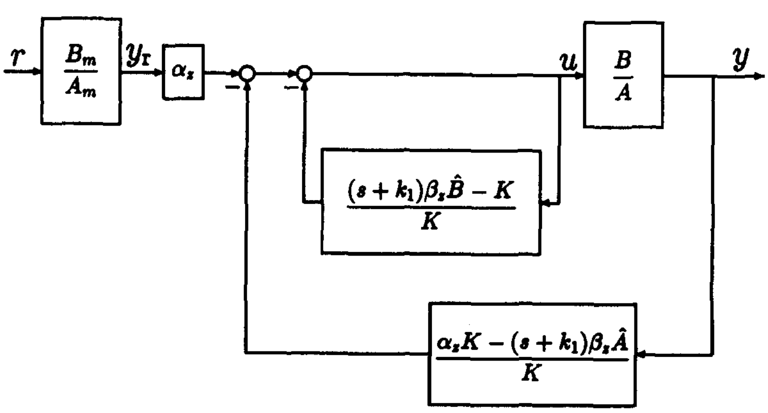

FIG. 4. The structure of the underlying closed-loop linear system.

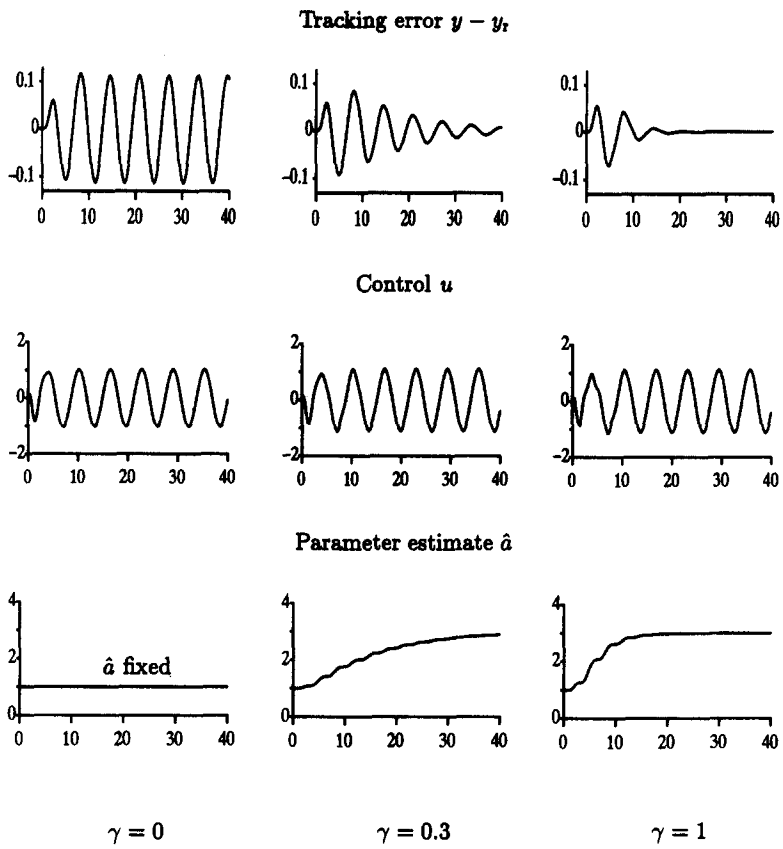

FIG. 5. Adaptation improves the tracking error transients without an increase in control effort. The plant is driven by $r(t) = \sin t$, and the plant parameter is $a = 3$.

increase to $c_1 = c_2 = c_3 = 5$, the system is stabilized. However, without adaptation, the tracking error, shown in Fig. 5, is about 12% of the reference input, which is not acceptable in most applications. The effectiveness of the new adaptive controller is demonstrated by the fact that even with slow adaptation ($\gamma = 0.3$), the tracking error is reduced to zero after a few periods of the reference input, as shown in Fig. 5. It is remarkable that even during the adaptation transients, the tracking error is smaller than in the nonadaptive system, while the control effort is about the same. When the adaptation gain is increased to $\gamma = 1$, the tracking performance is further improved with about the same control effort.

9. CONCLUSIONS

In this paper, we have continued the development of the theory of the new class of adaptive controllers proposed in our recent paper (Krstić *et al.*, 1994). We have shown that our recursive design achieves a strict passivity property of the main adaptation loop. Since the early days of adaptive control this property has been a desired feature of adaptive systems, seemingly unachievable for systems of relative degree higher than 2.

When we initially approached the adaptive control of linear systems as a nonlinear feedback problem it was not obvious whether with the adaptation switched off the resulting adaptive system would reduce to a linear system. We have now confirmed that this indeed is the case by revealing the structure of the underlying linear controller.

This linear controller has an important parametric robustness property. Its gains can be chosen sufficiently high to guarantee stability for any given bound on plant parameter uncertainty. However, this robustness property does not eliminate the need for adaptation when the plant parameter uncertainty is significant and results in an unacceptably large tracking error. The adaptive controller achieves better transients and guarantees that the tracking error converges to zero without an increase in control effort.

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APPENDIX A. PASSIVITY DEFINITIONS

We use the passivity definitions of (Willems, 1972; Hill and Moylan, 1980; Byrnes *et al.*, 1991) extended to time-varying nonlinear systems. Consider systems of the form

$$\begin{aligned} \dot{x} &= f(t,x) + g(t,x)u, \\ y &= h(t,x), \end{aligned} \tag{A.1}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in C^0(\mathbb{R}^m)$, and f, g, h piecewise continuous in t and smooth in x. Suppose $f(t,0) = 0$ and $h(t,0) = 0$ for all $t \ge 0$.

Definition A.1 The system (A.1) is said to be passive if there exists a continuous nonnegative ('storage') function $V: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, which satisfies $V(t,0) = 0$, $\forall t \ge 0$, such that for all $u \in C^0(\mathbb{R}^m)$, $x(0) \in \mathbb{R}^n$, $t \ge 0$

$$\int_0^t y^{\mathrm{T}}(\sigma)u(\sigma)\,\mathrm{d}\sigma \ge V(t,x(t)) - V(0,x(0)). \tag{A.2}$$

Definition A.2 The system (A.1) is said to be strictly passive if there exist a continuous nonnegative (storage) function $V: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, which satisfies $V(t,0) = 0$ $\forall t \ge 0$, and a positive definite function (dissipation rate) $\psi: \mathbb{R}^n \to \mathbb{R}_+$, such that for all $u \in C^0(\mathbb{R}^m)$, $x(0) \in \mathbb{R}^n$, $t \ge 0$

$$\int_0^t y^{\mathrm{T}}(\sigma)u(\sigma)\,\mathrm{d}\sigma \ge V(t,x(t)) - V(0,x(0)) + \int_0^t \psi(x(\sigma))\,\mathrm{d}\sigma. \tag{A.3}$$

Note that if V is positive definite, radially unbounded and decrescent (in x, uniformly in t), then, for $u = 0$, the equilibrium $x = 0$ of the (strictly) passive system (A.1) is globally uniformly (asymptotically) stable.

Proposition A.1 The negative feedback connection of a strictly passive system and a passive system with positive definite radially unbounded decrescent storage functions, has a globally uniformly stable equilibrium at the origin. Furthermore, the state of the strictly passive system converges to zero as $t \to \infty$.

The proof of the first part of Proposition A.1 is straightforward. The proof of the second part follows the standard invariance arguments for nonautonomous systems based on Barbalat's lemma.

APPENDIX B. PROPERNESS OF $C_y(s)$

To prove that

$$C_y(s) = \frac{\alpha_z K - (s + k_1)\beta_z \hat{A}}{K}$$

is proper, we examine the expression (43) for u which depends on y, y_r and the following variables:

$$v_{i,j} = \begin{cases} \dfrac{s^i(s^{j-1} + k_1 s^{j-2} + \cdots + k_{j-1})}{K(s)}u, & 1 \le j \le n-i \\[2ex] \dfrac{-s^{i+j-n-1}(k_j s^{n-j} + \cdots + k_n)}{K(s)}u, & n-i+1 \le j \le n \end{cases} \quad 0 \le i \le m, \tag{B.1}$$

$$\xi_{i,j} = \begin{cases} \dfrac{s^i(s^{j-1} + k_1 s^{j-2} + \cdots + k_{j-1})}{K(s)}y, & 1 \le y \le n-i \\[2ex] \dfrac{s^{i+j-n-1}(k_j s^{n-j} + \cdots + k_n)}{K(s)}y, & n-i+1 \le j \le n \end{cases} \quad 0 \le i \le n, \tag{B.2}$$

where (B.1) and (B.2) follow from (3). Using the definitions of z_i, $1 \le i \le \rho$, we can write (43) as

$$u = \sum_{i,j} p_{i,j} v_{i,j} + \sum_{i,j} q_{i,j} \xi_{i,j} + \bar{q}y + \sum_{i=0}^{\rho} r_i y_r^{(i)}$$

$$= \frac{p_{n-1}(s)}{K(s)} u + \frac{q_{n-1}(s)}{K(s)} y + \bar{q}y + r_\rho(s) y_r, \tag{B.3}$$

which shows that (65) has the following form:

$$\frac{K(s) - p_{n-1}(s)}{K(s)} u = \frac{q_n(s)}{K(s)} y + r_\rho(s) y_r, \tag{B.4}$$

where

$$\begin{aligned} q_n(s) &= q_{n-1}(s) + \bar{q}K(s) \\ &= -\alpha_z(s)K(s) + (s + k_1)\beta_z(s)\hat{A}(s). \end{aligned} \tag{B.5}$$

From

$$\sum_{i,j} q_{i,j} \xi_{i,j} = \frac{q_{n-1}(s)}{K(s)} y \tag{B.6}$$

and (B.2) it follows that $\deg q_{n-1}(s) \le n - 1$. Therefore $\deg q_n(s) = \deg [q_{n-1}(s) + \bar{q}K(s)] = n$. Hence, from (B.5) we obtain $\deg [\alpha_z(s)K(s) - (s + k_1)\beta_z(s)\hat{A}(s)] = n$. This means that all the terms of order higher than n are cancelled in $\alpha_z(s)K(s) - (s + k_1)\beta_z(s)\hat{A}(s)$, which is, of course, achieved by the construction of the error system. Thus, the transfer function $(\alpha_z K - (s + k_1)\beta_z \hat{A})/K$ is proper and the block diagram in Fig. 4 is implementable.