INT. J. CONTROL, 2003, VOL. 76, NO. 11, 1116–1124

Control of a non-linear PDE system arising from non-burning tokamak plasma transport dynamics

EUGENIO SCHUSTER†* and MIROSLAV KRSTIĆ†

The control of kinetic profiles is among the most important problems in fusion reactor research. It is strongly related to a great number of other problems in fusion energy generation such as burn control, transport reduction, confinement time improvement, MHD instability avoidance and high-β or high-confinement operating modes access. We seek a controller which is able to make the kinetic profiles converge to their desired equilibrium profiles. We are interested in constructing a stabilizing controller that achieves stability for unstable equilibrium profiles and increases performance for stable equilibrium profiles. As a first approach, we consider in this work a set of non-linear partial differential equations (PDEs) describing approximately the dynamics of the density and energy profiles in a non-burning plasma. This non-linear PDE model represents the one-dimensional transport equations for the kinetic variables, density and energy, in cylindrical geometry. The transport coefficients in this model are in turn non-linear functions of the kinetic variables. The original set of PDEs is discretized in space using a finite difference method which gives a high order set of coupled non-linear ordinary differential equations (ODEs). Applying a backstepping design we obtain a discretized coordinate transformation that transforms the original system into a properly chosen target system that is asymptotically stable in l^2-norm. To achieve such stability for the target system, convenient boundary conditions are chosen. Then, using the property that the discretized coordinate transformation is invertible for an arbitrary (finite) grid choice, we conclude that the discretized version of the original system is asymptotically stable and obtain a non-linear feedback boundary control law for the energy and density in the original set of coordinates. Numerical simulations show that the feedback control law designed using only one step of backstepping can successfully control the kinetic profiles.

1. Introduction

The regulation of the kinetic profiles is essential to achieving optimal fusion performance and making fusion an economically viable source of energy. Plasma behaviour is critically influenced, in several ways that are listed next, by the plasma density and temperature profiles.

The maintenance of certain profiles is expected to influence transport and consequently the energy confinement time. In-depth understanding of the processes ruling the tranport of both energy and particles provides information about the optimal profiles for operation. Thus a reliable profile control system is necessary to achieve those profiles that minimize transport. In fact, the inverse procedure should also be considered. The use of profile control in experimental devices could provide useful information of the transport process and conclusions about applicability of specific transport models.

The control of the kinetic profiles is also fundamental for MHD (magnetohydrodynamics) instability avoidance. The profile control can be used to achieve those plasma density, current density and temperature profiles which in turn achieve the pressure and safety factor profiles that are favourable for ideal MHD stability. The accesses to H-mode, where the energy confinement time is up to twice the value of the L-mode, and to high β modes free from instabilities have also been accomplished by modifying the density and temperature profiles.

One of the topics of most interest is burn control, the control of an ignited or subignited plasma. A D-T plasma may be thermally unstable in some regions of operation and a tight control is required for avoiding thermal excursion or quenching. Auxiliary heating, fuelling and impurity injection are among the most common actuators used to keep the density and temperature of the plasma at a desired working point. Among the problems related to the control of the kinetic variables, the problem of burn control is the most extensive found in the literature. This can be explained by the fact that the problem of controlling the burn instability can be approached considering a 0-D (zero-dimensional) model where spatially averaged quantities are considered. The availability of conventional control tools that are capable of dealing with this kind of model, where the dynamics of the average kinetic variables is described by ODEs, encourages the study of the problem. In previous works the problem is simplified by linearizing the non-linear 0-D model and putting the model in a standard control form for which simple linear control techniques can be used. Recently, we have introduced a new approach where the linearization of the model is avoided and much higher levels of performance and robustness are achieved (Schuster *et al.* 2001, 2002, 2003). However, the 0-D control of the burn instability using modulation of bulk heating, fuelling and impurity density does not take into account the 1-D (one-dimen-

Received 15 April 2002. Revised and accepted 15 April 2003.

* Author for correspondence. e-mail: schuster@mae.ucsd.edu

† Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093-0411, USA

International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2003 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/0020717031000135523

sional) effect of this modulation on the profiles. The heating, fuelling and impurity density are distributed throughout the plasma volume affecting the density and temperature profiles which in turn can change the transport mode, the energy confinement time and the plasma stability. In addition, the robustness of the resulting controller against uncertainties in some parameters of the model that are functions of the density and temperature profiles, like energy confinement time and reactivity rate, are very hard to evaluate using 0-D actuation.

All of these phenomena point out the importance of controlling the profiles of plasma density and temperature. To achieve this goal we need a control technique that can deal with the distributed and non-linear nature of those quantities, their coupling with one another, and their, at times conflicting, control objectives. The work we present here is inspired by Fuchs *et al.* (1983), Firestone and Kessel (1991) and Miley and Varadarajan (1992) and to some extent by Firestone *et al.* (1997 a, b). In all these mentioned works the 1-D model is represented by a set of non-linear PDEs. The reduction of the distributed parameter description of the system to a lumped parameter description is carried out using different methods. The resulting set of ODEs are linearized and conventional linear control methods are applied for the synthesis of the controller. In contrast to these previous works, the control method presented in this paper is based on the full non-linear model. As we showed for the 0-D case, the plasma dynamics is highly non-linear and fundamental information about the system is lost through the linearization, imposing in this way a limit on operability. Therefore, the linearization of the model should be avoided and this is central to our approach. We control the system by means of thermal and density actuation.

The goal of the controller is to make the kinetic profiles converge to their desired equilibrium profiles. We are interested in constructing a stabilizing controller that achieves stability for unstable equilibrium profiles and increases performance for stable equilibrium profiles. In order to simplify this initial approach to kinetic profile control in fusion reactors, we consider a non-burning plasma whose dynamics is described by a 1-D non-linear PDE model. This 1-D non-linear PDE model consists of the diffusion equations of the kinetic variables in cylindrical geometry where the diffusion coefficients, on the other hand, are non-linear functions of these kinetic variables. The original set of PDEs is discretized in space using a finite difference method which gives a high order set of coupled non-linear ODEs. Applying a backstepping design we obtain a discretized coordinate transformation that transforms the original system into a properly chosen target system that is asymptotically stable in l^2-norm. To achieve such stability for the target system, convenient boundary conditions are chosen. Then, using the property that the discretized coordinate transformation is invertible for an arbitrary (finite) grid choice, we conclude that the discretized version of the original system is asymptotically stable and obtain a non-linear feedback boundary control law for the energy and density in the original set of coordinates. This technique has been already applied successfully for other different physical applications (Boskovic and Krstic 2001, 2002). Numerical simulations show that the feedback control law designed on a very coarse grid (using just a few measurements of the energy and density in the core of the reactor) can successfully control the kinetic profiles.

The paper is organized as follows. In § 2 a non-linear one-dimensional PDE model that governs the dynamics of the density and energy profiles in a non-burning plasma is introduced. The control objective is stated in § 3. In § 4 a non-linear feedback control law that achieves asymptotic stabilization is presented, followed by the proof of stability for the target system in § 5. A feedback control law designed on a coarse grid is shown through a simulation study to successfully control the kinetic profiles of the plasma in § 6. Finally, some conclusions and suggestions are stated in § 7.

2. Model

The mathematical model used in this work is basically the set of transport equations in cylindrical geometry used by Firestone and Kessel (1991). The energy and density transport equations are given by

$$\frac{3}{2}\frac{\partial nT}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(\kappa\frac{\partial T}{\partial r} + \frac{3}{2}DT\frac{\partial n}{\partial r}\right) - P_{\mathrm{br}} + P_{\mathrm{aux}} \quad (1)$$

$$\frac{\partial n}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial n}{\partial r} - nV_p\right) + S \quad (2)$$

where n is the density, T is the temperature, $E = \frac{3}{2}nT$ is the energy, P_{aux} is the auxiliary heating power (actuator) and S is the fuelling rate (actuator). The radiation loss considered in this model is the bremsstrahlung loss

$$P_{\mathrm{br}} = A_b Z_{\mathrm{eff}} n_e^2 \sqrt{T}$$

where $Z_{\mathrm{eff}} = (\sum_i n_i Z_i^2)/n_e$, n_e is the electron density and n_i is the ion density. Since this model describes a non-burning plasma the alpha particle density is neglected. Therefore the quasi-neutrality condition $n_e = n_i Z_i$ implies that $n_e = n_i$ because the only ion present in the plasma is the deuterium–tritium ion ($Z_i = 1$). This implies in turn that $Z_{\mathrm{eff}} = 1$. The electron and ion temperatures are considered to be equal. The ohmic heating is neglected.

The thermal diffusivity coefficient is given by the empirical scaling relation (Becker *et al.* 1998)

$$\chi = \frac{n(0)}{n(r)} \frac{m^2}{s} \tag{3}$$

whereas the heat conduction coefficient is defined as

$$\kappa \equiv n\chi = n(0) \frac{m^2}{s} \tag{4}$$

implying that the heat conduction coefficient is constant and equal to the central density. The inward pinch velocity is given by the empirical scaling relation (Becker *et al.* 1988)

$$V_p = \frac{1}{2} \frac{D}{T} \frac{\partial T}{\partial r} \tag{5}$$

With the purpose of simplification, we write the diffusion coefficient as

$$D = \frac{2}{3}\chi = \frac{2}{3} \frac{n(0)}{n(r)} \frac{m^2}{s} \tag{6}$$

which is an approximation of the diffusion coefficient used in Firestone and Kessel (1991) and proposed in Becker *et al.* (1988). This approximation is not a requirement for the control method and its only purpose is the simplification of the presentation of the control method. The approximation simplifies the coupling between the temperature and density terms, reducing (1) and (2) to

$$\frac{\partial E}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[rD \frac{\partial E}{\partial r} \right] - P_{\mathrm{br}} + P_{\mathrm{aux}} \tag{7}$$

$$\frac{\partial n}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(D \frac{\partial n}{\partial r} - nV_p \right) \right] + S \tag{8}$$

We consider the arbitrary boundary conditions

$$\left. \frac{\partial E}{\partial r} \right|_{r=0} = 0 \tag{9}$$

$$\left. \frac{\partial n}{\partial r} \right|_{r=0} = 0 \tag{10}$$

$$\left. \frac{\partial E}{\partial r} \right|_{r=a} = k_E E(a) \tag{11}$$

$$\left. \frac{\partial n}{\partial r} \right|_{r=a} = k_n n(a) \tag{12}$$

3. Control objective

We write $E(r,t) = \bar{E}(r) + \tilde{E}(r,t)$ and $n(r,t) = \bar{n}(r) + \tilde{n}(r,t)$, where $\bar{E}(r)$ and $\bar{n}(r)$ are the equilibrium profiles which in turn are the solutions of the equilibrium equations

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left[r\bar{D} \frac{\partial \bar{E}}{\partial r} \right] - \bar{P}_{\mathrm{br}} + \bar{P}_{\mathrm{aux}} \tag{13}$$

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\bar{D} \frac{\partial \bar{n}}{\partial r} - \bar{n}\bar{V}_p \right) \right] + \bar{S} \tag{14}$$

with boundary conditions

$$\left. \frac{\partial \bar{E}}{\partial r} \right|_{r=0} = 0 \tag{15}$$

$$\left. \frac{\partial \bar{n}}{\partial r} \right|_{r=0} = 0 \tag{16}$$

$$\left. \frac{\partial \bar{E}}{\partial r} \right|_{r=a} = k_E \bar{E}(a) \tag{17}$$

$$\left. \frac{\partial \bar{n}}{\partial r} \right|_{r=a} = k_n \bar{n}(a) \tag{18}$$

It is clear that the equilibrium profile will depend not only on the boundary conditions but also on the auxiliary power and fuelling rate equilibrium profiles. With the boundary conditions chosen, a proper selection of the equilibrium profiles for the auxiliary power $\bar{P}_{\mathrm{aux}}$ and fuelling rate $\bar{S}$ allows us to achieve the desired equilibrium profiles for the energy and the density. It is important to note that in this approach to kinetic profile control we consider only density and thermal actuation at the edge of the plasma. Therefore, the fuelling rate $S = \bar{S}$ and the auxiliary power $P_{\mathrm{aux}} = \bar{P}_{\mathrm{aux}}$ are used only for the definition of the equilibrium profiles. The dynamics of the deviation variables $\tilde{E}(r,t)$ and $\tilde{n}(r,t)$ is given by

$$\begin{aligned}
\frac{\partial \tilde{E}}{\partial t} &= \frac{1}{r} \frac{\partial}{\partial r} \left[rD \frac{\partial (\bar{E} + \tilde{E})}{\partial r} \right] - P_{\mathrm{br}} + P_{\mathrm{aux}} \\
&= \frac{1}{r} \frac{\partial}{\partial r} \left[rD \frac{\partial \tilde{E}}{\partial r} \right] + \frac{1}{r} \frac{\partial}{\partial r} \left[rD \frac{\partial \bar{E}}{\partial r} \right] - P_{\mathrm{br}} + P_{\mathrm{aux}} \\
\frac{\partial \tilde{n}}{\partial t} &= \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(D \frac{\partial (\bar{n} + \tilde{n})}{\partial r} - nV_p \right) \right] + S \\
&= \frac{1}{r} \frac{\partial}{\partial r} \left[rD \left(\frac{3}{2} \frac{\partial \tilde{n}}{\partial r} - \frac{n}{2E} \frac{\partial \tilde{E}}{\partial r} \right) \right] \\
&\quad + \frac{1}{r} \frac{\partial}{\partial r} \left[rD \left(\frac{3}{2} \frac{\partial \bar{n}}{\partial r} - \frac{n}{2E} \frac{\partial \bar{E}}{\partial r} \right) \right] + S
\end{aligned}$$

where we have written

$$\begin{aligned}
V_p &= \frac{1}{2} \frac{D}{T} \frac{\partial T}{\partial r} = \frac{1}{2} \frac{D}{\left(\frac{2}{3}(E/n)\right)} \frac{\partial \left(\frac{2}{3}(E/n)\right)}{\partial r} \\
&= \frac{1}{2} D \left(\frac{1}{E} \frac{\partial E}{\partial r} - \frac{1}{n} \frac{\partial n}{\partial r} \right) \\
&= \frac{1}{2} D \left(\frac{1}{E} \frac{\partial \bar{E}}{\partial r} - \frac{1}{n} \frac{\partial \bar{n}}{\partial r} \right) + \frac{1}{2} D \left(\frac{1}{E} \frac{\partial \tilde{E}}{\partial r} - \frac{1}{n} \frac{\partial \tilde{n}}{\partial r} \right)
\end{aligned}$$

Taking into account that

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial(\cdot)}{\partial r}\right]=\frac{\partial}{\partial r}\left[D\frac{\partial(\cdot)}{\partial r}\right]+\frac{1}{r}D\frac{\partial(\cdot)}{\partial r}$$

and defining

$$g(E,n)=\frac{\partial}{\partial r}\left[D\frac{\partial\bar{E}}{\partial r}\right]+\frac{1}{r}D\frac{\partial\bar{E}}{\partial r}-P_{\mathrm{br}}+P_{\mathrm{aux}},\quad(19)$$

$$f(E,n)=\frac{3}{2}\left\{\frac{\partial}{\partial r}\left[D\frac{\partial\bar{n}}{\partial r}\right]+\frac{1}{r}D\frac{\partial\bar{n}}{\partial r}\right\}-\frac{1}{2}\left\{\frac{\partial}{\partial r}\left[\frac{Dn}{E}\frac{\partial\bar{E}}{\partial r}\right]+\frac{1}{r}\frac{Dn}{E}\frac{\partial\bar{E}}{\partial r}\right\}+S\quad(20)$$

we can rewrite the equations for the deviation variables as

$$\frac{\partial\tilde{E}}{\partial t}=\frac{\partial}{\partial r}\left[D\frac{\partial\tilde{E}}{\partial r}\right]+\frac{1}{r}D\frac{\partial\tilde{E}}{\partial r}+g(E,n)\quad(21)$$

$$\frac{\partial\tilde{n}}{\partial t}=\frac{3}{2}\left\{\frac{\partial}{\partial r}\left[D\frac{\partial\tilde{n}}{\partial r}\right]+\frac{1}{r}D\frac{\partial\tilde{n}}{\partial r}\right\}-\frac{1}{2}\left\{\frac{\partial}{\partial r}\left[\frac{Dn}{E}\frac{\partial\tilde{E}}{\partial r}\right]+\frac{1}{r}\frac{Dn}{E}\frac{\partial\tilde{E}}{\partial r}\right\}+f(E,n)\quad(22)$$

with boundary conditions

$$\left.\frac{\partial\tilde{E}}{\partial r}\right|_{r=0}=0\quad(23)$$

$$\left.\frac{\partial\tilde{n}}{\partial r}\right|_{r=0}=0\quad(24)$$

$$\left.\frac{\partial\tilde{E}}{\partial r}\right|_{r=a}=k_E\tilde{E}(a)+\Delta\tilde{E}_r\quad(25)$$

$$\left.\frac{\partial\tilde{n}}{\partial r}\right|_{r=a}=k_n\tilde{n}(a)+\Delta\tilde{n}_r\quad(26)$$

The objective is to stabilize $\tilde{E}(r,t)$ and $\tilde{n}(r,t)$, making them converge to zero, by using $\Delta\tilde{E}_r(t)$ and $\Delta\tilde{n}_r(t)$ as actuation at the edge of the plasma.

4. Controller design

Figure 1 summarizes the essence of the control method. We discretize the original set of PDEs in space using a finite difference method which gives a high order set of coupled non-linear ordinary differential equations (ODEs). Applying a backstepping design we obtain a discretized coordinate transformation that transforms the original system into a properly chosen target system that is asymptotically stable in l^2-norm. To achieve such stability for the target system, convenient boundary conditions are chosen. Then, using the property that the discretized coordinate transformation is invertible for an arbitrary (finite) grid choice, we conclude that the discretized version of the original system is asymptotically stable and obtain a non-linear feedback boundary control law for the energy and density in the original set of coordinates.

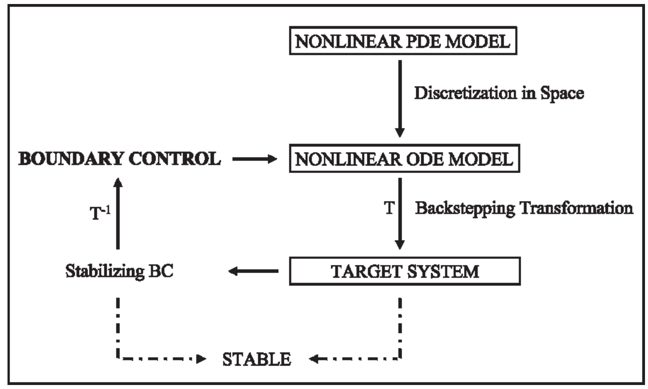


Figure 1. Control method scheme.

The idea is to design controllers using only a small number of steps of backstepping, or equivalently using only a small number of state measurements. The measurements are taken from the core of the plasma and the actuation is applied at the edge of the plasma.

To discretize the problem, let us start by defining $h=1/N$, where N is an integer. Then using the notation $x_i(t)=x(ih,t)$, $i=0,1,\ldots,N$, we write the discretized version of (21) and (22) as

$$\dot{\tilde{E}}_i=\frac{D_{i+\frac{1}{2}}\tilde{E}_{i+1}-2D_i\tilde{E}_i+D_{i-\frac{1}{2}}\tilde{E}_{i-1}}{h^2}+\frac{1}{ih}D_i\frac{\tilde{E}_{i+1}-\tilde{E}_i}{h}+g_i\quad(27)$$

$$g_i=\frac{D_{i+\frac{1}{2}}\bar{E}_{i+1}-2D_i\bar{E}_i+D_{i-\frac{1}{2}}\bar{E}_{i-1}}{h^2}+\frac{1}{ih}D_i\frac{\bar{E}_{i+1}-\bar{E}_i}{h}-(P_{\mathrm{br}})_i+(\bar{P}_{\mathrm{aux}})_i,\quad(P_{\mathrm{br}})_i=A_bn_i^2\sqrt{\frac{2}{3}\frac{E_i}{n_i}}$$

$$\dot{\tilde{n}}_i=\frac{3}{2}\left\{\frac{D_{i+\frac{1}{2}}\tilde{n}_{i+1}-2D_i\tilde{n}_i+D_{i-\frac{1}{2}}\tilde{n}_{i-1}}{h^2}+\frac{1}{ih}D_i\frac{\tilde{n}_{i+1}-\tilde{n}_i}{h}\right\}-\frac{1}{2}\left\{\frac{(Dn/E)_{i+\frac{1}{2}}\tilde{E}_{i+1}-2(Dn/E)_i\tilde{E}_i+(Dn/E)_{i-\frac{1}{2}}\tilde{E}_{i-1}}{h^2}+\frac{1}{ih}\left(\frac{Dn}{E}\right)_i\frac{\tilde{E}_{i+1}-\tilde{E}_i}{h}\right\}+f_i\quad(28)$$

$$f_i=\frac{3}{2}\left\{\frac{D_{i+\frac{1}{2}}\bar{n}_{i+1}-2D_i\bar{n}_i+D_{i-\frac{1}{2}}\bar{n}_{i-1}}{h^2}+\frac{1}{ih}D_i\frac{\bar{n}_{i+1}-\bar{n}_i}{h}\right\}-\frac{1}{2}\left\{\frac{(Dn/E)_{i+\frac{1}{2}}\bar{E}_{i+1}-2(Dn/E)_i\bar{E}_i+(Dn/E)_{i-\frac{1}{2}}\bar{E}_{i-1}}{h^2}+\frac{1}{ih}\left(\frac{Dn}{E}\right)_i\frac{\bar{E}_{i+1}-\bar{E}_i}{h}\right\}+\bar{S}_i$$

for $i = 1, \ldots, N-1$ and the discretized version of the boundary condition equations (23)–(26) as

$$\frac{\tilde{E}_1 - \tilde{E}_0}{h} = 0 \tag{29}$$

$$\frac{\tilde{n}_1 - \tilde{n}_0}{h} = 0 \tag{30}$$

$$\frac{\tilde{E}_N - \tilde{E}_{N-1}}{h} = k_E \tilde{E}_N + \Delta\tilde{E}_r \tag{31}$$

$$\frac{\tilde{n}_N - \tilde{n}_{N-1}}{h} = k_n \tilde{n}_N + \Delta\tilde{n}_r \tag{32}$$

where

$$D_{i-\frac{1}{2}} = D_i - \frac{D_i - D_{i-1}}{h}\frac{h}{2} = \frac{1}{2}D_i + \frac{1}{2}D_{i-1}$$

$$D_{i+\frac{1}{2}} = D_i + \frac{D_i - D_{i-1}}{h}\frac{h}{2} = \frac{3}{2}D_i - \frac{1}{2}D_{i-1}$$

$$D_i = \frac{2}{3}\frac{n_0}{n_i}\frac{m^2}{s}$$

$$\begin{aligned}\left(\frac{Dn}{E}\right)_{i-\frac{1}{2}} &= \left(\frac{Dn}{E}\right)_i - \frac{(Dn/E)_i - (Dn/E)_{i-1}}{h}\frac{h}{2}\\ &= \frac{1}{2}\left(\frac{Dn}{E}\right)_i + \frac{1}{2}\left(\frac{Dn}{E}\right)_{i-1}\\ \left(\frac{Dn}{E}\right)_{i+\frac{1}{2}} &= \left(\frac{Dn}{E}\right)_i + \frac{(Dn/E)_i - (Dn/E)_{i-1}}{h}\frac{h}{2}\\ &= \frac{3}{2}\left(\frac{Dn}{E}\right)_i - \frac{1}{2}\left(\frac{Dn}{E}\right)_{i-1}\\ \left(\frac{Dn}{E}\right)_i &= \frac{2}{3}\frac{n_0}{E_i}\frac{m^2}{s}\end{aligned}$$

The choice of a backward approximation for the derivatives of D and Dn/E at point i is key to our approach. In this way it is possible to write $D_{i+\frac{1}{2}}$ and $(Dn/E)_{i+\frac{1}{2}}$ as functions of the state variables at points i and $i-1$. This is a requirement for the backstepping procedure as will be discussed below.

We consider now the asymptotically stable (in L^2 norm) target system

$$\begin{aligned}\frac{\partial\tilde{F}}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial\tilde{F}}{\partial r}\right] - C_F\tilde{F}\\ &= \frac{\partial}{\partial r}\left[D\frac{\partial\tilde{F}}{\partial r}\right] + \frac{1}{r}D\frac{\partial\tilde{F}}{\partial r} - C_F\tilde{F}\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\partial\tilde{m}}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left[rD\left(\frac{3}{2}\frac{\partial\tilde{m}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial\tilde{F}}{\partial r}\right)\right] - C_m\tilde{m}\\ &= \frac{3}{2}\left\{\frac{\partial}{\partial r}\left[D\frac{\partial\tilde{m}}{\partial r}\right] + \frac{1}{r}D\frac{\partial\tilde{m}}{\partial r}\right\}\\ &\quad - \frac{1}{2}\left\{\frac{\partial}{\partial r}\left[\frac{Dn}{E}\frac{\partial\tilde{F}}{\partial r}\right] + \frac{1}{r}\frac{Dn}{E}\frac{\partial\tilde{F}}{\partial r}\right\} - C_m\tilde{m}\end{aligned} \tag{34}$$

where $C_F > 0$, $C_m > 0$ and the boundary conditions given by

$$\left.\frac{\partial\tilde{F}}{\partial r}\right|_{r=0} = 0 \tag{35}$$

$$\left.\frac{\partial\tilde{m}}{\partial r}\right|_{r=0} = 0 \tag{36}$$

$$\left.\frac{\partial\tilde{F}}{\partial r}\right|_{r=a} = -G\tilde{F}(a) \tag{37}$$

$$\left.\frac{\partial\tilde{m}}{\partial r}\right|_{r=a} = -G\tilde{m}(a) \tag{38}$$

with $G > 0$. The choice of the target system is based on the need to maintain the parabolic character of the partial differential equation (to keep the highest order derivatives) while removing the 'problematic' terms.

We write the discretized equations for the target system as

$$\begin{aligned}\dot{\tilde{F}}_i &= \frac{D_{i+\frac{1}{2}}\tilde{F}_{i+1} - 2D_i\tilde{F}_i + D_{i-\frac{1}{2}}\tilde{F}_{i-1}}{h^2}\\ &\quad + \frac{1}{ih}D_i\frac{\tilde{F}_{i+1} - \tilde{F}_i}{h} - C_F\tilde{F}_i\end{aligned} \tag{39}$$

$$\begin{aligned}\dot{\tilde{m}}_i &= \frac{3}{2}\left\{\frac{D_{i+\frac{1}{2}}\tilde{m}_{i+1} - 2D_i\tilde{m}_i + D_{i-\frac{1}{2}}\tilde{m}_{i-1}}{h^2} + \frac{1}{ih}D_i\frac{\tilde{m}_{i+1} - \tilde{m}_i}{h}\right\}\\ &\quad - \frac{1}{2}\left\{\frac{(Dn/E)_{i+\frac{1}{2}}\tilde{F}_{i+1} - 2(Dn/E)_i\tilde{F}_i + (Dn/E)_{i-\frac{1}{2}}\tilde{F}_{i-1}}{h^2}\right.\\ &\quad \left. + \frac{1}{ih}\left(\frac{Dn}{E}\right)_i\frac{\tilde{F}_{i+1} - \tilde{F}_i}{h}\right\} - C_m\tilde{m}_i\end{aligned} \tag{40}$$

with boundary conditions written as

$$\frac{\tilde{F}_1 - \tilde{F}_0}{h} = 0 \tag{41}$$

$$\frac{\tilde{m}_1 - \tilde{m}_0}{h} = 0 \tag{42}$$

$$\frac{\tilde{F}_N - \tilde{F}_{N-1}}{h} = -G\tilde{F}_N \tag{43}$$

$$\frac{\tilde{m}_N - \tilde{m}_{N-1}}{h} = -G\tilde{m}_N \tag{44}$$

Finally we look for a backstepping transformation of the discretized original system into the discretization of the target system. This coordinate transformation is sought in the form

$$\tilde{F}_i = \tilde{E}_i - \alpha_{i-1}(\tilde{E}_1, \ldots, \tilde{E}_{i-1}, \tilde{n}_1, \ldots, \tilde{n}_{i-1}) \tag{45}$$

$$\tilde{m}_i = \tilde{n}_i - \beta_{i-1}(\tilde{E}_1, \ldots, \tilde{E}_{i-1}, \tilde{n}_1, \ldots, \tilde{n}_{i-1}) \tag{46}$$

Subtracting (39) from (27) ((40) from (28)) we obtain $\dot{\alpha}_{i-1} = \dot{\tilde{E}}_i - \dot{\tilde{F}}_i$ ($\dot{\beta}_{i-1} = \dot{\tilde{n}}_i - \dot{\tilde{m}}_i$). Expressing the obtained equation in terms of $\alpha_{k-1} = \tilde{E}_k - \tilde{F}_k,\ k = i-1, i, i+1$ ($\beta_{k-1} = \tilde{n}_k - \tilde{m}_k,\ k = i-1, i, i+1$) we can obtain the expression for α_i (β_i)

$$\alpha_i = \frac{1}{D_{i+\frac{1}{2}} + (D_i/i)}\left[\left(2D_i + \frac{D_i}{i} + C_F h^2\right)\alpha_{i-1} - D_{i-\frac{1}{2}}\alpha_{i-2} - h^2 g_i - h^2 C_F \tilde{E}_i + h^2 \dot{\alpha}_{i-1}\right] \quad (47)$$

$$\beta_i = \frac{1}{\frac{3}{2}(D_{i+\frac{1}{2}} + (D_i/i))}\left\{\left[\frac{3}{2}\left(2D_i + \frac{D_i}{i}\right) + C_m h^2\right]\beta_{i-1} - \frac{3}{2}D_{i-\frac{1}{2}}\beta_{i-2} - h^2 f_i + r_i - h^2 C_m \tilde{n}_i + h^2 \dot{\beta}_{i-1}\right\} \quad (48)$$

starting with $\alpha_0 = \beta_0 = 0$ and where

$$r_i = \frac{1}{2}\left\{\left(\frac{Dn}{E}\right)_{i+\frac{1}{2}}\alpha_i - 2\left(\frac{Dn}{E}\right)_i \alpha_{i-1} + \left(\frac{Dn}{E}\right)_{i-\frac{1}{2}}\alpha_{i-2} + \frac{1}{i}\left(\frac{Dn}{E}\right)_i (\alpha_i - \alpha_{i-1})\right\} \quad (49)$$

$$\dot{\alpha}_{i-1} = \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \tilde{E}_k}\dot{\tilde{E}}_k + \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \tilde{n}_k}\dot{\tilde{n}}_k \quad (50)$$

$$\dot{\beta}_{i-1} = \sum_{k=1}^{i-1}\frac{\partial \beta_{i-1}}{\partial \tilde{E}_k}\dot{\tilde{E}}_k + \sum_{k=1}^{i-1}\frac{\partial \beta_{i-1}}{\partial \tilde{n}_k}\dot{\tilde{n}}_k \quad (51)$$

At this point we note the importance of the discretization method used to express $D_{i+\frac{1}{2}}$ and $(Dn/E)_{i+\frac{1}{2}}$. The avoidance of writing these terms as functions of the state variables at point $i+1$ is fundamental to achieve the desired backstepping transformation (45) and (46). However, it is important to emphasize that although the usage of this specific discretization method is a requirement for the backstepping procedure, it does not represent any limitation at all.

Similarly, subtracting (43) from (31) ((44) from (32)) and expressing the obtained equation in terms of $\alpha_{k-1} = \tilde{E}_k - \tilde{F}_k,\ k = i-1, i$ ($\beta_{k-1} = \tilde{n}_k - \tilde{m}_k,\ k = i-1, i$) we can define the control $\Delta\tilde{E}_r$ ($\Delta\tilde{n}_r$) as

$$\Delta\tilde{E}_r = \frac{\alpha_{N-1} - \alpha_{N-2}}{h} - k_E \tilde{E}_N - G(\tilde{E}_N - \alpha_{N-1}) \quad (52)$$

$$\Delta\tilde{n}_r = \frac{\beta_{N-1} - \beta_{N-2}}{h} - k_n \tilde{n}_N - G(\tilde{n}_N - \beta_{N-1}) \quad (53)$$

These expressions for $\Delta\tilde{E}_r$ and $\Delta\tilde{n}_r$ allow us to finally write the stabilizing laws for the modulation of the energy and the density at the edge of the plasma

$$\tilde{E}_N = \alpha_{N-1} + \frac{1}{(1+Gh)}[\tilde{E}_{N-1} - \alpha_{N-2}] \quad (54)$$

$$\tilde{n}_N = \beta_{N-1} + \frac{1}{(1+Gh)}[\tilde{n}_{N-1} - \beta_{N-2}] \quad (55)$$

5. Asymptotic stability of the discretized target system

To show stability of the target system (33) and (34), we take the Lyapunov function candidate

$$V = \frac{1}{2}\int_0^a r\left(\frac{\tilde{F}^2}{k^2} + \tilde{m}^2\right)\mathrm{d}r$$

with $k = 1.380\,662 \times 10^{-23}\ J/K$ (recall that $\tilde{F}$ is $O(10^5)$ and $\tilde{m}$ is $O(10^{20})$). Then we have

$$\begin{aligned}
\dot{V} &= \int_0^a r\left(\frac{\tilde{F}}{k^2}\dot{\tilde{F}} + \tilde{m}\dot{\tilde{m}}\right)\mathrm{d}r \\
&= \int_0^a r\left(\frac{\tilde{F}}{k^2}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \tilde{F}}{\partial r}\right] - C_F\tilde{F}\right\} + \tilde{m}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rD\left(\frac{3}{2}\frac{\partial \tilde{m}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \tilde{F}}{\partial r}\right)\right] - C_m\tilde{m}\right\}\right)\mathrm{d}r \\
&= \frac{\tilde{F}}{k^2}rD\frac{\partial \tilde{F}}{\partial r}\bigg|_0^a - \int_0^a \frac{1}{k^2}rD\left(\frac{\partial \tilde{F}}{\partial r}\right)^2\mathrm{d}r - \frac{C_F}{k^2}\int_0^a r\tilde{F}^2\,\mathrm{d}r + \tilde{m}rD\left(\frac{3}{2}\frac{\partial \tilde{m}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \tilde{F}}{\partial r}\right)\bigg|_0^a \\
&\quad - \int_0^a \frac{\partial \tilde{m}}{\partial r}rD\left(\frac{3}{2}\frac{\partial \tilde{m}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \tilde{F}}{\partial r}\right)\mathrm{d}r - C_m\int_0^a r\tilde{m}^2\,\mathrm{d}r \\
&= aD(a)\left[\frac{\tilde{F}(a)}{k^2}\tilde{F}_r(a) + \frac{3}{2}\tilde{m}(a)\tilde{m}_r(a) - \frac{1}{2}\frac{n(a)}{E(a)}\tilde{m}(a)\tilde{F}_r(a)\right] \\
&\quad - \int_0^a r\left[\frac{C_F}{k^2}\tilde{F}^2 + C_m\tilde{m}^2\right]\mathrm{d}r - \int_0^a r\frac{D}{k^2}\tilde{F}_r^2\,\mathrm{d}r - \frac{3}{2}\int_0^a rD\tilde{m}_r^2\,\mathrm{d}r + \frac{1}{2}\int_0^a rD\frac{n}{E}\tilde{F}_r\tilde{m}_r\,\mathrm{d}r
\end{aligned}$$

where we have used the notation $\partial(\cdot)/\partial r = (\cdot)_r$.

Taking into account the boundary conditions (37) and (38) and recalling that $E = \frac{3}{2}nT$, we can write

$$\begin{aligned}
\dot{V} = &- \int_0^a r\left[\frac{C_F}{k^2}\tilde{F}^2 + C_m\tilde{m}^2\right]\mathrm{d}r - \frac{1}{2}GaD(a)\tilde{m}^2(a) - \frac{1}{2}\int_0^a rD\tilde{m}_r^2\,\mathrm{d}r \\
&+ GaD(a)\left[-\frac{\tilde{F}^2(a)}{k^2} - \tilde{m}^2(a) + \frac{1}{3}\frac{\tilde{F}(a)\tilde{m}(a)}{T(a)}\right] + \int_0^a rD\left\{-\frac{\tilde{F}_r^2}{k^2} - \tilde{m}_r^2 + \frac{1}{3}\frac{\tilde{F}_r\tilde{m}_r}{T}\right\}\mathrm{d}r
\end{aligned}$$

and taking $C = \min(C_F, C_m)$ we can conclude

$$\dot{V} \le -CV + GaD(a)\left[-\frac{\tilde{F}^2(a)}{k^2} - \tilde{m}^2(a) + \frac{|\tilde{F}(a)||\tilde{m}(a)|}{T(a)}\right] + \int_0^a rD\left\{-\frac{\tilde{F}_r^2}{k^2} - \tilde{m}_r^2 + \frac{|\tilde{F}_r||\tilde{m}_r|}{T}\right\}\mathrm{d}r$$

Writing $T = kT^*$ where T^* is in K (Kelvin), while T is in J (Joule), and taking into account that $T^* >> 1$, we can state

$$\dot{V} \le -CV + GaD(a) \times \left[-\frac{\tilde{F}^2(a)}{k^2} - \tilde{m}^2(a) + \left|\frac{\tilde{F}(a)}{k}\right| |\tilde{m}(a)| \right] + \int_0^a rD \left\{ -\frac{\tilde{F}_r^2}{k^2} - \tilde{m}_r^2 + \left|\frac{\tilde{F}_r}{k}\right| |\tilde{m}_r| \right\} \mathrm{d}r$$

By Young's inequality we know that

$$\left[-\frac{\tilde{F}^2(a)}{k^2} - \tilde{m}^2(a) + \left|\frac{\tilde{F}(a)}{k}\right| |\tilde{m}(a)| \right] \le 0$$

$$\int_0^a rD \left\{ -\frac{\tilde{F}_r^2}{k^2} - \tilde{m}_r^2 + \left|\frac{\tilde{F}_r}{k}\right| |\tilde{m}_r| \right\} \mathrm{d}r \le 0$$

and we conclude that $\dot{V} \le -CV$ showing that the system is asymptotically stable.

The proof that the discretized target system (39) and (40) with boundary conditions (43) and (44) is asymptotically stable in l^2 norm would be completely analogous. The discrete Lyapunov function $V_d = \frac{1}{2}\sum_{i=0}^{N}\left((\tilde{F}_i^2/k^2) + \tilde{m}_i^2\right)$ would be considered instead and following an identical procedure the condition $\dot{V}_d \le -CV_d$ would be obtained.

6. Simulation results

The simulation presented in this section is run using the FTCS (forward in time, central in space) finite difference method for a time step $\Delta t = 0.001\,\mathrm{s}$, $a = 2.4\,\mathrm{m}$ and $N_s = 24 \Rightarrow h_s = 0.1$. The subscript 's' stands for simulation. In this way we differentiate the fine grid used for simulation purposes and the coarse grid used for control design purposes. The controller is designed using only one step of backstepping, i.e. for $N = 2 \Rightarrow h = 1.2$. We show that controllers of relatively low order, designed on a much coarser grid, which use the measurement of the energy and density fields only at a limited number of points, can successfully control the system. The choice of the simulation grid follows the standard guidelines for stability and accuracy of the numerical method used. However, the choice of the backstepping grid, i.e. the number of sensors, is driven by the objective to use the least number of sensors, minimizing in this way the implementation cost.

As shown in § 4, control laws for the energy (55) and the density (56) are given in terms of α_{N-1}, α_{N-2} and β_{N-1}, β_{N-2}, respectively, which can be obtained from the expressions (47) and (48) by using symbolic tools available.

For the considered non-burning plasma with boundary conditions (9)–(12), quadratic profiles $\bar{S} = S_0[1 - (r/a)^2)]$ and $\bar{P}_{\mathrm{aux}} = (P_{\mathrm{aux}})_0[(1 - (r/a)^2]$, the equilibrium profiles given by equations (13) and (14) are stable. However the rate of convergence to the equilibrium profiles from some initially perturbed profiles is very slow. Therefore, the main goal of the controller in this case is the improvement of performance. Considering $\tilde{E}(r, 0) = (-1 + 2r/a)10^5$ and $\tilde{n}(r, 0) = (-1 + 2r/a)10^{19}$, figure 2 shows the evolution of the kinetic profiles from these initial perturbed profiles to their equilibrium values. It is possible to note from the figures that the settling time is approximately 2 s. This represents an improvement of an order of magnitude with respect to the open loop settling time. Figure 3 show the actuation at the edge of the plasma that makes this possible.

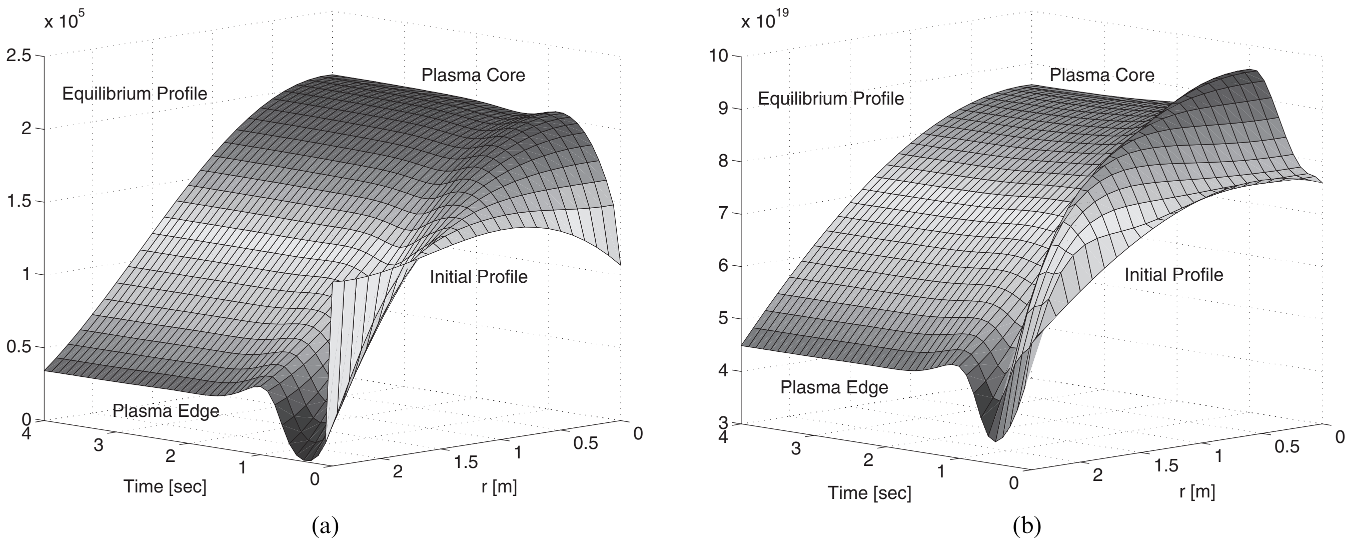
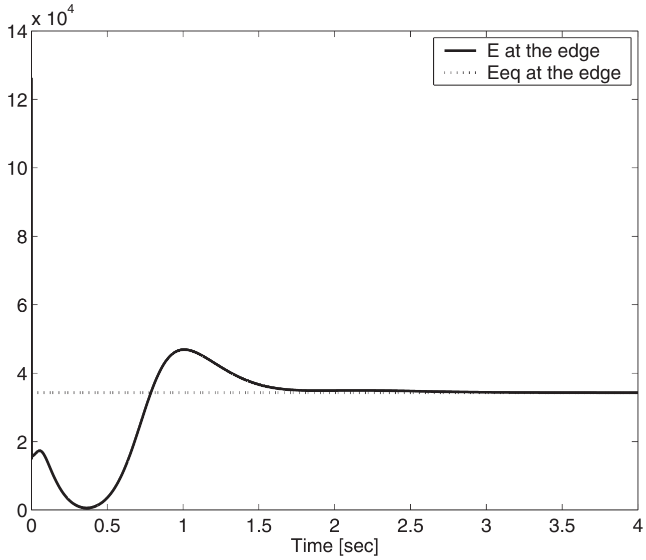
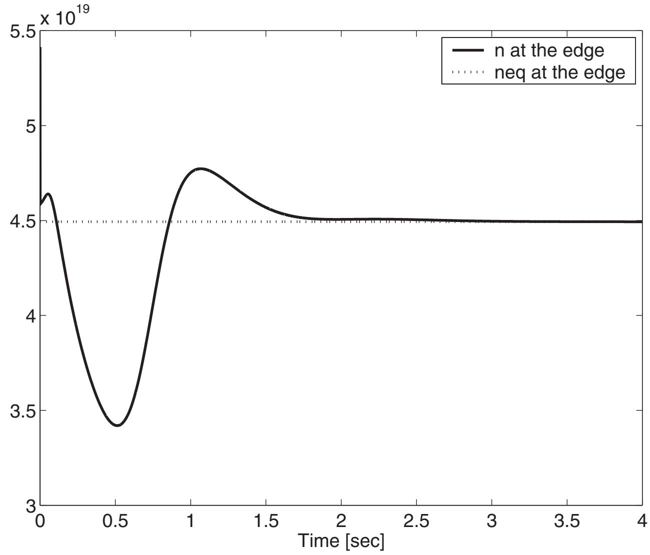


Figure 2. Profile evolution in time for E (a) and n (b).



(a)



(b)

Figure 3. Edge modulation for E (a) and n (b).

7. Conclusions and future work

A non-linear feedback controller based on Lyapunov backstepping design that achieves asymptotic stabilization of the equilibrium kinetic profiles in a cylindrical plasma has been synthesized. The result holds for any finite discretization in space of the original PDE model. The simulation study shows that the boundary controller designed using only one step of backstepping, i.e. using only one measurement from the interior of the reactor, can successfully control the kinetic profiles.

The control of the kinetic profiles by boundary control has been shown to be feasible. However, more study is necessary to find a way of modulating the kinetic variables at the edge of the plasma, i.e. achieving the desired values of $\Delta\tilde{E}_r(t)$ and $\Delta\tilde{n}_r(t)$ given by equations (53) and (54), through the modulation of physical properties of the scrape-off layer (SOL) such as gas puffing, gas pumping and impurity injection. In case the necessary modulation of the temperature and density at the edge of the plasma could not be achieved by physical means, actuation directly in the core of the plasma would be considered; approaching in this way a less challenging problem where the auxiliary power and fuelling rate is used not only for the definition of the equilibrium profiles but also for the stabilization of such profiles.

In the future a zero-dimensional model of the tokamak SOL will be used as a complement of the one-dimensional model for the core. In this way, we are going to be able not only to search for physical ways to achieve the modulation of the kinetic variables at the edge of the plasma required by our control method but also to work with kinetic profiles which are closer to the ones found in real reactors. This is due to the fact that the zero-dimensional model of the tokamak SOL will allow us to work with more realistic boundary conditions. Based on the fact that the one-dimensional model for the core and the zero-dimensional model of the tokamak SOL are connected through the values of the energy and density fluxes at the boundary, we have decided to use Neumann boundary conditions in this work as an anticipation of the future step.

In addition, a burning plasma model will be considered. In this way, it will be possible to test the control method for an inherently thermally unstable system. More updated correlations for the transport coefficients (D, κ, χ, V_p), if available, will also be considered. The research of transport coefficients is in a considerable state of flux. For this reason we synthesized a controller which does not depend on the particle diffusivity D. Note that the stability proof in § 5 is completely independent of D and we only assumed that $D > 0$. However, we approximated the relationship between the particle diffusivity D and the thermal diffusivity χ that modified the model and consequently the controller design. There are many models for thermal and particle transport and it is very difficult, if not impossible, to decide which one is the best. Since there is no clear evidence that the original relationship gives a better model of the system than the one given by the approximate relationship, we took this last one to simplify the design and therefore the presentation. However, if this were the case we would only have to study the robustness of the controller against this simplification and eventually in the worst case to redesign it taking into account the original relationship. Although there are several transport codes based on theoretical studies or experimental observations that succeed reproducing the transport behaviour of the plasma, none of them is suitable for control applications. At this stage of the fusion research, stronger emphasis must be put on the development of transport models suitable for control purposes.

Acknowledgements

This work was supported by grants from the National Science Foundation and the University of California Energy Institute.

References

Becker, G. Eberhagen, A., Fussmann, G., Gehre, O., Gernhardt, J., von Gierke, G., Glock, E., Gruber, O., Haas, G., Hofmann, J., Janeschitz, G., Karger, F., Kluber, O., Kornherr, M., Lackner, K., Lisitano, G., Mast, F., Mayer, H. M., McCormick, K., Meisel, D., Mertens, V., Muller, E. R., Murmann, H., Niedermayer, H., Poschenrieder, W., Rapp, H., Schneider, F., Siller, G., Soldner, F. X., Steuer, K.-H., Wagner, F., Speth, E., and Vollmer, O., 1988, Transport analysis of the observed T_e profile invariance in ASDEX. *Nuclear Fusion*, **28**, 139–146.

Boskovic, D., and Krstic, M., 2001, Stabilization of a solid propellant rocket instability by state feedback. *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, FL, USA, December.

Boskovic, D., and Krstic, M., 2002, Backstepping control of chemical tubular reactions. *Computers and Chemical Engineering*, **26**, 1077–1085.

Firestone, M. A., and Kessel, C. E., 1991, Plasma kinetics control in a tokamak. *IEEE Transactions on Plasma Science*, **19**, 29–41.

Firestone, M. A., Morrow-Jones, J. W., and Mau, T. K., 1997 a, Comprehensive feedback control of a tokamak fusion reactor. *Fusion Technology*, **32**, 390–403.

Firestone, M. A., Morrow-Jones, J. W., and Mau, T. K., 1999 b, Developing integrated tokamak dynamics models for next generation machine control. *Fusion Technology*, **32**, 526–544.

Fuchs, V., Shoucri, M. M., Thibaudeau, G., Harten, L., and Bers, A., 1983, High-Q thermally stable operation of a tokamak reactor. *IEEE Transactions on Plasma Science*, **PS-11**, 4–18.

Miley, G. H., and Varadarajan, V., 1992, On self-tuning control of tokamak thermokinetics. *Fusion Technology*, **22**, 425–438.

Schuster, E., Krstic, M., and Tynan, G., 2001, Nonlinear control of burn instability in fusion reactors. *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, FL, USA, December.

Schuster, E., Krstic, M., and Tynan, G., 2002, Nonlinear backstepping control for the fuel diffusion lag in fusion reactors. *Proceedings of the 15th IFAC World Congress on Automatic Control*, Barcelona, Spain, July.

Schuster, E., Krstic, M., and Tynan, G., 2003, Burn control in fusion reactors via nonlinear stabilization techniques. *Fusion Science and Technology*, **43**, 18–37.