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Observer-based schemes for adaptive nonlinear state-feedback control

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We combine recently proposed adaptive nonlinear controllers with two types of observer-based identifiers. The controllers guarantee an input-to-state stability property with respect to $\tilde{\theta}$ and $\dot{\tilde{\theta}}$, and the observer-based identifiers independently guarantee boundedness of $\tilde{\theta}$. A stability enhancement in the observer is crucial in establishing stability properties.

1. Introduction

After an initial success of Lyapunov-based adaptive nonlinear schemes (Kanellakopoulos *et al.* 1991, Jiang and Praly 1991, Krstić *et al.* 1992), a new class of adaptive controllers that guarantee input-to-state stability (ISS, as defined by Sontag 1989) with respect to $\tilde{\theta}$ and $\dot{\tilde{\theta}}$ allowed us to design nonlinear swapping schemes (Krstić and Kokotović 1993). They employ parameter identifiers with a wide variety of update laws—gradient and least-squares, normalized and unnormalized. The demonstrated strength of these controllers motivated us to re-examine observer-based identifiers (also known as equation error filtering) (Campion and Bastin 1990, Praly *et al.* 1991, Teel *et al.* 1991). Except for Teel (1992) and Ghanadan and Blankenship (1993), the state of the art of adaptive nonlinear observer-based schemes is still well represented by the analysis and systematization of Praly *et al.* (1991). In the absence of matching conditions, all the schemes of Praly *et al.* (1991) involve some growth restrictions.

For parametric-strict-feedback nonlinear systems, an overparametrized adaptive scheme without growth restrictions was designed by Kanellakopoulos *et al.* (1991). The overparametrization was completely removed using the 'tuning function' controller (Krstić *et al.* 1992). The first observer-based scheme for this class of systems was designed by Teel (1992), with the same overparametrization as used by Kanellakopoulos *et al.* (1991). Ghanadan and Blankenship (1993) developed an observer-based scheme for a larger class of approximately linearizable systems.

In this paper, we present two observer-based schemes for parametric-strict-feedback non-linear systems. The first scheme employs an observer for the error system which includes the controller, while the second scheme uses an observer for the plant. The observer-based identifiers independently guarantee boundedness of $\tilde{\theta}$ but not of $\dot{\tilde{\theta}}$. Therefore, the ISS property with respect to $\tilde{\theta}$ and $\dot{\tilde{\theta}}$ previously used by us (Krstić and Kokotović 1993) is not sufficient to prove the boundedness of all signals, and our proofs here differ from those in the previous

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work (Krstić and Kokotović 1993). There is also a difference in stability properties and proofs between the two schemes presented here.

An advantage of using observer-based identifiers over those based on tuning functions (Krstić *et al.* 1992) is a less involved derivation of the control law. This, however, comes at the expense of an increase in the dynamic order with an observer. On the other hand, an advantage of using observer-based identifiers over those based on nonlinear swapping (Krstić and Kokotović 1993) is a significant reduction in the dynamic order.

The error-observer scheme presented in this paper is readily extendable to the minimum phase nonlinear systems in the output-feedback canonical form, as well as to linear systems.

The paper is organized as follows. We introduce the class of uncertain nonlinear systems and state the control objective in § 2. In § 3, we present our controller design. The error-observer and plant-observer schemes are presented, along with an analysis of their stability and performance properties, in §§ 3 and 4, respectively.

2. Problem statement

The problem is adaptively to control nonlinear systems transformable into the *parametric-strict-feedback* form

$$\left.\begin{aligned} \dot{x}_i &= x_{i+1} + \theta^{\mathrm{T}}\varphi_i(x_1, \ldots, x_i), \quad 1 \leq i \leq n-1 \\ \dot{x}_n &= \beta_0(x)u + \theta^{\mathrm{T}}\varphi_n(x) \\ y &= x_1 \end{aligned}\right\} \tag{2.1}$$

where $\theta \in \mathbb{R}^p$ is the vector of unknown constant parameters, β_0, and the components of φ_i, $1 \leq i \leq n$, are smooth nonlinear functions in $\mathbb{R}^n$, $\varphi_1(0) = \cdots = \varphi_n(0) = 0$, and $\beta_0(x) \neq 0$, for all $x \in \mathbb{R}^n$.

The control objective is to force the output y of the system (2.1) asymptotically to track the output y_r of a known linear reference model of the form

$$\begin{aligned} \dot{x}_m &= \begin{bmatrix} 0 & & \\ \vdots & I_{n-1} & \\ 0 & & \\ -m_0 & \ldots & -m_{n-1} \end{bmatrix} x_m + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_m \end{bmatrix} r \\ y_r &= x_{m,1} \end{aligned} \tag{2.2}$$

where $M(s) = s^n + m_{n-1}s^{n-1} + \cdots + m_1 s + m_0$ is Hurwitz, $k_m > 0$, and $r(t)$ is bounded and piecewise continuous. An alternative objective, as used by Kanellakopoulos *et al.* (1991), is asymptotically to track a given reference signal $y_r(t)$ with the assumption that its first n derivatives are known, bounded and piecewise continuous.

Notation: For vectors we use $|x|_P \triangleq (x^{\mathrm{T}}Px)^{1/2}$ to denote the weighted euclidean norm of x. For matrices, $|X|_2$ denotes the induced 2-norm of X. The $\mathcal{L}_\infty$ and $\mathcal{L}_2$ norms for signals are denoted by $\|\cdot\|_\infty$ and $\|\cdot\|_2$, respectively. The spaces of all signals which are globally bounded, locally bounded and square-integrable on $[0, t_f)$, $t_f > 0$, are denoted by $\mathcal{L}_\infty[0, t_f)$, $\mathcal{L}_{\infty e}[0, t_f)$ and $\mathcal{L}_2[0, t_f)$, respectively. □

3. Controller and its error system

The adaptive nonlinear controller is recursively defined by

$$
\begin{aligned}
z_i &= x_i - x_{m,i} - \alpha_{i-1} \\
\alpha_i(x_1, \ldots, x_i, \hat{\theta}, x_m) &= -z_{i-1} - c_i z_i - \hat{\theta}^{\mathrm{T}} w_i + \sum_{k=1}^{i-1}\left(\frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial x_{m,k}} x_{m,k+1}\right) \\
&\quad - s_i(x_1, \ldots, x_i, \hat{\theta}, x_m) z_i \\
w_i(x_1, \ldots, x_i, \hat{\theta}, x_m) &= \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_k, \qquad i = 1, \ldots, n \\
u &= \frac{1}{\beta_0(x)}[\alpha_n(x, \hat{\theta}, x_m) - m_0 x_{m,1} - \cdots - m_{n-1} x_{m,n} + k_m r]
\end{aligned}
\tag{3.1}
$$

where $c_i > 0$, $i = 1, \ldots, n$, and, for notational convenience $z_0 \triangleq 0$, $\alpha_0 \triangleq 0$. The remaining design freedom is in the choice of the nonlinear dampling functions $s_i(x_1, \ldots, x_i, \hat{\theta}, x_m)$. The resulting system, called the *error system*, is

$$
\dot{z} = A_z(z, \hat{\theta}, t) z + W(z, \hat{\theta}, t)^{\mathrm{T}} \tilde{\theta} + D(z, \hat{\theta}, t) \dot{\hat{\theta}}, \quad z \in \mathbb{R}^n \tag{3.2}
$$

where $z_1 = x_1 - x_{m,1} = y - y_r$ represents the tracking error, and

$$
A_z(z, \hat{\theta}, t) = \begin{bmatrix}
-c_1 - s_1 & 1 & 0 & \cdots & 0 \\
-1 & -c_2 - s_2 & 1 & \ddots & \vdots \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & -1 & -c_n - s_n
\end{bmatrix}
$$

$$
W(z, \hat{\theta}, t)^{\mathrm{T}} = \begin{bmatrix} w_1^{\mathrm{T}} \\ w_2^{\mathrm{T}} \\ \vdots \\ w_n^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{n \times p}, \quad D(z, \hat{\theta}, t) = \begin{bmatrix} 0 \\ -\dfrac{\partial \alpha_1}{\partial \hat{\theta}} \\ \vdots \\ -\dfrac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \end{bmatrix} \in \mathbb{R}^{n \times p} \tag{3.3}
$$

Now we design the nonlinear damping functions $s_i(x_1, \ldots, x_i, \hat{\theta}, x_m)$ as

$$
s_i(x_1, \ldots, x_i, \hat{\theta}, x_m) = \kappa_i |w_i|^2 + g_i \left| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}^{\mathrm{T}} \right|^2 \tag{3.4}
$$

where κ_i, g_i, $i = 1, \ldots, n$ are positive scalar constants. The 'kappa-terms' $\kappa_i |w_i|^2$ have been introduced recently (Kanellakopoulos 1992). We have proved (Krstić and Kokotović 1993) that the controller with nonlinear damping functions s_i achieves the following ISS property of the error system (3.2) with respect to $\tilde{\theta}$ and $\dot{\hat{\theta}}$:

$$|z(t)| \leq \frac{1}{2(c_0)^{1/2}}\left(\frac{1}{\kappa_0}\|\tilde{\theta}\|_\infty^2 + \frac{1}{g_0}\|\dot{\hat{\theta}}\|_\infty^2\right)^{1/2} + |z(0)|\exp(-c_0 t) \tag{3.5}$$

4. Error-observer scheme

We implement an 'observer' for the error state z of (3.2) by dropping the $\tilde{\theta}$-term, i.e.

$$\dot{\hat{z}} = A_z(z, \hat{\theta}, t)\hat{z} + D(z, \hat{\theta}, t)\dot{\hat{\theta}} \tag{4.1}$$

With (4.1), the *observer error*

$$\varepsilon = z - \hat{z} \tag{4.2}$$

is governed by the equation in which the $\tilde{\theta}$-term reappears:

$$\dot{\varepsilon} = A_z(z, \hat{\theta}, t)\varepsilon + W(z, \hat{\theta}, t)^{\mathrm{T}}\tilde{\theta} \tag{4.3}$$

As the parameter update law we employ

$$\dot{\hat{\theta}} = \Gamma W\varepsilon, \quad \Gamma = \Gamma^{\mathrm{T}} > 0 \tag{4.4}$$

It is important to note that our closed-loop adaptive system with the controller (3.1) has two equivalent state representations (2.1), (4.1), (4.4) – and – (3.2), (4.3), (4.4).

Proposition 4.1—Stability and tracking: *The closed-loop adaptive system consisting of the plant* (2.1), *controller* (3.1), *observer* (4.1), *and update law* (4.4) *has a globally uniformly stable equilibrium at the origin* $z = 0$, $\varepsilon = 0$, $\tilde{\theta} = 0$, *and* $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \varepsilon(t) = 0$. *This means, in particular, that global asymptotic tracking is achieved*:

$$\lim_{t\to\infty}[y(t) - y_r(t)] = 0 \tag{4.5}$$

Proof: Starting from the update law (4.4), we obtain the following inequalities:

$$|\dot{\hat{\theta}}|^2 \leq \bar{\lambda}(\Gamma)^2|W\varepsilon|^2 = \bar{\lambda}(\Gamma)^2\left|\sum_{i=1}^{n} w_i\varepsilon_i\right|^2 \leq \bar{\lambda}(\Gamma)^2 n\sum_{i=1}^{n}|w_i|^2\varepsilon_i^2 \tag{4.6}$$

We make use of the following constants: $c_0 = \min_{1\leq i\leq n} c_i$, $\kappa_m = \min_{1\leq i\leq n}\kappa_i$,

$$\frac{1}{\kappa_0} = \sum_{i=1}^{n}\frac{1}{\kappa_i} \quad \text{and} \quad \frac{1}{g_0} = \sum_{i=1}^{n}\frac{1}{g_i}$$

and $\mu > 0$ to be chosen later. Along the solutions of (4.1), (4.3), (4.4), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}|\hat{z}|^2 + \frac{1}{2}|\varepsilon|^2 + \frac{1}{2}|\tilde{\theta}|^2_{\Gamma^{-1}}\right) \leq & -\mu\sum_{i=1}^{n}\left(c_i + \kappa_i|w_i|^2 + g_i\left|\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}^{\mathrm{T}}\right|^2\right)\hat{z}_i^2 \\ & - \mu\sum_{i=1}^{n}\hat{z}_i\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} \\ & - \sum_{i=1}^{n}\left(c_i + \kappa_i|w_i|^2 + g_i\left|\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}^{\mathrm{T}}\right|^2\right)\varepsilon_i^2 \\ & + \varepsilon^{\mathrm{T}}W^{\mathrm{T}}\tilde{\theta} - \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\hat{\theta}}\end{aligned}$$

$$\leqslant -\mu c_0|\hat{z}|^2 - \mu\sum_{i=1}^{n} g_i\left|\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}^{\mathrm{T}}\hat{z}_i + \frac{1}{2g_i}\dot{\hat{\theta}}\right|^2 + \frac{\mu}{4g_0}|\dot{\hat{\theta}}|^2$$

$$-c_0|\varepsilon|^2 - \kappa_m\sum_{i=1}^{n}|w_i|^2\varepsilon_i^2$$

$$\leqslant -\mu c_0|\hat{z}|^2 - c_0|\varepsilon|^2 - \left(\kappa_m - \mu\frac{\bar{\lambda}(\Gamma)^2 n}{4g_0}\right)\sum_{i=1}^{n}|w_i|^2\varepsilon_i^2 \tag{4.7}$$

Choosing $\mu < 4g_0\kappa_m/n\bar{\lambda}(\Gamma)^2$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}|\hat{z}|^2 + \frac{1}{2}|\varepsilon|^2 + \frac{1}{2}|\tilde{\theta}|^2_{\Gamma^{-1}}\right) \leqslant -\mu c_0|\hat{z}|^2 - c_0|\varepsilon|^2 \tag{4.8}$$

which proves that $z = 0$, $\varepsilon = 0$, $\tilde{\theta} = 0$ is g.u.s. From LaSalle's invariance theorem, it further follows that $z(t)$, $\varepsilon(t) \to 0$ as $t \to \infty$. □

Note that the stability enhancing terms $\kappa_i|w_i|^2$ in the matrix A_z of the observer error equation (4.3) are crucial for counteracting the destabilizing effects of $\dot{\hat{\theta}}$.

Next, we give $\mathscr{L}_2$ and $\mathscr{L}_\infty$ tracking performance bounds. Without loss of generality, we assume that $\hat{z}(0) = z(0)$ and $\Gamma = \gamma I$.

Proposition 4.2—Performance: *In the adaptive system* (2.1), (3.1), (4.1), (4.4), *the following inequalities hold*:

(i) $$\|z\|_2 \leqslant \frac{|\tilde{\theta}(0)|}{(2c_0\gamma)^{1/2}}\left[1 + \left(\frac{n\gamma^2}{2g_0\kappa_m}\right)^{1/2}\right] + \frac{1}{(2c_0)^{1/2}}|z(0)| \tag{4.9}$$

(ii) $$|z(t)| \leqslant \frac{|\tilde{\theta}(0)|}{2(c_0\kappa_0)^{1/2}}\left[1 + \left(\frac{2n\gamma^2}{g_0\kappa_m}\right)^{1/2}\right] + |z(0)|\exp(-c_0 t) \tag{4.10}$$

Proof:

(i) Along the solutions of (4.3)–(4.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|\varepsilon|^2 + \frac{1}{2\gamma}|\tilde{\theta}|^2\right) \leqslant -c_0|\varepsilon|^2 \tag{4.11}$$

Since $\varepsilon(0) = z(0) - \hat{z}(0) = 0$, this implies that $\|\tilde{\theta}\|_\infty = |\tilde{\theta}(0)|$ and

$$\|\varepsilon\|_2 \leqslant \frac{1}{(2c_0\gamma)^{1/2}}|\tilde{\theta}(0)| \tag{4.12}$$

Now from (4.8), for $\mu < 4g_0\kappa_m/n\gamma^2$, we get

$$\mu\|\hat{z}\|_2^2 + \|\varepsilon\|_2^2 \leqslant \frac{1}{c_0}\left(\frac{\mu|\hat{z}(0)|^2 + |\varepsilon(0)|^2}{2} + \frac{1}{2\gamma}|\tilde{\theta}(0)|^2\right) \tag{4.13}$$

and, since $\hat{z}(0) = z(0)$, then

$$\|\hat{z}\|_2 \leqslant \frac{1}{(2c_0\gamma\mu)^{1/2}}|\tilde{\theta}(0)| + \frac{1}{(2c_0)^{1/2}}|z(0)| \tag{4.14}$$

Letting $\mu = 2g_0\kappa_m/n\gamma^2$ and adding (4.12) and (4.14) in $\|z\|_2 \leqslant \|\varepsilon\|_2 + \|\hat{z}\|_2$, we arrive at (4.9).

(ii) In a fashion similar to (4.7), we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu|z|^2 + |\varepsilon|^2}{2}\right) \leq -c_0(\mu|z|^2 + |\varepsilon|^2) - \mu\sum_{i=1}^{n}\kappa_i|w_i|^2 z_i^2 - \sum_{i=1}^{n}\kappa_i|w_i|^2\varepsilon_i^2 + \mu\sum_{i=1}^{n} z_i w_i^{\mathrm{T}}\tilde{\theta} + \sum_{i=1}^{n}\varepsilon_i w_i^{\mathrm{T}}\tilde{\theta} + \frac{\mu}{4g_0}|\dot{\hat{\theta}}|^2$$

$$\leq -c_0(\mu|z|^2 + |\varepsilon|^2) + \frac{\mu}{4\kappa_0}|\tilde{\theta}|^2 + \frac{1}{2\kappa_0}|\tilde{\theta}|^2 - \left(\frac{\kappa_m}{2} - \mu\frac{n\gamma^2}{4g_0}\right)\sum_{i=1}^{n}|w_i|^2\varepsilon_i^2 \tag{4.15}$$

Choosing $\mu = g_0\kappa_m/n\gamma^2$, we get

$$\mu|z(t)|^2 + |\varepsilon(t)|^2 \leq (\mu|z(0)|^2 + |\varepsilon(0)|^2)\exp(-2c_0 t) + \frac{\mu + 2}{2\kappa_0}\int_0^t \exp(-2c_0[t - \tau])\,|\tilde{\theta}(\tau)|^2\,\mathrm{d}\tau \tag{4.16}$$

which implies

$$|z(t)| \leq \frac{1}{2(c_0\kappa_0)^{1/2}}\left[1 + \left(\frac{2}{\mu}\right)^{1/2}\right]\|\tilde{\theta}\|_\infty + |z(0)|\exp(-c_0 t) \tag{4.17}$$

The last inequality proves (4.10) and also establishes an ISS property from $\tilde{\theta}$ to z. It is easy to see that $|\varepsilon(t)| \leq 1/2(c_0\kappa_0)^{1/2}\|\tilde{\theta}\|_\infty$ describes the ISS property from $\tilde{\theta}$ to ε. □

Remark 4.1: Although the initial states $z_2(0), \ldots, z_\rho(0)$ may depend on c_i, κ_i, g_i, this dependence can be removed by setting $z(0) = 0$ with the following initialization of the reference model:

$$x_{m,i}(0) = x_i(0) - \alpha_{i-1}(x_1(0), \ldots, x_{i-1}(0), \hat{\theta}(0), x_{m,1}(0), \ldots, x_{m,i-1}(0)) \tag{4.18}$$

It can also be proven that in this initialization $x_m(0)$ does not depend on c_i, κ_i, g_i. Therefore the bounds (4.9), (4.10) can be made as small as desired by a choice of c_0. □

5. Plant-observer scheme

For the plant (2.1) rewritten in the form

$$\dot{x} = Ex + e_n u + \phi(x)^{\mathrm{T}}\theta \tag{5.1}$$

where

$$E = \begin{bmatrix} 0 & & I_{n-1} & \\ \vdots & & & \\ 0 & & \ldots & 0 \end{bmatrix} \quad \text{and} \quad \phi^{\mathrm{T}} = \begin{bmatrix} \varphi_1^{\mathrm{T}} \\ \vdots \\ \varphi_n^{\mathrm{T}} \end{bmatrix}$$

we implement an 'observer'

$$\dot{\hat{x}} = (\bar{A} - \lambda\phi(x)^{\mathrm{T}}\phi(x)\bar{P})(\hat{x} - x) + Ex + e_n u + \phi(x)^{\mathrm{T}}\hat{\theta} \tag{5.2}$$

where $\bar{A}$ satisfies $\bar{P}\bar{A} + \bar{A}^{\mathrm{T}}\bar{P} = -qI$, $\bar{P} = \bar{P}^{\mathrm{T}} > 0$, and λ, $q > 0$. The *observer error*

$$\varepsilon_x = x - \hat{x} \tag{5.3}$$

is governed by

$$\dot{\varepsilon}_x = (\bar{A} - \lambda\phi(x)^{\mathrm{T}}\phi(x)\bar{P})\ \varepsilon_x + \phi(x)^{\mathrm{T}}\tilde{\theta} \tag{5.4}$$

The stability enhancing matrix $-\lambda\phi(x)^{\mathrm{T}}\phi(x)\bar{P}$ plays a crucial role in counteracting the destabilizing effect of $\hat{\theta}$. The update law is

$$\dot{\hat{\theta}} = \Gamma\phi\bar{P}\varepsilon_x, \quad \Gamma = \Gamma^{\mathrm{T}} > 0 \tag{5.5}$$

Lemma 5.1: *If* $x \in \mathscr{L}_{\infty e}[0, t_f)$, *then the update law* (5.5) *guarantees that* $\tilde{\theta} \in \mathscr{L}_\infty[0, t_f)$ *and* $\varepsilon_x \in \mathscr{L}_\infty[0, t_f) \cap \mathscr{L}_2[0, t_f)$.

Proof: Standard, using

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\varepsilon_x|^2_{\bar{P}} + |\tilde{\theta}|^2_{\Gamma^{-1}}) \leqslant -q|\varepsilon_x|^2 \qquad \square$$

Proposition 5.1—Boundedness and tracking: *All the signals in the closed-loop adaptive system consisting of the plant* (2.1), *controller* (3.1), *observer* (5.2), *and the update law* (5.5), *are globally uniformly bounded, and* $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \varepsilon_x(t) = 0$. *This means, in particular, that global asymptotic tracking is achieved*:

$$\lim_{t\to\infty}[y(t) - y_r(t)] = 0 \tag{5.6}$$

Proof: Owing to the continuity of $x_m(t)$ and the smoothness of the nonlinearities in (2.1), the solution of the closed-loop adaptive system exists and is unique. Let its maximum interval of existence be $[0, t_f)$.

For $\mu > 0$ we readily obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}|z|^2 + |\varepsilon_x|^2_{\bar{P}}\right) &\leqslant -c_0\mu|z|^2 + \frac{\mu}{4\kappa_0}|\tilde{\theta}|^2 + \frac{\mu}{4g_0}|\dot{\hat{\theta}}|^2 - q|\varepsilon_x|^2 \\ &\quad - 2\lambda|\phi\bar{P}\varepsilon_x|^2 + 2\tilde{\theta}^{\mathrm{T}}\phi\bar{P}\varepsilon_x \\ &\leqslant -c_0\mu|z|^2 - q|\varepsilon_x|^2 + \frac{\mu}{4\kappa_0}|\tilde{\theta}|^2 + \frac{1}{\lambda}|\tilde{\theta}|^2 \\ &\quad - \left(\lambda - \mu\frac{\bar{\lambda}(\Gamma)^2}{4g_0}\right)|\phi\bar{P}\varepsilon_x|^2\end{aligned} \tag{5.7}$$

Choosing $\mu < 4g_0\lambda/\bar{\lambda}(\Gamma)^2$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{2}|z|^2 + |\varepsilon_x|^2_{\bar{P}}\right) \leqslant -c_0\mu|z|^2 - q|\varepsilon_x|^2 + \left(\frac{\mu}{4\kappa_0} + \frac{1}{\lambda}\right)|\tilde{\theta}|^2 \tag{5.8}$$

which, in view of Lemma 5.1, implies that $z \in \mathscr{L}_\infty[0, t_f)$.

We have thus shown that all of the signals of the closed-loop adaptive system are bounded on $[0, t_f)$ by constants depending only on the initial conditions. Hence $t_f = \infty$.

To prove convergence of z to zero, we recall first that from Lemma 5.1 we know that $\varepsilon_x \in \mathscr{L}_2$. In view of the boundedness of ϕ, this guarantees that $\dot{\hat{\theta}} \in \mathscr{L}_2$. Factoring the regressor matrix W as follows:

$$W^{\mathrm{T}}(z, \hat{\theta}, t) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -\frac{\partial \alpha_1}{\partial x_1} & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ -\frac{\partial \alpha_{n-1}}{\partial x_1} & -\frac{\partial \alpha_{n-1}}{\partial x_2} & \dots & 1 \end{bmatrix} \phi^{\mathrm{T}}(x) \triangleq M(z, \hat{\theta}, t)\phi^{\mathrm{T}}(x) \tag{5.9}$$

we consider $\zeta \triangleq z - M\varepsilon_x$ and obtain

$$\dot{\zeta} = A_z(z, \hat{\theta}, t)\zeta + [\dot{M} + A_z(z, \hat{\theta}, t)M - M(\bar{A} - \lambda\phi^{\mathrm{T}}\phi\bar{P})]\varepsilon_x + D(z, \hat{\theta}, t)\dot{\hat{\theta}} \tag{5.10}$$

where $\dot{M} + A_z(z, \hat{\theta}, t)M - M(\bar{A} - \lambda\phi^{\mathrm{T}}\phi\bar{P})$ is bounded. It is now straightforward to derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|\zeta|^2\right) \leq -\frac{c_0}{2}|\zeta|^2 + \frac{1}{2c_0}|\dot{M} + A_z(z, \hat{\theta}, t)M - M(\bar{A} - \lambda\phi^{\mathrm{T}}\phi\bar{P})|_2^2|\varepsilon_x|^2 + \frac{1}{4g_0}|\dot{\hat{\theta}}|^2 \tag{5.11}$$

and since ε_x, $\dot{\hat{\theta}} \in \mathscr{L}_2$, it follows (see, e.g., Lemma A.1 of Krstić and Kokotović 1993) that $\zeta \in \mathscr{L}_2$. Therefore $z \in \mathscr{L}_2$. We recall that z, $\varepsilon_x \in \mathscr{L}_\infty$ and note that (3.2) implies $\dot{z} \in \mathscr{L}_\infty$ and (5.4) implies $\dot{\varepsilon}_x \in \mathscr{L}_\infty$. Therefore, by Barbalat's lemma, $z(t)$, $\varepsilon_x(t) \to 0$ as $t \to \infty$. □

For the plant–observer scheme, $\mathscr{L}_\infty$ tracking performance bounds can be derived as in Proposition 4.2. In the case $\bar{A} = -c_0 I$, $\bar{P} = \frac{1}{2}I$, $q = c_0$, $\lambda = 2\kappa_0$, and, without loss of generality, $\hat{x}(0) = x(0)$, $\Gamma = 2\gamma I$, by proceeding from (5.8), as in the proof of Proposition 4.2, we get

$$|z(t)| \leq \frac{|\tilde{\theta}(0)|}{2(c_0\kappa_0)^{1/2}}\left[1 + \left(\frac{2n\gamma^2}{g_0\kappa_m}\right)^{1/2}\right] + |z(0)|\exp(-c_0 t) \tag{5.12}$$

It is not clear, however, how to derive a useful $\mathscr{L}_2$ tracking performance bound similar to (4.10).

6. Conclusions

With the observer-based design presented in this paper, we enlarged the class of adaptive schemes for nonlinear systems which neither satisfy matching nor growth conditions. The strength of our controllers is evident not only from the fact that they guarantee stability with different identifiers, but also from their ability to guarantee similar performance bounds with different identifiers.

Although they have a similar structure, the two observer-based schemes presented here have different stability properties and proofs. While for the error–observer scheme we prove stability of the origin in the sense of Lyapunov, for the plant–observer scheme we only prove boundedness and convergence. The lack of a Lyapunov stability proof is the main reason that an $\mathscr{L}_2$ performance bound explicit in design parameters and initial conditions is not available for the plant–observer scheme.

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