

Explicit State and Output Feedback Boundary Controllers for Partial Differential Equations

Andrey Smyshlyaev and Miroslav Krstic

Abstract—In this paper the explicit (closed form) solutions to several application-motivated parabolic problems are presented. The boundary stabilization problem is converted to a problem of solving a specific linear hyperbolic partial differential equation (PDE). This PDE is then solved for several particular cases. Closed loop solutions to the original parabolic problem are also found explicitly. Output feedback problem under boundary measurement is explicitly solved with both anti-collocated and collocated sensor/actuator locations. It is shown how closed form frequency domain compensators based on the closed form observers and controllers can be designed.

Index Terms—parabolic PDE, boundary control, backstepping.

I. INTRODUCTION

METHODS for boundary control of linear parabolic partial differential equation (PDEs) are well established (see, e.g., [1]–[10]). However, even in the simplest case of constant coefficients the existing results are not explicit and require numerical solution, e.g., solving an operator Riccati equation in case of the linear quadratic regulator (LQR) method.

In this paper we present the explicit (closed-form) solutions to several application-motivated parabolic problems including the very important case of constant coefficients. The method we use to get those solutions is essentially the backstepping technique developed originally to handle nonlinear finite-dimensional systems [11]. Here we apply it to a linear infinite-dimensional plant. The approach exploits the structure of the parabolic PDE, namely, its "tri-diagonal form". As a result, the boundary stabilization problem is converted to a problem of solving a specific linear hyperbolic PDE. In several cases the solution to this PDE can be found explicitly; we show some of these cases.

We also give explicit solutions for the output feedback problem under boundary measurement. Both anti-collocated and collocated sensor/actuator pairs are considered. Duality of the observer design problem to the stabilization problem is shown, specifically meaning that the observer gains can be derived from the control gains. The closed form observers and controllers make it possible to design closed form frequency domain compensators.

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II. STATE-FEEDBACK DESIGNS

We will present the solutions to four distinct parabolic PDEs. For the first problem the procedure will be described in details and for the remaining cases most of the details will be omitted.

A. Unstable Heat Equation

Let us consider the following plant:

$$\begin{aligned} u_t(x, t) &= \varepsilon u_{xx}(x, t) + \lambda_0 u(x, t), \quad x \in (0, 1), \quad (1) \\ u(0, t) &= 0, \quad (2) \end{aligned}$$

where $\varepsilon > 0$ and λ_0 are arbitrary constants. This equation is controlled at $x = 1$ using $u(1, t)$ (Dirichlet actuation) or $u_x(1, t)$ (Neumann actuation) as a control input. The open-loop system (1)–(2) (with $u(1, t) = 0$ or $u_x(1, t) = 0$) is unstable with arbitrarily many unstable eigenvalues (for large λ_0/ε). The methods for the boundary control of (1)–(2) include pole placement [9], LQR [3], and finite-dimensional backstepping [12]. In [13], two stabilizing controllers (backstepping and pole placement) were constructed in a closed form, but only for $\lambda_0/\varepsilon < 3\pi^2/4$, i.e., in case of one unstable eigenvalue. However, the *explicit* (closed-form) boundary stabilization result in the case of *arbitrary* ε , λ_0 is not available in the literature even for this benchmark constant coefficient case.

We start with the following coordinate transformation

$$w(x, t) = u(x, t) - \int_0^x k(x, y) u(y, t) dy \quad (3)$$

that transforms system (1)–(2) into the system

$$\begin{aligned} w_t(x, t) &= \varepsilon w_{xx}(x, t) - cw(x, t), \quad x \in (0, 1), \quad (4) \\ w(0, t) &= 0, \quad (5) \\ w(1, t) &= 0 \quad \text{or} \quad w_x(1, t) = 0, \quad (6) \end{aligned}$$

which is exponentially stable for $c \geq 0$. Once we find the transformation (3) (namely $k(x, y)$), the boundary condition (6) gives the controller in the form

$$u(1, t) = \int_0^1 k_1(y) u(y, t) dy \quad (7)$$

for the Dirichlet actuation, $k_1(y) = k(1, y)$, and

$$u_x(1, t) = k_2(1) u(1, t) + \int_0^1 k_2(y) u(y, t) dy \quad (8)$$

for the Neumann actuation, $k_2(y) = k_x(1, y)$.

We are now going to derive $k(x, y)$. From (1), (3), and (4) we get

$$\begin{aligned} w_t(x, t) &= u_t(x, t) - \int_0^x k(x, y)(\varepsilon u_{yy}(y, t) + \lambda_0 u) dy \\ &= \varepsilon u_{xx}(x, t) + \lambda_0 u(x, t) - \varepsilon k(x, x)u_x(x, t) \\ &\quad + \varepsilon k(x, 0)u_x(0, t) + \varepsilon k_y(x, x)u(x, t) \\ &\quad - \int_0^x (\varepsilon k_{yy}(x, y) + \lambda_0 k(x, y))u(y, t) dy, \end{aligned} \quad (9)$$

$$\begin{aligned} w_{xx}(x, t) &= u_{xx}(x, t) - k(x, x)u_x(x, t) - k_x(x, x)u(x, t) \\ &\quad - \frac{d}{dx}k(x, x) - \int_0^x k_{xx}(x, y)u(y, t) dy. \end{aligned} \quad (10)$$

Combining (9) and (10) gives

$$\begin{aligned} 0 &= w_t(x, t) - \varepsilon w_{xx}(x, t) + cw(x, t) \\ &= \left(2\varepsilon \frac{d}{dx}k(x, x) + \lambda_0 + c \right) u(x, t) + \varepsilon k(x, 0)u_x(0, t) \\ &\quad + \int_0^x \{ \varepsilon k_{xx}(x, y) - \varepsilon k_{yy}(x, y) \\ &\quad - (\lambda_0 + c)k(x, y) \} u(y, t) dy \end{aligned} \quad (11)$$

We can see now that the kernel $k(x, y)$ must satisfy the following hyperbolic PDE:

$$k_{xx}(x, y) - k_{yy}(x, y) = \lambda k(x, y), \quad (x, y) \in \mathcal{T}, \quad (12)$$

$$k(x, 0) = 0, \quad (13)$$

$$k(x, x) = -\frac{\lambda x}{2}. \quad (14)$$

where $\mathcal{T} = \{x, y : 0 < y < x < 1\}$ and $\lambda = (\lambda_0 + c)/\varepsilon$. To find the solution of this PDE, following Liu [14] and Colton [15], we use the transformation to an integral equation and the method of successive approximations. Introducing the change of variables

$$\xi = x + y, \quad \eta = x - y, \quad (15)$$

and denoting

$$G(\xi, \eta) = k(x, y) = k\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right), \quad (16)$$

we get the PDE

$$G_{\xi\eta} = \frac{\lambda}{4}G(\xi, \eta), \quad (17)$$

$$G(\xi, \xi) = 0, \quad (18)$$

$$G(\xi, 0) = -\frac{\lambda}{4}\xi. \quad (19)$$

Integrating (17) with respect to η from 0 to η and using (19) we obtain

$$G_\xi(\xi, \eta) = -\frac{\lambda}{4} + \frac{\lambda}{4} \int_0^\eta G(\xi, s) ds. \quad (20)$$

Integrating (20) with respect to ξ from η to ξ and using (18) gives the integral equation for $G(\xi, \eta)$

$$G(\xi, \eta) = -\frac{\lambda}{4}(\xi - \eta) + \frac{\lambda}{4} \int_\eta^\xi \int_0^\eta G(\tau, s) ds d\tau. \quad (21)$$

Now set

$$\begin{aligned} G_0(\xi, \eta) &= -\frac{\lambda}{4}(\xi - \eta), \\ G_{n+1}(\xi, \eta) &= \frac{\lambda}{4} \int_\eta^\xi \int_0^\eta G_n(\tau, s) ds d\tau. \end{aligned} \quad (22)$$

We can find the general term G_n in closed form (which can be proved by induction):

$$G_n(\xi, \eta) = -\frac{(\xi - \eta) \xi^n \eta^n}{(n!)^2 (n+1)} \left(\frac{\lambda}{4}\right)^{n+1}. \quad (23)$$

By the method of successive approximations, the sum of all the terms G_n gives the solution to (21):

$$G(\xi, \eta) = \sum_{n=0}^{\infty} G_n(\xi, \eta) = -\frac{\lambda(\xi - \eta)}{2} \frac{I_1(\sqrt{\lambda\xi\eta})}{\sqrt{\lambda\xi\eta}}, \quad (24)$$

where I_i is a modified Bessel function of order i . Writing (24) in terms of x, y gives the following solution for $k(x, y)$

$$k(x, y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}}, \quad (25)$$

so, for Dirichlet actuation,

$$k_1(y) = k(1, y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(1 - y^2)}\right)}{\sqrt{\lambda(1 - y^2)}}, \quad (26)$$

and for Neumann actuation,

$$k_2(y) = k_x(1, y) = -\lambda y \frac{I_2\left(\sqrt{\lambda(1 - y^2)}\right)}{1 - y^2}. \quad (27)$$

In Figure 1 the kernel $k_1(y)$ is plotted for several values of λ . We see that the maximum of the absolute value of the kernel moves to the left as λ grows. We can actually calculate the area under the curves and estimate an amount of total gain effort required:

$$\begin{aligned} E &= \int_0^1 |k_1(y)| dy = \lambda \int_0^1 y \frac{I_1\left(\sqrt{\lambda(1 - y^2)}\right)}{\sqrt{\lambda(1 - y^2)}} dy \\ &= \int_0^{\sqrt{\lambda}} I_1(z) dz = I_0\left(\sqrt{\lambda}\right) - 1. \end{aligned} \quad (28)$$

Thus $E \sim \lambda$ for small λ and $E \sim e^{\sqrt{\lambda}}/(\sqrt{2\pi} \lambda^{1/4})$ for large λ .

For the case of a homogeneous Neumann boundary condition at $x = 0$ for the equation (1) it is easy to repeat all the steps we have done for the Dirichlet case and get the following closed form solution for the kernel:

$$k(x, y) = -\lambda x \frac{I_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}}. \quad (29)$$

Note that the leading factor here is x , versus y in (25). The maximum of the absolute value of the kernel is reached at $x = 0$. This makes sense because the control has to

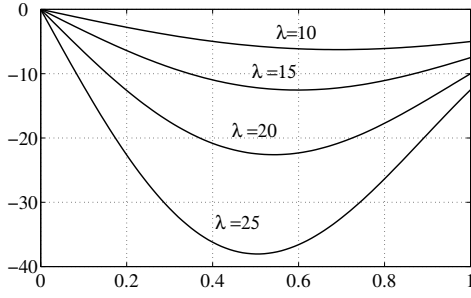


Fig. 1. Dependence of the gain kernel on the level of instability.

react the most aggressively to perturbations that are the farthest from it. The peak value is equal to $|k(1, 0)| = \sqrt{\lambda} I_1(\sqrt{\lambda}) \sim e^{\sqrt{\lambda}}/\lambda^{1/4}$ as $\lambda \rightarrow \infty$.

In order to prove stability we need to prove that the transformation (3) is invertible. Let us write the inverse transformation in the form

$$u(x, t) = w(x, t) + \int_0^x l(x, y)w(y, t) dy. \quad (30)$$

Substituting (30) into equations (4)–(6) and using (1)–(2) we obtain the following PDE governing $l(x, y)$:

$$l_{xx}(x, y) - l_{yy}(x, y) = -\lambda l(x, y), \quad (x, y) \in \mathcal{T}, \quad (31)$$

$$l(x, 0) = 0, \quad (32)$$

$$l(x, x) = -\frac{\lambda x}{2}. \quad (33)$$

Noticing that in this case $l(x, y) = -k(x, y)$ when λ is replaced by $-\lambda$, we immediately obtain

$$l(x, y) = -\lambda y \frac{J_1(\sqrt{\lambda(x^2 - y^2)})}{\sqrt{\lambda(x^2 - y^2)}} \quad (34)$$

where J_1 is the usual (non-modified) Bessel function of the first order, which, incidentally, is a bounded function of $\lambda \geq 0$. The smoothness of the both $k(x, y)$ and $l(x, y)$ in x, y establishes the equivalence of norms of u and w in both L_2 and H_1 . From the properties of the damped heat equation (4)–(6) exponential stability in both L_2 and H_1 follows.

Furthermore, the system (1)–(2), (7) is not only well posed but its solution is explicitly available. Let us show how it can be obtained. First we solve the dumped heat equation (4)–(6):

$$w(x, t) = 2 \sum_{n=1}^{\infty} e^{-\frac{(c+\pi^2 n^2)t}{\varepsilon}} \sin(\pi n x) \int_0^1 w_0(\xi) \sin(\pi n \xi) d\xi. \quad (35)$$

The initial condition w_0 can be calculated explicitly from u_0 via (3), (25). Substituting the result into (30), (34), changing order of integration and calculating some integrals we obtain the explicit solution to closed loop system (1)–(2), (7):

$$u(x, t) = \sum_{n=1}^{\infty} e^{-\frac{(c+\pi^2 n^2)t}{\varepsilon}} \phi_n(x) \int_0^1 \psi_n(\xi) u_0(\xi) d\xi, \quad (36)$$

where

$$\begin{aligned} \psi_n(x) &= \sin(\pi n x) + \int_x^1 \lambda x \frac{I_1(\sqrt{\lambda(\xi^2 - x^2)})}{\sqrt{\lambda(\xi^2 - x^2)}} \sin(\pi n \xi) d\xi, \\ \phi_n(x) &= \frac{2\pi n}{\sqrt{\lambda + \pi^2 n^2}} \sin(\sqrt{\lambda + \pi^2 n^2} x). \end{aligned} \quad (37)$$

Several things can be noticed here. One can directly see from (36) that the backstepping controller has moved the eigenvalues from their open-loop (unstable) locations $\lambda_0 - \pi^2 n^2$ into locations of the damped heat equation $-(c + \pi^2 n^2)$. It is interesting to note, that although infinitely many eigenvalues cannot be arbitrarily assigned, our controller is able to assign all of them to the particular location $-(c + \pi^2 n^2)$. The eigenfunctions of the closed-loop system are assigned to $\phi_n(x)$. We can see from (37) that the controller has reduced the amplitude and increased the frequency of the open-loop eigenfunctions $2 \sin(\pi n x)$.

We come to the following result.

Theorem 1: For any $u_0 \in L_2(0, 1)$ the system (1)–(2), (7) with the kernel $k_1(y)$ given by (26) has a unique classical solution $u(x, t) \in C^{2,1}((0, 1) \times (0, \infty))$ and is exponentially stable at the origin, $u(x, t) \equiv 0$, in the $L_2(0, 1)$ and $H_1(0, 1)$ norms. The same result holds in case of the Neumann type of actuation with $k_2(y)$ given by (27).

For the rest of the paper we will not repeat the procedure of derivation of a PDE for the kernel and the arguments of previous paragraph proving stability and well posedness.

B. Heat Equation with Destabilizing Boundary Condition

We now consider a more complicated system

$$u_t(x, t) = \varepsilon u_{xx}(x, t) + \lambda_0 u(x, t), \quad (38)$$

$$u_x(0, t) = qu(0, t), \quad (39)$$

$$u(1, t) = \int_0^1 k(1, y)u(y) dy, \quad (40)$$

where the boundary condition on the uncontrolled end is mixed and can cause instability for $q < 0$. We use the transformation (3) to map this system into the target system

$$w_t(x, t) = \varepsilon w_{xx}(x, t) - cw(x, t), \quad (41)$$

$$w(0, t) = qw(0, t), \quad (42)$$

$$w(1, t) = 0, \quad (43)$$

which is exponentially stable for $c > \max\{0, -\varepsilon q|q|\}$. It can be shown that the gain kernel should satisfy the following PDE:

$$k_{xx}(x, y) - k_{yy}(x, y) = \lambda k(x, y), \quad (44)$$

$$k_y(x, 0) = qk(x, 0), \quad (45)$$

$$k(x, x) = -\frac{\lambda x}{2}. \quad (46)$$

where $\lambda = (\lambda_0 + c)/\varepsilon$. We propose to search a solution in the following form

$$k(x, y) = -\lambda x \frac{I_1(\sqrt{\lambda(x^2 - y^2)})}{\sqrt{\lambda(x^2 - y^2)}} + \int_0^{x-y} I_0(\sqrt{\lambda(x+y)(x-y-\tau)}) \sigma(\tau) d\tau. \quad (47)$$

Here the first term is a solution to (44)–(46) with $q = 0$ which have been obtained in the previous subsection. Second term is just one of the solutions to (44), σ being an arbitrary function. We can see now that (47) is a solution to (44), (46) and we need only to choose $\sigma(\tau)$ so that (45) is satisfied. Substituting (47) into (45) we obtain the following integral equation for $\sigma(x)$:

$$\int_0^x \sigma(\tau) \left(\frac{\lambda}{2} \tau \frac{I_1(\sqrt{\lambda x(x-\tau)})}{\sqrt{\lambda x(x-\tau)}} + q I_0(\sqrt{\lambda x(x-\tau)}) \right) d\tau + \sigma(x) = \sqrt{\lambda} q I_1(\sqrt{\lambda} x). \quad (48)$$

To solve this equation we apply Laplace transform with respect to x to both sides of (48) and get:

$$\sigma(s) + \int_0^\infty e^{-s\xi} \int_0^x \sigma(\tau) \left(\frac{\lambda}{2} \tau \frac{I_1(\sqrt{\lambda x(x-\tau)})}{\sqrt{\lambda x(x-\tau)}} + q I_0(\sqrt{\lambda x(x-\tau)}) \right) d\tau d\xi = q \frac{s - \sqrt{s^2 - \lambda}}{\sqrt{s^2 - \lambda}}. \quad (49)$$

After changing the order of integration, calculating of the inner integral, and introducing $s' = (s + \sqrt{s^2 - \lambda})/2$ we obtain:

$$\int_0^\infty \sigma(\tau) e^{-s'\tau} d\tau = q \frac{s - \sqrt{s^2 - \lambda}}{q + \sqrt{s^2 - \lambda}}, \quad (50)$$

Now using the relation $s = s' + \lambda/(4s')$ we get

$$\sigma(s') = \frac{2q\lambda}{(2s' + q)^2 - (\lambda + q^2)}. \quad (51)$$

Taking inverse Laplace transform we get

$$\sigma(x) = \frac{q\lambda}{\sqrt{\lambda + q^2}} e^{-qx/2} \sinh \frac{\sqrt{\lambda + q^2}}{2} x. \quad (52)$$

So, the final solution to (44)–(46) is

$$k(x, y) = -\lambda x \frac{I_1(\sqrt{\lambda(x^2 - y^2)})}{\sqrt{\lambda(x^2 - y^2)}} + \frac{q\lambda}{\sqrt{\lambda + q^2}} \int_0^{x-y} e^{-q\tau/2} \sinh \left(\frac{\sqrt{\lambda + q^2}}{2} \tau \right) \times I_0(\sqrt{\lambda(x+y)(x-y-\tau)}) d\tau. \quad (53)$$

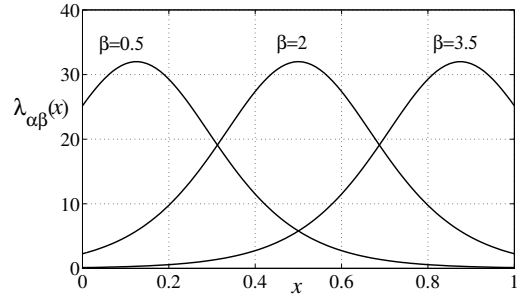


Fig. 2. "One-peak" $\lambda_{\alpha\beta}(x)$ for $\alpha = 4$ and $\varepsilon = 1$.

C. "One-Peak" Family – Chemical Tubular Reactor

Consider the system

$$u_t(x, t) = \varepsilon u_{xx}(x, t) + \lambda_{\alpha\beta}(x) u(x, t), \quad x \in (0, 1), \quad (54)$$

$$u(0, t) = 0, \quad (55)$$

where $\lambda_{\alpha\beta}(x)$ is given by

$$\lambda_{\alpha\beta}(x) = \frac{2\varepsilon\alpha^2}{\cosh^2(\alpha x - \beta)}. \quad (56)$$

This function parametrizes a family of "one-peak" functions. The maximum of $\lambda(x)$ is $2\varepsilon\alpha^2$ and is achieved at $x = \beta/\alpha$. The parameters α and β can be chosen to give the maximum an arbitrary value and location. Examples of $\lambda_{\alpha\beta}(x)$ for different values of α and β are shown in Figure 2. The "sharpness" of the peak is not arbitrary and is given by $\lambda''_{\max} = -\lambda_{\max}^2/\varepsilon$. Despite the strange-looking expression for $\lambda_{\alpha\beta}(x)$, the system (54)–(55) can approximate very well the linearized model of chemical tubular reactor (see [16] and references therein) which is open loop unstable. Our result on stabilization of (54)–(55) is given by the following theorem.

Theorem 2: The controller

$$u(1, t) = -\int_0^1 \alpha \varepsilon^{\alpha \tanh \beta(1-y)} [\tanh \beta - \tanh(\beta - \alpha y)] u(y, t) dy \quad (57)$$

exponentially stabilizes the zero solution of the system (54)–(55).

Proof: Using the same procedure as in the previous subsection we map (54)–(55) into the system (4)–(6) using the transformation (3). Without repeating all the steps (9)–(11) we show the resulting PDE for the kernel $k(x, y)$:

$$k_{xx}(x, y) - k_{yy}(x, y) = \varepsilon^{-1} \lambda_{\alpha\beta}(y) k(x, y), \quad (58)$$

$$k(x, 0) = 0, \quad (59)$$

$$k(x, x) = -\frac{1}{2\varepsilon} \int_0^x \lambda_{\alpha\beta}(\tau) d\tau. \quad (60)$$

Postulating $k(x, y) = X(x)Y(y)$, we have the following set of ODEs:

$$X''(x) = \mu X(x), \quad (61)$$

$$Y''(y) = Y(y)(\mu + 2X(y)Y'(y) + 2X'(y)Y(y)), \quad (62)$$

with the additional conditions

$$Y(0) = 0, \quad (X(x)Y(x))' = -\lambda_{\alpha\beta}(x)/(2\varepsilon), \quad (63)$$

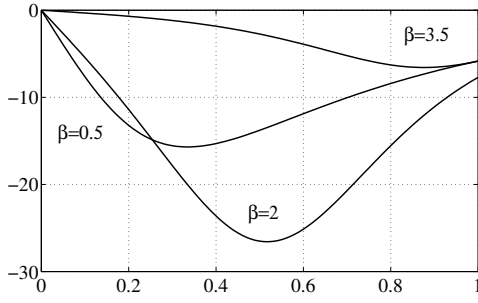


Fig. 3. Gain kernel $k_1(y)$ for (54)–(55) with $\alpha = 4$ and $\varepsilon = 1$.

where μ is an arbitrary parameter. Let us choose $X(x) = e^{\sqrt{\mu}x}$ and substitute it into (62). We get

$$Y''(y) = \mu Y(y) + 2e^{\sqrt{\mu}y} Y'(y) Y(y) + 2\sqrt{\mu} e^{\sqrt{\mu}y} Y^2(y). \quad (64)$$

Changing variables $Y = f(y)e^{-\sqrt{\mu}y}$ we arrive at the following ODE:

$$f''(y) - 2f'(y)f(y) - 2\sqrt{\mu}f'(y) = 0 \quad (65)$$

$$f(0) = 0, \quad (66)$$

$$f'(0) = \mu - \alpha^2. \quad (67)$$

with the additional condition

$$f'(x) = -\frac{\lambda_{\alpha\beta}(x)}{2\varepsilon}. \quad (68)$$

The solution to the problem (65)–(67) is

$$f(y) = -\alpha (\tanh(\alpha y - \beta) + \tanh \beta), \quad (69)$$

where $\tanh \beta = \sqrt{\mu}/\alpha$. We can check that this solution satisfies (68) and gives the kernel

$$k(x, y) = -\alpha e^{\alpha \tanh \beta (x-y)} (\tanh \beta - \tanh(\beta - \alpha y)). \quad (70)$$

Setting $x = 1$ in (70) concludes the proof. \square

In Figure 3 the stabilizing kernels corresponding to $\lambda_{\alpha\beta}(x)$ from Figure 2 are shown. We can see that the controller effort depends very much on the location of the peak of $\lambda(x)$, which has an obvious explanation. When the peak is close to $x = 1$, the controller's influence is very high, when it is close to $x = 0$, the boundary condition helps to stabilize, so the worst case is the peak somewhere in the middle of the domain.

D. Solid Propellant Rocket Model

Consider the following system

$$u_t(x, t) = u_{xx}(x, t) + g e^{\gamma x} u(0, t), \quad x \in (0, 1) \quad (71)$$

$$u_x(0, t) = 0. \quad (72)$$

Here g and γ are arbitrary constants. This equation represents a model of unstable burning in solid propellant rockets (for more details see [17] and references therein). This system is unstable (with $u(1) = 0$) for any $g > 2$, $\gamma \geq 0$.

PDE for the gain kernel can be obtained using the same procedure as in Section II-A except that now the transformation (3) is used to convert the plant into system (4)–(6) with $w_x(0, t) = 0$ instead of $w(0, t) = 0$:

$$k_{xx}(x, y) - k_{yy}(x, y) = 0, \quad (x, y) \in \mathcal{T}, \quad (73)$$

$$k_y(x, 0) = g e^{\gamma x} - g \int_0^x k(x, y) e^{\gamma y} dy, \quad (74)$$

$$k(x, x) = 0. \quad (75)$$

Note the non-local character of the boundary condition (74). The structure of (73)–(75) suggests to search for the solution in the following form:

$$k(x, y) = C_1 e^{\gamma_1(x-y)} + C_2 e^{\gamma_2(x-y)}. \quad (76)$$

Substituting (76) into (73)–(75) we determine the constants C_1 , C_2 , γ_1 , γ_2 and thus obtain the solution

$$k(x, y) = -\frac{g}{g_0} e^{\frac{\gamma}{2}(x-y)} \sinh(g_0(x-y)), \quad (77)$$

$$g_0 = \sqrt{g + \frac{\gamma^2}{4}}.$$

We arrive at the following result:

Theorem 3: The controller

$$u(1, t) = -\int_0^1 \frac{g e^{\frac{\gamma}{2}(1-y)}}{g_0} \sinh(g_0(1-y)) u(y, t) dy \quad (78)$$

exponentially stabilizes the zero solution of the system (71)–(72).

E. Combining Previous Results

The solutions presented in subsections II-A– II-B can be combined to obtain the explicit result for the more complex systems. Consider the system

$$u_t(x, t) = \varepsilon u_{xx}(x, t) + (\lambda_{\alpha\beta}(x) + \lambda_0) u(x, t), \quad (79)$$

$$u(0, t) = 0, \quad (80)$$

$$u(1, t) = \int_0^1 k_c(y) u(y) dy. \quad (81)$$

where $k_c(y)$ is sought to stabilize this system. Denote by $k^{\alpha\beta}(x, y)$ and $k^\lambda(x, y)$ the control gains for the equations (54)–(55) and (1)–(2), respectively. The transformation (3) with $k(x, y) = k^{\alpha\beta}(x, y)$ maps (79)–(81) into the system

$$w_t(x, t) = \varepsilon w_{xx}(x, t) + \lambda_0 w(x, t), \quad (82)$$

$$w(0, t) = 0, \quad (83)$$

$$w(1, t) = \int_0^1 k_1^\lambda(y) w(y) dy. \quad (84)$$

which is stabilized by $k_1^\lambda(y)$. Thus we can get for $k_c(y)$ the expression in quadratures in terms of $k^{\alpha\beta}(x, y)$ and $k^\lambda(x, y)$:

$$k_c(y) = k_1^\lambda(y) + k_1^{\alpha\beta}(y) - \int_y^1 k_1^\lambda(\xi) k^{\alpha\beta}(\xi, y) d\xi \quad (85)$$

For example for $\beta=0$ one can get the closed-form solution

$$k_c(y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}} - \alpha \tanh(\alpha y) I_0\left(\sqrt{\lambda(1-y^2)}\right) \quad (86)$$

In the same fashion one can obtain the explicit stabilizing controllers for even more complicated plants. Here is the most general plant for which we have been able to do it:

$$\begin{aligned} u_t(x, t) &= \varepsilon u_{xx}(x, t) + bu_x(x, t) + \lambda_0 u(x, t) \\ &\quad + ge^{\gamma x} u(0, t), \quad (87) \\ u_x(0, t) &= qu(0, t). \quad (88) \end{aligned}$$

This is a system with 6 parameters, each of them can contribute to instability. It is remarkable that it can be stabilized by closed form control law.

III. OUTPUT-FEEDBACK DESIGNS

The stabilizing controllers developed thus far require complete measurements from the interior of the domain which are usually unavailable. So we look for the observers that estimate $u(x, t)$ inside the domain. We assume that sensing is available only at the boundary and thus consider two cases: anti-collocated, when the sensor is placed on the opposite boundary to the actuator (so $u(0, t)$ is measured, $u(1, t)$ is controlled), and collocated, when the sensor and the actuator are set at the same boundary (so $u(1, t)$ is measured, $u_x(1, t)$ is controlled).

A. Unstable Heat Equation

Consider the following equation

$$u_t(x, t) = \varepsilon u_{xx}(x, t) + \lambda_0 u(x, t), \quad (89)$$

$$u_x(0, t) = 0, \quad (90)$$

with only $u(0)$ measured. We propose the following Luenberger-type observer motivated by a finite dimensional backstepping type observer of Krener and Wang [18]:

$$\hat{u}_t(x, t) = \varepsilon \hat{u}_{xx}(x, t) + \lambda_0 \hat{u}(x, t) + p_1(x)[u(0, t) - \hat{u}(0, t)], \quad (91)$$

$$\hat{u}_x(0, t) = p_{10}[u(0, t) - \hat{u}(0, t)], \quad (92)$$

$$\hat{u}(1, t) = \int_0^1 k_1(y) \hat{u}(y, t) dy, \quad (93)$$

and the controller

$$u(1, t) = \int_0^1 k_1(y) \hat{u}(y, t) dy. \quad (94)$$

Here $p_1(x)$ and p_{10} are output injection functions (p_{10} is a constant) to be designed. Note, that we introduce output injection not only in the equation (91) but also at the boundary where measurement is available. The observer

error $\tilde{u}(x, t) = u(x, t) - \hat{u}(x, t)$ satisfies the following PDE:

$$\tilde{u}_t(x, t) = \varepsilon \tilde{u}_{xx}(x, t) + \lambda_0 \tilde{u}(x, t) - p_1(x) \tilde{u}(0, t), \quad (95)$$

$$\tilde{u}_x(0, t) = -p_{10} \tilde{u}(0, t), \quad (96)$$

$$\tilde{u}(1, t) = 0. \quad (97)$$

Observer gains $p_1(x)$ and p_{10} should now be chosen to stabilize the system (95)–(97). For linear finite dimensional systems the problem of finding the observer gains is dual to the problem of finding the control gains. This motivates us to try to solve the problem of stabilization of (95)–(97) by the same integral transformation approach as the (state feedback) boundary control problem. We look for a backstepping-like transformation

$$\tilde{u}(x, t) = \tilde{w}(x, t) - \int_0^x p(x, y) \tilde{w}(y, t) dy \quad (98)$$

that transforms system (95)–(97) into the exponentially stable system

$$\tilde{w}_t(x, t) = \varepsilon \tilde{w}_{xx}(x, t) - c \tilde{w}(x, t), \quad x \in (0, 1), \quad (99)$$

$$\tilde{w}_x(0, t) = 0, \quad (100)$$

$$\tilde{w}(1, t) = 0. \quad (101)$$

By substituting (98) into (95)–(97) and using (99)–(101) it can be shown that the kernel $p(x, y)$ must satisfy the following hyperbolic PDE:

$$p_{yy}(x, y) - p_{xx}(x, y) = \lambda p(x, y), \quad (102)$$

$$\frac{d}{dx} p(x, x) = \frac{\lambda}{2}, \quad (103)$$

$$p(1, y) = 0. \quad (104)$$

In addition, the following conditions must be satisfied:

$$p_1(x) = \varepsilon p_y(x, 0), \quad p_{10} = p(0, 0). \quad (105)$$

Let us make a change of variables

$$\check{x} = 1 - y, \quad \check{y} = 1 - x, \quad \check{p}(\check{x}, \check{y}) = p(x, y). \quad (106)$$

It can be verified that in these new variables the problem (102)–(104) becomes exactly the same as (12)–(14) for $k(x, y)$ and we get

$$\check{p}(\check{x}, \check{y}) = -\lambda \check{y} \frac{I_1\left(\sqrt{\lambda(\check{x}^2 - \check{y}^2)}\right)}{\sqrt{\lambda(\check{x}^2 - \check{y}^2)}}. \quad (107)$$

Using (105) we obtain the following result.

Theorem 4: The controller (94) with the observer (91)–(93) where $k_1(x)$ and $p_1(x)$ are given by

$$\begin{aligned} k_1(y) &= -\lambda \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}, \\ p_1(x) &= \frac{\lambda(1-x)}{x(2-x)} I_2\left(\sqrt{\lambda x(2-x)}\right), \quad (108) \end{aligned}$$

and $p_{10} = -\lambda/2$, exponentially stabilizes the zero solution of the system (89)–(90).

The above result can be easily extended for the Neumann type of actuation as well.

Since both the controller and the observer are explicit in our design we can derive again (following the same procedure as in section II-A) the closed loop solution of (89)–(94). The presence of the second PDE (the observer) makes it more complicated:

$$u(x, t) = \sum_{n=0}^{\infty} e^{-\frac{(c+\mu_n^2)t}{\varepsilon}} \phi_n(x) \left\{ \int_0^1 \psi_n(\xi) u_0(\xi) d\xi - (-\mu_n)^n \left(C_n t + \sum_{m \neq n} C_m \frac{1 - e^{-\frac{(\mu_n^2 - \mu_m^2)t}{\varepsilon}}}{\mu_n^2 - \mu_m^2} \right) \right\} \quad (109)$$

where $\mu_n = \pi(n + 1/2)$,

$$C_n = \int_0^1 \lambda \frac{I_1(\sqrt{\lambda(1-\xi^2)})}{\sqrt{\lambda(1-\xi^2)}} \psi_n'(1-\xi) d\xi \times \int_0^1 \frac{\phi_n'(1-\xi)}{\lambda + \mu_n^2} (u_0(\xi) - \hat{u}_0(\xi)) d\xi, \quad (110)$$

$$\psi_n(x) = \cos(\mu_n x) + \int_0^x \lambda \xi \frac{I_1(\sqrt{\lambda(x^2 - \xi^2)})}{\sqrt{\lambda(x^2 - \xi^2)}} \cos(\mu_n \xi) d\xi, \\ \phi_n(x) = 2 \cos(\sqrt{\lambda + \mu_n^2} x). \quad (111)$$

B. Explicit Solution for a "One-Peak" Family of $\lambda(x)$

We show now how the observer for the collocated case can be designed. Of course, for the problem to make sense we assume that the actuation is Neumann type. Consider the problem (54)–(56) with only $u(1, t)$ measured. We propose the following observer

$$\hat{u}_t(x, t) = \varepsilon \hat{u}_{xx}(x, t) + \lambda_{\alpha\beta}(x) \hat{u}(x, t) + p_1(x)[u(1, t) - \hat{u}(1, t)], \quad (112)$$

$$\hat{u}(0, t) = 0, \quad (113)$$

$$\hat{u}_x(1, t) = -p_{10}[u(1, t) - \hat{u}(1, t)] + k_1(1)\hat{u}(1, t) + \int_0^1 k_2(y)\hat{u}(y, t) dy \quad (114)$$

with the controller

$$u_x(1, t) = k_1(1)\hat{u}(1, t) + \int_0^1 k_2(y)\hat{u}(y, t) dy, \quad (115)$$

Control gains k_1 and k_2 have been determined in Section II-C.

The observer error $\tilde{u}(x)$ satisfies the equation

$$\tilde{u}_t(x, t) = \varepsilon \tilde{u}_{xx}(x, t) + \lambda_{\alpha\beta}(x)\tilde{u}(x, t) - p_1(x)\tilde{u}(1, t) \quad (116)$$

$$\tilde{u}(0, t) = 0, \quad (117)$$

$$\tilde{u}_x(1, t) = p_{10}\tilde{u}(1, t). \quad (118)$$

We are looking for the transformation:

$$\tilde{u}(x, t) = \tilde{w}(x, t) - \int_x^1 p(x, y) \tilde{w}(y, t) dy \quad (119)$$

that transforms (116)–(118) into the exponentially stable target system

$$\tilde{w}_t(x, t) = \varepsilon \tilde{w}_{xx}(x, t), \quad x \in (0, 1), \quad (120)$$

$$\tilde{w}(0, t) = 0, \quad (121)$$

$$\tilde{w}_x(1, t) = 0. \quad (122)$$

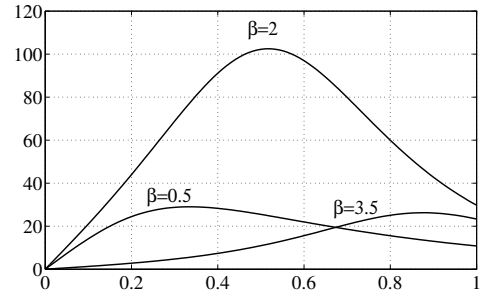


Fig. 4. Observer gain for $\alpha = 4$.

Note, that the transformation (119) is in upper-triangular form. By substituting (119) into (116)–(118) and using (120)–(122) it can be shown that the kernel $p(x, y)$ must satisfy the following hyperbolic PDE:

$$\varepsilon p_{yy}(x, y) - \varepsilon p_{xx}(x, y) = \lambda_{\alpha\beta}(x)p(x, y), \quad (123)$$

$$p_x(0, y) = qp(0, y), \quad (124)$$

$$p(x, x) = -\frac{1}{2\varepsilon} \int_0^x (\lambda(\xi) + c) d\xi. \quad (125)$$

In addition, the following conditions must be satisfied:

$$p_1(x) = -\varepsilon p_y(x, 1), \quad p_{10} = p(1, 1). \quad (126)$$

Once the solution $p(x, y)$ to the problem (123)–(125) is found, the observer gains can be obtained from (126). It can be verified that the solution of this PDE is $p(x, y) = k(x, y)$ where $k(x, y)$ is given by (70). We immediately get $p_1(x) = \varepsilon k_2(x)$, $p_{10} = k_1(1)$ and the following result.

Theorem 5: The controller (115) with the observer (112)–(114) where $k_2(x)$ and $p_1(x)$ are given by

$$k_2(x) = p_1(x) = \varepsilon \alpha^2 \tanh(\beta) \times e^{(1-x)\alpha \tanh \beta} (\tanh \beta - \tanh(\beta - \alpha x)), \quad (127)$$

stabilizes the zero solution of the system (54)–(55).

In Figure 4 the observer gains corresponding to $\lambda_{\alpha\beta}(x)$ from Figure 2 are shown.

C. Combining Results

The solutions from Sections III-A and III-B can be combined to obtain an explicit solution to (79)–(81). Denote by $p^{\alpha\beta}(x, y)$ and $p^\lambda(x, y)$ the observer gains for (54)–(55) and (1)–(2), respectively. In the same fashion as it was done in Section II-E we can obtain the observer gain for the equation (79)–(81):

$$p_1(x) = p_1^\lambda(x) + p_1^{\alpha\beta}(x) + \varepsilon p_{10}^\lambda p^{\alpha\beta}(x, 1) - \int_x^1 p^{\alpha\beta}(x, \xi) p_1^\lambda(\xi) d\xi, \quad (128)$$

$$p_{10} = p_{10}^\lambda + p_{10}^{\alpha\beta}. \quad (129)$$

For example for $\beta = 0$ one can get the closed-form solution

$$p_1(x) = \frac{\varepsilon \lambda}{1-y^2} I_2(\sqrt{\lambda(1-x^2)}) + \varepsilon \lambda \alpha \tanh(\alpha x) \frac{I_1(\sqrt{\lambda(1-x^2)})}{\sqrt{\lambda(1-x^2)}}. \quad (130)$$

IV. EXPLICIT COMPENSATOR DESIGN (FREQUENCY DOMAIN)

The solutions obtained in previous sections can be used to get the explicit compensator transfer functions (treating $u(0, t)$ or $u(1, t)$ as an input and $u(1, t)$ or $u_x(1, t)$ as an output). We will illustrate this point with the system from Section II-D with $\gamma = 0$ and $u(0, t)$ measured:

$$u_t(x, t) = u_{xx}(x, t) + gu(0, t), \quad (131)$$

$$u_x(0, t) = 0. \quad (132)$$

Consider the following observer:

$$\hat{u}_t(x, t) = \hat{u}_{xx}(x, t) + gu(0, t), \quad (133)$$

$$\hat{u}_x(0, t) = 0, \quad (134)$$

$$\hat{u}(1, t) = -\sqrt{g} \int_0^1 \sinh(\sqrt{g}(1-y)) \hat{u}(y, t) dy. \quad (135)$$

The controller (and the output for the observer) is (see (78)):

$$u(1, t) = -\sqrt{g} \int_0^1 \sinh(\sqrt{g}(1-y)) \hat{u}(y, t) dy. \quad (136)$$

We want to find a transfer function from the input $u(0, t)$ to the output $u(1, t)$, i.e., $u(1, s) = -C(s)u(0, s)$. Taking the Laplace transform of (133)–(135), setting the initial condition to zero, $\hat{u}(x, 0) = 0$, we have (for simplicity of notation we denote by $\hat{u}(x, s)$ and $u(0, s)$ the Laplace transforms of $\hat{u}(x, t)$ and $u(0, t)$, respectively):

$$s\hat{u}(x, s) = \hat{u}_{xx}(x, s) + gu(0, s), \quad (137)$$

$$\hat{u}_x(0, s) = 0, \quad (138)$$

$$\hat{u}(1, s) = -\sqrt{g} \int_0^1 \sinh(\sqrt{g}(1-y)) \hat{u}(y, s) dy. \quad (139)$$

The equation (137) with boundary conditions (138)–(139) is an ODE with respect to x (we regard s as a parameter). The solution of (137) satisfying (138) is:

$$\hat{u}(x, s) = \hat{u}(0, s) \cosh(\sqrt{s}x) + \frac{g}{s} (1 - \cosh(\sqrt{s}x)) u(0, s) \quad (140)$$

Using boundary condition (139) we obtain $\hat{u}(0, s)$:

$$\hat{u}(0, s) = \frac{\cosh(\sqrt{s}) - \cosh(\sqrt{g})}{s \cosh(\sqrt{s}) - g \cosh(\sqrt{g})} gu(0, s) \quad (141)$$

Substituting now (141) into (140) with $x = 1$ we obtain the following result:

Theorem 6: The transfer function of the system (133)–(136) with $u(0, t)$ as an input and $u(1, t)$ as an output is

$$C(s) = \frac{g}{s} \left(-1 + \frac{(s-g) \cosh(\sqrt{s}) \cosh(\sqrt{g})}{s \cosh(\sqrt{s}) - g \cosh(\sqrt{g})} \right). \quad (142)$$

The validation of application of the above procedure for linear parabolic PDEs (which proves that $C(s)$ is indeed a transfer function) can be found in [1, Chapter 4]. Note that $s = 0$ is not the pole:

$$C(0) = \frac{g}{2} + \frac{1}{\cosh(\sqrt{g})} - 1 \quad (143)$$

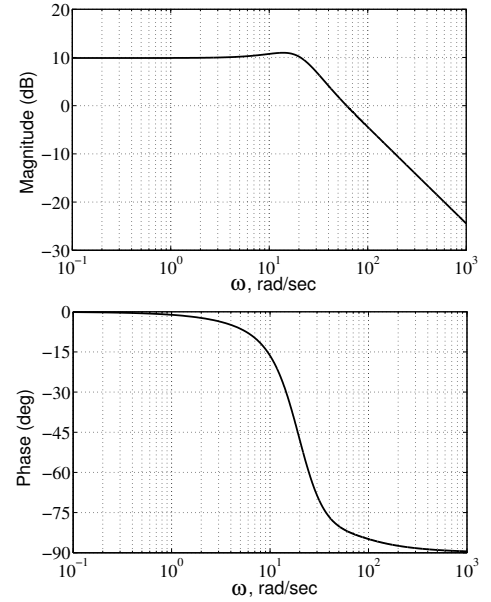


Fig. 5. Bode plot of $C(j\omega)$ for $g = 8$.

The transfer function (142) has infinitely many poles, all of them are real and negative. The Bode plots of $C(s)$ for $g = 8$ are presented in Figure 5. It is evident from the Bode plots that $C(s)$ can be approximated by a second order, relative degree one transfer function. For example, a rough estimate would be

$$C(s) \approx 60 \frac{s + 17}{s^2 + 25s + 320}. \quad (144)$$

The relative degree one nature of the compensator is the result of employing a full order (rather than a reduced order) observer.

V. CONCLUSION

We presented closed form solutions to a boundary stabilization problem for a class of parabolic PDEs. We should note that the described method is not limited to plants for which an explicit solution can be found. It can be applied to a general class of parabolic PDEs

$$u_t(x, t) = \varepsilon u_{xx}(x, t) + \lambda(x) u(x, t) + g(x) u(0, t) + \int_0^x f(x, y) u(y, t) dy. \quad (145)$$

It turns out that in this case the gain kernel should satisfy the following hyperbolic Klein-Gordon type PDE [19]

$$\varepsilon k_{xx}(x, y) - \varepsilon k_{yy}(x, y) = (\lambda(y) + c)k(x, y) - f(x, y) + \int_y^x k(x, \tau) f(\tau, y) d\tau, \quad (146)$$

with boundary conditions

$$\varepsilon k_y(x, 0) = \varepsilon qk(x, 0) + g(x) - \int_0^x k(x, y) g(y) dy, \quad (147)$$

$$k(x, x) = -\frac{1}{2\varepsilon} \int_0^x (\lambda(y) + c) dy. \quad (148)$$

This PDE can be solved either by the method of successive approximations much like in Section II-A (see (15)–(24)) or by using a numerical scheme for Klein-Gordon type of equations.

The approach can also be modified to obtain a controller that minimizes a reasonable cost functional that puts penalty on both state and control (so-called inverse optimal controller) giving stability margins and robustness [19].

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