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Stabilization of a solid propellant rocket instability by state feedback

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SUMMARY

In this paper a globally stabilizing feedback boundary control law for an arbitrarily fine discretization of a one-dimensional nonlinear PDE model of unstable burning in solid propellant rockets is presented. The PDE has a destabilizing boundary condition imposed on one part of the boundary. We discretize the original nonlinear PDE model in space using finite difference approximation and get a high order system of coupled nonlinear ODEs. Then, using backstepping design for parabolic PDEs, properly modified to accommodate the imposed destabilizing nonlinear boundary condition at the burning end, we transform the original system into a target system that is asymptotically stable in l^2-norm with the same type of boundary condition at the burning end, and homogeneous Dirichlet boundary condition at the control end. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: backstepping; boundary control; solid propellant rockets

1. INTRODUCTION

1.1. Motivation and problem background

This work on control of combustion instabilities in solid propellants comes in the wake of high activity in control of liquid fuel combustion [1–3].

Solid propellant rockets come in many different types and sizes. Applications vary from anti-aircraft, anti-tank, and anti-missile-missiles, to ballistic missiles and large space launch vehicle booster [4]. Compared to liquid rockets, solid propellant rockets are usually simpler in design, easy to apply, can be hermetically sealed for long-time (5–10 year) storage, and require little servicing. On the other hand, they cannot be fully checked before they are used and thrust cannot be randomly varied.

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Solid propellant rocket engines are sometimes subject to combustion instabilities that can cause engine failure either through excess pressure, increased wall heat transfer, or a combination of the two. Although always undesirable, combustion instabilities are especially dangerous to larger rockets [5]. Current remedy for reducing instabilities occurring in solid propellants is either by placing of an irregular rod of non-burning material within the burning volume, or drilling radial holes at intervals along the grain. Non-steady burning and combustion instabilities appearing in solid propellants have been investigated extensively both theoretically and experimentally. Combustion models used, although numerous and different in various aspects of the analysis, commonly treat the system as a three region problem: the solid phase, the gas–solid interface region, and the gas phase. The interface is generally collapsed to a plane and is used as a matching condition between the two phases. Most of the models published agree in the treatment of the solid phase as a single homogeneous region, up to the interface. The point where main differences occur is the treatment of the gas phase. The model that we are using is based on the work of Denison and Baum [6]. The model derived there assumes that characteristic times for all processes involved in solid propellant combustion are short compared to the characteristic time for heat conduction in solid. The combustion mechanism involves inert heat conduction in the solid, surface gasification by an Arrhenius process and a gas-phase deflagration with a high non-dimensional activation energy. Properties of solid are assumed to be constant.

1.2. Model

Although models differ importantly in regard to handling the gas phase, most of them arrive at a two parameter response function of mass flux with respect to pressure disturbances in their stability analysis, as summarized in an extensive review by Culick [7]. For more details on the validity of the assumptions used in the model by Denison and Baum, and a more in depth insight in the work on unsteady combustion in solid propellants, interested readers are referred to References [7, 8].

The type of instability that we are interested in are self-excited modes of burning rate response appearing at frequencies which are not acoustic modes of the combustion chamber. For a more detailed description and classification of combustion instabilities see Reference [9]. Stability analysis of the model shows that the steady state solution for the uncontrolled case is unstable.

1.3. Control objective

The objective is to stabilize the steady state of the model using boundary control in temperature on the non-reacting boundary (see Figure 1).[‡] While there may be more practical ways of actuating this problem than heating/cooling at the boundary opposite to the burning end (distributed heating/cooling would provide more control authority in the case when the heat conductivity of the solid propellant is low), the boundary actuation considered here offers the greatest challenge from the point of view of control synthesis.

[‡] Our idea of actively controlling combustion instability by heating/cooling the rocket fuel was motivated by a proposition made by T'ien and Sirignano [12] for liquid fuel rockets. They suggested that a remedy to instability in the case of practical liquid rocket could be heating of fuel by the wall of the combustion chamber by regenerative cooling (cooling of the chamber wall is in addition beneficial on its own).

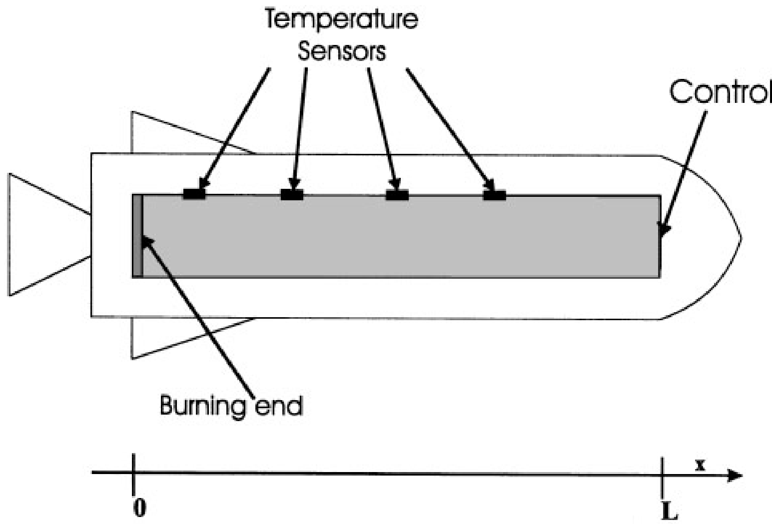

Figure 1. Solid propellant rocket.

It is important to note that, due to burning, the length of the domain $L(t)$ is time varying, with $\dot{L}(t)<0$. A physically realistic assumption is made that the rate of change of $L(t)$ is much slower than the rate of temperature dynamics in the solid propellant. As we explain in the stability analysis, the negative sign of $\dot{L}(t)$ enhances stability, and for sufficiently short solid propellants the dynamics become stable, allowing to turn off the controller during the final stage of burning.

As model parameters are changing and temperature sensors are lost in the process of burning/shortening of the solid propellant, the controller can be rescheduled to accommodate these relatively slow changes.

1.4. Results

We start by discretizing the original PDE model in space using finite differences which gives a high order system of coupled nonlinear ODEs. Applying the backstepping design [10], appropriately adapted to parabolic PDEs [11] we obtain a discretized co-ordinate transformation that transforms the original system into a properly chosen target system that is asymptotically stable in l^2-norm. Then, using the property that the discretized co-ordinate transformation is invertible for an arbitrary (finite) grid choice, we conclude that the discretized version of the original system is *globally asymptotically stable* and obtain a nonlinear feedback boundary control law for the temperature in the original set of co-ordinates.

A major novelty of the backstepping design in this paper is that it controls an instability caused by a boundary condition that the control cannot alter (even in the target system) but it dominates it in an indirect way, through damping introduced in the target system. This type of destabilizing boundary condition is a peculiarity of the solid propellant rocket model and is not found in chemical reactor models, which otherwise have similarities with the rocket model.

It is intuitively clear that the number of sensing points needed to stabilize the system is close to the number of open-loop unstable eigenvalues (provided the sensors are properly positioned for the given eigenfunctions of the system). For the solid propellant rocket model we show that the number of unstable eigenvalues is always one. In simulations we show that the controller we propose works not only with densely distributed sensors (this result is proved by Lyapunov analysis and therefore skipped in the simulation section) but also with a very low number of sensors.

1.5. Organization

In Section 2 a nonlinear one-dimensional PDE model that governs heat transfer inside the solid propellant is introduced, followed by the stability analysis of the open loop system. A nonlinear feedback control law that achieves global asymptotic stabilization is presented in Section 3, followed by the proof of stability for the target system in Section 4. Finally, a feedback control law designed on a very coarse grid is shown to successfully stabilize the system for a variety of different simulation settings in Section 5.

2. MATHEMATICAL MODEL

In this section we introduce a model of transient heat conduction in solid propellants. The model that we use is based on the model introduced by Denison and Baum [6]. The main difference is that we are considering a realistic model of a finite length rod, as opposed to the assumption of an infinitely long rod in Reference [6] introduced to simplify the stability analysis. We present only the key steps necessary for understanding the model here. For a detailed derivation see Reference [6]. It is assumed that the characteristic time for heat conduction in the solid is at least an order of magnitude greater than the characteristic times for the gas phase transport processes and chemical reactions. That means that we assume quasi-steady-state in the gas phase that affects the heat conduction in solid phase through boundary condition on gas–solid interface only. In addition, the principal assumptions include homogeneous propellant with constant properties, Lewis number of unity, single step combustion reaction of any order, no reactions in the solid state, vaporization according to an Arrhenius law and no erosive burning. Under given assumptions the differential equation governing heat transfer in the solid phase is

$$\rho_s C_s \frac{\partial T}{\partial t} = m_w C_s \frac{\partial T}{\partial x} + K_s \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where T stands for the temperature of the solid defined for $x \in [0, x_{\max}(t)]$, ρ_s, C_s, and K_s, respectively, for density, specific heat, and thermal conductivity of the solid, and m_w for the mass flux relative to the burning surface. Note that the length of the solid propellant stick is a function of time and it continuously decreases, due to deflagration, once the solid propellant is ignited. The mass flux is governed by $m_w = e^{-[E_w/RT(0,t)]}$, where R stands for gas constant, and E_w for activation energy for vaporization and decomposition at the wall ($x = 0$). Equation (1) satisfies boundary condition

$$T(x_{\max}, t) = T_{NB} \tag{2}$$

where T_{NB} stands for the temperature at the non-reacting boundary. The other boundary condition at the burning end ($x = 0$) is obtained from the heat balance at the solid surface. The heat conducted into solid is given by

$$K_s \frac{\partial T(0,t)}{\partial x} = \left(K \frac{\partial T(0,t)}{\partial x} \right)_g + m_w (C_s T(0,t) - C_p T(0,t) - L_{vd}) \tag{3}$$

where C_p stands for specific heat of gas, L_{vd} for heat absorbed by vaporization and decomposition, and subscript g refers to properties of gas. In addition, by simultaneously integrating energy and species conservation equations in the gas phase we obtain a relation for the temperature gradient at any point in the gas phase as

$$\frac{K}{C_p m_w} \frac{\partial T}{\partial x} = \left(T_f - T - \frac{Q_r}{C_p} \varepsilon_r \right) \tag{4}$$

where T_f stands for flame temperature, Q_r for heat of combustion, and ε_r for fraction of total mass flux associated with reactant species. Evaluating (4) at $x = 0$ and substituting it in (3) we obtain the boundary condition at $x = 0$ as

$$\frac{\partial T(0,t)}{\partial x} = \frac{e^{-[E_w/RT(0,t)]}}{K_s} [C_p T_f + \varepsilon_{rw} Q_r + L_{vd} - C_s T(0,t)] \tag{5}$$

System (1) with boundary conditions (2) and (5) has a steady-state

$$\bar{T}(x) = \bar{T}(0) - \frac{\bar{T}(0) - T_{NB}}{1 - e^{-(\bar{m}_w C_s/K_s) x_{max}}} [1 - e^{-(\bar{m}_w C_s/K_s) x}] \tag{6}$$

where $\bar{m}_w = m_w(\bar{T}(0))$. Introducing non-dimensional spatial variable, time, and, non-dimensional temperature deviation from the steady-state (6) respectively as $x' = (\bar{m}_w C_s/K_s) x$, $t' = (\bar{m}_w^2 C_s/\rho_s K_s) t$, $u(x',t') = [T(x',t') - \bar{T}(x')]/\bar{T}(0)$, and omitting superscripts $'$ for convenience, we obtain a non-dimensionalized system

$$u_t(x,t) = u_{xx}(x,t) + f_1(u(0,t)) u_x(x,t) + B_1 e^{-x} [1 - f_1(u(0,t))] \tag{7}$$

$$\begin{aligned} u_x(0,t) &= -f_2(u(0,t)) \\ u(L,t) &= \text{control} \end{aligned} \tag{8}$$

where we denote $f_1(u) = e^{B_2[u/(1+u)]}$, $f_2(u) = B_1[f_1(u) - 1] + f_1(u)u$, and $L = (\bar{m}_w C_s/K_s) x_{max'}$

$$B_1 = \frac{1 - [T_{NB}/T(0)]}{1 - e^{-L}}, \quad B_2 = \frac{E_w}{R\bar{T}(0)}$$

Note that since $u(0,t) > -1 (u(0,t) = -1$ corresponds to absolute zero), both f_1 and f_2 are well defined. Although it appears that B_1 might start increasing as L slowly changes, it actually remains constant. Since both $\bar{T}(0)$ and $\bar{T}_f$ are constant, from (5) we conclude that $\partial \bar{T}(0)/\partial x$ does not change. Taking a derivative of $\bar{T}(x)$ and evaluating obtained expression for $x = 0$ we get

$$\frac{\partial \bar{T}(0)}{\partial x} = -\frac{\bar{T}(0) - T_{NB}}{1 - e^{-L}} = -B_1 \bar{T}(0) \tag{9}$$

Since both $\bar{T}(0)$ and $\partial \bar{T}(0)/\partial x$ remain unchanged, we conclude that B_1 does not change either. Physically, as the solid propellant burns it becomes shorter, the influence of the burning end becomes more pronounced, and T_{NB} increases in a fashion that keeps B_1 constant. As shown in Reference [6], depending on the current steady-state, small pressure disturbances in gas pressure can lead to instability, i.e. the departure of the surface temperature, and hence mass flux, from the steady-state value. Linearizing (7) around the equilibrium at the origin we obtain the linearized system as

$$u_t = u_{xx} + u_x - Au(0,t)e^{-x} \tag{10}$$

$$u_x(0,t) = -qu(0,t) \tag{11}$$

$$u(L,t) = 0 \tag{12}$$

where $A = B_1B_2$ and $q = 1 + A$. To prove that the linearized system (10)–(12) can become unstable we show that there exists a particular solution that grows exponentially for some choice of A, q, and L. We start by introducing a coordinate change $z(x,t) = e^{-(1/2)x+(1/4)t}u(x,t)$ that transforms (10)–(12) into

$$z_t = z_{xx} - Az(0,t)e^{-(1/2)x} \tag{13}$$

$$z_x(0,t) = -\left(q - \frac{1}{2}\right)u(0,t) \tag{14}$$

$$z(L,t) = 0 \tag{15}$$

Postulating a solution of the z-system as $z(x,t) = f(x)z(0,t)$, where by definition

$$f(0) = 1 \tag{16}$$

and substituting it in (13), we get

$$z_t(x,t)f(x) = z(0,t)[f_{xx}(x) - Ae^{-(1/2)x}] \tag{17}$$

Now it becomes obvious from (17) that if we can construct $f(x)$ that satisfies

$$f_{xx} - Ae^{-(1/2)x} = k^2 f, \quad k^2 > \frac{1}{4} \tag{18}$$

boundary conditions (14) and (15), and condition (16), system (10)–(12) will be unstable. We indeed find such a solution as

$$u(x,t) = \left(C(k)e^{(k-(1/2))x} + C(-k)e^{-(k+(1/2))x} + \frac{A}{(1/4) - k^2}e^{-x}\right)e^{(k^2-(1/4))t}u(0,0) \tag{19}$$

where

$$C(k) = \frac{(k + (1/2))^2 - q(k + (1/2)) + A}{2k(k + (1/2))} \tag{20}$$

and k satisfies

$$\begin{aligned} k^3 + k^2\left(\frac{1}{2} - q\right) + k\left(A - \frac{1}{4}\right) + \left(\frac{q}{4} - \frac{1}{8} - \frac{A}{2}\right) \\ = -e^{-2kL}\left[k^3 + k^2\left(q - \frac{1}{2}\right) + k\left(A - \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{A}{2} - \frac{q}{4}\right)\right] \\ + 2Ake^{-(k+(1/2))L} \end{aligned} \tag{21}$$

Note that in the limit case $L \to \infty$, following the same approach as for the finite L, for the linearized system to be unstable k has to satisfy

$$k^2 + (1-q)k + \left(A + \frac{1}{4} - \frac{q}{2}\right) = 0 \tag{22}$$

If $q > 1$ and $q^2 > 4A$, we find that a sufficient condition for $k^2 > \frac{1}{2}$ becomes

$$q^2 - q - 2A > 0 \tag{23}$$

that is exactly the same stability condition that Denison and Baum arrived at in Reference [6].

Numerically finding the eigenvalues of (10)–(12), we were able to show that regardless of the values for the system parameters B_1, B_2, and L, or in other words regardless of the level of the open-loop instability, the system could have only one unstable eigenvalue. As the open-loop system becomes more unstable the magnitude of the unstable eigenvalue increases, but the total number of unstable eigenvalues remains one. The dependence of the magnitude of the largest eigenvalue $\lambda_{\max}$ of system (10)–(12) for a particular choice $A = 1.0$ is shown in Figure 2. What is interesting is that our analysis shows that a longer rocket will tend to be more unstable than a shorter one, given all the system parameters same. This means that if we can stabilize the system at the beginning we will be able to do so at any other time instant since the solid propellant burns and its length reduces due to deflagration. In addition, from the point of view of implementation, the control algorithm can be turned off after the remaining length of the solid propellant stick becomes smaller than the critical length L_{crit}, defined as value for which $\lambda_{\max}$ reaches zero, i.e. $\lambda_{\max}(L_{\text{crit}}) = 0$.

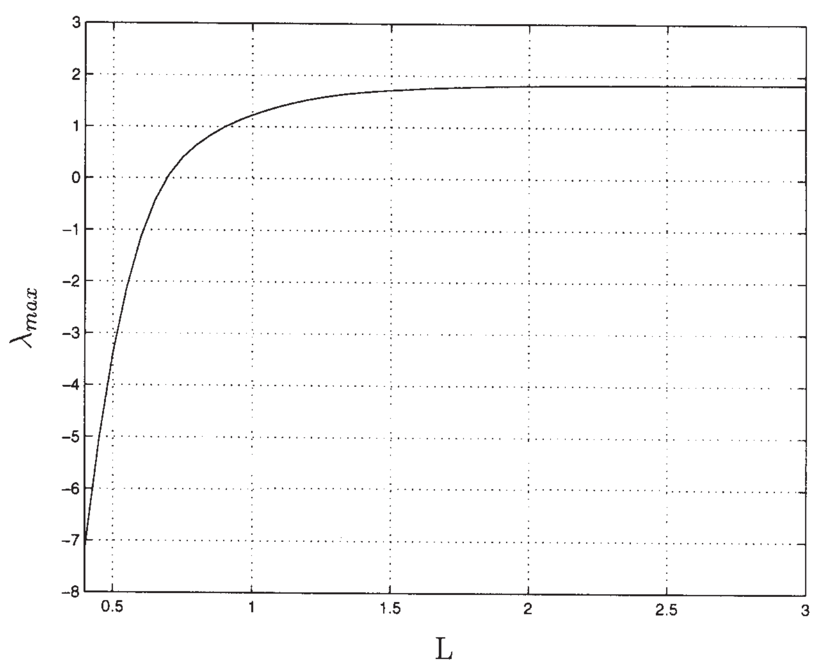

Figure 2. Dependence of the largest open-loop eigenvalue $\lambda_{\max}$ on the non-dimensionalized rocket length L for $A = 1.0$.

3. CONTROL LAW

In this section we present a feedback boundary control design for unstable burning in solid rocket propellants.

Unlike most control problems for PDEs that assume the freedom for the designer in prescribing boundary conditions everywhere on the boundary, this 1D nonlinear PDE model has an unfavourable nonlinear boundary condition imposed at one of the boundaries by the nature of the system itself. This fact presents a design challenge since the standard backstepping procedure for parabolic PDEs of transforming the original system into a target system with homogeneous boundary conditions, and then proving asymptotic stability using Poincaré inequality (see Reference [13] for details) cannot work. We present here a modified version of the standard backstepping design for parabolic PDEs that can accommodate the imposed destabilizing boundary condition at the burning end. The first aspect of the modification consists in choosing a target system that has the same type of boundary condition at $x = 0$ as the original system, and homogeneous Dirichlet boundary condition at $x = L$. We then prove asymptotic stability of the transformed system using Agmon's instead of Poincaré inequality. As it will become obvious from the stability proof, the proposed technique can be extended to a more general class of nonlinear parabolic PDE systems with imposed boundary conditions.

To discretize the problem, let us start by denoting $h = L/N$, where N is an integer. Then, with u_i defined as $u_i(t) = u(ih, t)$, $i = 0, \ldots, N$, we represent the non-dimensional system (7) as

$$\begin{aligned}\dot{u}_i = &\frac{1}{h^2} u_{i+1} - \left[\frac{2}{h^2} - \frac{f_1(u_0)}{h}\right] u_i \\ &+ \left[\frac{1}{h^2} - \frac{f_1(u_0)}{h}\right] u_{i-1} + B_1 \mathrm{e}^{-ih}[1 - f_1(u_0)]\end{aligned} \tag{24}$$

with boundary condition at zero end expressed as $(u_0 - u_{-1})/h = -f_2(u_0)$, where u_{-1} stands for an artificially introduced phantom node. Since $u(L, t)$ is the control in the PDE, the control input to the discretized system is u_N. We now suggest a backstepping controller that transforms the original system into the discretization of the system

$$w_t(x, t) = w_{xx}(x, t) + f_1(w(0, t)) w_x(x, t) - cw(x, t) \tag{25}$$

$$w_x(0, t) = -f_2(w(0, t)) \tag{26}$$

$$w(L, t) = 0 \tag{27}$$

where $c > k_2^2$, k_2 being a positive constant that satisfies $f_2(w(0))w(0) < k_2 w(0)^2$ for $\forall w(0)$. Under given conditions system (25)–(27) can be shown to be asymptotically stable. We should stress that the choice of the target system is one of the key issues here. When transforming the original system we should try to keep its parabolic character, i.e. keep the second spatial derivative in the transformed co-ordinates. Even when applied for linear parabolic PDEs, the control laws obtained using standard backstepping would have gains that grow unbounded as $N \to \infty$. The problem with standard backstepping is that it would not only attempt to stabilize the equation, but also place its poles, and thus as $N \to \infty$, change its parabolic character. The co-ordinate transformation is sought in the form

$$w_0 = u_0 \tag{28}$$

$$w_1 = u_1 - \alpha_0(u_0) \tag{29}$$

$$w_i = u_i - \alpha_{i-1}(u_0, \ldots, u_{i-1}), \quad i = 2, \ldots, N-1 \tag{30}$$

$$u_N = \alpha_{N-1}(u_0, \ldots, u_{N-1}) \tag{31}$$

where $w_i(t) = w(ih, t)$. The discretized form of Equation (25) is

$$\dot{w}_i = \frac{1}{h^2} w_{i+1} - \left[\frac{2}{h^2} - \frac{f_1(w_0)}{h} + c\right] w_i + \left[\frac{1}{h^2} - \frac{f_1(w_0)}{h}\right] w_{i-1} \tag{32}$$

with $(w_0 - w_{-1})/h = -f_2(w_0)$, $w_N = 0$, w_{-1} being a phantom node. By combining the above expressions, namely subtracting (32) from (24), expressing the obtained equation in terms of $u_k - w_k$, $k = i-1$, i, $i+1$, and applying (30), we obtain

$$\begin{aligned} \alpha_i = h^2 \Bigg\{ & \left[\frac{2}{h^2} - \frac{f_1(u_0)}{h} + c\right] \alpha_{i-1} - \left[\frac{1}{h^2} - \frac{f_1(u_0)}{h}\right] \alpha_{i-2} \\ & - c u_i - B_1 e^{-ih}[1 - f_1(u_0)] + \sum_{k=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial u_k} \left(\frac{1}{h^2} u_{k+1} - \left[\frac{2}{h^2} - \frac{f_1(u_0)}{h}\right] u_k \right. \\ & \left. \left. + \left[\frac{1}{h^2} - \frac{f_1(u_0)}{h}\right] u_{k-1} + B_1 e^{-kh}[1 - f_1(u_0)] \right) \right\} \end{aligned} \tag{33}$$

for $i = 0, \ldots, N-1$. We then set $\alpha_{-1} = \alpha_{-2} = 0$ and use $u_N = \alpha_{N-1}$ as control. By inspection of the recursive control design algorithm one can verify that the co-ordinate transformation is invertible (which implies global asymptotic stability of the discretized system) and that the control law is smooth.

We should stress that the control law u_N obtained by recursively applying (33) is of a Dirichlet type. The only reason for giving the control law as a temperature control law, although flux actuation would be both more realistic and more effective, is because the model from Reference [6] that we are using gives boundary condition at $x = L$ as Dirichlet boundary condition. Extension from Dirichlet to Neumann boundary condition at $x = L$ would be performed in exactly the same fashion as we did for the model of thermal convection loop in Reference [13].

4. ASYMPTOTIC STABILITY OF THE DISCRETIZED SYSTEM IN MODIFIED COORDINATES

In this section we prove global asymptotic stability for (25)–(27) in L^2-norm. The effect of the moving boundary is resolved by utilizing the approach of Armaou and Christofides (see Reference [14] and references therein) for control of parabolic PDEs with time-dependent spatial domains. Note that we do not require an actuator whose position changes with time since the co-ordinate system in the model of Denison and Baum [6] is selected so that the solid–gas interface is maintained at $x = 0$. In real application the actuator would be placed at the nonreacting end of the solid propellant stick that does not move. We start with a Lyapunov function

$$V = \frac{1}{2} \int_0^{L(t)} w(\xi, t)^2 \, d\xi \tag{34}$$

and find its derivative with respect to time, along the trajectories of system (25) (27), to be

$$\begin{aligned}\dot{V} &= \int_0^{L(t)} w w_t \, d\xi + \frac{1}{2}\dot{L}(t)w(L)^2 \leqslant \int_0^{L(t)} w w_t \, d\xi \\ &= \int_0^{L(t)} f_1(w(0,t)) w w_\xi \, d\xi + \int_0^{L(t)} w w_{\xi\xi} \, d\xi - c\int_0^{L(t)} w^2 \, d\xi \\ &= \int_0^{L(t)} \frac{f_1(w(0,t))}{2} d(w^2) + w_x w\Big|_0^{L(t)} - \int_0^{L(t)} w_\xi^2 \, d\xi - c\int_0^{L(t)} w^2 \, d\xi \end{aligned} \tag{35}$$

where we have used the fact that $\dot{L} < 0$. Since

$$0 \leqslant f_1 \leqslant e^{B_2} \tag{36}$$

$$f_2(w(0,t))w(0,t) \leqslant k_2 w(0,t)^2, \quad k_2 > 0 \tag{37}$$

we get

$$\begin{aligned}\dot{V} &\leqslant -\frac{e^{B_2}}{2} w(0,t)^2 - w_x(0,t)w(0,t) - \int_0^{L(t)} w_\xi^2 \, d\xi - c\int_0^{L(t)} w^2 \, d\xi \\ &\leqslant f_2(w(0,t))w(0,t) - \int_0^{L(t)} w_\xi^2 \, d\xi - c\int_0^{L(t)} w^2 \, d\xi \\ &\leqslant k_2 w(0,t)^2 - \int_0^{L(t)} w_\xi^2 \, d\xi - c\int_0^{L(t)} w^2 \, d\xi \end{aligned} \tag{38}$$

Now using Agmon's inequality

$$\max_{x\in[0,L]} w^2(x) \leqslant w^2(L) + 2\sqrt{\int_0^{L(t)} w(\xi)^2 \, d\xi}\sqrt{\int_0^{L(t)} w_\xi(\xi)^2 \, d\xi} \tag{39}$$

we get

$$\begin{aligned}\dot{V} &\leqslant 2k_2\sqrt{\int_0^{L(t)} w^2 \, d\xi}\sqrt{\int_0^{L(t)} w_\xi^2 \, d\xi} - \int_0^{L(t)} w_\xi^2 \, d\xi - c\int_0^{L(t)} w^2 \, d\xi \\ &\leqslant k_2^2 \int_0^{L(t)} w^2 \, d\xi + \int_0^{L(t)} w_\xi^2 \, d\xi - \int_0^{L(t)} w_\xi^2 \, d\xi - c\int_0^{L(t)} w^2 \, d\xi \\ &= -[c - k_2^2]\int_0^{L(t)} w(\xi,t)^2 \, d\xi = -2[c-k_2^2]V, \end{aligned} \tag{40}$$

which implies that system (25)–(27) is asymptotically stable in L^2-norm. The proof that (32) is asymptotically stable in l^2-norm with $(w_0 - w_{-1})/h = -f_2(w_0)$ and $w_N = 0$ would be completely analogous. We would start with Lyapunov function $V_d = \frac{1}{2}\sum_{i=0}^{N(t)} w_i^2$, use the fact that $\dot{N}(t) < 0$, use discretized version of the Agmon's inequality (39), and obtain $\dot{V}_d \leqslant -2[c-k_2^2]V_d$.

5. SIMULATION STUDY

In this section we present simulation results for the unstable burning in homogeneous solid propellants.

As shown in Section 3, control law (33) is given in a recursive form that can be easily applied using symbolic tools available. Once the final expression for temperature control is obtained, for some particular choice of N, one would have to use full state feedback to stabilize the system, i.e. the complete knowledge of the temperature field is necessary. Instead, we show that controllers of relatively low order (designed on a much coarser grid) can successfully stabilize the system for a variety of different simulation settings.

The reasoning behind the idea of using low order backstepping controllers is based on intuition and is given here without any formal proof. The idea of using controllers designed using only a small number of steps of backstepping to stabilize the system for a certain range of the open-loop instability is based on the fact that in most real life systems only a finite number of open-loop eigenvalues is unstable. The conjecture is then to apply a low order backstepping controller (controller that uses only a small number of state measurements) that is capable of detecting the occurrence of instability from a limited number of measurements, and stabilize the system. Indeed, extensive simulation study for thermal convection loop [13] suggested that to accommodate a higher level of open-loop instability one would have to increase the order of controller, i.e. use a controller designed using more steps of backstepping. The situation is even more promising for the solid propellant instability. Unlike in the case of the heat convection loop, the increased level of open-loop instability for the solid propellant will not result in the increased number of open-loop unstable eigenvalues. Instead, only the magnitude of the single unstable eigenvalue will increase as the system becomes more unstable (the level of open-loop instability increases as either L, B_1 or B_2 increases). This fact suggested that a fixed low order controller might be capable of stabilizing the system for a variety of simulation settings.

Indeed, a controller designed using only one step of backstepping (using only two temperature measurements $u(0,t)$ and $u(L/2,t)$) was capable of successfully stabilizing the system for a variety of different simulation settings. By controller designed using only one step of backstepping we assume controller designed on a very coarse grid, namely on a grid with just two points. In this case the control is implemented by using α_1 for control, where α_1 is obtained from expression (33) for $i = 1$ with $h = L/2$ and $u_1 = u(L/2,t)$. Simulations were run with initial temperature distribution $u(x,0) = 0.01(\cos(\pi x/2L) + \cos(\pi x/3L) + \cos(\pi x/5L))$ using BTCS finite difference method for $N = 200$ and the time step equal to 0.01 s. The effect of moving boundary due to deflagration was not taken into account in any of the simulations in this simulation study. Although we have tested the controller for several different combinations of A and L, we only present a result for $A = 1$ ($B_1 = B_2 = 1.0$) and $L = 2.0$, and briefly summarize results for various other combinations. The controller was capable of stabilizing solid propellant of length $L = 0.5$ up to $A = 5$, $L = 1.0$ up to $A = 3$, $L = 1.5$ up to $A = 2$, $L = 2.0$ up to $A = 1.5$, and $L = 2.5$ up to $A = 0.5$. A trend of a decreasing range of A for which we could stabilize the system as L increases is obvious. We come to the same conclusion as in the eigenvalue analysis for the open loop system: longer rockets tend to be harder to stabilize.

The simulation results presented here are for a solid propellant rocket of non-dimensional length $L = 2.0$ with $B_1 = 1.0$ and $B_2 = 1.0$. The temperature response for the uncontrolled case is shown in Figure 3. As it can be seen from Figure 3, the particular choice of L, B_1, and B_2 corresponds to a highly unstable solid propellant system. The system goes to an undesired

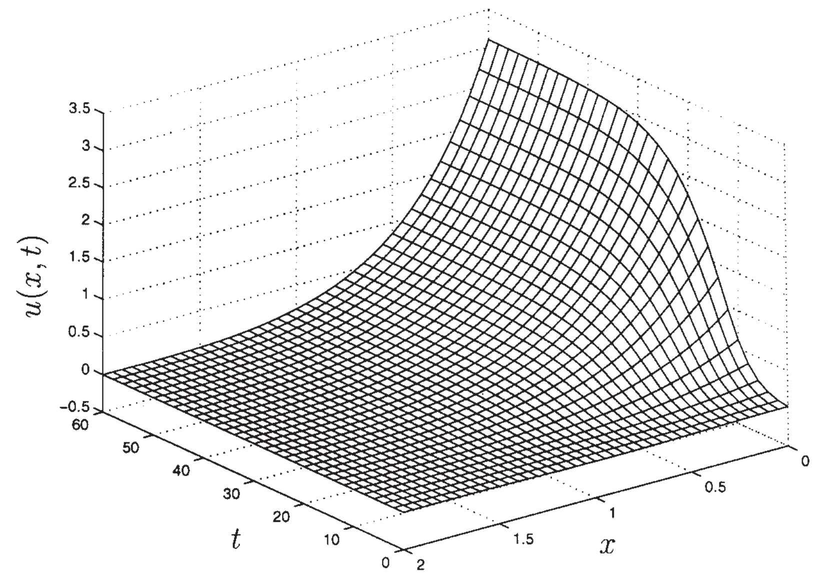

Figure 3. Open loop temperature $u(x,t)$ for $B_1 = B_2 = 1.0$ and $L = 2.0$.

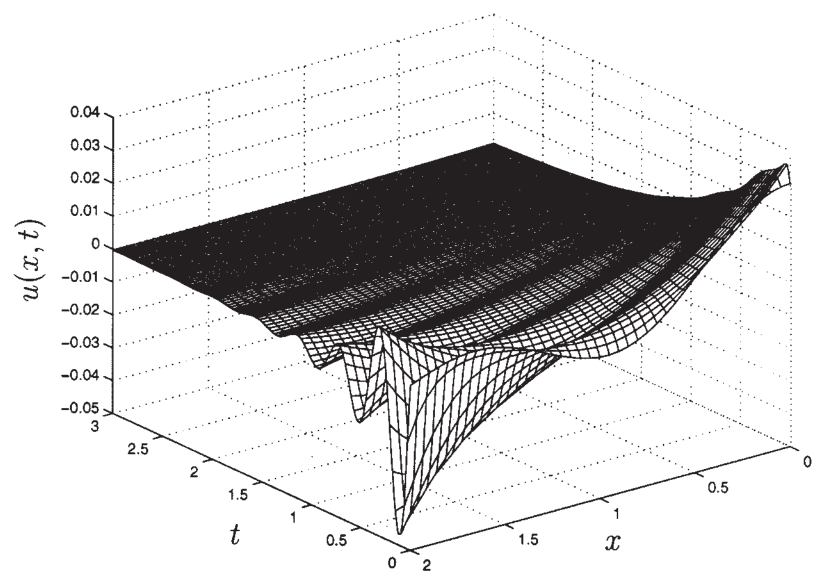

Figure 4. Closed loop temperature $u(x,t)$ for $B_1 = B_2 = 1.0$ and $L = 2.0$ with feedback controller designed using only one step of backstepping.

equilibrium profile where the burning temperature is about four-times higher than the nominal temperature, and therefore would result in a destructive failure. The corresponding temperature $u(x,t)$ for the controlled case is shown in Figure 4. The controller was capable of stabilizing the system very fast using relatively small control effort.

6. CONCLUSIONS

A nonlinear feedback controller based on Lyapunov backstepping design that achieves global asymptotic stabilization of the unstable burning in solid propellants has been derived. The result holds for any finite discretization in space of the original PDE model.

The simulation study indicates that a feedback control law designed using only one step of backstepping can be successfully used to stabilize the system for a variety of different simulation settings.

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