of dimension n onto the first n' coordinates. Thus the triangular structure simplifies the evaluation of this nested family of models for model order selection.

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Boundary Control of an Unstable Heat Equation Via Measurement of Domain-Averaged Temperature

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***Abstract*—In this note, a feedback boundary controller for an unstable heat equation is designed. The equation can be viewed as a model of a thin rod with not only the heat loss to a surrounding medium (stabilizing) but also the heat generation inside the rod (destabilizing). The heat generation adds a destabilizing linear term on the right-hand side of the equation. The boundary control law designed is in the form of an integral operator with a known, continuous kernel function but can be interpreted as a backstepping control law. This interpretation provides a Lyapunov function for proving stability of the system. The control is applied by insulating one end of the rod and applying either Dirichlet or Neumann boundary actuation on the other.**

***Index Terms*—Backstepping, boundary control, distributed parameter systems, heat equation, stabilization.**

I. Introduction

In this note, a problem of temperature stabilization by means of feedback boundary control is addressed for a model of a thin rod that takes into account not only the loss of heat to a surrounding medium but also the destabilizing heat generation inside the rod. The main result is the development of the first *backstepping* [6] control law involving infinitely many steps for a PDE. An inherent danger in applying infinitely many steps of backstepping is that the feedback gains may go to infinity. This is prevented here by choosing the transformed system in a special way which not only makes the feedback kernel a continuous function but also a known, closed-form function.

The idea of applying boundary conditions in the form of state feedback is not new. Some of the results on feedback stabilization of parabolic equations include work of Triggiani [11] who analyzed the case of a general parabolic equation defined on a bounded domain. Employing a semigroup approach, Triggiani obtained feedback boundary conditions, expressed as a specified feedback of the solution, that guarantee exponential decay of the solution as $t \to \infty$ even for the case when the open loop system is unstable. The obtained result holds in higher dimensions and the only assumptions made are that an algebraic (full rank) condition at the unstable eigenvalues is assumed to hold, and that either Dirichlet or mixed boundary conditions are prescribed everywhere on the boundary. The result by Triggiani [11] can be extended to the case of mixed boundary conditions without much difficulty [1]. We use such an extended version of the controller from [11] in our comparison study.

Motivated by models appearing in quasistatic theory of thermoelasticity (entropy of the system satisfies the heat equation), Day [5] analyzed the behavior of solutions of the one-dimensional heat equation (and more general types of one-dimensional parabolic equations) with boundary conditions given as weighted integrals of the state variable

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over the entire domain. The boundary conditions, although not explicitly stated by the author, were actually given in the form of the state feedback. Although the final result is given for a more general parabolic equation than the one we analyze in this note, it is always assumed that the coefficient multiplying the linear term has a favorable sign (enhances the overall stability of the system), which is not necessary in our case.

More recent results on the subject of feedback control of systems described by parabolic partial differential equations, and particularly thermal processes, include the work of Burns and Rubio [2]. Using the results from Burns, Rubio, and King [3], they analyzed the feedback operators obtained as solutions of algebraic Ricatti equations arising from infinite dimensional LQR/LQG control problems. Based on the numerical results for the case of a two-dimensional (2-D) heat equation, Burns and Rubio concluded that functional gains, i.e., kernel functions appearing in integral representations of feedback operators, exist and have compact support near the boundary where the control is applied. The idea is then to use that information as a guidance for optimal placement of discrete sensors.

We analyze the most general case when the effects of the heat loss and the heat generation are significant and have to be modeled. In this case the system can have only one constant temperature distribution along the rod which can be either stable or unstable, depending whether the heat loss dominates the effect of heat generation or not. In this note we focus on the "unstable" heat equation (heat generation dominates the heat loss) for which we will be able to design a control law that stabilizes the system.

The control objective is achieved by applying either Dirichlet or Neumann boundary control on one end and insulating the other. In addition, for the unstable heat equation analyzed, which in terms of dimensionless temperature $u(x, t)$ has the form $u_t = u_{xx} + \lambda u$, where subscripts denote partial derivatives, an exact range of the *positive* parameter λ multiplying the linear term for which the system can be stabilized is found. An appropriately constructed nonsingular coordinate transformation, a special application of infinite dimensional backstepping [6], will allow us to convert the original system into a new set of coordinates where we can design a control law that achieves *stabilization* using homogeneous boundary control.

II. Problem Statement

Let us consider the problem of heat conduction in a rod of small cross-section. It is assumed that the rod is so thin that the temperature at all points of the section may be taken to be the same. The homogeneous rod has length L, constant area of cross-section A, perimeter p, density ρ, specific heat c, conductivity K and diffusivity k. We assume that each element of the surface of the rod loses the heat to a surrounding medium by radiation and, in addition, the heat is generated inside the rod due to constant electric current flowing through the rod. Let H be the surface conductance (emissivity) of the rod, i the strength of the current and ρ_e electrical resistivity, i.e., the resistance per unit cross-section per unit length. The temperature of the surrounding medium and all the properties of the system, except the electric resistivity ρ_e, are assumed to be constant. The electric resistivity ρ_e is changing linearly with the temperature as $\rho_e(T) = \rho_e(T_1)(1 - \alpha_e(T - T_1))$, where T_1 stands for the temperature around which the ρ_e is linearized, and α_e is thermal coefficient of electric resistivity. It should be noted that, depending on the nature of the material the rod is manufactured from (conductor, semi-conductor, superconductor, composite, etc.) and the operating temperature range, the parameter α_e can be either negative or positive. The heat equation now becomes (see [4, Ch. IV])

$$T(l, t)_t = kT(l, t)_{ll} - \nu(T(l, t) - T_0) + B(1 - \alpha_e(T(t, l) - T_1)) \tag{2.1}$$

where T_0 is the temperature of the surrounding medium into which the rod radiates, $k = K/\rho c$, $\nu = Hp/c\rho A$ and $B = (i^2/c\rho A^2)\rho_e(T_1)$.

Define the dimensionless length, time and temperature variables, respectively, as

$$x = \frac{l}{L}, \quad \tau = \frac{t}{\frac{L^2}{k}}, \quad u = \frac{T - T_e}{T_e},$$

where

$$T_e = T_0 + B\frac{\alpha_e(T_1 - T_0) - 1}{B\alpha_e - \nu}$$

stands for the constant equilibrium temperature distribution along the rod and $x \in [0, 1]$. Finally, the nondimensional form of the equation (2.1) becomes

$$u(x, \tau)_\tau = u(x, \tau)_{xx} + \lambda u(x, \tau) \tag{2.2}$$

where

$$\lambda = \frac{L^2(B\alpha_e - \nu)}{k}.$$

Since the thermal coefficient of electric resistivity is approximately

$$\alpha_e \approx -\frac{1}{250}\,\frac{1}{K}$$

(see [10, Ch. VI]) for the most conductors at the room temperature (resistivity for the most conductors linearly increases with temperature), we will always have $\lambda < 0$ for that case. For semiconductors, on the other hand, dependence of the electric resistivity on temperature is governed by an exponential relation (resistivity at the temperatures close to room temperature exponentially decreases with temperature), which when linearized gives $\alpha_e > 0$ and typically 10–100 times as big as for conductors. Therefore, depending on the geometry of the rod and the magnitude of the current i, λ can be both positive or negative for the rod made of semiconductor material. For the case when heat generation inside the rod can be neglected the nondimensional form of the system is just a special case of the equation (2.2) for $\lambda = 0$ (see [9, Ch. I] for details).

III. Main Result

Consider the nondimensionalized heat equation (2.2) with boundary condition prespecified at $x = 0$ only

$$\begin{cases} u_t = u_{xx} + \lambda u & \text{in } (0, 1) \times (0, \infty), \\ u_x(0, t) = 0 & \text{in } (0, \infty), \\ u(x, 0) = u^0(x) & \text{in } (0, 1) \end{cases} \tag{3.1}$$

where the constant $\lambda \geq 0$ is a constant parameter, $u^0(x)$ denotes the initial data and the nondimensional time variable τ has been replaced with t for convenience. Under the homogeneous Dirichlet boundary condition at $x = 1$ ($u(1, t) = 0$ in $(0, \infty)$), equation (3.1) is unstable if $\lambda > \pi^2/4$ since, for $\lambda = 0$, $\pi^2/4$ is the first eigenvalue of (3.1) with $u(1, t) = 0$. This becomes obvious if we introduce a new variable $v(x, t) = u(x, t)e^{-\lambda t}$. Therefore, a natural question to ask is: Can one find a Dirichlet boundary feedback control law $u(1, t)$ that exponentially stabilizes the system (3.1) if $\lambda > \pi^2/4$?

Using a Lyapunov design, we indeed obtain a Dirichlet boundary feedback law that achieves exponential stability of the closed-loop system

$$\begin{cases} u_t = u_{xx} + \lambda u & \text{in } (0, 1) \times (0, \infty), \\ u_x(0, t) = 0 & \text{in } (0, \infty), \\ u(1, t) = -a\tan(a)\displaystyle\int_0^1 u(\xi, t)\, d\xi & \text{in } (0, \infty), \\ u(x, 0) = u^0(x),\ u_x^0(x) = 0 & \text{in } (0, 1). \end{cases} \tag{3.2}$$

Theorem 3.1: Assume that

$$\lambda \in \left[0,\, \frac{3\pi^2}{4}\right)$$

and

$$a \in \left(\max\left\{0,\, \mathrm{sgn}\left(\frac{\lambda}{2} - \frac{\pi^2}{8}\right)\sqrt{\left|\frac{\lambda}{2} - \frac{\pi^2}{8}\right|}\right\},\quad \frac{\pi}{2}\right).$$

1) For arbitrary initial data $u^0(x) \in C(0, 1)$, $u_x^0(0) = 0$, (3.2) has a unique classical solution that satisfies the following L^2 exponential stability estimate:

$$\|u(t)\| \le M\|u^0\|e^{-((\pi^2/4)+2a^2-\lambda)t} \tag{3.3}$$

where M is a positive constant independent of u^0.

2) For arbitrary initial data $u^0(x) \in H^1(0, 1)$, $u_x^0(0) = 0$, equation (3.2) has a unique strong solution that satisfies the following H^1 exponential stability estimate:

$$\|u(t)\|_{H^1} \le M\|u^0\|_{H^1}e^{-((\pi^2/4)+2a^2-\lambda)t/2} \tag{3.4}$$

where M is a positive constant independent of u^0.

IV. Proof of the Main Result

Proof of Theorem 3.1: To prove the Theorem 3.1 let us first consider the equation

$$\begin{cases} w_t = w_{xx} - c(x)w & \text{in } (0, 1) \times (0, \infty), \\ w_x(0, t) = 0,\ w(1, t) = 0 & \text{in } (0, \infty), \\ w(x, 0) = w^0(x),\ w_x^0(x) = 0 & \text{in } (0, 1) \end{cases} \tag{4.1}$$

with

$$c(x) = -\lambda + 2\frac{a^2}{\cos^2(ax)}. \tag{4.2}$$

It can be shown that the system (4.1) is exponentially stable if $\min_{0\le x\le 1}\, c(x) = c(0) = -\lambda + 2a^2 > -(\pi^2/4)$, which is satisfied under the conditions on λ and a stated in the theorem. Thus, if we can find an invertible coordinate transformation to transform (3.2) into (4.1), then the Theorem 3.1 is proven.

Lemma 4.1: The coordinate transformation

$$w(x, t) = u(x, t) + a\tan(ax)\int_0^x u(\xi, t)\, d\xi \tag{4.3}$$

defined for $x \in [0, 1]$ and $0 < a < \pi/2$, has an inverse

$$u(x, t) = w(x, t) - a\sin(ax)\int_0^x \frac{w(\xi, t)}{\cos(a\xi)}\, d\xi \tag{4.4}$$

and converts the system (3.2) into (4.1) with the initial distribution $w^0(x)$ related to $u^0(x)$ as $w^0(x) = u^0(x) + a\tan(ax)\int_0^x u^0(\xi)\, d\xi$.

Proof of Lemma 4.1: To prove that (4.4) is the inverse of (4.3) we start from

$$\begin{aligned} w(x, t) &= u(x, t) - \beta(x, t) \\ \beta(x, t) &= -a\tan(ax)\int_0^x u(\xi, t)\, d\xi. \end{aligned} \tag{4.5}$$

For $x = 0$ we get $\beta(0, t) = 0$ since $w(0, t) = u(0, t)$. For $x \in (0, 1]$, we start by finding $\beta_x(x, t)$ from (4.5), use the variation of constants formula with $\beta(0, t) = 0$, and get

$$\beta(x, t) = -a\sin(ax)\int_0^x \frac{w(\xi, t)}{\cos(a\xi)}\, d\xi. \tag{4.6}$$

This proves the first part of the lemma. To prove the second part let us first write (4.3) as

$$\begin{aligned} w(x, t) &= u(x, t) - k(x)\int_0^x u(\xi, t)\, d\xi \\ k(x) &= -a\tan(ax). \end{aligned} \tag{4.7}$$

We now look for conditions that $k(x)$ and $c(x)$ should satisfy such that if u satisfies equation (3.2), then w satisfies equation (4.1). Taking one partial derivative of the expression (4.7) with respect to t, two derivatives with respect to x, and substituting the obtained expressions in (4.1) gives

$$[\lambda + 2k'(x) + c(x)]u(x, t) + [k''(x) - \lambda k(x) - c(x)k(x)] \cdot \int_0^x u(\xi, t)\, d\xi = 0. \tag{4.8}$$

Therefore, if $k(x)$ and $c(x)$ satisfy

$$\begin{aligned} \lambda + 2k'(x) + c(x) &= 0 \\ k''(x) - \lambda k(x) - c(x)k(x) &= 0 \end{aligned} \tag{4.9}$$

the theorem is proven. By substitution of (4.2) and (4.7) we verify that (4.9) is indeed satisfied. Finally, the boundary condition for $w_x(0, t)$ is obtained by differentiating (4.3) and substituting $u_x(0, t) = 0$, while the Dirichlet feedback boundary control $u(1, t)$ is obtained by substituting $x = 1$ in (4.3) together with the fact that $w(1, t) = 0$. This concludes the proof. □

Before continuing, we first remark that problem (3.2) is well posed since transformation (4.3) is invertible and the problem defined by (4.1) is well posed (see, e.g., [7, Ch. IV]). Also, by (4.4), there exists a positive constant $\mu > 0$ such that $\|u(t)\|_{L^2} \le \mu\|w(t)\|_{L^2}$, $\|u(t)\|_{H^1} \le \mu\|w(t)\|_{H^1}$, and by (4.3) there exists a positive constant $\nu > 0$ such that $\|w(t)\|_{L^2} \le \nu\|u(t)\|_{L^2}$, $\|w(t)\|_{H^1} \le \nu\|u(t)\|_{H^1}$. Therefore, it is sufficient to prove (3.3) and (3.4) for the solution w of (4.1).

i) Define

$$E(w, t) = \tfrac{1}{2}\int_0^1 w(x, t)^2\, dx. \tag{4.10}$$

Using the fact that for all functions with $w_x(0) = w(1) = 0$ inequality $(\pi^2/4)\|w\|^2 \le \|w_x\|^2$ holds ($\pi^2/4$ is the smallest eigenvalue of the operator $-(\partial^2/\partial x^2)$ with the same boundary conditions[1]), and that $\min_{0\le x\le 1}\, c(x) > -(\pi^2/4)$, we get

$$\dot{E}(w, t) \le -2\left(\frac{\pi^2}{4} + 2a^2 - \lambda\right)E(w, t) \tag{4.11}$$

which implies

$$E(w, t) \le E(w, 0)e^{-2((\pi^2/4)+2a^2-\lambda)t}, \quad \text{for } t \ge 0. \tag{4.12}$$

ii) Set

$$V(t) = \int_0^1 w_x(x, t)^2\, dx. \tag{4.13}$$

Using the definition (4.10) of E, we deduce that (the following Cs denote various positive constants that may vary from line to line)

$$\dot{E}(t) + V(t) \le CE(t). \tag{4.14}$$

Multiplying (4.14) by $e^{((\pi^2/4)+2a^2-\lambda)t}$ and then integrating the obtained expression from 0 to t gives

$$e^{((\pi^2/4)+2a^2-\lambda)t}E(t) + \int_0^t e^{((\pi^2/4)+2a^2-\lambda)s}V(s)\, ds \le CE(0). \tag{4.15}$$

[1]This result is well known for Dirichlet boundary conditions (see [8, Ch. I]). For the mixed boundary conditions it can be proved by representing $w(x) = \sum_{k=0}^{\infty} a_k\cos((2k+1)(\pi/2)x)$, noting by integration by parts that $-\int_0^1 w_{xx}w\, dx = \int_0^1 w_x^2\, dx$, and showing by simple calculation that $-\int_0^1 w_{xx}w\, dx = \sum_{k=0}^{\infty} a_k^2(2k+1)^2(\pi^2/4) \ge (\pi^2/4)\sum_{k=0}^{\infty} a_k^2 = (\pi^2/4)\int_0^1 w^2\, dx$.

Multiplying the first equation of (4.1) by w_{xx} and integrating from 0 to 1 by parts we obtain

$$\begin{aligned}\dot{V}(t) &\le -2\int_0^1 w_{xx}^2\,dx + \int_0^1 w_{xx}^2\,dx + C\int_0^1 w^2\,dx \\ &\le C\int_0^1 w^2\,dx \le CE(t), \end{aligned} \tag{4.16}$$

which implies that

$$\frac{d}{dt}\left(V(t)e^{((\pi^2/4)+2a^2-\lambda)t}\right) \le C[E(t)+V(t)]e^{((\pi^2/4)+2a^2-\lambda)t}. \tag{4.17}$$

Integrating (4.17) from 0 to t, together with (4.12) and (4.15) finally gives

$$V(t)e^{((\pi^2/4)+2a^2-\lambda)t} \le C[V(0)+E(0)]. \tag{4.18}$$

This shows that (3.4) holds.

□

V. Extension to Neumann Boundary Control Case

In this section, we extend the results from Section III to Neumann boundary control. The main ideas for this case are similar to those for the Dirichlet case. We start from the fact that equation (3.1) with Neumann boundary condition $u_x(1,t)=0$ is unstable for $\lambda > 0$. We propose a Neumann boundary feedback control law such that the closed-loop system

$$\begin{cases} u_t = u_{xx} + \lambda u & \text{in } (0,1)\times(0,\infty),\\ u_x(0,t)=0 & \text{in } (0,\infty),\\ u_x(1,t) = -(\alpha + a\tan(a))u(1,t) & \\ \qquad -\left(\alpha a\tan(a) + \dfrac{a^2}{\cos^2 a}\right) & \\ \qquad \cdot \displaystyle\int_0^1 u(\xi,t)\,d\xi & \text{in } (0,\infty),\\ u(x,0)=u^0(x),\ u_x^0(x)=0, & \text{in } (0,1) \end{cases} \tag{5.19}$$

is exponentially stable for $\alpha > 2$.

Theorem 5.1: Assume that

$$\lambda \in \left[0,\ 1+\frac{\pi^2}{2}\right)$$

$$a \in \left(\max\left\{0,\ \operatorname{sgn}\left(\frac{\lambda}{2}-\frac{1}{2}\right)\sqrt{\left|\frac{\lambda}{2}-\frac{1}{2}\right|}\right\},\quad \frac{\pi}{2}\right)$$

and $\alpha > 2$.

1) For arbitrary initial data $u^0(x)\in C(0,1)$, $u_x^0(0)=0$, equation (5.19) has a unique classical solution that satisfies the following L^2 exponential stability estimate:

$$\|u(t)\| \le M\|u^0\|e^{-(1+2a^2-\lambda)t} \tag{5.20}$$

where M is a positive constant independent of u^0.

2) For arbitrary initial data $u^0(x)\in H^1(0,1)$, $u_x^0(0)=0$, equation (5.19) has a unique strong solution that satisfies the following H^1 exponential stability estimate:

$$\|u(t)\|_{H^1} \le M\|u^0\|_{H^1}e^{-(1+2a^2-\lambda)t/2}. \tag{5.21}$$

The proof of Theorem 5.1 is similar to that of Theorem 3.1. We outline only the differences. Instead of (4.1) we consider

$$\begin{cases} w_t = w_{xx} - c(x)w & \text{in } (0,1)\times(0,\infty),\\ w_x(0,t)=0, & \\ w_x(1,t) = -\alpha w(1,t) & \text{in } (0,\infty),\\ w(x,0)=w^0(x),\ w_x^0(x)=0 & \text{in } (0,1). \end{cases} \tag{5.22}$$

It can be shown that the system (5.22) is exponentially stable if $c(x)$ satisfies $\min_{0\le x\le 1} c(x) = c(0) > -1$. Using inequality

$$\begin{aligned}\int_0^1 w(x,t)^2\,dx &\le 2w(1,t)^2 \\ &\quad + 2\int_0^1(1-x)\,dx\int_0^1 w_x(x,t)^2\,dx \\ &\le 2w(1,t)^2 + \int_0^1 w_x(x,t)^2\,dx \end{aligned} \tag{5.23}$$

and definition of $c(0)$, we get that

$$\dot{E}(w,t) \le (2-\alpha)w(1,t)^2 - 2(1+2a^2-\lambda)E(w,t) \tag{5.24}$$

which implies

$$E(w,t) \le E(w,0)e^{-2(1+2a^2-\lambda)t}, \quad \text{for } t\ge 0. \tag{5.25}$$

The remainder is the same except $(\pi^2/4)+2a^2-\lambda$ replaced by $1+2a^2-\lambda$ and V by $V(t)=\alpha w(1,t)^2 + \int_0^1 w_x(x,t)^2\,dx$.

VI. Simulation Study

In this section, a simulation study that addresses the most relevant aspects of the proposed feedback boundary control scheme is conducted. The study consists of two distinct parts. In the first part we present results that put the emphasis on the main features of the proposed feedback boundary control scheme, while the second part includes a comparison with a controller based on pole-placement feedback design for parabolic PDEs from [11]. In both parts of the study we present the results for Dirichlet feedback control law (3.2) only. The behavior of the closed loop system for Neumann case (5.19) is completely analogous.

We start with the unstable heat equation $u_t = u_{xx} + 3u$, with $u(x,0) = 1-9x^2+8x^3$. As shown in the eigenvalue analysis, the case with $\lambda=3$, which corresponds to one unstable eigenvalue, cannot be stabilized using homogeneous boundary conditions $u_x(0,t)=0$ and $u(1,t)=0$. Although we do not show the simulation results for this open-loop case, we mention that the nondimensional temperature $u(x,t)$ grew exponentially above 50 in less than 10 s.

As the first step we compare the two Dirichlet feedback boundary control designs for the system $u_t = u_{xx}+3u$. The only difference between the two proposed feedback designs is the value of the adjustable control gain that was chosen as $a=5\pi/20$ and $a=9\pi/20$ respectively. In both cases Dirichlet controllers are able to stabilize the unstable heat equation. The first row of Fig. 1 shows the nondimensional temperature at the uncontrolled end $x=0$. The temperature at $x=0$ is the most representative of the controller performance since the point $x=0$ is the farthest from the end $x=1$ at which the control is applied, and therefore it decays at the slowest rate. As expected, the controller with higher control gain achieves much faster convergence. The fast response is paid for with a significantly higher control effort. Control signals for both $a=5\pi/20$ and $a=9\pi/20$ are shown in the second row of the same figure.

It is important to understand how conservative are the estimates on the range of the parameter λ for which the stabilization of the system is possible, and determine the lower bound on control gain a that renders the closed loop system stable. Indeed, simulation results suggest that we can stabilize the system for a much wider range of λ than the theory predicts. Fig. 2 shows the closed loop temperature $u(x,t)$ and the temperature control $u(1,t)$ for the system with $\lambda=10$, which is roughly 35% above the predicted upper bound $\lambda = 3\pi^2/4$. The simulation was performed for the same initial distribution $u(x,0)=1-9x^2+8x^3$ with $a=0.95(\pi/2)$. As it can be seen from Fig. 2 it takes the controller significantly longer to stabilize the system. Simulation results also suggest that the lower bound on control gain $k=a\tan(a)$, or alternatively on a, is not optimal and that stabilization can be achieved with smaller

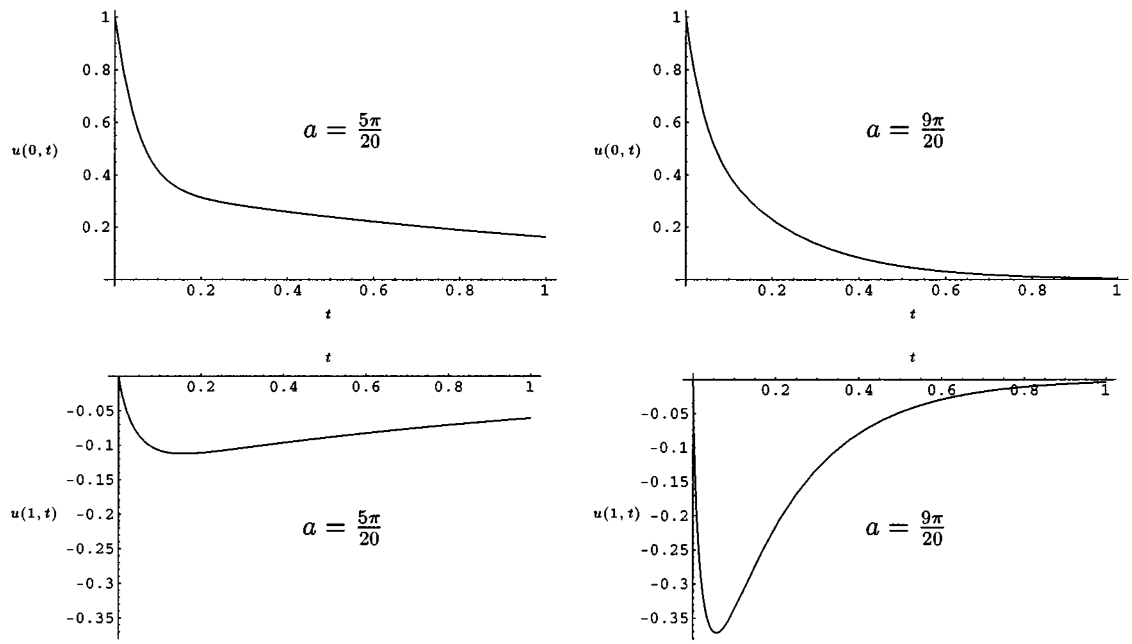

Fig. 1. Closed-loop response of the system with the Dirichlet feedback boundary control $u(1, t) = -a\tan(a)\int_0^1 u(\xi, t)\,d\xi$ for $u(x, 0) = 1 - 9x^2 + 8x^3$, $\lambda = 3$ (one unstable eigenvalue), and two different values of a. [First row: The evolution of the uncontrolled end, $u(0, t)$; Second row: The control effort $u(1, t)$.]

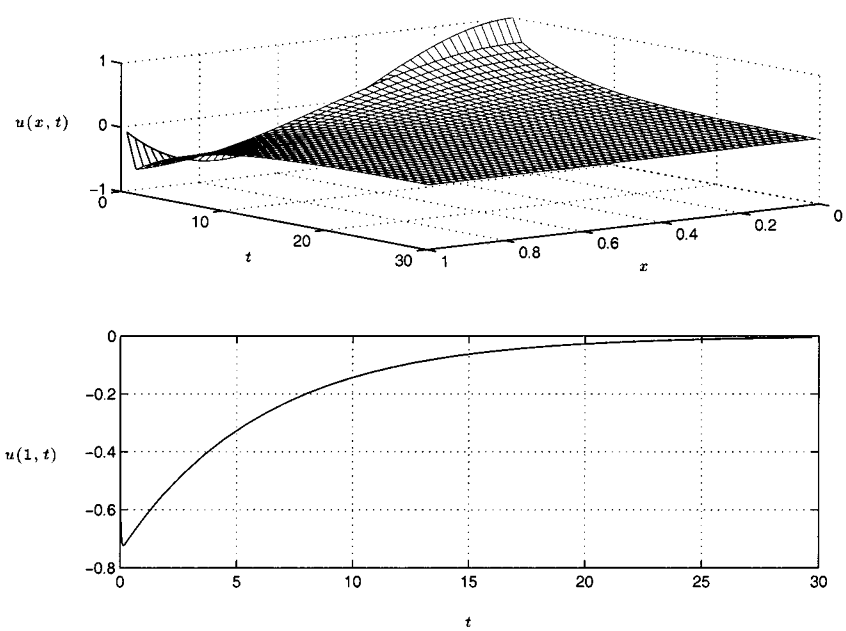

Fig. 2. Closed-loop temperature $u(x, t)$ and the temperature control $u(1, t)$ for the *unstable* heat equation ($\lambda = 10$) with the Dirichlet feedback boundary control $u(1, t) = -a\tan(a)\int_0^1 u(\xi, t)\,d\xi$, $a = 0.95(\pi/2)$, and initial distribution $u(x, 0) = 1 - 9x^2 + 8x^3$.

control gain. Fig. 3 shows the closed loop temperature $u(x, t)$ and the temperature control $u(1, t)$ for the system with $\lambda = \pi^2/2$, and a control gain equal to 80% of the minimum required gain, i.e.,

$$a = 0.80\sqrt{\frac{\lambda}{2} - \frac{\pi^2}{8}} = 0.80\frac{\sqrt{2}\pi}{4}.$$

The same type of behavior with respect to the range of the parameter λ and the lower bound on control gain a was observed for various different combinations of initial distributions and parameters λ and a.

In this second part of the simulation study we show a comparison between the controller presented in this note and a feedback controller based on [11]. The algorithm from [11] is extended in a straightforward manner to accommodate the case of mixed Dirichlet–Neumann boundary conditions, namely Neumann boundary condition at 0-end and Dirichlet at 1-end. Before we proceed to the simulation results we briefly go over the assumptions and relevant details of the controller design from [11]. The idea employed by Triggiani in [11] was to separate the system into a finite-dimensional unstable part and an infinite-dimensional stable part. The feedback control that stabilizes the unstable part, while leaving the stable part stable, is then applied.

The controller is designed under the assumption of a single unstable eigenvalue. Applying the algorithm outlined in [11] to our 1-D system, we get that $\overline{A}_u^T = \Lambda + W^T\overline{P}^T$, where $\overline{A}_u$, Λ, W, and $\overline{P}$, respectively,

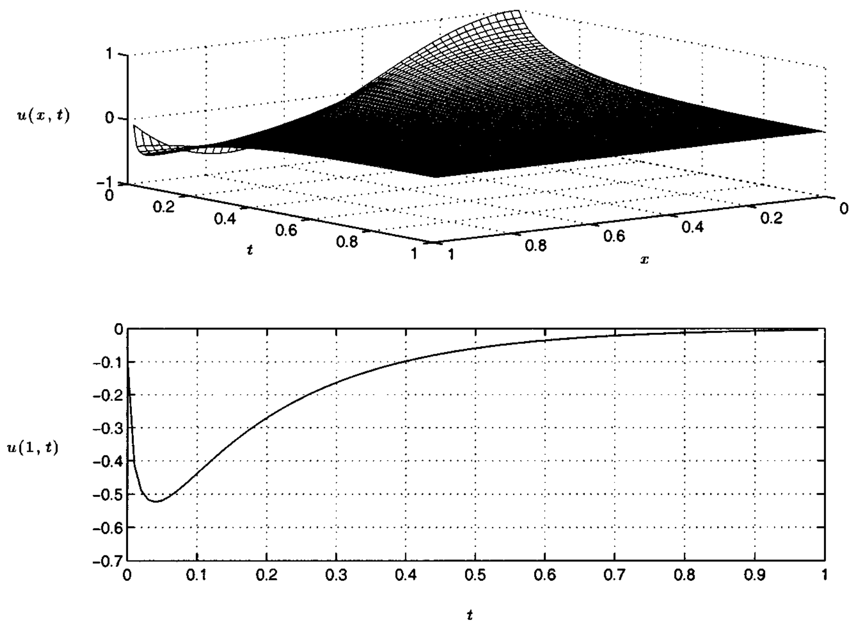

Fig. 3. Closed-loop temperature $u(x, t)$ and the temperature control $u(1, t)$ for the *unstable* heat equation ($\lambda = \pi^2/2$) with the Dirichlet feedback boundary control $u(1, t) = -a\tan(a)\int_0^1 u(\xi, t)\,d\xi$, $a = 0.8\sqrt{(\lambda/2) - (\pi^2/8)}$, and initial distribution $u(x, 0) = 1 - 9x^2 + 8x^3$.

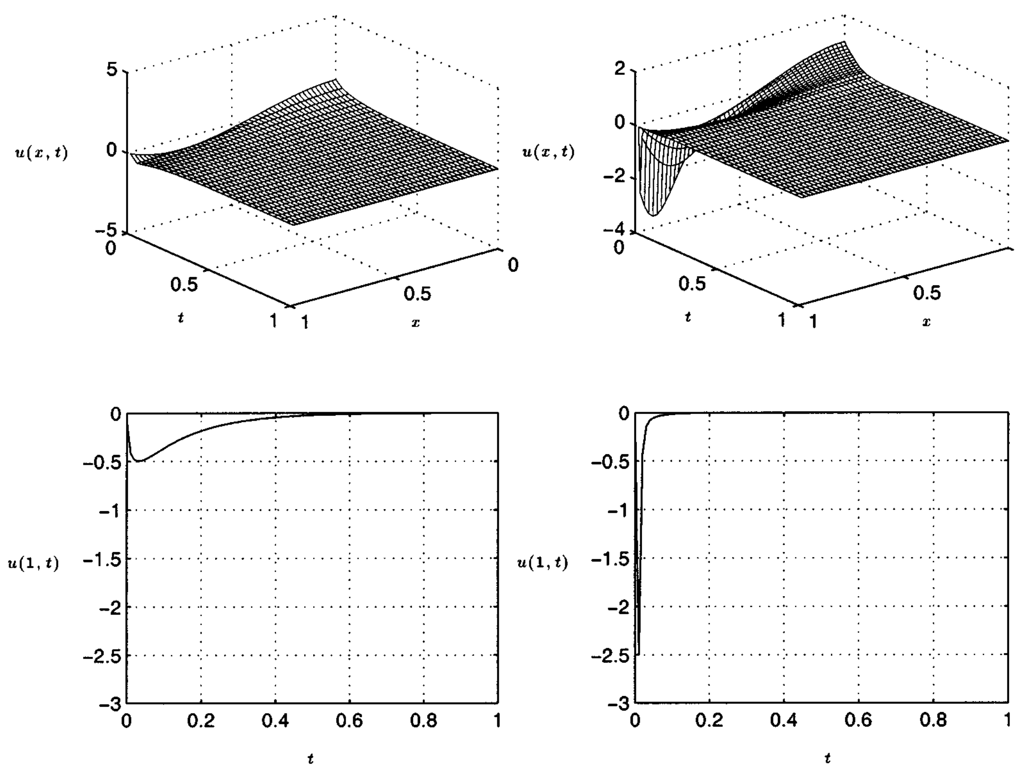

Fig. 4. Closed-loop response of the system with two different Dirichlet feedback boundary control laws [left $u(1, t) = -a\tan(a)\int_0^1 u(\xi, t)\,d\xi$, $a = 0.95(\pi/2)$, and right $u(1, t) = (\lambda_{\rm new} + \lambda - (\pi^2/4)/\pi)\int_0^1 \cos((\pi/2)\xi)u(\xi, t)\,d\xi$, $\lambda_{\rm new} = 0.1$] for $u(x, 0) = 1 - 9x^2 + 8x^3$, and $\lambda = 3$ (one unstable eigenvalue). [First row: $u(x, t)$; Second row: The control effort $u(1, t)$.]

stand for the new target subspace of the original unstable subspace A_u, the block-diagonal matrix associated with unstable eigenvalues, the feedback matrix, and the matrix associated with interior vectors w_k (see [11] for details), reduces to

$$-\lambda_{\rm new} = \lambda_1 + w_1 p_1, \quad \lambda_{\rm new} > 0. \tag{6.26}$$

The objective, as implied by the equation (6.26), is to design a feedback control law that places the unstable eigenvalue λ_1 at $-\lambda_{\rm new}$. Finding the spectral decomposition of the open loop system, and following the approach outlined in [11], we finally obtain

$$u(1, t) = \frac{\lambda_{\rm new} + \lambda - \dfrac{\pi^2}{4}}{\pi}\int_0^1 \cos\left(\frac{\pi}{2}\xi\right) u(\xi, t)\,d\xi. \tag{6.27}$$

Although we have conducted comparison study of the two designs for several different combinations of initial distributions, system parameter λ, and control gains (a in our case and λ_{new} for the controller from [11]), we only present a result for $\lambda = 3$, $u(x, 0) = 1 - 9x^2 + 8x^3$, $a = 0.95(\pi/2)$, and $\lambda_{\text{new}} = 0.1$, and briefly summarize results for other settings that we have tested. The behavior of the closed loop system was simulated using BTCS finite difference method for $N = 200$ and the time step equal to $1e$-6 s.

Fig. 4 shows the closed loop temperatures $u(x, t)$ and the temperature controls $u(1, t)$ for this particular setting. Note that we have chosen λ_{new} that achieves a good trade-off between the rate of convergence and the size of the control effort. Placing $-\lambda_{\text{new}}$ further left on the real axis would insignificantly improve convergence but would result in much higher control effort. What is apparent from Fig. 4 is that the controller from [11] is faster, but better performance had to be paid for by much higher control effort. As a rule, the controller from [11] was achieving faster convergence for all the settings we had tested, but it required much more aggressive control effort (approximately 2–7 times higher). Finally, as expected, none of the controllers could stabilize the system with two unstable eigenvalues.

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Remarks on the Robust Output Regulation Problem for Nonlinear Systems

Jie Huang

***Abstract*—The robust nonlinear output regulation problem was first solved under a polynomial condition on an input feedforward function. Another condition was given later which appears less restrictive than the first one. In this note, we will show that both these two conditions lead to the same sufficient condition that the input feedforward function along the trajectories of the exosystem is a sum of finitely many harmonics, or what is called trigonometric polynomial.**

***Index Terms*—Nonlinear systems, output regulation, servomechanism problem.**

I. Introduction

The robust servomechanism (alternatively, structurally stable output regulation) problem for linear systems has been thoroughly studied in the 1970s in [4], and [7], [8], among others. Briefly, this problem is concerned with designing a control law for a plant such that the output of the plant asymptotically tracks a class of reference inputs and rejects a class of disturbances in the presence of certain plant parameter perturbations. For the class of nonlinear systems, the same problem was first treated for the special case in which the exogenous signals are constant [8], [6], and [12]. The nonlinear output regulation problem with time varying exogenous signals was first studied in 1990 by Isidori and Byrnes without considering the parameter uncertainty [13]. Subsequently, the robust version of the same problem has been pursued in [10], [19], [5], [11], [9], [15], and [2]. It is shown in [10] and [19] that, under some standard assumptions, the robust output regulation problem is solvable if certain input feedforward function is a polynomial in the exogenous signal. Another condition was given later in [2] which requires the input feedforward function to satisfy a partial differential equation. It was also shown there that the input feedforward function satisfies that partial differential equation if it is a polynomial in the exogenous signal. Thus the later condition is considered less restrictive than the first one. In this note, we will show that both of these two conditions lead to the same sufficient condition that the input feedforward function along the trajectories of the exosystem is a sum of finitely many sinusoidal functions. The result not only links the two existing conditions, but also leads to a clear-cut method to synthesize a minimal dimension internal model needed for designing the desirable controller.

II. Problem Description

The robust nonlinear output regulation problem deals with a plant described by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), v(t), w), \quad x(0) = x_0 \\ y(t) &= h(x(t), u(t), v(t), w), \quad t \geq 0 \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the plant state, $u(t) \in R^m$ the plant input, $y(t) \in R^p$ the plant output representing the tracking error, $w \in R^N$ the plant uncertain parameters, and $v(t) \in R^q$ the exogenous signal

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