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Automatica 37 (2001) 2033–2040

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Brief Paper

Nonlinear stabilization of a thermal convection loop by state feedback☆

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Received 3 April 2000; revised 26 April 2001; received in final form 13 June 2001

Abstract

A nonlinear feedback control law that achieves global asymptotic stabilization of a 2D thermal convection loop (widely known for its "Lorenz system" approximation) is presented. The loop consists of viscous Newtonian fluid contained in between two concentric cylinders standing in a vertical plane. The lower half of the loop is heated while the upper half is cooled, which makes the no-motion steady state for the uncontrolled case unstable for values of the non-dimensional Rayleigh number $R_a > 1$. The objective is to stabilize that steady state using boundary control of velocity and temperature on the outer cylinder. We discretize the original nonlinear PDE model in space using finite difference method and get a high order system of coupled nonlinear ODEs in 2D. Then, using backstepping design, we transform the original coupled system into two uncoupled systems that are asymptotically stable in l^2-norm with homogeneous Dirichlet boundary conditions. The resulting boundary controls actuate velocity and temperature in the original coordinates. The control design is accompanied by an extensive simulation study which shows that the feedback control law designed on a very coarse grid (using just a few measurements of the flow and temperature fields) can successfully stabilize the actual system for a very wide range of the Rayleigh number. © 2001 Published by Elsevier Science Ltd.

Keywords: Convection loop; Boundary control; Backstepping

1. Introduction

A feedback boundary control law that globally stabilizes the no-motion steady state is designed for a closed convection loop that is created by heating the lower half of the loop and cooling the upper half. The imposed temperature gradient induces density difference between the lower and upper portions of the loop and acts as a driving force in this system. This motion is opposed by the damping effects of viscosity and thermal diffusivity.

Natural convection loops have been extensively studied and used in solar energy heating and cooling systems, geothermal power production, greenhouses, permafrost protection, emergency reactor cooling systems, turbine blade cooling, engine and computer cooling applications, and in process industries. Their extensive use is primarily due to the fact that they provide a means for circulating the fluid without the use of pumps. Some of the studies of such loops include work of Welander (1967) who analyzed flow in a rectangular vertical loop with point heat source on the bottom, point heat sink on the top, and two vertical branches. Welander concluded that under given assumptions the system had one steady solution, with warm fluid rising in one branch and cold fluid sinking in the other, that may become unstable in an oscillatory manner. The work of Creveling, De Paz, Baladi, and Schoenhals (1975) focused on toroidal loop heated from below by uniform heat flux and cooled from above using concentric cooling jacket with high coolant flow rate. They demonstrated, both experimentally and analytically, that for the heat transfer rates in the range between low and high the flow becomes unstable. Bau and Torrance (1981) also introduced and experimentally validated a model of an open, symmetrically heated convection loop. Unlike Welander (1967), Creveling, De Paz, Baladi, and Schoenhals (1975), and Bau and Torrance (1981) who utilized one-dimensional approach by averaging the governing equations over the cross section of the loop, Mertol, Greif, and Zvirin (1982) derived two-dimensional model assuming axial symmetry of the

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor C. Canudas de Wit under the direction of Editor Hassan Khalil.

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0005-1098/01/$ - see front matter © 2001 Published by Elsevier Science Ltd.
PII: S0005-1098(01)00169-8

model and neglecting axial conduction, viscous dissipation, and the effects of curvature.

One of the ways to suppress instabilities occurring in convection loops and change the nature of the flow is through feedback control. Based on the model from Bau and Torrance (1981), Singer and Bau (1991) and Wang, Singer, and Bau (1992) demonstrated, both analytically and experimentally, that the heat convection in a toroidal vertical loop can be successfully controlled by suppressing or enhancing disturbance occurring in the flow. In their work, Wang et al. (1992) noticed that the model from Bau and Torrance (1981), that implicitly assumes the friction and heat transfer laws similar to those of laminar, fully developed, Poiseuille flow, is not the most adequate choice and suggested that a more realistic model might be used. In fact, they have observed in their experiments the development of a secondary circulation that may significantly modify both the friction and heat transfer laws. Using celebrated Lorenz (1963) equations as a simplified model of fluid convection Bewley (1999) examined the application of linear control to a low-order nonlinear chaotic convection problems. Global stabilization of the Lorenz equations has been also investigated by Wan and Bernstein (1995) and Janković (1997) by means of nonlinear feedback control. More recently, Burns, King, and Rubio (1999) have designed an LQG controller, the first for a general two-dimensional model of a circular pipe, that achieves local stability enhancement.

In this paper, we use a model of a closed thermal convection loop consisting of viscous Newtonian fluid contained in between two concentric cylinders standing in a vertical plane. The equations governing velocity and temperature distribution in this system are the same as those used by Burns et al. (1999). The only assumptions made in the model are that the system parameters are constant except for the density in the buoyancy term (Boussinesq approximation), the gap between the cylinders is narrow compared to the size of the loop, and that the azimuthal velocity can be neglected. Note that the narrow gap assumption implies that the axial velocity depends on radial coordinate only, which in turn implies absence of the secondary circulation in this model.

Our objective is to stabilize the unstable no-motion steady state using boundary control of velocity and temperature on the outer cylinder. To achieve that, we first discretize the original PDE model in space using finite difference method (both in radial and axial directions) which gives a high order system of coupled nonlinear ordinary differential equations in 2D. Then, using backstepping design (Krstić, Kanellakopoulos, & Kokotović, 1995), we obtain a discretized coordinate transformation that transforms the original coupled system into two uncoupled systems that are asymptotically stable in l^2-norm with homogeneous Dirichlet boundary conditions. The fact that the discretized coordinate transformation is invertible, for an arbitrary (finite) grid choice, implies *global asymptotic stability* of the discretized version of the original system. The coordinate transformation is then used to obtain nonlinear feedback boundary control laws for velocity and temperature in the original set of coordinates.

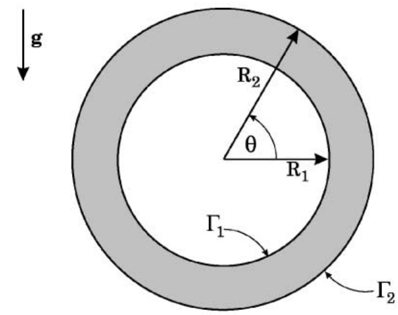

Fig. 1. Thermal convection loop description.

The paper is organized as follows. In Section 2, a two-dimensional PDE model for the closed thermal convection loop is derived and stability analysis of the no-motion steady state in terms of the loop Rayleigh number is presented. A nonlinear feedback control law that achieves global asymptotic stabilization is presented in Section 3, followed by the stability proof for the discretized system in modified coordinates in Section 4. Finally, the feedback control law designed on a very coarse grid is shown to successfully stabilize the system for a wide range of Rayleigh number in a simulation study presented in Section 5.

2. Mathematical model

In this section, we derive a mathematical model for the closed thermal convection loop. The loop consists of viscous Newtonian fluid contained in between two concentric cylinders standing in a vertical plane (see Fig. 1). We assume that the gap between the cylinders is narrow compared to the size of the loop, i.e. $R_2 - R_1 \ll R_1 < R_2$. The narrow gap assumption allows us to assume that the velocity of the fluid inside the loop depends on time and radial coordinate only. The azimuthal velocity, i.e. velocity of the fluid in a direction perpendicular to the vertical plane, is assumed to be zero. The properties of the fluid are assumed to be constant except for the density in the buoyancy term (Boussinesq approximation). This assumption, in addition, implies that the continuity equation is the same as for the incompressible fluid.

The thermal convection loop to be analyzed represents a problem of coupled velocity and temperature fields. The lower half of the loop is heated while the upper half is cooled. Since the fluid in the upper half of the loop is cooled it becomes more dense and tends to move downwards. Conversely, fluid in the lower half is heated, becomes lighter and rises upwards. The imposed average temperature gradient therefore acts as a driving force in this system. This motion is opposed by the damping effects of viscosity and thermal diffusivity.

The Boussinesq equations governing velocity and temperature in this system are those used by Burns et al. (1999). The spatial domain, in terms of cylindrical coordinates, is $(r,\theta)\in[R_1,R_2]\times[0,2\pi]$, where r stands for the radial coordinate, θ for the axial, and $R_2 > R_1 > 0$ respectively stand for radii of the outer and inner cylinders. Representing the system from Burns et al. (1999) in cylindrical coordinates and integrating the momentum equation along the circular path at fixed radius r eliminates the pressure term and gives the final form of the governing equations in cylindrical coordinates as

$$v_t(r,t) = \frac{\gamma}{2\pi}\int_0^{2\pi} T\cos\theta\,\mathrm{d}\theta + \nu\left(-\frac{v}{r^2}+\frac{v_r}{r}+v_{rr}\right), \qquad (1)$$

$$T_t(r,\theta,t) = -\frac{v}{r}T_\theta + \chi\left(\frac{T_{\theta\theta}}{r^2}+\frac{T_r}{r}+T_{rr}\right), \qquad (2)$$

with boundary conditions $v(R_1,t)=v(R_2,t)=0$, $T(R_1,\theta,t)=KR_1\sin(\theta)$, and $T(R_2,\theta,t)=KR_2\sin(\theta)$. In the above system v denotes velocity vector, T stands for temperature, ν and χ are respectively coefficients of kinematic viscosity and thermal diffusivity, β is the coefficient of thermal expansion, $\gamma=|\mathbf{g}|\beta$, $\mathbf{g}$ being the vector of gravitational acceleration, K is a constant parameter, while subscripts denote partial derivatives with respect to corresponding variables. We should emphasize that we could have prescribed Neumann instead of Dirichlet boundary conditions for the steady-state boundary temperatures, and later designed the feedback control in terms of Neumann boundary conditions. The difference for that case would be in modifying the transformed temperature system ($-cz$ term added on the RHS of the z_t equation, $c>1/4R_2$), and then using the Agmon's instead of Poincaré inequality for proving asymptotic stability in modified coordinates. The main reason for choosing Dirichlet boundary conditions is the fact that the most of the work published on stabilization of convective problems uses Dirichlet boundary conditions for actuation. Under given conditions, the system consisting of Eqs. (1) and (2) has a no-motion steady state of the form $(\bar{v},\bar{T})=(0,Kr\sin\theta)$. Introducing a new temperature variable $\tau=T-\bar{T}=T-Kr\sin\theta$ we shift the equilibrium to $(v,\tau)=(0,0)$, which is open loop unstable for sufficiently high value of K. The objective is to stabilize $v(r,t)$ and $\tau(r,\theta,t)$ for that case to zero while keeping $v(R_1,t)\equiv 0$ and $\tau(R_1,\theta,t)\equiv 0$, and using $v(R_2,t)$ and $\tau(R_2,\theta,t)$ for actuation. From a physical point of view, this implies that the total temperature control on the outer boundary will consist of steady-state component $KR_2\sin(\theta)$ modulated by an unsteady control $\tau(R_2,\theta,t)$. Note that, since $v(R_2,t)$ is a scalar independent of θ and $\tau(R_2,\theta,t)$ is a function of θ, this actuation is infinite-dimensional. Even though actuation is infinite-dimensional, this is not a simple problem with distributed (body-force) actuation because the boundary actuation is one dimensional, while the spatial domain is 2D. The local LQG design in Burns et al. (1999) (our objective is *global* stabilization) uses only $\tau(R_2,\theta,t)$ for control, while keeping $v(R_2,t)=0$. The design that we present here uses also the scalar quantity $v(R_2,t)$ for control. Physically, this means that we are not only heating the "outer boundary" of the flow domain but also rotating the boundary.

Before proceeding to stability analysis of the origin, we nondimensionalize the system by introducing a set of nondimensional variables as $r'=r/d$, $t'=t/(d^2/\chi)$, $v'=v/(\chi/d)$, and $\tau'=\tau/\Delta T$, where $d=R_2-R_1$ stands for the channel width and $\Delta T=-(4/\pi)K(R_1+R_2/2)$ is the average temperature difference between the upper and lower half of the loop. Omitting superscripts $'$ for convenience, we finally obtain the nondimensional form of Eqs. (1) and (2)

$$v_t = \frac{1}{\pi}PR_aC\int_0^{2\pi}\tau\cos\theta\,\mathrm{d}\theta + P\left(-\frac{v}{r^2}+\frac{v_r}{r}+v_{rr}\right), \qquad (3)$$

$$\tau_t = \frac{\mathrm{d}\pi}{2(R_1+R_2)}v\cos\theta - \frac{v}{r}\tau_\theta + \frac{\tau_{\theta\theta}}{r^2}+\frac{\tau_r}{r}+\tau_{rr}, \qquad (4)$$

where we have introduced dimensionless Prandtl and Rayleigh numbers respectively as $P=\nu/\chi$ and $R_a=(1/C)\gamma\Delta T d^3/2\nu\chi$, C being a nondimensional scaling factor, to be defined later, that is going to be used for normalization of the stability criterion. To prove that the origin of the system (3), (4) can become unstable for sufficiently large values of the non-dimensional Rayleigh number we start by expanding the temperature in Fourier series in terms of the angle θ as $\tau(r,\theta,t)=\sum_{n=0}^{\infty}S_n(r,t)\sin(n\theta)+C_n(r,t)\cos(n\theta)$. The stability analysis that we are going to perform goes along the same lines as the one done by Wang et al. (1992) for the simplified averaged model. The only significant difference is that in our case the system matrix has differential operators at some of its entries, as opposed to real numbers in the case of simplified model analyzed by Wang et al. (1992).

Substituting $\tau(r,\theta,t)$ into the governing Eqs. (3), (4) and requiring that these equations be satisfied in the sense of weighted residuals, we obtain an infinite set of partial differential equations in t and r. Three equations will decouple from the rest of the set (ones involving $S_1(r,t)$ and $C_1(r,t)$) and can be solved independently. Linearizing those three equations in the vicinity of $(v,c,s)=(0,0,0)$ (we have labeled $S_1(r,t)$ and $C_1(r,t)$, respectively as

$s(r,t)$ and $c(r,t)$ for convenience), we obtain a decoupled system

$$\begin{bmatrix} v_t \\ c_t \end{bmatrix} = \begin{bmatrix} P\Delta_r & PR_aCI \\ \dfrac{d\pi}{2(R_1+R_2)}I & \Delta_r \end{bmatrix} \begin{bmatrix} v \\ c \end{bmatrix}, \tag{5}$$

$$s_t = \Delta_r s, \tag{6}$$

where the operator Δ_r is defined as

$$\Delta_r = -1/r^2 + 1/r\partial/\partial r + \partial^2/\partial r^2.$$

Denoting the eigenvalues of Δ_r with homogeneous Dirichlet boundary conditions by $-\lambda_i$, we find the eigenvalues μ_k of the system (5) to satisfy the equation

$$\mu_k^2 + (\lambda_j + P\lambda_i)\mu_k + P\lambda_i\lambda_j - PR_aC\frac{d\pi}{2(R_1+R_2)} = 0. \tag{7}$$

It can be shown that the eigenvalue problem for Δ_r with homogeneous Dirichlet boundary conditions is a special case of a regular Sturm–Liouville problem. As shown in Churchill and Brown (1987), it can be proven that $\lambda_i \geqslant 0$. In addition, for $\lambda = 0$ the corresponding eigenfunction is of the form $C_1 r + C_2 r^{-1}$ and can satisfy homogeneous Dirichlet boundary conditions only for the trivial case $C_1 = C_2 \equiv 0$. Therefore, we have that $\lambda_i > 0$. Using the fact, we conclude that under the given assumptions the system becomes unstable if

$$R_a > \frac{\lambda_i\lambda_j}{(d\pi/2(R_1+R_2))C}. \tag{8}$$

Since the function on the RHS of (8) reaches its minimum for $i = j = 1$, choosing $C = (2\lambda_1^2/\pi d)(R_1 + R_2)$ will normalize the stability condition (8) into $R_a > 1$. Numerically finding the first three eigenvalues of the differential operator Δ_r to be -1019.77, -4077.75, and -9174.34, we get that the number of unstable eigenvalues increases to two, three, and four when the value of the Rayleigh number becomes greater than 3.999, 8.996, and 15.989 respectively. Note that the higher the Rayleigh number of the system, the more λ_i, λ_j combinations will potentially satisfy (8), and each of those combinations satisfying (8) corresponds to an additional unstable root $\mu_k(\lambda_i, \lambda_j)$ of (7).

3. Control law

To discretize the problem, let us start by denoting $h = (R_2 - R_1)/N$ and $g = 2\pi/M$, where N and M are integers. Then, with v_i and t_{ij} respectively defined as $v_i(t) = v(R_1 + ih, t)$ and $\tau_{ij}(t) = \tau(R_1 + ih, jg, t)$, $i = 0, \dots, N$ and $j = 0, \dots, M-1$, we represent the non-dimensional system (3) and (4) as

$$\begin{aligned} \dot{v}_i = {} & \frac{g}{\pi} PR_aC \sum_{j=0}^{M-1} \tau_{i,j}\cos(jg) \\ & + P\left(-\frac{v_i}{(R_1+ih)^2} + \frac{v_{i+1} - v_{i-1}}{2(R_1+ih)h} \right. \\ & \left. + \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} \right), \end{aligned} \tag{9}$$

$$\begin{aligned} \dot{\tau}_{i,j} = {} & \frac{d\pi}{2(R_1+R_2)} v_i \cos(jg) - \frac{v_i}{R_1+ih}\frac{\tau_{i,j+1} - \tau_{i,j-1}}{2g} \\ & + \frac{\tau_{i,j+1} - 2\tau_{i,j} + \tau_{i,j-1}}{g^2(R_1+ih)^2} + \frac{\tau_{i+1,j} - \tau_{i-1,j}}{2h(R_1+ih)} \\ & + \frac{\tau_{i+1,j} - 2\tau_{i,j} + \tau_{i-1,j}}{h^2} \end{aligned} \tag{10}$$

with $v_0 = 0$ and $\tau_{0,j} = 0$ for $j = 0, \dots, M-1$. Since $v(R_2, t)$ and $\tau(R_2, \theta, t)$ are the controls in the PDE, the control inputs to the discretized system are v_N and $\tau_{N,j}$ for $j = 0, \dots, M-1$. We now suggest a backstepping controller which transforms the original system into the discretization of the system $\omega_t = P(-\omega/r^2 + \omega_r/r + \omega_{rr})$, $z_t = z_{\theta\theta}/r^2 + z_r/r + z_{rr}$ with boundary conditions $\omega(t, R_1) \equiv \omega(t, R_2) \equiv 0$, $z(t, R_1, \theta) \equiv z(t, R_2, \theta) \equiv 0$, which is asymptotically stable in L^2-norm. We should stress that the choice of the target system is one of the key issues. When transforming the original system, we should try to keep its parabolic character, i.e., keep the second spatial derivative in the transformed coordinates. Even when applied for linear parabolic PDEs, the control laws obtained using standard backstepping would have gains that grow unbounded as $n \to \infty$. The problem with standard backstepping is that it would not only attempt to stabilize the equation, but also place all of its poles, and thus as $n \to \infty$, change its parabolic character. The coordinate transformation is sought in the form

$$\omega_i = v_i - \alpha_{i-1}(v_1, \dots, v_{i-1}, \tau_{1,k=0,\dots,M-1}, \tau_{i-1,k=0,\dots,M-1}); \tag{11}$$

$$z_{i,j} = \tau_{i,j} - \beta_{i-1,j}(v_1, \dots, v_{i-1}, \tau_{1,k=0,\dots,M-1}, \tau_{i-1,k=0,\dots,M-1}), \tag{12}$$

where $\omega_i(t) = \omega(R_1 + ih, t)$, $z_{ij}(t) = z(R_1 + ih, jg, t)$, and $\omega_0 = \omega_N = z_{0,j=0,\dots,M-1} = z_{N,j=0,\dots,M-1} = 0$. The discretized form of the target system is

$$\begin{aligned} \dot{\omega}_i = {} & P\left(-\frac{\omega_i}{(R_1+ih)^2} + \frac{\omega_{i+1} - \omega_{i-1}}{2(R_1+ih)h} \right. \\ & \left. + \frac{\omega_{i+1} - 2\omega_i + \omega_{i-1}}{h^2} \right), \end{aligned} \tag{13}$$

$$\begin{aligned} \dot{z}_{i,j} = {} & \frac{z_{i,j+1} - 2z_{i,j} + z_{i,j-1}}{g^2(R_1+ih)^2} + \frac{z_{i+1,j} - z_{i-1,j}}{2h(R_1+ih)} \\ & + \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{h^2}. \end{aligned} \tag{14}$$

By combining the above expressions, namely subtracting (13) from (9), expressing the obtained equation in terms of $v_k - \omega_k,\ k = i-1, i, i+1$, and applying (11) (analogously (14) from (10) for the temperature subsystem, and then using (12)) we obtain

$$\alpha_i = \left(1+\frac{h}{2(R_1+ih)}\right)^{-1}\Bigg\{\left(2+\frac{h^2}{(R_1+ih)^2}\right)\alpha_{i-1} + \left(\frac{h}{2(R_1+ih)}-1\right)\alpha_{i-2} - \frac{g}{\pi}R_a C h^2 \sum_{j=0}^{M-1}\tau_{ij}\cos(jg) + h^2\sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial v_l}\left[\frac{g}{\pi}R_a C\sum_{m=0}^{M-1}\tau_{l,m}\cos(mg) + \frac{v_{l+1}-2v_l+v_{l-1}}{h^2} + \frac{v_{l+1}-v_{l-1}}{2(R_1+lh)h} - \frac{v_l}{(R_1+lh)^2}\right] + \frac{h^2}{P}\sum_{l=1}^{i-1}\sum_{k=0}^{M-1}\frac{\partial\alpha_{i-1}}{\partial\tau_{lk}}\left[\frac{\tau_{l+1,k}-2\tau_{l,k}+\tau_{l-1,k}}{h^2} + \frac{\tau_{l+1,k}-\tau_{l-1,k}}{2h(R_1+lh)} + \frac{\tau_{l,k+1}-2\tau_{l,k}+\tau_{l,k-1}}{g^2(R_1+lh)^2} - \frac{v_l}{R_1+lh}\frac{\tau_{l,k+1}-\tau_{l,k-1}}{2g}\frac{\mathrm{d}\pi}{2(R_1+R_2)}\cos(kg)v_l\right]\Bigg\}, \tag{15}$$

$$\beta_{i,j} = \left(1+\frac{h}{2(R_1+ih)}\right)^{-1} \times\Bigg\{2\beta_{i-1,j} + \left(\frac{h}{2(R_1+ih)}-1\right)\beta_{i-2,j} - \frac{h^2}{(R_1+ih)^2g^2}(\beta_{i-1,j+1}-2\beta_{i-1,j}+\beta_{i-1,j-1}) - \frac{h^2\mathrm{d}\pi}{2(R_1+R_2)}\cos(jg)v_i + \frac{h^2 v_i}{2(R_1+ih)g}(\tau_{i,j+1}-\tau_{i,j-1}) + h^2\sum_{l=1}^{i-1}\frac{\partial\beta_{i-1,j}}{\partial v_l}\left[\frac{g}{\pi}PR_aC\sum_{m=0}^{M-1}\tau_{l,m}\cos(mg) + P\left(\frac{v_{l+1}-2v_l+v_{l-1}}{h^2} + \frac{v_{l+1}-v_{l-1}}{2(R_1+lh)h} - \frac{v_l}{(R_1+lh)^2}\right)\right] + h^2\sum_{l=1}^{i-1}\sum_{k=0}^{M-1}\frac{\partial\beta_{i-1,j}}{\partial\tau_{lk}} \times\left[\frac{\tau_{l+1,k}-2\tau_{l,k}+\tau_{l-1,k}}{h^2} + \frac{\tau_{l+1,k}-\tau_{l-1,k}}{2h(R_1+lh)} + \frac{\tau_{l,k+1}-2\tau_{l,k}+\tau_{l,k-1}}{g^2(R_1+lh)^2} - \frac{v_l}{R_1+lh}\frac{\tau_{l,k+1}-\tau_{l,k-1}}{2g} + \frac{\mathrm{d}\pi}{2(R_1+R_2)}\cos(kg)v_l\right]\Bigg\}, \tag{16}$$

starting with $\alpha_0 = \beta_{0,j} \equiv 0$. The controls are defined as $v_N = \alpha_{N-1}$ and $\tau_{N,j} = \beta_{N-1,j}$. By inspection of the recursive control design algorithm, one can verify that the coordinate transformation is invertible (which implies global asymptotic stability of the discretized system) and that the control law is smooth.

4. Asymptotic stability of the discretized system in modified coordinates

In this section, we prove global asymptotic stability for (13) and (14) in l^2-norm with zero Dirichlet boundary conditions $\omega_0 = \omega_N = z_{0,j=0,\ldots,M-1} = z_{N,j=0,\ldots,M-1} = 0$. Note that by definition $\omega_k \equiv 0$ and $z_{k,j} \equiv 0$ if $k > N$ or $k < 0$, and that due to the periodicity in the axial direction $z_{k,M+l} = z_{k,l}$ for any $l = -(M-1),\ldots,0,\ldots,M-1$. To prove the stability of (13), we start with Lyapunov function $V_{1d} = (1/2P)\sum_{i=0}^{N}\omega_i^2$, and find its derivative with respect to time, along the trajectories of the system (13), to be

$$\dot V_{1d} = -\sum_{i=0}^{N}\frac{\omega_i^2}{(R_1+ih)^2} + \frac{1}{2}\sum_{i=0}^{N}\frac{\omega_i\omega_{i+1}}{(R_1+ih)(R_1+(i+1)h)} - \frac{1}{h^2}\sum_{i=0}^{N}(\omega_{i+1}-\omega_i)^2. \tag{17}$$

Applying discretized version of the Poincaré inequality $-\sum_{i=0}^{N}(\omega_{i+1}-\omega_i)^2 \leqslant -(2/N^2)\sum_{i=0}^{N}\omega_i^2$, identity

$$\sum_{i=0}^{N}\frac{\omega_i\omega_{i+1}}{(R_1+ih)(R_1+(i+1)h)} = \sum_{i=0}^{N}\frac{\omega_i^2}{(R_1+ih)^2} - \frac{1}{2}\sum_{i=0}^{N}\left(\frac{\omega_{i+1}}{R_1+(i+1)h} - \frac{\omega_i}{R_1+ih}\right)^2, \tag{18}$$

and substituting h in terms of N in (17) will finally give

$$\dot V_{1d} \leqslant -\frac{1}{2}\sum_{i=0}^{N}\frac{\omega_i^2}{(R_1+ih)^2} - \frac{1}{4}\sum_{i=0}^{N}\left(\frac{\omega_{i+1}}{R_1+(i+1)h} - \frac{\omega_i}{R_1+ih}\right)^2 - \frac{2}{(R_2-R_1)^2}\sum_{i=0}^{N}\omega_i^2 \leqslant -P\left(\frac{1}{R_2^2} + \frac{4}{(R_2-R_1)^2}\right)V_{1d}, \tag{19}$$

which implies that the system (13) is asymptotically stable in l^2-norm. Following an analogous approach for system (14), we take Lyapunov function candidate

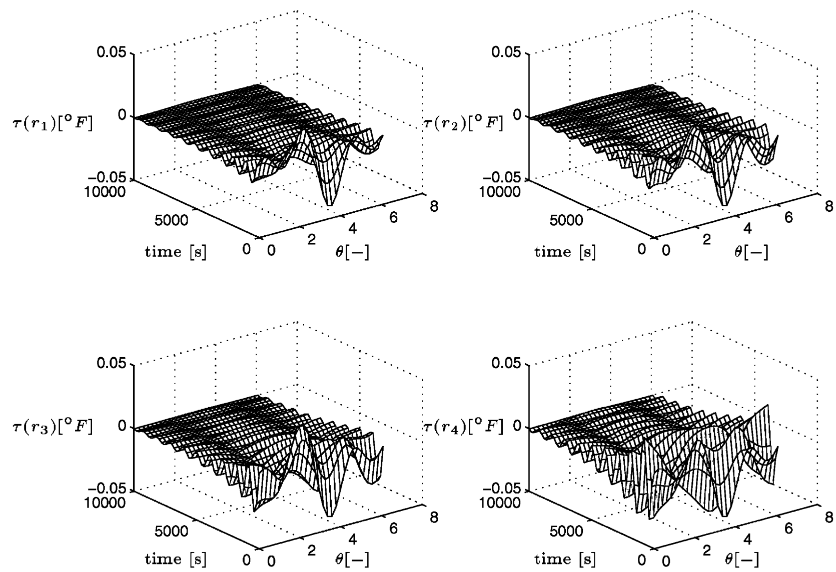

Fig. 2. Temperatures $\tau(r_i,\theta,t)$ in °F at fixed radius $r_i = R_1 + id/5$ $(i = 1,2,3,4)$ for $R_a = 11.394$ with controller designed for $N_c = 3$ and $M_c = 6$.

$V_{2d} = \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{M-1} z_{i,j}^2$ and find its derivative to satisfy

$$\dot{V}_{2d} \leqslant -\left(\frac{4}{(R_2 - R_1)^2} - \frac{1}{R_1^2}\right)V_{2d}.$$

This implies global asymptotic stability of (14) in l^2-norm since $4R_1^2 - (R_2 - R_1)^2 > R_1^2 - (R_2 - R_1)^2 > 0$ due to the narrow gap assumption. We therefore conclude that the equilibrium $(\omega_i, z_{i,j}) = (0,0)$, $i = 1,\dots,N-1$ and $j = 0,\dots,M-1$, of the system (13), (14) with boundary conditions $\omega_0 = \omega_N = z_{0,j=0,\dots,M-1} = z_{N,j=0,\dots,M-1} = 0$ is globally asymptotically stable in l^2-norm.

5. Simulation study

In this section, we present simulation results for the narrow gap convection loop consisting of water confined in between two long cylinders standing in a vertical plane. The simulation setup is the same as one used by Burns et al. (1999) (water at 60°F), with system parameters given as $R_1 = 1.1975$ ft, $R_2 = 1.2959$ ft, $\beta = 8\times10^{-5}\,1/^\circ$F, $\chi = 1.514\times10^{-6}$ ft$^2/s$, and $v = 1.22\times10^{-5}$ ft$^2/s$. All simulations are run with the same initial distribution in velocity and temperature $v(r,0) = 0$, $\tau(r,0) = 0.01(\sin\theta + \cos\theta - \sin 2\theta - \cos 2\theta + \sin 3\theta + \cos 3\theta)$, in ft/$s$ and °F respectively, using BTCS finite difference method for $N = 30$, $M = 180$ and the time step equal to 0.1 s. As shown in Section 3, control laws for velocity (15) and temperature (16) are given in a recursive form that can be easily applied using symbolic tools available. Once the final expressions for velocity and temperature control are obtained, for some particular choice of N and M, one would have to use full state feedback to stabilize the system, i.e. the complete knowledge of velocity and temperature fields is necessary. Instead, we show that controllers of relatively low order (designed on a much coarser grid) can successfully stabilize the system for a wide range of nondimensional Rayleigh number. In general, simulation results suggest that to accommodate the flows with higher Rayleigh number one would have to increase the order of controller by applying recursive expressions (15) and (16) for higher N_c and M_c, where the subscripts "c" stand for controller. From now on we will use N_c and M_c to refer to a coarse grid discretization used in controller design, and N and M to refer to a fine grid used to simulate the behavior of the system described by Eqs. (9) and (10).

Although we have designed and tested controllers for both $N_c = 2$ and $N_c = 3$, and general M_c, we only present results for the latter case and go briefly over the results for $N_c = 2$. The controller designed for $N_c = 2$ was capable of stabilizing the system with one unstable eigenvalue with $M_c = 2$, and the system with two unstable eigenvalues with $M_c = 4$. For the case of three unstable eigenvalues we could not stabilize the system even with $M_c = M = 180$.

We now proceed to deriving control laws for $N_c = 3$ and general M_c by introducing $h_c = (R_2 - R_1)/N_c$ and $g_c = 2\pi/M_c$. Starting with $\alpha_0 = \beta_{0,j} \equiv 0$ and using (15) and (16) we find expressions for α_1, α_2, $\beta_{1,j}$, and $\beta_{2,j}$, where α_2 and $\beta_{2,j}$ are used as controls. The control signals are dependent on $v_i(t) = v(R_1 + ih_c, t)$, and $\tau_{ij}(t) = \tau(R_1 + ih_c, jg_c, t)$ for $i = 1,2$ and $j = 0,\dots,M_c - 1$ only, which means that we use only two velocity measurements inside the channel (v_1 at $R_1 + d/3$ and

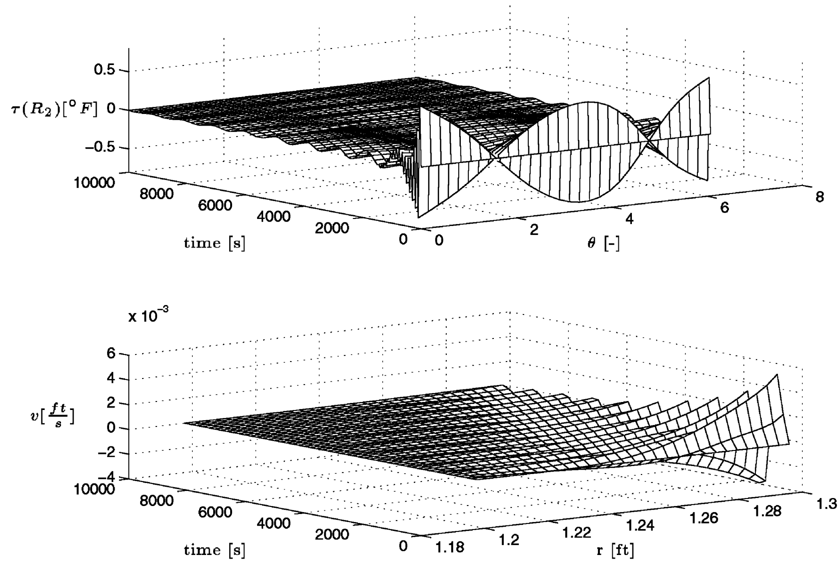

Fig. 3. Temperature boundary control $\tau(R_2,\theta,t)$ and velocity $v(r,t)$ for $R_a = 11.394$ with controller designed for $N_c = 3$ and $M_c = 6$.

v_2 at $R_1 + 2d/3$) and corresponding $2M_c$ temperature measurements ($\tau_{1,j=0,\dots,M_c-1}$ and $\tau_{2,j=0,\dots,M_c-1}$) to compute control laws. The temperature control law is computed directly for M_c equidistant points along the circumference of the outer cylinder, and using cubic spline interpolation for the remaining points.

As expected, by refining the grid in controller design from $N_c = 2$ to $N_c = 3$ we were able to extend the range of the nondimensional Rayleigh number for which we can stabilize the system. We are now able to stabilize the system with three unstable eigenvalues ($R_a = 11.394$) with $M_c = 6$. The temperature evolution in time at fixed distance r_i from the center of the loop ($r_i = R_1 + id/5$, $i = 1,2,3,4$) is shown in Fig. 2. The velocity $v(r,t)$ and temperature control $\tau(R_2,\theta,t)$ are shown in Fig. 3. The corresponding temperature and velocity responses for the uncontrolled case are respectively of the order of 0.3°F and 0.01 ft/s even after 10 000 s. Although we do not show those simulation results, the proposed control law was capable of stabilizing the "less critical" cases with one and two unstable eigenvalues with smaller control effort in less than 5000 s with $M_c = 6$.

If we try to further increase the Rayleigh number to $R_a = 19.43$, which corresponds to four unstable eigenvalues, we see that the controller designed with $N_c = 3$ cannot stabilize the origin even with $M_c = M = 180$. However, the controller does reduce the state error, especially in the middle of the loop.

To summarize, controller designed for $N_c = 2$ was able to stabilize up to two unstable eigenvalues, and the one designed for $N_c = 3$ up to three unstable eigenvalues. Without any intention to state it as a "proven" fact, we notice that a pattern of a certain kind is repeating for both $N_c = 2$ and $N_c = 3$ cases.

6. Conclusions

A nonlinear feedback controller based on Lyapunov backstepping design that achieves global asymptotic stabilization of the unstable no-motion steady state for a 2D thermal convection loop has been derived. The result holds for any finite discretization in space of the original PDE model.

The simulation study indicates that the feedback control laws designed on a very coarse grid can be successfully used to sustain the no-motion steady state well beyond the critical Rayleigh number associated with the onset of instability in the uncontrolled system.

Several key questions present a challenge for future research. It would be of interest to extend this result from the case of an arbitrary finite discretization of the model in space to the continuous model itself. This would, among other things, involve the proof that the proposed coordinate transformation remains bounded in the limit when the spatial grid becomes infinitely fine, i.e. when N and M tend to infinity. Another key question regarding the applicability of the proposed approach is how one would measure the velocity/temperature field in a flow domain. In a CFD setting this would, of course, not be a problem. However, in experimental application, one would typically have access only to quantities near the wall. This would make it necessary to develop observers and output-feedback control design for these flow models.

Acknowledgements

This work was supported by grants from AFOSR, ONR, and NSF. The authors would also like to thank

Andras Balogh, Peter Blossey, and Tom Bewley for their valuable feedback.

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