

The sets of events that operators 1 and 2 can trigger are, respectively,  $\Sigma_1^t = \{\alpha_1, \beta_1, \gamma_1\}$  and  $\Sigma_2^t = \{\alpha_2, \beta_2, \gamma_2\}$ . Similarly, the sets of events operators 1 and 2 can disable are, respectively,  $\Sigma_1^d = \{\alpha_1, \beta_1, \gamma_1\}$  and  $\Sigma_2^d = \{\alpha_2, \beta_2, \gamma_2, \gamma_1\}$ .

If user 1 wants to insure safety (i.e., prevent the system from reaching any of the illegal states) and user 2 is not cooperative, then  $\Sigma_{s_1} = \Sigma_1^d$ . The legal language in this case is not controllable and the supremal controllable sublanguage is empty. However, if user 2 is cooperative, then  $\Sigma_{s_1} = \Sigma$ , and the legal language is controllable. In contrast, If user 2 wants to insure safety, then  $\Sigma_{s_2} = \Sigma_2^d$  in case user 1 is uncooperative, and  $\Sigma_{s_2} = \Sigma$  in case user 1 is cooperative. In both cases the legal language is controllable.

Let us now assume that the initial state IEI is the only marked state of the system, and that in addition to safety, the controller must satisfy the liveness condition specified by the marked language that consists of all the event strings that lead the system to this marked state.

Let us now consider the two-user control problem with the requirement that both safety and liveness must be satisfied. In case user 1 wants to achieve safety and liveness, only the situation where user 2 is cooperative with respect to safety is relevant. Let us further assume that user 2 is cooperative also with respect to liveness, in which case,  $EA = \emptyset$  and  $FA = \{1, 2\}$ .

In case 1 (where  $\Sigma_{l_i} = \Sigma_i^t - (\cup_{j \neq i} \Sigma_j^d)$ ), we obtain  $\Sigma_{l_1} = \{\alpha_1, \beta_1\}$ . By our synthesis algorithm, the resulting safe and live system consists of states IEI and REI.

In case 2 (where  $\Sigma_{l_i} = \Sigma_i^t - (\cup_{j \in EA} \Sigma_j^d)$ ), we obtain  $\Sigma_{l_1} = \{\alpha_1, \beta_1, \gamma_1\}$ . By our synthesis algorithm, the resulting safe and live system consists of states IEI and REI.

In case 3 (where  $\Sigma_{l_i} = (\cup_{i \in FA} \Sigma_i^t) - (\cup_{j \in EA} \Sigma_j^d)$ ), we obtain  $\Sigma_{l_1} = \Sigma$ . By our synthesis algorithm, the resulting safe and live system consists of all the legal states.

In case 4 (where  $\Sigma_{l_i} = \Sigma$ ), we obtain  $\Sigma_{l_1} = \Sigma$ . By our synthesis algorithm, the resulting safe and live system consists of all the legal states.

In a similar fashion, we can discuss how user 2 can achieve safety and liveness.

## VII. CONCLUSION

We have introduced an extended framework for discrete-event control where, in addition to the events that can be triggered by the environment, the user has at his/her disposal a set of events that he/she can trigger. Both the user and the environment can each disable certain events of the other. We examined the control problem where both safety and liveness requirements can be specified in a somewhat more general setting than in the traditional discrete-event control framework. A particularly interesting generalization is obtained when the environment consists of (or includes) one or more additional users. This leads to a variety of interesting scenarios where the users have each their own control objectives (specifications) and capabilities.

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## Extremum Seeking Control for Discrete-Time Systems

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**Abstract**—We present an extremum seeking control algorithm for discrete-time systems applied to a class of plants that are represented as a series combination of a linear input dynamics, a static nonlinearity with an extremum, and a linear output dynamics. By using the two-time scale averaging theory, we derive a mild sufficient condition under which the plant output exponentially converges to an  $O(\alpha^2)$  neighborhood of the extremum value, where  $\alpha$  is the magnitude of modulation signal. The sufficient condition is related to positive realness of linear parts of the plant but only at the modulation frequency. The algorithm is illustrated with a brief simulation study.

**Index Terms**—Averaging, discrete-time systems, extremum seeking.

## I. INTRODUCTION

Extremum seeking, a nonmodel based method of adaptive control, deals with systems where the reference-to-output map is uncertain but is known to have an extremum. The objective of extremum seeking is to find the set point that achieves the extremum.

Krstić and Wang [1] presented the first stability analysis for an extremum seeking system applied to a general nonlinear dynamical plant. Their analysis used averaging and singular perturbations. Krstić [2]

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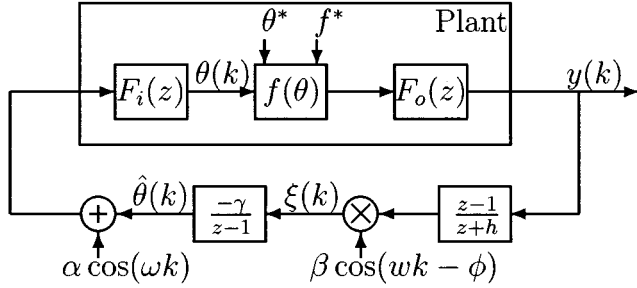


Fig. 1. Extremum seeking control scheme for discrete-time systems.

presented a tighter linearized analysis and proposed dynamic compensation for providing stability guarantees and fast tracking of changes in the operating point. Several valuable extensions of extremum seeking followed [3]–[6].

In this note, we present an extremum seeking scheme for discrete-time systems. The plant model and control algorithm have the same structure as in [2]. Nevertheless, it turns out that the stability analysis of the discrete-time case is quite different from that of the continuous-time case. By applying the two-time scale averaging theory [7] to stability analysis in the discrete time case, we derive a sufficient condition under which the plant output exponentially converges to an  $O(\alpha^2)$  neighborhood of the extremum value, where  $\alpha$  is the magnitude of modulation signal.

This note is organized as follows. Section II describes the discrete-time extremum seeking algorithm. Section III gives several preliminary lemmas on linear time-periodic systems. Section IV organizes the equations of the closed-loop system in a way convenient for stability analysis. Section V states and proves stability, and derives ultimate bounds on error signals. Section VI provides simulation results and discussions. Section VII offers conclusions.

## II. DISCRETE-TIME EXTREMUM SEEKING CONTROL

The implementation is depicted in Fig. 1 and takes the same structure as that of continuous time extremum seeking algorithm in [2]. Both of the linear blocks,  $F_i(z)$  and  $F_o(z)$ , are required to be exponentially stable. The high-pass filter  $((z-1)/(z+h))$  is designed as  $0 < h < 1$ , and the modulation frequency  $\omega$  is selected such that  $\omega = a\pi$ ,  $0 < |a| < 1$ , and  $a$  is rational. Without loss of generality, the static nonlinear block  $f(\theta)$  is assumed to have a minimum at  $\theta = \theta^*$ , and to be of the form

$$f(\theta) = f^* + (\theta - \theta^*)^2. \quad (1)$$

Cubic and higher order terms are omitted for notational convenience as they are negligible in local stability analysis via averaging.

## III. PRELIMINARY LEMMAS

In the subsequent discussion, the following notation and definitions will be used. A transfer function in front of a bracketed time function, such as  $G(z)[u(k)]$ , means a time-domain signal obtained as an output of  $G(z)$  driven by  $u(k)$ . For a square matrix  $A$ ,  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  are the smallest and the largest eigenvalues, respectively.  $\varepsilon^{-k}$  denotes exponentially decaying terms. The following lemmas are used to facilitate the extremum seeking system analysis.

*Lemma 1:* If the transfer function  $H(z)$  has all of its poles inside the unit circle and real-valued impulse response, then, for any real  $\psi$

$$\begin{aligned} H(z)[\cos(\omega k - \psi)] &= \text{Re} \left\{ H(e^{j\omega}) e^{j(\omega k - \psi)} \right\} + \varepsilon^{-k} \\ &= |H(e^{j\omega})| \cos(\omega k - \psi + \psi_H) + \varepsilon^{-k} \end{aligned}$$

where  $\psi_H = \angle(H(e^{j\omega}))$ .

*Lemma 2:* If the transfer functions  $G(z)$  and  $H(z)$  have all of their poles inside the unit circle, the following statement is true for any real  $\phi$  and any uniformly bounded  $v(k)$ :

$$\begin{aligned} G(z)[(H(z)[\cos(\omega k - \phi)])v(k)] \\ = \text{Re} \left\{ e^{j(\omega k - \phi)} H(e^{j\omega}) G(e^{j\omega}) [v(k)] \right\} + \varepsilon^{-k}. \quad (2) \end{aligned}$$

*Proof:* See the Appendix.  $\square$

*Lemma 3:* For any two rational functions  $A(\cdot)$  and  $B(\cdot, \cdot)$ , the following is true:

$$\begin{aligned} \text{Re} \left\{ e^{j(\omega k - \psi)} A(e^{j\omega}) \right\} \text{Re} \left\{ e^{j(\omega k - \phi)} B(z, e^{j\omega}) [v(k)] \right\} \\ = \frac{1}{2} \text{Re} \left\{ e^{j(\psi - \phi)} A(e^{-j\omega}) B(z, e^{j\omega}) [v(k)] \right\} \\ + \frac{1}{2} \text{Re} \left\{ e^{j(2\omega k - \psi - \phi)} A(e^{j\omega}) B(z, e^{j\omega}) [v(k)] \right\}. \end{aligned}$$

*Lemma 4:* For any rational function  $B(\cdot, \cdot)$ , the following is true:

$$\begin{aligned} \text{Re} \left\{ e^{j(\omega k - \phi)} B(z, e^{j\omega}) [v(k)] \right\} \\ = \cos(\omega k - \phi) \text{Re} \{ B(z, e^{j\omega}) [v(k)] \} \\ - \sin(\omega k - \phi) \text{Im} \{ B(z, e^{j\omega}) [v(k)] \}. \end{aligned}$$

*Lemma 5:* Suppose that the transfer functions  $H(z)$  and  $G(z)$  have all of their poles inside the unit circle, and have minimal state space realizations  $(A_1, B_1, C_1, D_1)$  and  $(A_2, B_2, C_2, D_2)$ , respectively. Then,  $G(z)[\cos(\omega k - \phi)H(z)[v(k)]]$  can be represented in a state-space form as

$$\begin{aligned} (A(k), B(k), C(k), D(k)) \\ = \left( \left[ \begin{array}{c|c} \frac{A_1}{c(\omega k)B_2C_1} & \mathbf{0} \\ \hline & A_2 \end{array} \right], \left[ \begin{array}{c} B_1 \\ c(\omega k)B_2D_1 \end{array} \right], \right. \\ \left. \left[ c(\omega k)D_2C_1 \mid C_2 \right], c(\omega k)D_2D_1 \right) \end{aligned}$$

where  $c(\omega k) = \cos(\omega k - \phi)$  and  $A(k)$  is exponentially stable.

*Proof:* See the Appendix.  $\square$

## IV. CLOSED-LOOP SYSTEM

The extremum seeking system depicted in Fig. 1 is governed by the following equations:

$$y(k) = F_o(z)[f^* + (\theta(k) - \theta^*)^2] \quad (3)$$

$$\theta(k) = F_i(z) \left[ \alpha \cos(\omega k) - \frac{\gamma}{z-1} [\xi(k)] \right] \quad (4)$$

$$\xi(k) = \beta \cos(\omega k - \phi) \frac{z-1}{z+h} [y(k)]. \quad (5)$$

For the convenience of analysis, the following terms are defined:

$$\theta_0(k) = F_i(z)[\alpha \cos(\omega k)] \quad (6)$$

$$\tilde{\theta}(k) = \theta^* - \theta(k) + \theta_0(k) \quad (7)$$

$$\tilde{y}(k) = y(k) - F_o(z)[f^*] \quad (8)$$

where  $\tilde{\theta}(k)$  is the tracking error and  $\tilde{y}(k)$  is the output error. Substitution of (4) and (6) in (7) yields

$$\tilde{\theta}(k) = \theta^* + \frac{\gamma}{z-1} F_i(z)[\xi(k)]$$

which can be transformed into a difference equation

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + \gamma F_i(z)[\xi(k)].$$

Further, substitution for  $\xi$  from (5) and for  $y$  from (3) yields

$$\begin{aligned} \tilde{\theta}(k+1) - \tilde{\theta}(k) \\ = \gamma F_i(z) \left[ \beta c(\omega k) \frac{z-1}{z+h} F_o(z) [f^* + (\theta - \theta^*)^2] \right] \end{aligned}$$

where  $c(\omega k) = \cos(\omega k - \phi)$ . Using  $\theta - \theta^* = \theta_0 - \tilde{\theta}$  by rearrangement of (7), we obtain

$$\begin{aligned} & \tilde{\theta}(k+1) - \tilde{\theta}(k) \\ &= \gamma F_i(z) \left[ \beta c(\omega k) \frac{z-1}{z+h} F_o(z) [f^* + (\theta_0 - \tilde{\theta})^2] \right] \\ &= \epsilon F_i(z) \left[ c(\omega k) \frac{z-1}{z+h} F_o(z) [\tilde{\theta}^2 - 2\theta_0 \tilde{\theta}] \right] \\ &+ \epsilon F_i(z) \left[ c(\omega k) \frac{z-1}{z+h} F_o(z) [f^* + \theta_0^2] \right] \end{aligned} \quad (9)$$

where  $\epsilon = \gamma\beta$ . Applying the modulation Lemmas 2, 3, 4 in succession to the term containing  $2\theta_0\tilde{\theta}$  in (9), we obtain

$$\begin{aligned} & \epsilon F_i(z) \left[ c(\omega k) \frac{z-1}{z+h} F_o(z) [-2\theta_0 \tilde{\theta}] \right] \\ &= -2\alpha \epsilon F_i(z) \left[ \mathbf{Re} \left\{ e^{j(\omega k - \phi)} \right\} \frac{z-1}{z+h} \right. \\ &\quad \times F_o(z) [F_i(z) [\cos(\omega k)] \tilde{\theta}] \left. \right] \quad (\text{from (6)}) \\ &= -2\alpha \epsilon F_i(z) \left[ \mathbf{Re} \left\{ e^{j(\omega k - \phi)} \right\} \right. \\ &\quad \times \mathbf{Re} \left\{ e^{j\omega k} M(z, e^{j\omega}) [\tilde{\theta}] \right\} \left. \right] \quad (\text{from Lemma 2}) \\ &= \alpha \epsilon F_i(z) [s(2\omega k) \mathbf{Im} \{ M(z, e^{j\omega}) [\tilde{\theta}] \} \\ &\quad - c(2\omega k) \mathbf{Re} \{ M(z, e^{j\omega}) [\tilde{\theta}] \} \\ &\quad - \alpha \epsilon F_i(z) [\mathbf{Re} \{ e^{j\phi} M(z, e^{j\omega}) [\tilde{\theta}] \} + \varepsilon^{-k}], \\ & \quad (\text{from Lemma 3 and 4}) \end{aligned} \quad (10)$$

where  $M(z, e^{j\omega}) = F_i(e^{j\omega})((e^{j\omega}z - 1)/(e^{j\omega}z + h))F_o(e^{j\omega}z)$ ,  $s(2\omega k) = \sin(2\omega k - \phi)$  and  $c(2\omega k) = \cos(2\omega k - \phi)$ . Finally, substituting (10) in (9), we obtain the whole closed-loop system

$$\tilde{\theta}(k+1) - \tilde{\theta}(k) = \epsilon(L(z)[\tilde{\theta}] + \Phi_1(k) + \Phi_2(k)) + \delta(k) \quad (11)$$

where

$$\begin{aligned} L(z) &= -\frac{\alpha}{2} F_i(z) (e^{j\phi} M(z, e^{j\omega}) + e^{-j\phi} M(z, e^{-j\omega})) \\ \Phi_1(k) &= \alpha F_i(z) [s(2\omega k) \mathbf{Im} \{ M(z, e^{j\omega}) [\tilde{\theta}] \} \\ &\quad - c(2\omega k) \mathbf{Re} \{ M(z, e^{j\omega}) [\tilde{\theta}] \}] \\ \Phi_2(k) &= F_i(z) \left[ c(\omega k) \frac{z-1}{z+h} F_o(z) [\tilde{\theta}^2] \right] \\ \delta(k) &= \epsilon F_i(z) \left[ c(\omega k) \frac{z-1}{z+h} F_o(z) [f^* + \theta_0^2] + \varepsilon^{-k} \right]. \end{aligned}$$

The various terms in (11) can be characterized in view of  $\tilde{\theta}$  as follows:  $L(z)[\tilde{\theta}]$  is the linear time-invariant;  $\Phi_1(k)$  is linear time-varying;  $\Phi_2(k)$  is nonlinear time-varying; and  $\delta(k)$  is time-varying, independent of  $\tilde{\theta}$ , and found to satisfy the following property.

*Lemma 6:*  $\delta(k)$  exponentially converges to an  $O(\epsilon\alpha^2)$  neighborhood of zero:

$$|\delta(k)| \leq \varepsilon^{-k} + \kappa_1 \epsilon \alpha^2 \quad (12)$$

where  $\kappa_1$  is a constant.

*Proof:* See the Appendix.  $\square$

From Lemma 6, it is clear that the bound on  $\delta(k)$  can be adjusted by the magnitude of the modulation signal  $\alpha$  independently of  $\epsilon$ . By exploiting this property of  $\delta(k)$ , we present a stability analysis for the system (11) with two steps. At first, regarding  $\delta(k)$  as a perturbation, we analyze the system (11) without  $\delta(k)$ . Then, we consider the whole system including  $\delta(k)$ .

## V. STABILITY ANALYSIS

First, we consider the homogeneous part of the  $\tilde{\theta}$ -error system (11)

$$\tilde{\theta}(k+1) - \tilde{\theta}(k) = \epsilon(L(z)[\tilde{\theta}] + \Phi_1(k) + \Phi_2(k)) \quad (13)$$

which depends on time  $k$  periodically. The following theorem presents a sufficient condition under which the  $\tilde{\theta}$ -error system (13) is locally exponentially stable at the origin:

*Theorem 1:* If  $F_i(1) \mathbf{Re} \{ e^{j\phi} F_i(e^{j\omega}) ((e^{j\omega} - 1)/(e^{j\omega} + h)) F_o(e^{j\omega}) \} > 0$ , then there exists a positive constant  $\epsilon^*$  such that the state-space realization of the  $\tilde{\theta}$ -error system (13) is locally exponentially stable at the origin for all  $0 < \epsilon (= \gamma\beta) \leq \epsilon^*$ .

*Proof:* In order to prove this theorem, we employ the two-time scale averaging theory rather than the one-time scale averaging theory because the right-hand side of (13) is related to dynamics with the inputs  $\tilde{\theta}(k)$  and  $\tilde{\theta}^2(k)$  [7]. Since  $\Phi_1(k)$  and  $\Phi_2(k)$  in (13) take the same structure as  $G(z)[\cos(\omega k - \phi)H(z)[v(k)]]$  in Lemma 5, we can choose minimal state space realizations of  $L(z)$ ,  $\Phi_1(k)$ , and  $\Phi_2(k)$  as  $(A_1, B_1, C_1, D_1)$ ,  $(A_2(k), B_2(k), C_2(k), D_2(k))$ , and  $(A_3(k), B_3(k), C_3(k), D_3(k))$ , respectively. Moreover, since all of the poles in  $L(z)$ ,  $\Phi_1(k)$ , and  $\Phi_2(k)$  are inside the unit circle,  $A_1$ ,  $A_2(k)$ , and  $A_3(k)$  are exponentially stable. Now, the  $\tilde{\theta}$ -error system (13) can be transformed into a state-space form

$$x'(k+1) = A(k)x'(k) + h(k, \tilde{\theta}(k)) \quad (14)$$

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + \epsilon f'(k, \tilde{\theta}(k), x'(k)) \quad (15)$$

where

$$A(k) = \begin{bmatrix} A_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2(k) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_3(k) \end{bmatrix}$$

$h(k, \tilde{\theta}(k)) = [B_1^T \tilde{\theta} | B_2^T(k) \tilde{\theta} | B_3^T(k) \tilde{\theta}^2]^T$ , and  $f'(k, \tilde{\theta}(k), x'(k)) = D_1 \tilde{\theta} + D_2 \tilde{\theta} + D_3 \tilde{\theta}^2 + [C_1 | C_2(k) | C_3(k)] x'(k)$ . Since  $A(k)$  is exponentially stable, the state space form (14) and (15) is adequate for the application of the two-time scale averaging theory [7]. Define the function

$$w(k, \tilde{\theta}) = \sum_{i=0}^{k-1} \Psi(k, i+1) h(i, \tilde{\theta})$$

where  $\Psi(k, i) = \prod_{l=i}^{k-1} A(l+k-1-l)$ , and construct the transformation

$$x(k) = x'(k) - w(k, \tilde{\theta}).$$

Then, the transformed system is represented as

$$x(k+1) = A(k)x(k) + \epsilon g(k, \tilde{\theta}, x) \quad (16)$$

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + \epsilon f(k, \tilde{\theta}, x) \quad (17)$$

where

$$\begin{aligned} & \epsilon g(k, \tilde{\theta}, x) \\ &= A(k)w(k, \tilde{\theta}(k)) + h(k, \tilde{\theta}(k)) - w(k+1, \tilde{\theta}(k+1)) \\ &= w(k+1, \tilde{\theta}(k)) - w(k+1, \tilde{\theta}(k+1)) \\ &= - \left( \int_0^1 \frac{\partial w}{\partial \tilde{\theta}}(k+1, s\tilde{\theta}(k+1) + (1-s)\tilde{\theta}(k)) ds \right) \\ &\quad \cdot \epsilon f'(k, \tilde{\theta}, x + w(k, \tilde{\theta})) \\ & f(k, \tilde{\theta}, x) = f'(k, \tilde{\theta}(k), x + w(k, \tilde{\theta})). \end{aligned}$$

The averaged system of (17) is defined by

$$\tilde{\theta}_{\text{av}}(k+1) = \tilde{\theta}_{\text{av}}(k) + \epsilon f_{\text{av}}(\tilde{\theta}_{\text{av}}(k)) \quad (18)$$

where  $f_{\text{av}}$  is calculated by the averaging operator  $\text{AVG}\{\cdot\}$  [7] defined as

$$\begin{aligned} f_{\text{av}}(\tilde{\theta}) &= \text{AVG}\{f(k, \tilde{\theta}, 0)\} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{k=s+1}^{s+T} f(k, \tilde{\theta}, 0). \end{aligned}$$

On the other hand,  $f(k, \tilde{\theta}, 0)$  can be reconverted into  $\mathcal{Z}$ -domain as follows:

$$\begin{aligned} f(k, \tilde{\theta}, 0) &= f'(k, \tilde{\theta}, w(k, \tilde{\theta})) \\ &= D_1 \tilde{\theta} + D_2 \tilde{\theta} + D_3 \tilde{\theta}^2 + [C_1 | C_2(k) | C_3(k)] \\ &\quad \times \sum_{i=0}^{k-1} \Psi(k, i+1) \left[ B_1^T \tilde{\theta} | B_2^T(i) \tilde{\theta} | B_3^T(i) \tilde{\theta}^2 \right]^T \\ &= L(z)[\tilde{\theta}] + \Phi_1(k) + \Phi_2(k) \end{aligned}$$

where  $\tilde{\theta}$  is regarded as a constant. Hence  $f_{\text{av}}(\tilde{\theta})$  can be reformulated as

$$f_{\text{av}}(\tilde{\theta}) = \text{AVG}\{L(z)[\tilde{\theta}] + \Phi_1(k) + \Phi_2(k)\}. \quad (19)$$

Using Lemma 1 and regarding  $\tilde{\theta}$  as a constant lead to the following derivations:

$$\begin{aligned} &\text{AVG}\{\Phi_1(k)\} \\ &= \text{AVG}\{aF_i(z)[s(2\omega k) \mathbf{Im}\{M(z, e^{j\omega})[u_s(k)]\} \\ &\quad - c(2\omega k) \mathbf{Re}\{M(z, e^{j\omega})[u_s(k)]\}] \tilde{\theta}\} = 0 \\ &\text{AVG}\{\Phi_2(k)\} \\ &= \text{AVG}\left\{F_i(z) \left[ c(\omega k) \frac{z-1}{z+h} F_o(z)[u_s(k)] \right] \tilde{\theta}^2 \right\} = 0 \end{aligned}$$

where  $u_s(k)$  denotes the unit step sequence,  $s(2\omega k) = \sin(2\omega k - \phi)$ ,  $c(2\omega k) = \cos(2\omega k - \phi)$ , and  $c(\omega k) = \cos(\omega k - \phi)$ . Hence

$$\begin{aligned} f_{\text{av}}(\tilde{\theta}) &= \text{AVG}\{L(z)[\tilde{\theta}]\} \\ &= \text{AVG}\left\{ -\frac{\alpha}{2} F_i(z)(e^{j\omega} M(z, e^{j\omega}) \right. \\ &\quad \left. + e^{-j\omega} M(z, e^{-j\omega})) [u_s(k)] \tilde{\theta} \right\} \\ &= -\kappa_2 \alpha \tilde{\theta} \end{aligned} \quad (20)$$

where  $\kappa_2 = (1/2)F_i(1) \mathbf{Re}\{e^{j\phi} F_i(e^{j\omega})((e^{j\omega} - 1)/(e^{j\omega} + h)) F_o(e^{j\omega})\} = (1/2)F_i(1)|F_i(e^{j\omega})((e^{j\omega} - 1)/(e^{j\omega} + h))F_o(e^{j\omega})| \cos(\psi_M + \phi)$  and  $\psi_M = \angle(F_i(e^{j\omega})((e^{j\omega} - 1)/(e^{j\omega} + h))F_o(e^{j\omega}))$ . Substituting (20) into (18) results in the averaged system

$$\tilde{\theta}_{\text{av}}(k+1) = (1 - \kappa_2 \epsilon \alpha) \tilde{\theta}_{\text{av}}(k)$$

where if  $\kappa_2 > 0$ ,  $\tilde{\theta}_{\text{av}}$  is exponentially stable for all  $0 < \epsilon < (2/\kappa_2 \alpha)$ . Consequently, according to Theorem 2.2.4 in [7], this theorem is proved.  $\square$

It is observed from the sufficient condition of Theorem 1 that the local exponential stability of (13) is closely related to positive realness of linear parts of the plant but only at the modulation frequency  $\omega$ . This is a very mild condition.

Now, we consider the stability of the overall system (11). For this purpose, it is necessary to investigate the perturbed averaged system

$$\tilde{\theta}_{\text{av}}(k+1) = (1 - \kappa_2 \epsilon \alpha) \tilde{\theta}_{\text{av}}(k) + \delta(k). \quad (21)$$

Since  $|\delta(k)| \leq \epsilon^{-k} + \kappa_1 \epsilon \alpha^2$  from the Lemma 6, it is obvious that  $\tilde{\theta}_{\text{av}}(k)$  in (21) exponentially converges to an  $O(\alpha)$  neighborhood of

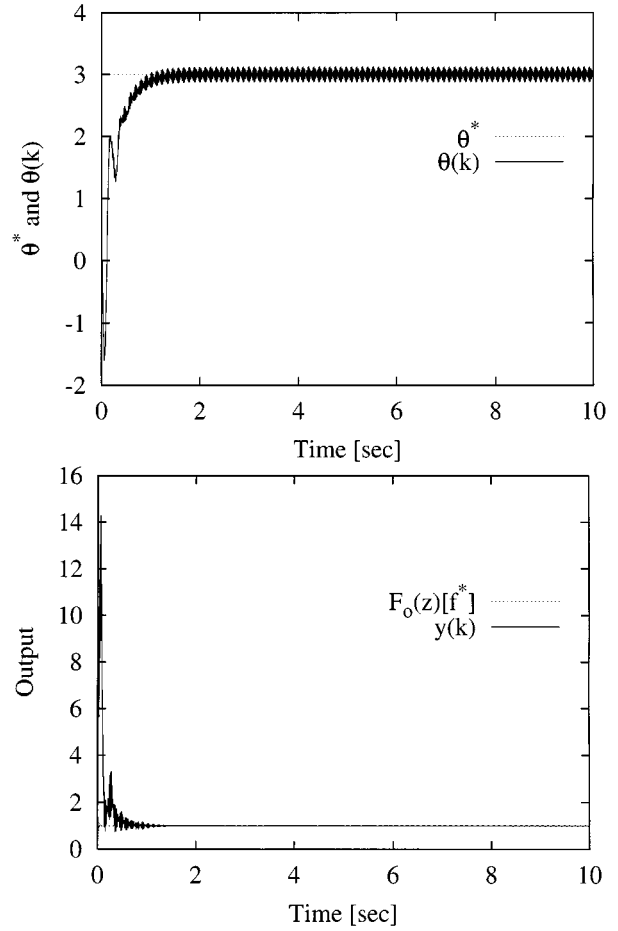


Fig. 2. Responses for  $\omega = (\pi/1.1)$  rad/sample and  $\gamma = 0.6$ .

zero. On the other hand, it is known from [7] and [8] that the exponential convergence rate of  $\tilde{\theta}$  in the original system (11) tends to that of  $\tilde{\theta}_{\text{av}}$  in the averaged system, as  $\epsilon$  tends to zero. Therefore, we can conclude the following theorem.

**Theorem 2:** Suppose that the conditions of Theorem 1 are satisfied. Then, for sufficiently small  $\alpha$ , there exists  $\epsilon_1^*, 0 < \epsilon_1^* \leq \epsilon^*$ , such that  $\tilde{\theta}$  in the original system (11) locally exponentially converges to an  $O(\alpha)$  neighborhood of zero for all  $0 < \epsilon \leq \epsilon_1^*$ .

With the result of Theorem 2, the convergence property of the output error  $\tilde{y}(k)$  is described as:

**Corollary 1:** Under the conditions of Theorem 2, the output error  $\tilde{y}(k)$  defined in (8) locally exponentially converges to an  $O(\alpha^2)$  neighborhood of zero.

*Proof:* We have that

$$\tilde{y}(k) = F_o(z)[(\tilde{\theta} - \theta^*)^2] = F_o(z)[(\tilde{\theta} - \theta_0)^2] \quad (22)$$

where  $\tilde{\theta}$  locally exponentially converges to an  $O(\alpha)$  neighborhood of zero from Theorem 2 and  $\theta_0$  exponentially converges to an  $O(\alpha)$  neighborhood of zero. Hence,  $\tilde{y}(k)$  locally exponentially converges to an  $O(\alpha^2)$  neighborhood of zero.  $\square$

## VI. SIMULATION RESULTS

In order to test the feasibility of the proposed extremum seeking algorithm, we conduct a simulation study for a plant with the transfer functions

$$F_i(z) = \frac{z + 0.4}{(z + 0.5)(z + 0.6)} \quad \text{and} \quad F_o(z) = \frac{z - 0.2}{z + 0.6}.$$

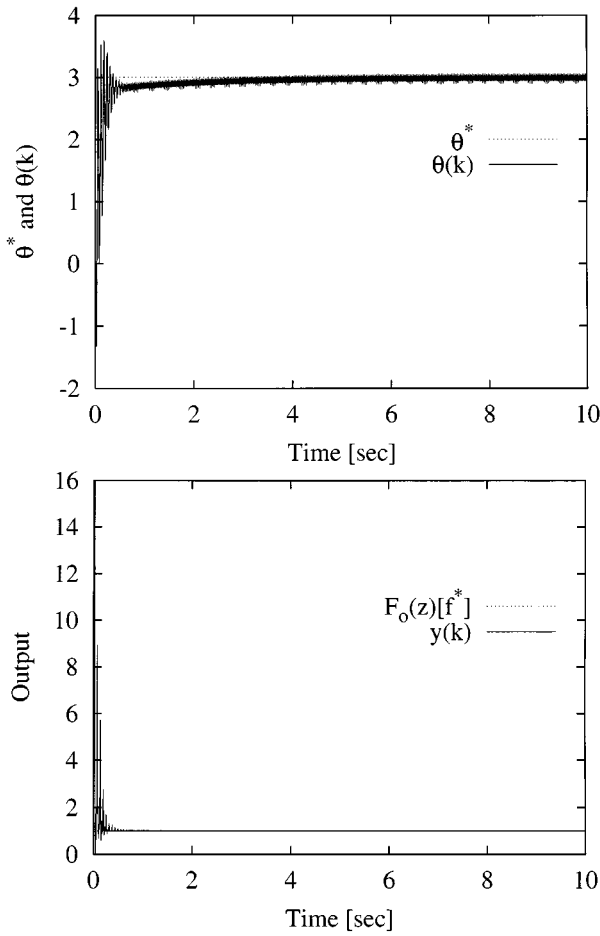


Fig. 3. Responses for  $\omega = (\pi/1.5)$  rad/sample and  $\gamma = 2.1$ .

Other design parameters are selected as:  $\theta^* = 3$ ,  $f^* = 2$ ,  $h = 0.9$ ,  $\alpha = 0.05$ ,  $\beta = 0.05$ , and  $\phi = 0$ . Simulation is conducted for  $\omega = (\pi/1.1)$  and  $\omega = (\pi/1.5)$ , so that it can be calculated that  $|M(e^{j(\pi/1.1)})| = 4.57$ ,  $\angle(M(e^{j(\pi/1.1)})) = -0.75$  rad,  $|M(e^{j(\pi/1.5)})| = 2.68$ ,  $\angle(M(e^{j(\pi/1.5)})) = 0.93$  rad, and  $F_i(1) = 0.58$ , where  $M(e^{j\omega}) = F_i(e^{j\omega})(e^{j\omega} - 1)/(e^{j\omega} + h)F_o(e^{j\omega})$ . Since  $\cos(\angle(M(e^{j(\pi/1.1)}))) > 0$ ,  $\cos(\angle(M(e^{j(\pi/1.5)}))) > 0$ , and  $F_i(1) > 0$ , the sufficient condition of Theorem 1 is satisfied for both  $\omega = (\pi/1.1)$  and  $(\pi/1.5)$ . Accordingly, it is certain that the system is exponentially stable, which is illustrated in Figs. 2 and 3. It is also shown from Figs. 2 and 3 that  $\theta(k)$  converges to  $\theta^*$  with larger magnitude of oscillation than that in the convergence of  $y(k)$  to  $F_o(z)[f^*]$ . This observation illustrates the results of Theorem 2 and Corollary 1 that  $\hat{\theta}(k)$  and  $\hat{y}(k)$  locally exponentially converge to an  $O(\alpha)$  and  $O(\alpha^2)$  neighborhood of zero, respectively.

## VII. CONCLUSION

We have presented an extremum seeking control algorithm for discrete-time systems. By using the two-time scale averaging theory [7], we derived a very mild sufficient condition under which the system output exponentially converges to an  $O(\alpha^2)$  neighborhood of the extremum value. The sufficient condition is related to positive realness of linear parts of the plant but only at the modulation frequency  $\omega$ . The simulation study demonstrates the validity of the extremum seeking algorithm.

Future study subjects include: development of a method to improve and analyze the transient performance; rejection of measurement

noises; tracking of time-varying  $f^*$  and  $\theta^*$ ; and practical design guidelines for selecting modulation signal frequency  $\omega$ , phase shift of demodulation signal  $\phi$ , and various other gains.

## APPENDIX

*Proof of Lemma 2:* The lemma is proved using the following straightforward calculation:

$$\begin{aligned} & G(z)[(H(z)[\cos(\omega k - \phi)])v(k)] \\ &= G(z) \left[ \text{Re} \left\{ H(e^{j\omega})e^{j(\omega k - \phi)} \right\} v(k) + \varepsilon^{-k} \right] \\ & \quad \text{(by Lemma 1)} \\ &= \text{Re} \left\{ e^{-j\phi} \mathcal{Z}^{-1} \{ G(z)H(e^{j\omega})V(e^{-j\omega}z) \} \right\} + \varepsilon^{-k} \\ &= \text{Re} \left\{ e^{j(\omega k - \phi)} H(e^{j\omega}) \mathcal{Z}^{-1} \{ G(e^{j\omega}z)V(z) \} \right\} + \varepsilon^{-k} \\ &= \text{Re} \left\{ e^{j(\omega k - \phi)} H(e^{j\omega})G(e^{j\omega}z)[v(k)] \right\} + \varepsilon^{-k}. \end{aligned}$$

□

*Proof of Lemma 5:* Let  $x_1(k)$  and  $x_2(k)$  be the state vectors of  $H(z)$  and  $G(z)$  respectively. Then,  $G(z)[\cos(\omega k - \phi)H(z)[v(k)]]$  is represented in the state-space form as

$$\begin{aligned} x_1(k+1) &= A_1 x_1(k) + B_1 v(k) \\ y_1(k) &= C_1 x_1(k) + D_1 v(k) \\ x_2(k+1) &= A_2 x_2(k) + B_2 \cos(\omega k - \phi) y_1(k) \\ y_2(k) &= C_2 x_2(k) + D_2 \cos(\omega k - \phi) y_1(k) \end{aligned}$$

where  $y_1(k) = H(z)[v(k)]$  and  $y_2(k) = G(z)[\cos(\omega k - \phi)y_1(k)]$ . Combining the above two state-space forms yields

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)v(k) \\ y_2(k) &= C(k)x(k) + D(k)v(k) \end{aligned}$$

where  $x^T(k) = [x_1^T(k) \mid x_2^T(k)]$

$$A(k) = \left[ \begin{array}{c|c} A_1 & \mathbf{0} \\ \hline c(\omega k)B_2C_1 & A_2 \end{array} \right]$$

$B(k) = [(B_1/(c(\omega k)B_2D_1))]$ ,  $C(k) = [c(\omega k)D_2C_1 \mid C_2]$ ,  $D(k) = c(\omega k)D_2D_1$ , and  $c(\omega k) = \cos(\omega k - \phi)$ . Therefore,  $(A(k), B(k), C(k), D(k))$  can be a state space realization of  $G(z)[\cos(\omega k - \phi)H(z)[v(k)]]$ .

Since  $A_1$  and  $A_2$  in  $A(k)$  are exponentially stable, given any  $Q_1 = Q_1^T > 0$  and  $Q_2 = Q_2^T > 0$ , there exist  $P_1 = P_1^T > 0$  and  $P_2 = P_2^T > 0$ , which are the unique solutions of the following linear equations, respectively:

$$A_1^T P_1 A_1 - P_1 = -Q_1 \quad \text{and} \quad A_2^T P_2 A_2 - P_2 = -Q_2. \quad (23)$$

By constructing a block diagonal matrix  $P = \left[ \begin{array}{c|c} P_1 & \mathbf{0} \\ \hline \mathbf{0} & P_2 \end{array} \right]$ ,  $P = P^T > 0$ , we obtain

$$A^T(k)PA(k) - P = \left[ \begin{array}{c|c} -Q_{11}(k) & R^T(k) \\ \hline R(k) & -Q_2 \end{array} \right] \quad (24)$$

where  $Q_{11}(k) = Q_1 - c^2(\omega k)C_1^T B_2^T P_2 B_2 C_1$  and  $R(k) = c(\omega k)A_2^T P_2 B_2 C_1$ . For any given  $Q_2$ , we can choose  $Q_1$  such that

$$\begin{aligned} \lambda_{\min}(Q_1) &> \lambda_{\max} \left( c^2(\omega k)C_1^T B_2^T P_2 B_2 C_1 \right) \\ &\quad + \frac{\lambda_{\max}(R(k)R^T(k))}{\lambda_{\min}(Q_2)} \end{aligned}$$

for all  $k > 0$ . This  $Q_1$  enables  $-Q_{11}(k)$  and  $-Q_2 + R(k)Q_{11}^{-1}(k)R^T(k)$  to be negative definite for all  $k > 0$ , and we obtain the following decomposition of (24):

$$\begin{aligned} A^T(k)PA(k) - P &= \left[ \begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline -R(k)Q_{11}^{-1}(k) & \mathbf{I} \end{array} \right] \\ &\quad \times \left[ \begin{array}{c|c} -Q_{11}(k) & \mathbf{0} \\ \hline \mathbf{0} & \Delta \end{array} \right] \left[ \begin{array}{c|c} \mathbf{I} & -Q_{11}^{-1}(k)R^T(k) \\ \hline \mathbf{0} & \mathbf{I} \end{array} \right] \quad (25) \end{aligned}$$

where  $\Delta = -Q_2 + R(k)Q_{11}^{-1}(k)R^T(k) < 0$  for all  $k > 0$ . Consequently,  $A^T(k)PA(k) - P < 0$  for all  $k > 0$ , and  $A(k)$  is exponentially stable from the Lyapunov stability theory.  $\square$

*Proof of Lemma 6:* The term  $\theta_0^2(k)$  in  $\delta(k)$  is calculated as

$$\theta_0^2(k) = \frac{1}{2}\alpha^2 |F_i(e^{j\omega})|^2 (1 + \cos(2k\omega + 2\psi_1)) + \varepsilon^{-k} \quad (26)$$

where  $\psi_1 = \angle(F_i(e^{j\omega}))$ . Then,  $\delta(k)$  is rearranged as

$$\delta(k) = \delta_1(k) + \delta_2(k) \quad (27)$$

where

$$\begin{aligned} \delta_1(k) &= \varepsilon F_i(z) \left[ \cos(\omega k - \phi) \frac{z-1}{z+h} \right. \\ &\quad \left. \times F_o(z) \left[ f^* + \frac{1}{2}\alpha^2 |F_i(e^{j\omega})|^2 \right] \right] + \varepsilon^{-k} \\ \delta_2(k) &= \frac{1}{2}\varepsilon\alpha^2 |F_i(e^{j\omega})|^2 F_i(z) \\ &\quad \times \left[ \cos(\omega k - \phi) \frac{z-1}{z+h} F_o(z) [\cos(2k\omega + 2\psi_1)] \right]. \end{aligned}$$

Since the high-pass filter  $(z - 1/z + h)$  has zero DC gain,  $\delta_1(k)$  in (27) contains only exponentially decaying terms. On the other hand, by using Lemma 1,  $\delta_2(k)$  is calculated as

$$\begin{aligned} \delta_2(k) &= \frac{1}{2}\varepsilon\alpha^2 c_1 F_i(z) [\cos(\omega k - \phi) \cos(2\omega k + \psi_2)] \\ &= \frac{1}{4}\varepsilon\alpha^2 c_1 F_i(z) [\cos(3\omega k - \phi + \psi_2) \\ &\quad + \cos(\omega k + \phi + \psi_2)] \\ &= \frac{1}{4}\varepsilon\alpha^2 c_1 (|F_i(e^{j3\omega})| \cos(3\omega k - \phi + \psi_3) \\ &\quad + |F_i(e^{j\omega})| \cos(\omega k + \phi + \psi_4)) \\ &\leq \kappa_1 \varepsilon\alpha^2, \end{aligned}$$

where  $c_1 = |F_i(e^{j\omega})|^2 ((e^{j2\omega} - 1)/(e^{j2\omega} + h)) |F_o(e^{j2\omega})|$ ,  $\psi_2 = 2\psi_1 + \angle(F_o(e^{j2\omega})) + \angle(e^{j2\omega} - 1/e^{j2\omega} + h)$ ,  $\psi_3 = \psi_2 + \angle(F_i(e^{j3\omega}))$ ,  $\psi_4 = \psi_2 + \angle(F_i(e^{j\omega}))$  and  $\kappa_1 = (1/4)c_1 (|F_i(e^{j3\omega})| + |F_i(e^{j\omega})|)$ . Q.E.D.

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## Analyzing Wide-Band Noise Processes With Application to Control and Filtering

Agamirza E. Bashirov and Sevin Uğural

**Abstract**—In this note, the concept of wide-band noise is analyzed via a certain integral representation. It is proved that there are infinitely many wide-band noise processes represented in integral form which correspond to the same autocovariance function. Based on this integral representation, a technique of reduction of a wide-band noise driven system to a white noise driven system is presented. This technique is used to modify the separation principle and the Kalman–Bucy filtering to wide-band noise driven systems.

**Index Terms**—Linear stochastic system, optimal control, white noise, wide-band noise.

## I. INTRODUCTION

The modern stochastic optimal control and filtering theories use white noise driven systems. The results such as the separation principle and the Kalman–Bucy filtering are based on the white noise model. Indeed, white noise being a mathematical idealization gives only an approximate description to real noise. In some fields the parameters of real noise are near to the parameters of white noise and, so, the mathematical methods of control and filtering for white noise driven systems can be satisfactorily applied in them. However, in many fields white noise is a crude approximation to real noise. Consequently, the theoretical optimal controls and the theoretical optimal filters for white noise driven systems become not optimal and, indeed, might be quite far from being optimal. It becomes important to develop the control and estimation theories for the systems driven by noise models which describe real noise more adequately.

The issue is that real noise has a property which ensures the correlation of its values within a small time interval, i.e., if we denote it by  $\varphi$ , then

$$\text{cov}(\varphi(t+s), \varphi(t)) = \begin{cases} \lambda(t, s), & 0 \leq s < \varepsilon \\ 0, & s \geq \varepsilon \end{cases}, \quad t \geq 0 \quad (1)$$

where  $\varepsilon > 0$  is a small value and  $\lambda$  is a nonzero function. A random process  $\varphi$  with the property (1) is called a wide-band noise process and it is said to be stationary (in wide sense) if the function  $\lambda$  depends only on  $s$  (see Fleming and Rishel [1]). If  $\varepsilon$  is so small that it is normally assumed to be 0, then the wide-band noise process  $\varphi$  is transformed into white noise. As it was mentioned, in many fields, for example, in gravimetry (see Bashirov *et al.*[2]) such a substitution of wide-band noise by white noise gives rise to tangible distortions.

Kushner has applied an approach to wide-band noise that is based on approximations (see [3] and the references there). A different approach which is based on a certain integral representation was suggested in Bashirov [4]. Using this approach, in [4] and [5] different estimation problems were studied for linear partially observable systems disturbed by wide-band noise. The respective proofs in [4] and

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