Global boundary stabilization of the Korteweg-de Vries-Burgers equation*

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Abstract. The problem of global exponential stabilization by boundary feedback for the Korteweg-de Vries-Burgers equation on the domain $[0, 1]$ is considered. We derive a control law of the form $u(0) = u_x(1) = u_{xx}(1) - k[u(1)^3 + u(1)] = 0$, where $k$ is a sufficiently large positive constant, and prove that it guarantees $L^2$-global exponential stability, $H^3$-global asymptotic stability, and $H^3$-semiglobal exponential stability. Our decay rate estimates depend not only on the diffusion coefficient but also on the dispersion coefficient. The closed-loop system is shown to be well posed.

Mathematical subject classification: 35B35, 35Q53.

Key words: Korteweg-de Vries-Burgers equation; boundary stabilization.

1 Introduction

Motivation and background. In this article, we shall be concerned with the problem of global boundary stabilization of the Korteweg-de Vries-Burgers equation

$$u_t - \varepsilon u_{xx} + \delta u_{xxx} + uu_x = 0, \quad 0 < x < 1, \quad t > 0, \quad (1.1)$$

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where $\varepsilon$ and $\delta$ are positive parameters. When $\varepsilon = 0$, equation (1.1) becomes the Korteweg-de Vries equation [10]

$$u_t + \delta u_{xxx} + uu_x = 0,$$

(1.2)

which has been proposed as a model for one directional long waves of small-amplitude in many different physical systems such as shallow water waves, sound waves and magnetosonic waves. When $\delta = 0$, equation (1.1) becomes the Burgers equation

$$u_t - \varepsilon u_{xx} + uu_x = 0.$$

(1.3)

The KdV equation incorporates effects of dispersion and of nonlinear convection, yielding good qualitative predictions of various phenomena observed in waves. However, in many real situations, one must include energy dissipation mechanisms, accounted for in the KdVB equation through the term $\varepsilon u_{xx}$, although for many physical systems modeled by (1.1) these mechanisms are disparate and not always well understood.

Recently, extensive attention has been paid to the problems of controllability, stability and stabilization for the KdV and KdVB equations. With periodic boundary conditions, the results were obtained by Bona et al [6], Russell and Zhang [19, 20, 21, 22], Komornik et al [11], Zhang [27], and Rosier [18]. In the case where the domain of the equation is the whole real line, important results were established by Amick et al [2], Biler [5], Bona et al [7], Naumkin [13, 14], Nishihara [15], Schonbek et al [23, 24] and Zhang [25].

The objective of the present work is to study the global boundary stabilization of the KdVB equation on a finite spatial interval $[0, 1]$ with non-periodic boundary feedbacks. The uncontrolled system (1.1) is not globally stable. This was shown by Balogh [4] numerically by identifying equilibrium profiles other than the trivial equilibrium at zero.\footnote{Balogh's paper [4] was a follow up to the conference version of the present paper. He was inspired to propose a more aggressive control law that performed excellent in simulations, however his paper guarantees only $H^1$ stability. Another follow-up paper that solves the problem in a discretized form using finite dimensional nonlinear control was Ammou and Christofides [3], which received the prestigious O. Hugo Schuck Best Paper Award.} We show that one can find a smooth function $g(\cdot)$ with $g(0) = 0$ such that the boundary conditions

$$u(0, t) = u_x(1, t) = 0, \quad u_{xx}(1, t) = g(u(1, t))$$

(1.4)
guarantee global asymptotic stability in terms of not only the $L^2$ norm but also
the $H^3$ norm, which ensures boundedness of solution for $H^3$ initial data of any
size.

An example of a physical problem where our control law would be
implementable is the water channel setup with boundary actuation discussed in
Rosier [18], where the height or curvature of the water profile at ends of the
channel can be actuated by standard mechanical means applied at the ends of
the bottom of the channel. However, it would be wrong to view the result of
the paper only in the context of the water channel. KdVB equation also models
other physical forms of waves such as sound waves and magnetosonic waves. In
those case the sensed quantity would be the pressure at the walls of the domain,
and the boundary actuation would be via a loudspeaker in one case, and via
electromagnetic actuation in the other.

It is equally important to understand that the consideration of the KdVB equa-
tion on the finite interval does not reduce the physical relevance of the problem
but, on the contrary, makes it more realistic. It is the equation on the infinite real
line (and even more so the one with periodic boundary conditions) that represents
the mathematical idealization of the problem, not the one on the finite interval.
In applications these waves evolve on finite domains.

Feedback design. We seek a boundary feedback law in the form (1.4). The
energy

$$E(t) = \int_0^1 u(x, t)^2 dx$$

(1.5)
of the system (1.1), (1.4) has a time derivative

$$\dot{E}(t) = 2 \int_0^1 u(x, t) u_t(x, t) dx$$

$$= 2 \int_0^1 u(x, t) [\varepsilon u_{xx} - \delta u_{xxx} - uu_x] dx$$

$$= 2 \varepsilon u_x^1_0 - 2\varepsilon \int_0^1 u_x^2 dx - 2\delta uu_{xx}^1_0 + \delta u_x^2 - \frac{2}{3} u_x^3_0$$

(1.6)

$$= -2\varepsilon \int_0^1 u_x(x, t)^2 dx - 2\delta u(1, t) g(u(1, t))$$
\[- \delta u_x(0, t)^2 - \frac{2}{3} u(1, t)^3.\]

We select the invertible feedback function
\[g(u(1, t)) = k_1 u(1, t)^3 + k_2 u(1, t),\]  
with positive constants \(k_1\) and \(k_2\) satisfying
\[k_1, k_2 > \frac{1}{6 \delta},\]  
to get
\[
\dot{E}(t) = -2\epsilon \int_0^1 u_x^2 dx - 2\delta u(1, t)[k_1 u(1, t)^3 + k_2 u(1, t)]
- \delta u_x(0, t)^2 - \frac{2}{3} u(1, t)^3
\leq -2\epsilon \int_0^1 u_x^2 dx - 2\delta u(1, t)[k_1 u(1, t)^3 + k_2 u(1, t)]
+ \frac{1}{3}[u(1, t)^2 + u(1, t)^4]
\leq -2\epsilon E(t),
\]  
where the last inequality follows from Poincaré's inequality (see [1, p.158]). Thus \(E(t)\) tends to zero exponentially as \(t \to \infty\).

Our choice of the feedback law as an invertible function allows that it be implemented as
\[u(1, t) = g^{-1}(u_{xx}(1, t))\]
if physics dictate that \(u(1, t)\) can be actuated and \(u_{xx}(1, t)\) can be measured, and also provides a level of robustness to uncertain \(\delta\).

**Summary of results.** We will see that exponential stability holds not only for the \(L^2\) energy \(E(t)\) but also for the higher order energy
\[V(t) = \int_0^1 u_x(x, t)^2 dx,\]

which is very significant because, by embedding theorem (see [1, p.97]), we have
\[
\max_{0 \leq x \leq 1} u(x, t)^2 \leq V(t).
\]
In fact, we prove that boundedness and exponential decay holds for $\int_0^1 u_{xxx}^2 \, dx$. Furthermore, we derive decay estimates that depend not only on the diffusion coefficient but also on the dispersion coefficient, the first estimates of such type in the literature.

**Organization of the paper.** The closed-loop system considered in the rest of the paper is

$$
u_t - \varepsilon \nu_{xx} + \delta \nu_{xxx} + \nu \nu_x = 0, \quad 0 < x < 1, \quad t > 0, \quad (1.12)$$

$$\nu(0, t) = \nu_x(1, t) = 0, \quad t > 0, \quad (1.13)$$

$$\nu_{xx}(1, t) = k_1 \nu(1, t)^3 + k_2 \nu(1, t), \quad t > 0, \quad (1.14)$$

$$\nu(x, 0) = \nu^0(x), \quad 0 < x < 1. \quad (1.15)$$

We present our main results in section 2. Using the Lyapunov method, we prove in section 3 that the KdVB equation with the above boundary feedbacks is $L^2$-globally exponentially stable, $H^3$-globally asymptotically stable, and $H^3$-semiglobally exponentially stable, with decay rates depending on both diffusion and dispersion coefficients, and establish the global existence and uniqueness of the solutions with the help of the Banach fixed point theorem and the theory of semigroups.

## 2 Main Result

In what follows, we denote by $H^s(0, 1)$ the usual Sobolev space (see, e.g., [1, 12]) for any $s \in \mathbb{R}$. For $s \geq 0$, $H_0^s(0, 1)$ denotes the completion of $C_0^\infty(0, 1)$ in $H^s(0, 1)$, where $C_0^\infty(0, 1)$ denotes the space of all infinitely differentiable functions on $(0, 1)$ with compact support in $(0, 1)$. We denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and scalar product of $L^2(0, 1)$, respectively. For the need of problem (1.12)-(1.15), we introduce additional function spaces with boundary conditions as follows:

$$
\mathcal{H}_0^1(0, 1) = \{ \varphi \in H^1(0, 1) : \varphi(0) = 0 \}; 
(2.1)
$$

$$
\mathcal{H}_0^2(0, 1) = \{ \varphi \in H^2(0, 1) : \varphi(0) = \varphi_x(1) = 0 \}; 
(2.2)
$$

$$
\mathcal{H}_{00}(0, 1) = \{ \varphi \in H^2(0, 1) : \varphi(0) = \varphi_x(0) = 0 \}; 
(2.3)
$$

$$
\mathcal{H}_0^3(0, 1) = \{ \varphi \in H^3(0, 1) : \varphi(0) = \varphi_x(1) = \varphi_{xx}(1) = 0 \}. 
(2.4)
$$

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For a given initial condition \( u^0(x) \), in order for problem (1.12)--(1.15) to have a classical solution, \( u^0 \) must satisfy the following compatibility conditions

\[
\begin{align*}
  u^0(0) &= u^0_x(1) = 0, \\  u^0_{xx}(1) &= k_1 u^0(1)^3 + k_2 u^0(1),
\end{align*}
\]

which can be derived from (1.13)--(1.14). Thus we introduce one more function space with boundary conditions (2.5)--(2.6) as follows

\[
  H^3_{bc}(0, 1) = \{ u^0 \in H^3(0, 1) : \text{u^0 satisfies (2.5)--(2.6)} \}. 
\]

It is easy to see that \( H^3_0(0, 1) \subset H^3_{bc}(0, 1) \). We note that \( H^3_{bc}(0, 1) \) is not a vector space.

Let \( X \) be a Banach space and \( T > 0 \). We denote by \( C^k([0, T]; X) \) the space of all \( k \) times continuously differentiable functions defined on \([0, T]\) with values in \( X \), and write \( C([0, T]; X) \) for \( C^0([0, T]; X) \).

In [17, p.274], the weak solution is defined as the limit of classical solutions (quote): "A function \( y(x, t) \) is said to be a weak solution if there exists a sequence of ordinary solutions \( y_n(x, t) \) such that \( y(x, t) = \lim_{n \to \infty} y_n(x, t) \)". Following this line, we define a weak solution of equation (1.12)--(1.15) as follows.

**Definition 2.1.** A function \( u \in C([0, T]; H^3_0(0, 1)) \) is said to be a weak solution of equation (1.12)--(1.15) if \( u(x, 0) = u^0(x) \) and there exists a sequence \( \{ u_n \} \) of classical solutions of (1.12)--(1.15) with the initial conditions \( u^0_n \) such that \( u_n \) converges to \( u \) in \( C([0, T]; H^3_0(0, 1)) \).

**Remark 2.1.** The weak solution in the sense of Definition 2.1 is actually stronger than the weak solution in the sense of satisfying weak form

\[
\int_0^T \left[ (u, \varphi_t) - \varepsilon(u_x, \varphi_x) - \delta(u_x, \varphi_{xx}) - (uu_x, \varphi) \right] dt
\]

\[
= \delta \int_0^T \left[ k_1 u(1, t)^3 + k_2 u(1, t) \right] \varphi(1, t) dt - (u^0, \varphi(0)),
\]

for any \( \varphi \in C^\infty([0, T]; H^2_{00}(0, 1)) \) with \( \varphi(x, T) = 0 \). Indeed, for the classical solution \( u_n \), multiplying (1.12) by \( \varphi \) and integrating over \((0, 1) \times (0, T)\), we
obtain
\[
\int_0^T [(u_n, \varphi_t) - \varepsilon(u_n x, \varphi_x) - \delta(u_n x, \varphi_{xx}) - (u_n u_n x, \varphi)]dt = \delta \int_0^T [k_1 u_n(1, t)^3 + k_2 u_n(1, t)]\varphi(1, t)dt - (u_0^0, \varphi(0)).
\]

(2.9)

Since \( u_n \) converges to \( u \) in \( C([0, T]; \mathcal{H}_0^1(0, 1)) \), \( u_n^0 \) converges to \( u^0 \) in \( H^1(0, 1) \) and, by the imbedding theorem (see, e.g., [1, p.97]), \( u_n(1, t) \) converges to \( u(1, t) \) uniformly in \( C[0, T] \). Thus we can pass to the limit in (2.9) and obtain (2.8). \( \square \)

If we take (2.8) as the definition of weak solution, we will have difficulty in proving the uniqueness of such defined weak solution. To see this, let \( u_1 \) and \( u_2 \) be two functions satisfying (2.8). To prove that \( u_1 = u_2 \), as usual, we need to take \( \varphi(t) = u_1(t) - u_2(t) \) in (2.8). However, we can not do so since \( u_1(t) \), \( u_2(t) \) and \( \varphi(t) \) are in the different spaces \( \mathcal{H}_0^1(0, 1) \) and \( \mathcal{H}_{00}^1(0, 1) \). Therefore, we did not take (2.8) as the definition of weak solution. Next we state our two main theorems. In both theorems, \( c = c(\varepsilon, \delta, k_1, k_2) \) represents a finite positive constant which, however, tends to infinity as \( \varepsilon \to 0 \) or \( \delta \to 0 \).

**Theorem 2.1.** Suppose that \( k_1, k_2 > \frac{1}{6\delta} \) and denote
\[
M_1 = \frac{2\varepsilon + 2\delta k_2^2}{\varepsilon},
\]
\[
M_2 = \frac{12\delta k_1^2\varepsilon^2 + 6\delta k_1 - 1}{\varepsilon^2(6\delta k_1 - 1)}.
\]

(i) For the initial data \( u^0(x) \in \mathcal{H}_0^1(0, 1) \), problem (1.12)–(1.15) has a unique global weak solution. Moreover, the solution satisfies the following \( L^2 \) global-exponential stability estimate:
\[
\|u(t)\|^2 \leq \|u^0\|^2 e^{-2\varepsilon t}, \quad \forall \ t \geq 0,
\]

and the \( H^1 \) global-asymptotic and semiglobal-exponential stability estimate:
\[
\max_{0 \leq x \leq 1} u(x, t)^2 \leq \|u_x(t)\|^2 \\
\leq M_1 \|u^0_x\|^2 \exp\left(M_2 \|u^0\|^2\right)e^{-\varepsilon t}, \quad \forall \ t \geq 0.
\]

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(ii) For the initial data \( u^0(x) \in H^3_{bc}(0, 1) \), problem (1.12)-(1.15) has a global classical solution satisfying

\[
u \in C([0, \infty); H^3_{bc}(0, 1)) \cap C^1([0, \infty); L^2(0, 1)). \tag{2.14}\]

Moreover, the solution satisfies the following \( H^3 \) global-asymptotic and semiglobal-exponential stability estimate:

\[
\|u(t)\|_{H^3}^2 \leq c\|u^0\|_{H^3}^2 \exp \left(c\|u^0\|_{H^3}^2\right)e^{-\varepsilon t}, \quad \forall \ t \geq 0. \tag{2.15}\]

Note that the decay rates depend on only \( \varepsilon \), not \( \delta \). But it is obvious that the dispersion term \( \delta u_{xxx} \) should make contribution to the decay rates. Indeed, we have the following decay rate estimates.

**Theorem 2.2.** Suppose that \( k_1, k_2 > \frac{1}{6\delta} \) and set

\[
\omega = \frac{2}{9}(4\delta + \varepsilon), \tag{2.16}\]

\[
K = 2 \max \left\{1, \left(\frac{2}{\delta(4\delta + \varepsilon)}\right)^{\frac{\omega}{1}}\right\}, \tag{2.17}\]

\[
F(r) = r^2 + r^{4+\frac{5\omega}{2\varepsilon}}. \tag{2.18}\]

(i) For the initial data \( u^0(x) \in \mathcal{H}_0^1(0, 1) \), the solution of (1.12)-(1.15) satisfies the global-asymptotic and semiglobal-exponential stability estimates:

\[
\|u(t)\|^2 \leq K\left(\|u^0\|^2 + \|u^0\|^{2+\frac{2\omega}{r}}\right)e^{-2\omega t}, \quad \forall \ t \geq 0, \tag{2.19}\]

and

\[
\max_{0 \leq x \leq 1} |u(x, t)|^2 \leq \|u_x(t)\|^2 \leq cF(\|u^0\|) \exp \left[cF(\|u^0\|)\right]e^{-\omega t}, \quad \forall \ t \geq 0. \tag{2.20}\]

(ii) For the initial data \( u^0(x) \in H^3_{bc}(0, 1) \), the solution of (1.12)-(1.15) satisfies the following \( H^3 \) global-asymptotic and semiglobal-exponential stability estimate:

\[
\|u\|_{H^3}^2 \leq c \sum_{i=1}^3 F^i(\|u^0\|_{H^3}) \exp \left[cF(\|u^0\|)\right]e^{-\omega t}, \quad \forall \ t \geq 0. \tag{2.21}\]
Remark 2.2. It is of interest to compare the two sets of estimates (in Theorems 2.1 and 2.2). For instance, consider (2.12) and (2.19). It is clear that for \( \delta \ll \varepsilon \) the estimate (2.12) is tighter. Consider \( \delta \gg \varepsilon \) and \( \delta \geq 1/\sqrt{2} \), in which case the two estimates (after simplification of these estimates) are
\[
\|u(t)\| \leq \|u^0\|e^{-\varepsilon t},
\]
\[
\|u(t)\| \leq \sqrt{2}\|u^0\|\left(1 + \|u^0\|^\delta\right)e^{-\frac{\delta}{9}t}.
\]
The former estimate has a tighter dependence on \( \|u^0\| \) while the latter estimate has a tighter dependence on time. A good tradeoff estimate would be a minimum of the two. Similar considerations are possible for \( H^1 \) and \( H^3 \) estimates.

Remark 2.3. The decay rate \( \omega \) can be enhanced to any \( \omega_0 \) such that \( \omega_0 < \frac{1}{4}(4\delta + \varepsilon) \). However, the factors before \( e^{-2\sigma t} \) in (2.19) will depend on \( \omega_0 \) and tend to \( \infty \) as \( \omega_0 \to \frac{1}{4}(4\delta + \varepsilon) \). Similarly, the decay rate \( \omega \) in (2.20) and (2.21) can be improved to any \( \sigma \) such that \( \sigma < 2\omega \) with the same problem. This remark applies to the decay rates in (2.13) and (2.15).

Remark 2.4. The estimate (2.20) is referred to as global-asymptotic and semi-global-exponential by analogy to stability definitions for finite dimensional systems [9]. An estimate that decays to zero for all initial conditions, possibly at a rate slower than exponential, is said to be global-asymptotic.\(^2\) An estimate that decays exponentially and is linear in the norm of the initial condition is global-exponential. An estimate that can be made linear in the norm of the initial condition for any ball of initial conditions (but not necessarily with a uniform coefficient) is semi-global.

The methods used in this paper are the Lyapunov method, the Banach contraction fixed point theorem and some differential inequality of Gronwall type. Note that if we just choose the energy function \( E \) as a Lyapunov function, then we can obtain only the decay rate estimates of Theorem 2.1. In order to obtain the decay rate estimates of Theorem 2.2, we need to introduce the following Lyapunov function – a weighted energy function
\[
E_w(t) = \int_0^1 (x + 1)u^2(x, t) \, dx.
\]

\(^2\)There are additional properties that the estimate needs to satisfy. A reader unfamiliar with these definitions is urged to consult [9].

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To see why we need the weighted energy function, let us compare \( \dot{E}(t) \) with \( \dot{E}_w(t) \). By (1.9), we have

\[
\dot{E}(t) = -2\varepsilon \int_0^1 u_x^2 \, dx - 2\delta u(1, t) [k_1 u(1, t)^3 + k_2 u(1, t)]
- \delta u_x(0, t)^2 - \frac{2}{3} u(1, t)^3. \tag{2.23}
\]

By calculation (see (3.74) below), we have

\[
\dot{E}_w(t) = -4\delta [k_1 u(1, t)^4 + k_2 u(1, t)^2] - \delta u_x(0, t)^2 - \frac{4}{3} u(1, t)^3
- 3\delta \int_0^1 u_x^2 \, dx - 2\varepsilon \int_0^1 (x + 1) u_x^2 \, dx
- 2\varepsilon \int_0^1 u u_x \, dx + \frac{2}{3} \int_0^1 u^3 \, dx. \tag{2.24}
\]

Note that the term \(-3\delta \int_0^1 u_x^2 \, dx\) in the above, which does not appear in \( \dot{E}(t) \). It is this term that produces the improved decay rate \( \omega \).

3 Proof of Theorems 2.1 and 2.2

We first establish the existence and uniqueness of a local classical solution of problem (1.12)–(1.15) with the help of the Banach fixed point theorem and the theory of semigroups. To use the theory of semigroups, we define the linear operator \( L \) by

\[
L \varphi = -\delta \varphi_{xxx} + \varepsilon \varphi_{xx} \tag{3.1}
\]

with the domain

\[
D(L) = H_0^3(0, 1). \tag{3.2}
\]

One can easily verify that the adjoint operator \( L^* \) of \( L \) is given by

\[
L^* \varphi = \delta \varphi_{xxx} + \varepsilon \varphi_{xx} \tag{3.3}
\]

with the domain

\[
D(L^*) = \{ \varphi \in H^3(0, 1) : \varphi(0) = \varphi_x(0) = \delta \varphi_{xx}(1) + \varepsilon \varphi_x(1) = 0 \}. \tag{3.4}
\]
It is easy to verify that $L$ and $L^*$ are dissipative and densely defined closed linear operators on $L^2(0,1)$. Hence, by the theory of semigroups (see [16, p.14, Corollary 4.4]), $L$ is the infinitesimal generator of a $C_0$ (i.e., strongly continuous) semigroup of contraction on $L^2(0,1)$.

For a given initial condition $u^0(x)$, in order for problem (1.12)–(1.15) to have a sufficiently regular solution, $u^0$ must satisfy certain compatibility conditions which can be derived from (1.12)–(1.14) as follows. Let us denote

$$I_1(u^0) = \varepsilon u^0_{xx}(x) - \delta u^0_{x,xx}(x) - u^0(x)u^0_x(x).$$

(3.5)

By (1.12), we have

$$u_t(x,0) = I_1(u^0).$$

(3.6)

Therefore, differentiating (1.13)–(1.14) with respect to $t$, we obtain

$$I_1(u^0(0)) = u_t(x,0) = 0,$$

(3.7)

$$\frac{d}{dx}(I_1(u^0))_{x=1} = u_{tx}(1,0) = 0,$$

(3.8)

$$\frac{d^2}{dx^2}(I_1(u^0))_{x=1} = u_{txx}(1,0) = [3k_1u^0(1)^2 + k_2]I_1(u^0(1)).$$

(3.9)

Denote

$$I_2(u^0) = \varepsilon \frac{d^2}{dx^2}(I_1(u^0)) - \delta \frac{d^3}{dx^3}(I_1(u^0))$$

$$- I_1(u^0)u^0_x(x) - u^0(x) \frac{d}{dx}(I_1(u^0)).$$

(3.10)

Differentiating (1.12) with respect to $t$, we obtain

$$u_{tt}(x,0) = I_2(u^0).$$

(3.11)

Therefore, differentiating (1.13)–(1.14) twice with respect to $t$, we obtain

$$I_2(u^0(0)) = u_{tt}(0,0) = 0,$$

(3.12)

$$\frac{d}{dx}(I_2(u^0))_{x=1} = u_{ttx}(1,0) = 0,$$

(3.13)

$$\frac{d^2}{dx^2}(I_2(u^0))_{x=1} = u_{ttxx}(1,0) = 6k_1u^0(1)I_1(u^0(1))$$

$$+ [3k_1u^0(1)^2 + k_2]I_2(u^0(1)).$$

(3.14)
Thus we introduce function spaces with certain boundary conditions. Set
\[
H^4_{bc}(0, 1) = \left\{ u^0 \in H^4(0, 1) : u^0 \text{ satisfies (2.5)–(2.6)} \right. \\
\left. \text{and (3.7)} \right\},
\]
(3.15)
\[
H^7_{bc}(0, 1) = \left\{ u^0 \in H^7(0, 1) : u^0 \text{ satisfies (2.5)–(2.6)} \right. \\
\left. \text{and (3.7)–(3.12)} \right\},
\]
(3.16)
\[
H^9_{bc}(0, 1) = \left\{ u^0 \in H^9(0, 1) : u^0 \text{ satisfies (2.5)–(2.6)} \right. \\
\left. \text{and (3.7)–(3.14)} \right\}.
\]
(3.17)

It is easy to see that \( H^4_{bc}(0, 1) \subset H^4_{bc}(0, 1), \ H^7_{bc}(0, 1) \subset H^7_{bc}(0, 1) \) and \( H^9_{bc}(0, 1) \subset H^9_{bc}(0, 1) \). We note that \( H^4_{bc}(0, 1), H^7_{bc}(0, 1) \) and \( H^9_{bc}(0, 1) \) are not vector spaces.

For a given initial condition \( u^0(x) \in H^2_{bc}(0, 1) \), let \( I_1(u^0) \) and \( I_2(u^0) \) be given by (3.5) and (3.10), respectively. We define the subset \( W \) of the function space \( C^2([0, T]; \mathcal{H}^1_0(0, 1)) \) by
\[
W = \left\{ w \in C^2([0, T]; \mathcal{H}^1_0(0, 1)) : w(x, 0) = u^0(x), \right.
\]
\[
w_t(x, 0) = I_1(u^0), \ w_{tt}(x, 0) = I_2(u^0) \right\}.
\]
(3.18)

\( W \) is not a linear space. We note that \( W \neq \emptyset \) since
\[
w = u^0(x) + t I_1(u^0) + \frac{t^2}{2} I_2(u^0) \in W.
\]

Here we need (3.7) and (3.12) to show that \( w \in C^2([0, T]; \mathcal{H}^1_0(0, 1)) \). We impose the condition
\[
w(x, 0) = u^0(x), \ w_t(x, 0) = I_1(u^0), \ w_{tt}(x, 0) = I_2(u^0) \quad \text{on} \ W
\]
since the solution of (1.12)–(1.15) satisfies this condition and the fixed point in \( W \) of the nonlinear mapping defined by (3.29) below will be the solution of (1.12)–(1.15).

**Lemma 3.1.** The subset \( W \) is closed in \( C^2([0, T]; \mathcal{H}^1_0(0, 1)) \), where the norm of \( C^2([0, T]; \mathcal{H}^1_0(0, 1)) \) is defined by
\[
\|w\|_W = \left( \max_{0 \leq t \leq T} (\|w_x(t)\|^2 + \|w_{xt}(t)\|^2 + \|w_{xxt}(t)\|^2) \right)^{1/2}.
\]
(3.19)
Proof. Let \( \{ w_n \} \subset W \) and \( w_n \to w \) in \( C^2([0, T]; \mathcal{H}^1_0(0, 1)) \) as \( n \to \infty \). Then we have \( w_n(x, t) \to w(x, t) \), \( w_{nt}(x, t) \to w_t(x, t) \) and \( w_{ntt}(x, t) \to w_{tt}(x, t) \) for every \( (x, t) \in [0, 1] \times [0, T] \). Hence
\[
\begin{align*}
 w(x, 0) &= \lim_{n \to \infty} w_n(x, 0) = \lim_{n \to \infty} u^0(x) = u^0(x), \\
 w_t(x, 0) &= \lim_{n \to \infty} w_{nt}(x, 0) = \lim_{n \to \infty} I_1(u^0) = I_1(u^0), \\
 w_{tt}(x, 0) &= \lim_{n \to \infty} w_{n tt}(x, 0) = \lim_{n \to \infty} I_2(u^0) = I_2(u^0).
\end{align*}
\]
This shows that \( w \in W \).

In order to apply the density argument for the initial condition \( u^0 \), we need to show that \( H^3_{bc}(0, 1) \) is dense in \( \mathcal{H}^1_0(0, 1) \) and \( H^3_{bc}(0, 1) \). We prove this for more general boundary conditions. Let \( m_1, m_2 \) be integers with \( 0 \leq m_1 < m_2 \) and \( f_{ik}(s_1, s_2, \ldots, s_{m_1}) \ (i = 0, 1; \ k = m_1, \ldots, m_2 - 1) \) any given functions. Let \( H^{m_1}_{gbc}(0, 1) \) be a subset of \( H^{m_1}(0, 1) \) with certain boundary conditions such as (2.5)-(2.6). We define
\[
\begin{align*}
H^{m_2}_{gbc}(0, 1) &= \{ \varphi \in H^{m_1}_{gbc}(0, 1) \cap H^{m_2}(0, 1) : \varphi^{(k)}(i) \\
&= f_{ik}(\varphi(i), \varphi'(i), \varphi^{(m_1-1)}(i)), \ i = 0, 1; \ k = m_1, \ldots, m_2 - 1 \}.
\end{align*}
\]

Lemma 3.2. For any integers \( m_1 \) and \( m_2 \) with \( 0 \leq m_1 < m_2 \), \( H^{m_2}_{gbc}(0, 1) \) is dense in \( H^{m_1}_{gbc}(0, 1) \).

Proof. For any \( \varphi \in H^{m_1}_{gbc}(0, 1) \), by the trace theorem (see, e.g., [12, p.39, Theorem 8.3]), there exists a function \( \psi \in H^{m_2}(0, 1) \) such that
\[
\begin{align*}
\psi^{(k)}(i) &= \varphi^{(k)}(i), \ i = 0, 1; \ k = 0, 1, \ldots, m_1 - 1, \\
\psi^{(k)}(i) &= f_{ik}(\varphi(i), \varphi'(i), \varphi^{(m_1-1)}(i)), \ i = 0, 1; \ k = m_1, \ldots, m_2 - 1.
\end{align*}
\]
It is clear that \( \varphi - \psi \in H^0_0(0, 1) \). Therefore there exists a sequence \( \{ \xi_n \} \subset C^\infty_0(0, 1) \) such that \( \xi_n \) converges to \( \varphi - \psi \) in \( H^{m_1}(0, 1) \). Denote
\[
\varphi_n = \xi_n + \psi.
\]
Then \( \varphi_n \) converges to \( \varphi \) in \( H^{m_1}(0, 1) \). Moreover, it is easy to see that \( \varphi_n \in H^{m_2}(0, 1) \). In addition, we have

\[
\begin{align*}
\varphi_n^{(k)}(i) &= \psi^{(k)}(i) = \varphi^{(k)}(i), \quad i = 0, 1; \quad k = 0, 1, \ldots, m_1 - 1, \quad (3.23) \\
\varphi_n^{(k)}(i) &= \psi^{(k)}(i) \\
&= f_{ik}(\varphi(i), \varphi'(i), \varphi^{(m_1-1)}(i)) \\
&= f_{ik}(\varphi_n(i), \varphi'_n(i), \varphi_n^{(m_1-1)}(i)), \quad i = 0, 1; \\
&\quad k = m_1, \ldots, m_2 - 1. \quad (3.24)
\end{align*}
\]

Hence we deduce that \( \varphi_n \in H_{gbc}^{m_2}(0, 1) \).

In order to prove our main results, we first prove that problem (1.12)–(1.15) has a unique local classical solution.

**Proposition 3.1.** For the initial data \( u^0(x) \in H_{bc}^7(0, 1) \), there exists a finite time \( T > 0 \) such that problem (1.12)–(1.15) has a unique local classical solution satisfying

\[
u \in C([0, T]; H_{bc}^7(0, 1)) \cap C^1([0, T]; H_{bc}^4(0, 1)) \cap C^2([0, T]; H_{bc}^1(0, 1)).
\]

**Proof.** The idea of the proof is as follows. We first prove that the linearized boundary value problem

\[
\begin{align*}
y_t - \varepsilon y_{xx} + \delta y_{xxx} + w w_x &= 0, \quad 0 < x < 1, \quad 0 < t < T, \quad (3.25) \\
y(0, t) &= y_x(1, t) = 0, \quad 0 < t < T, \quad (3.26) \\
y_{xx}(1, t) &= k_1 w(1, t)^3 + k_2 w(1, t), \quad 0 < t < T, \quad (3.27) \\
y(x, 0) &= u^0(x), \quad 0 < x < 1, \quad (3.28)
\end{align*}
\]

has a unique solution \( y \in W \) for any fixed \( w \in W \). We then define a nonlinear mapping \( A \) by

\[
A w = y,
\]

and show that \( A \) has a unique fixed point \( u^* \) by using the Banach contraction fixed point theorem. This \( u^* \) is the unique solution of problem (1.12)–(1.15) as required.

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In order to show that the linear problem (3.25)–(3.28) is well posed, we transform it into a homogeneous boundary value problem. Set
\[
\psi = \frac{1}{2} x(x - 1)^2 [k_1 w(1, t)^3 + k_2 w(1, t)], \quad (3.30)
\]
\[
v = y - \psi. \quad (3.31)
\]
Then it is clear that \(v\) satisfies
\[
v_t - \varepsilon v_{xx} + \delta v_{xxx} = f, \quad 0 < x < 1, \quad 0 < t < T, \quad (3.32)
\]
\[
v(0, t) = v_x(1, t) = v_{xx}(1, t) = 0, \quad 0 < t < T, \quad (3.33)
\]
\[
v(x, 0) = v^0(x), \quad 0 < x < 1, \quad (3.34)
\]
where
\[
f(x, t) = -\psi_t + \varepsilon \psi_{xx} - \delta \psi_{xxx} - w w_x, \quad (3.35)
\]
and (use (2.6) and \(w(x, 0) = u^0(x)\))
\[
v^0(x) = u^0(x) - \frac{1}{2} x(x - 1)^2 [k_1 w(1, 0)^3 + k_2 w(1, 0)] \in \mathcal{H}^3_0(0, 1). \quad (3.36)
\]
Using the linear operator \(L\) defined by (3.1), problem (3.25)–(3.28) can be written as an abstract Cauchy problem
\[
v_t = Lv + f, \quad (3.37)
\]
\[
v(0) = v^0. \quad (3.38)
\]
For the time being, we assume that \(u^0(x) \in H^2_{bc}(0, 1)\) and
\[
w \in W^4 = \{ w \in W : w \in C^4([0, T]; \mathcal{H}^3_0(0, 1)) \}. \quad (3.39)
\]
Then \(f\) is differentiable with respect to \(t\). Since we have proved that \(L\) is the infinitesimal generator of a \(C_0\) semigroup of contraction on \(L^2(0, 1)\), it follows from the theory of semigroups (see [16, p.107, Corollary 2.5]) that the problem (3.32)–(3.34) has a unique classical solution
\[
v \in C^1([0, T]; L^2(0, 1)) \cap C([0, T]; \mathcal{H}^3_0(0, 1))
\]
and then the problem (3.25)–(3.28) has a unique classical solution
\[
y = v + \psi \in C^1([0, T]; L^2(0, 1)) \cap C([0, T]; \mathcal{H}^3_0(0, 1) \cap H^3(0, 1)).
\]

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To show that $y$ has more regularity, we differentiate (3.37) twice with respect to $t$ and obtain

$$v_{tt} = Lv_t + f_t.$$  

(3.40)

If we can show that $v_{tt}(0) \in D(L) = \mathcal{H}_0^3(0, 1)$, as above, we deduce

$$v_{tt} \in C^1([0, T]; \mathcal{L}^2(0, 1)) \cap C([0, T]; \mathcal{H}_0^3(0, 1))$$

and then

$$y \in C^3([0, T]; \mathcal{L}^2(0, 1)) \cap C^2([0, T]; \mathcal{H}_0^2(0, 1) \cap \mathcal{H}^3(0, 1)).$$

This regularity of the solutions justifies the validity of the calculations performed below.

To prove that $v_{tt}(0) \in D(L)$, we need to show that $v_{tt}(0) \in \mathcal{H}^3(0, 1)$ and $v_{tt}(0, 0) = v_{ttx}(1, 0) = v_{ttxx}(1, 0) = 0$. For this, we first calculate $f$ and, by (3.35), obtain

$$f(x, t) = -\frac{1}{2} x(x - 1)^2[3k w(1, t)^2 + k_2]w_t(1, t)$$

$$+ \epsilon (3x - 2)[k_1 w(1, t)^3 + k_2 w(1, t)]$$

$$- 3\delta[k_1 w(1, t)^3 + k_2 w(1, t)] - w(x, t)w_x(x, t).$$  

(3.41)

We then calculate $v_t(x, 0)$ and, by (3.37), obtain

$$v_t(x, 0) = Lv^0(x) + f(x, 0)$$

$$= -\delta u^0_{xxx}(x) + \epsilon u^0_{xx}(x) + f(x, 0) \text{ (use (3.36))}$$

$$= -\delta u^0_{xxx}(x) + 3\delta[k_1 w(1, 0)^3 + k_2 w(1, 0)]$$

$$+ \epsilon u^0_{xx}(x) - \epsilon (3x - 2)[k_1 w(1, 0)^3 + k_2 w(1, 0)]$$

$$- \frac{1}{2} x(x - 1)^2[3k w(1, 0)^2 + k_2]w_t(1, 0)$$

$$+ \epsilon (3x - 2)[k_1 w(1, 0)^3 + k_2 w(1, 0)]$$

$$- 3\delta[k_1 w(1, 0)^3 + k_2 w(1, 0)] - w(x, 0)w_x(x, 0)$$

(since $w(x, 0) = u^0(x)$ and $w_t(1, 0) = I_1(u^0(1))$)

$$= -\delta u^0_{xxx}(x) + \epsilon u^0_{xx}(x) - u^0(x)u^0_x(x)$$

$$- \frac{1}{2} x(x - 1)^2 I_1(u^0(1))[3k w^0(1)^2 + k_2] \text{ (use (3.5))}$$

$$= I_1(u^0) - \frac{1}{2} x(x - 1)^2 I_1(u^0(1))[3k w^0(1)^2 + k_2],$$

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Finally, differentiating (3.37) with respect to $t$ gives

$$v_{tt}(x, 0) = Lu_t(x, 0) + f_t(x, 0)$$

$$= -\delta \frac{d^3}{dx^3}(I_1(u^0)) + \varepsilon \frac{d^2}{dx^2}(I_1(u^0))$$

$$+ 3\delta I_1(u^0(1))[3k_1u^0(1)^2 + k_2]$$

$$- \varepsilon(3x - 2)I_1(u^0(1))[3k_1u^0(1)^2 + k_2]$$

$$- 3k_1x(x - 1)^2w(1, 0)w_t(1, 0)^2$$

$$- \frac{1}{2}x(x - 1)^2[3k_1w(1, 0)^2 + k_2]w_{tt}(1, 0)$$

$$+ \varepsilon(3x - 2)w_t(1, 0)[3k_1w(1, 0)^2 + k_2]$$

$$- 3\delta w_t(1, 0)[3k_1w(1, 0)^2 + k_2]$$

$$- w_t(x, 0)w_x(x, 0) - w(x, 0)w_{tx}(x, 0)$$

(since $w(x, 0) = u^0(x)$, $w_t(1, 0) = I_1(u^0(1))$)

and $w_{tt}(1, 0) = I_2(u^0(1))$

$$= -\delta \frac{d^3}{dx^3}(I_1(u^0)) + \varepsilon \frac{d^2}{dx^2}(I_1(u^0))$$

$$- I_1(u^0)u^0_x(x) - u^0(x) \frac{d}{dx}(I_1(u^0))$$

$$- 3k_1x(x - 1)^2u^0(1)I_1(u^0(1))^2$$

$$- \frac{1}{2}x(x - 1)^2[3k_1u^0(1)^2 + k_2]I_2(u^0(1))$$

(use (3.10))

$$= I_2(u^0) - 3k_1x(x - 1)^2u^0(1)I_1(u^0(1))^2$$

$$- \frac{1}{2}x(x - 1)^2[3k_1u^0(1)^2 + k_2]I_2(u^0(1))$$

$$\in H^3(0, 1).$$

Moreover, by compatibility conditions (3.12)–(3.14), we deduce that

$$v_{tt}(0, 0) = I_2(u^0(0)) = 0,$$

$$v_{txx}(1, 0) = \frac{d}{dx}(I_2(u^0))_{x=1} = 0,$$

$$v_{txtx}(1, 0) = \frac{d^2}{dx^2}(I_2(u^0))_{x=1} - 6k_1u^0(1)I_1(u^0(1))^2$$

$$- [3k_1u^0(1)^2 + k_2]I_2(u^0(1)) = 0.$$
Thus we have shown that $v_{tt}(0) \in D(L)$.

Continuing with the assumption that $u_0^0(x) \in H_{bc}^0(0, 1)$ and $w \in W^4$, we proceed to establish a priori estimates that allow us to relax this assumption and prove that $A$ is a contractive mapping. Let $y_1$ and $y_2$ be two solutions of the problem (3.25)–(3.28) corresponding to $w_1$ and $w_2$ and initial data $u_1^0$ and $u_2^0$, respectively, and set

$$z = y_1 - y_2, \quad z^0 = u_1^0 - u_2^0, \quad \eta = w_1 - w_2.$$  

(3.47)

Then we have

$$z_t - \varepsilon z_{xx} + \delta z_{xxx} + w_1 \eta_x + \eta w_2 x = 0, \quad 0 < x < 1, \quad 0 < t < T, \quad (3.48)$$

$$z(0, t) = z_x (1, t) = 0, \quad 0 < t < T, \quad (3.49)$$

$$z_{xx} (1, t) = k_1 [w_1 (1, t)^3 - w_2 (1, t)^3] + k_2 \eta (1, t), \quad 0 < t < T, \quad (3.50)$$

$$z(x, 0) = z^0, \quad 0 < x < 1. \quad (3.51)$$

In what follows, we will frequently use the inequalities

$$\max_{0 \leq x \leq 1} |u(x)| \leq \|u_x\|, \quad \forall u \in \mathcal{H}_0^1 (0, 1), \quad (3.52)$$

$$\|u\| \leq \frac{1}{\sqrt{2}} \|u_x\|, \quad \forall u \in \mathcal{H}_0^1 (0, 1). \quad (3.53)$$

The function $C(r_1, r_2)$ below denotes a generic positive continuous function that may vary from line to line.

Using (3.48) and integrating by parts, we obtain

$$\frac{d}{dt} \int_0^1 z(t)^2 dx = 2 \int_0^1 z (\varepsilon z_{xx} - \delta z_{xxx} - w_1 \eta_x - \eta w_2 x) dx$$

$$\leq -2 \delta z (1, t) z_{xx} (1, t) - \delta z_x^2 (0, t) - 2 \varepsilon \|z_x\|^2$$

$$+ 2 \|z\| (\|w_1\|_W \|\eta\|_W + \|\eta\|_W \|w_2\|_W)$$

$$\leq 4 \delta \|z_x\| \|\eta\|_W (k_1 (\|w_1\|_W^2 + \|w_2\|_W^2) + k_2)$$

$$- \delta z_x^2 (0, t) - 2 \varepsilon \|z_x\|^2$$

$$+ 2 \|z\| (\|w_1\|_W \|\eta\|_W + \|\eta\|_W \|w_2\|_W)$$

$$\leq \|\eta\|_W^2 C(\|w_1\|_W, \|w_2\|_W), \quad (3.54)$$

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Integrating the inequality from 0 to $t$ gives
\begin{equation}
\|z(t)\|^2 \leq T \|\eta\|_w C(\|w_1\|_w, \|w_2\|_w) + \|z^0\|^2, \quad 0 \leq t \leq T.
\end{equation}

Using (3.48) and integrating by parts, we obtain
\begin{align*}
\frac{d}{dt} \int_0^1 z_t(t)^2 dx &= 2 \int_0^1 z_x(t)z_{xt}(t) dx = -2 \int_0^1 z_{xx}(t)z_x(t) dx \\
&= 2 \int_0^1 z_{xx}(-\varepsilon z_{xx} + \delta z_{xxx} + w_1 \eta_x + \eta w_2) dx \\
&\leq \delta z^2_{xx}(1, t) - \delta z^2_{xx}(0, t) - 2\varepsilon \|z_{xx}\|^2 \\
&+ 2\|z_{xx}\|\|\eta\|_w(\|w_1\|_w + \|w_2\|_w) \\
&\quad \text{(note that } z_{xx}(1, t)^2 \leq C(\|w_1\|_w, \|w_2\|_w)\|\eta\|^2_w) \\
&\leq \|\eta\|^2_w C(\|w_1\|_w, \|w_2\|_w).
\end{align*}

Integrating the inequality from 0 to $t$ gives
\begin{equation}
\|z_x(t)\|^2 \leq T \|\eta\|_w C(\|w_1\|_w, \|w_2\|_w) + \|z^0_x\|^2, \quad 0 \leq t \leq T.
\end{equation}

Differentiating equation (3.48) and repeating the above procedure, we obtain
\begin{align*}
\frac{d}{dt} \int_0^1 z_t(t)^2 dx &= 2 \int_0^1 z_t(\varepsilon z_{t,xx} - \delta z_{t,xxx} - w_1 t \eta_x \\
&\quad - w_1 \eta_{xt} - \eta_t w_2 - \eta w_{2,xt}) dx \\
&\leq -2\delta z_t(1, t)z_{xt}(1, t) - \delta z^2_{xt}(0, t) - 2\varepsilon \|z_{xt}\|^2 \\
&+ 2\|z_{xt}\|(\|w_1\|_w \|\eta\|_w + \|w_1\|_w \|\eta\|_w \\
&\quad + \|\eta\|\|w_2\|_w + \|\eta\|\|w_2\|_w) \\
&\quad \text{(note that } \|z_t(1, t)\| \leq \|z_{xt}\| \text{ cancelled by} \\
&\quad - 2\varepsilon \|z_{xt}\|^2 \text{ and } z_{xt}(1, t)^2 \\
&\quad \leq C(\|w_1\|_w, \|w_2\|_w)\|\eta\|^2_w) \\
&\leq \|\eta\|^2_w C(\|w_1\|_w, \|w_2\|_w).
\end{align*}

Integrating the inequality from 0 to $t$ gives
\begin{equation}
\|z_t(t)\|^2 \leq T \|\eta\|_w^2 C(\|w_1\|_w, \|w_2\|_w) + \|z_t(0)\|^2, \quad 0 \leq t \leq T.
\end{equation}
Likewise, we have
\[
\frac{d}{dt} \int_0^1 z_{xt}(t)^2 dx = 2 \int_0^1 z_{xt}(t)z_{xxt}(t) dx = -2 \int_0^1 z_{xxx}(t)z_{tt}(t) dx
\]
\[
= 2 \int_0^1 z_{xxx}(\epsilon z_{xt} + \delta z_{xxt} + w_1 t \eta_x \\
+ w_1 \eta_x + \eta_t w_2 + \eta \eta_{2xt}) dx
\]
\[
\leq \delta z_{xxx}^2 (1, t) - \delta z_{xxt}^2 (0, t) - 2\epsilon \|z_{xxt}\|^2
\]
\[
+ 2\|z_{xxt}\| \|w_1\|_W \|\eta\|_W + \|w_1\|_W \|\eta\|_W
\]
\[
+ \|\eta\|_W \|w_2\|_W + \|\eta\|_W \|w_2\|_W
\]
\[
\text{(note that } z_{xxt}(1, t)^2 \leq C(\|w_1\|_W, \|w_2\|_W) \|\eta\|^2_W)\]
\[
\leq \|\eta\|^2_W C(\|w_1\|_W, \|w_2\|_W).
\]  

Integrating the inequality from 0 to \(t\) gives
\[
\|z_{xt}(t)\|^2 \leq T \|\eta\|^2_W C(\|w_1\|_W, \|w_2\|_W) + \|z_{xt}(0)\|^2, \quad 0 \leq t \leq T. \quad (3.61)
\]

Repeating the above procedure again, we obtain
\[
\|z_{tt}(t)\|^2 \leq T \|\eta\|^2_W C(\|w_1\|_W, \|w_2\|_W) + \|z_{tt}(0)\|^2, \quad (3.62)
\]

and
\[
\|z_{xxt}(t)\|^2 \leq T \|\eta\|^2_W C(\|w_1\|_W, \|w_2\|_W) + \|z_{xxt}(0)\|^2. \quad (3.63)
\]

Now we relax the assumption that \(u^0(x) \in H^9_{bc}(0, 1)\) and \(w \in W^4\). Since \(W^4\) is dense in \(W\) and, by Lemma 3.2, \(H^9_{bc}(0, 1)\) is dense in \(H^7_{bc}(0, 1)\), using (3.55), (3.57), (3.59), (3.61), (3.62) and (3.63), by a density argument, we conclude that, for \(u^0 \in H^7_{bc}(0, 1)\) and \(w \in W\), the solution \(y\) of (3.25)—(3.28) is in \(C^2([0, T]; H^1_0(0, 1))\). Moreover, since
\[
y(x, 0) = u^0(x),
\]
\[
y_t(x, 0) = \varepsilon u_{xx}^0(x) - \delta u_{xxx}^0(x) - w(x, 0)w_x(x, 0)
\]
\[
= \varepsilon u_{xx}^0(x) - \delta u_{xxx}^0(x) - u^0(x)u_x^0(x) \quad \text{(use (3.5))}
\]
\[
= I_1(u^0),
\]
\[ y_{tt}(x, 0) = \varepsilon \frac{d^2}{dx^2}(I_1(u^0)) - \delta \frac{d^3}{dx^3}(I_1(u^0)) - w_t(x, 0)w_x(x, 0) \]
\[ - w(x, 0)w_{tx}(x, 0) \]
\[ = \varepsilon \frac{d^2}{dx^2}(I_1(u^0)) - \delta \frac{d^3}{dx^3}(I_1(u^0)) - I_1(u^0)u_x^0(x) \]
\[ - u^0(x)\frac{d}{dx}(I_1(u^0)) \text{ (use (3.10))} \]
\[ = I_2(u^0), \]

we have \( y \in W \). Thus \( A \) maps \( W \) into itself.

Taking \( u_2^0 = 0 \) and \( w_2 = 0 \) (note that \( 0 \in W \) for \( u^0 = 0 \)) in (3.55), (3.57), (3.59), (3.61), (3.62) and (3.63) we obtain

\[
\| Aw \|_W^2 \leq \| y_{ttt}(0) \|^2 + \| y_{tt}(0) \|^2 + \| y_{xt}(0) \|^2 + \| y_t(0) \|^2 + \| u^0_x \|^2 + \| u^0_t \|^2 + T \| w \|_W^2 C(\| w \|_W) \] (3.64)
\[
\leq R^2 + T \| w \|_W^2 C(\| w \|_W),
\]

where \( R = R(\| u^0 \|_{H^7}) \) is a positive constant depending on \( \| u^0 \|_{H^7} \). Let \( B(0, 2R) \) denote the ball in \( C^2([0, T]; J^0_{1}(0, 1)) \)

\[
B(0, 2R) = \{ w \in C^2([0, T]; J^0_{1}(0, 1)) : \| w \|_W \leq 2R \}. \] (3.65)

For \( w \in W \cap B(0, 2R) \), it follows from (3.64) that if \( T \) small enough then

\[
\| Aw \|_W^2 \leq R^2 + TR^2C(R) \leq 4R^2. \] (3.66)

Hence \( A \) maps \( W \cap B(0, 2R) \) into \( W \cap B(0, 2R) \). On the other hand, it follows from (3.57), (3.59), (3.61), (3.62) and (3.63) that if \( T \) is so small that

\[
TC(R) < 1, \] (3.67)

then \( A \) is a contractive mapping (note that \( z(x, 0) = z_t(x, 0) = z_{tt}(x, 0) = 0 \) since we have \( \eta(x, 0) = \eta_t(x, 0) = 0 \) for \( w_1, w_2 \in W \)). Moreover, by Lemma 3.1, \( W \cap B(0, 2R) \) is closed in \( C^2([0, T]; J^0_{1}(0, 1)) \). Therefore, by the Banach contraction fixed point theorem, \( A \) has a unique fixed point \( u^* \in W \). So the problem (1.12)–(1.15) has a unique solution \( u^* \) for \( T \) small enough.
To prove that

\[ u^* \in C([0, T]; H_{bc}^7(0, 1)) \cap C^1([0, T]; H_{bc}^4(0, 1)), \]  

(3.68)

we set \( \theta = u^*_{xx} \). Then by (1.12), we have

\[ \theta_x = \frac{1}{\delta} (\varepsilon \theta - u^*_t - u^*_x u^*_x), \]  

(3.69)

\[ \theta(1) = k_1 u^*(1, t)^3 + k_2 u^*(1, t). \]  

(3.70)

Solving the equation, we obtain

\[ u^*_{xx} = \frac{1}{\delta} \int_0^{1-x} \left[ u^*_t(1 - \xi, t) + u^*(1 - \xi, t) u^*_x(1 - \xi, t) \right] e^{\frac{\xi}{\delta}(x-1+\xi)} d\xi \]  

(3.71)

\[ + \left[ k_1 (u^*(1, t)^3 + k_2 u^*(1, t) \right] e^{\frac{\xi}{\delta}(x-1)}, \]

which shows that \( u^*_{xx} \in C([0, T]; L^2(0, 1)) \) and then

\[ u^*_{xxx} = \frac{1}{\delta} (\varepsilon u^*_{xx} - u^*_t - u^*_x u^*_x) \in C([0, T]; L^2(0, 1)), \]  

(3.72)

\[ u^*_{xxxx} = \frac{1}{\delta} (\varepsilon u^*_{xxx} - u^*_x u^*_x - u^*_x^2 - u^*_x u^*_x) \in C([0, T]; L^2(0, 1)). \]  

(3.73)

In the same way, we can prove that

\[ u_{xxt}, u_{xxxx}, u_{xxxxx} \in C([0, T]; L^2(0, 1)). \]

It therefore follows from (1.12) that

\[ u_{xxxxx}, u_{xxxxxxx}, u_{xxxxxx} \in C([0, T]; L^2(0, 1)). \]

Moreover, boundary conditions (2.5)–(3.12) for \( u^* \) follows from equations (1.12)–(1.15). Hence (3.68) holds.

We are now ready to prove Theorems 2.1 and 2.2.

**Proof of Theorems 2.1 and 2.2.** In order to prove the existence and uniqueness of a global solution, we need to establish a priori estimates for the solution of (1.12)–(1.15). We first assume that \( u^0 \in H_{bc}^7(0, 1) \). Then by Proposition 3.1, the problem (1.12)–(1.15) has a unique local classical solution with

\[ u \in C([0, T]; H_{bc}^7(0, 1)) \cap C^1([0, T]; H_{bc}^4(0, 1)) \cap C^2([0, T]; H_{bc}^4(0, 1)), \]
Hence, the calculations performed in the following are valid.

**Step 1: Stability.** We mainly prove the stability estimates of Theorem 2.2. The proof of the stability estimates of Theorem 2.1 is simpler and therefore we just give an outline.

Firstly, the estimate (2.12) follows easily from (1.9).

In order to prove (2.19), we introduce the Lyapunov function

\[ \int_0^1 (x + 1)u(t)^2 \, dx. \]

Using (1.12), we obtain

\[
\frac{d}{dt} \int_0^1 (x + 1)u(t)^2 \, dx = 2 \int_0^1 (x + 1)u(\varepsilon u_x - \delta u_{xxx} - uu_x) \, dx
\]

\[
= -4\delta[k_1 u(1, t)^4 + k_2 u(1, t)^2] - \delta u_x(0, t)^2 - \frac{4}{3} u(1, t)^3
\]

\[
- 3\delta \int_0^1 u_x^2 \, dx - 2\varepsilon \int_0^1 (x + 1)u_x^2 \, dx
\]

\[
- 2\varepsilon \int_0^1 uu_x \, dx + \frac{2}{3} \int_0^1 u^3 \, dx \leq \left( \frac{2}{3} - 4k_1 \delta \right) u(1, t)^4
\]

\[
+ \left( \frac{2}{3} - 4k_2 \delta \right) u(1, t)^2 - \delta u_x(0, t)^2 - (3\delta + \varepsilon) \int_0^1 u_x^2 \, dx
\]

\[
+ \frac{2}{3} \| u_x \| \int_0^1 u^2 \, dx + \varepsilon \int_0^1 u^2 \, dx \leq \left( \frac{2}{3} - 4k_1 \delta \right) u(1, t)^4
\]

\[
+ \left( \frac{2}{3} - 4k_2 \delta \right) u(1, t)^2 - \delta u_x(0, t)^2 - (2\delta + \varepsilon) \int_0^1 u_x^2 \, dx
\]

\[
+ \frac{1}{9\delta} \left( \int_0^1 u^2 \, dx \right)^2 + \varepsilon \int_0^1 u^2 \, dx.
\]

With (3.53), we have

\[
\int_0^1 (x + 1)u^2 \, dx \leq 2 \int_0^1 u^2 \, dx \leq \int_0^1 u_x^2 \, dx.
\]

(3.75)
It therefore follows from (3.74) that
\[
\frac{d}{dt} \int_0^1 (x + 1)u(t)^2 \, dx \leq \left(\frac{2}{3} - 4k_1\delta\right)u(1, t)^4
\]
\[
+ \left(\frac{2}{3} - 4k_2\delta\right)u(1, t)^2 - \delta u_x(0, t)^2 - 2(2\delta + \varepsilon) \int_0^1 u^2 \, dx
\]
\[
+ \frac{1}{9\delta} \left(\int_0^1 u^2 \, dx\right)^2 + \varepsilon \int_0^1 u^2 \, dx \leq \left(\frac{2}{3} - 4k_1\delta\right)u(1, t)^4
\]
\[
+ \left(\frac{2}{3} - 4k_2\delta\right)u(1, t)^2 - \delta u_x(0, t)^2
\]
\[
- \frac{1}{2}(4\delta + \varepsilon) \int_0^1 (x + 1)u^2 \, dx
\]
\[
+ \frac{1}{9\delta} \left(\int_0^1 u^2 \, dx\right)^2 \leq \left(\frac{2}{3} - 4k_1\delta\right)u(1, t)^4
\]
\[
+ \left(\frac{2}{3} - 4k_2\delta\right)u(1, t)^2 - \delta u_x(0, t)^2 \quad \text{(use (2.12))}
\]
\[
- \frac{9\delta(4\delta + \varepsilon) - 2\|u^0\|^2 e^{-2\varepsilon t}}{18\delta} \int_0^1 (x + 1)u^2 \, dx.
\]

Set
\[
T_0 = \begin{cases} 
0, & \|u^0\|^2 \leq \frac{1}{2}\delta(4\delta + \varepsilon), \\
\frac{1}{2\varepsilon} \ln \left(\frac{2\|u^0\|^2}{\delta(4\delta + \varepsilon)}\right), & \|u^0\|^2 > \frac{1}{2}\delta(4\delta + \varepsilon).
\end{cases}
\]

Then for \( t \geq T_0 \) we have
\[
2\|u^0\|^2 e^{-2\varepsilon t} \leq \delta(4\delta + \varepsilon).
\]

It therefore follows from (3.76) that
\[
\frac{d}{dt} \int_0^1 (x + 1)u(t)^2 \, dx \leq -2\omega \int_0^1 (x + 1)u^2 \, dx, \quad \forall t \geq T_0,
\]

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where \( \omega \) is defined by (2.16). Hence we obtain
\[
\int_0^1 u(t)^2 \, dx \leq \int_0^1 (x + 1)u(t)^2 \, dx \\
\leq e^{-2\omega(t-T_0)} \int_0^1 (x + 1)u(x, T_0)^2 \, dx \quad \text{(use (2.12))} 
\]
\[
\leq 2\|u^0\|^2 e^{-2\omega T_0} e^{2\omega T_0} e^{-2\omega t} \\
\leq 2\|u^0\|^2 e^{2\omega T_0} e^{-2\omega t}, \quad \forall t \geq T_0. 
\]

For \( 0 \leq t \leq T_0 \), by (2.12), we have
\[
\int_0^1 u(t)^2 \, dx \leq \|u^0\|^2 \\
\leq \|u^0\|^2 e^{2\omega T_0} e^{-2\omega t}. 
\]

By (3.77), we have
\[
e^{2\omega T_0} \leq \max \left\{ 1, \left( \frac{2\|u^0\|^2}{\delta(4\delta + \epsilon)} \right)^{\frac{\omega}{\epsilon}} \right\} \\
\leq \max \left\{ 1, \left( \frac{2\|u^0\|^2}{\delta(4\delta + \epsilon)} \right)^{\frac{\omega}{\epsilon}} \right\} \left( 1 + \|u^0\|^2 \frac{2\omega}{\epsilon} \right). 
\]

Thus (2.19) follows from (3.79), (3.80) and (3.81).

To prove (2.20), we first estimate the following
\[
B(t) = \left( 4k_1 \delta - \frac{2}{3} \right) u(1, t)^4 + \left( 4k_2 \delta - \frac{2}{3} \right) u(1, t)^2 \\
+ \delta u_x(0, t)^2 + (2\delta + \epsilon) \int_0^1 u_x^2 \, dx. 
\]

By (3.74), we deduce that
\[
\frac{d}{dt} \int_0^1 (x + 1)u(t)^2 \, dx + B(t) \\
\leq \frac{1}{9\delta} \left( \int_0^1 u^2 \, dx \right)^2 + \epsilon \int_0^1 u^2 \, dx, \quad \forall t \geq 0. 
\]
In what follows, we denote by $c = c(\varepsilon, \delta, k_1, k_2)$ a generic positive constant that may vary from line to line. Multiplying (3.83) by $e^{\varepsilon t}$, adding $\omega e^{\varepsilon t} \int_0^1 (x + 1)u(t)^2dx$ to both sides of (3.83) and using (2.19), we obtain

$$
\frac{d}{dt} \left( e^{\varepsilon t} \int_0^1 (x + 1)u(t)^2dx \right) + e^{\varepsilon t} B(t)
\leq \omega e^{\varepsilon t} \int_0^1 (x + 1)u(t)^2dx + \frac{1}{9\delta} e^{\varepsilon t} \left( \int_0^1 u^2 dx \right)^2 + \varepsilon e^{\varepsilon t} \int_0^1 u^2 dx
\leq c \left( \| u^0 \|^2 + \| u^0 \|^4 \right) e^{-\varepsilon t},
$$

(3.84)

where we have used that

$$r^\alpha \leq r^2 + r^{4 + \frac{4\omega}{\varepsilon}}, \text{ for all } r \geq 0 \text{ and } 2 \leq \alpha \leq 4 + \frac{4\omega}{\varepsilon}.
$$

Integrating from 0 to $\infty$, we obtain

$$
\int_0^\infty e^{\varepsilon s} B(s)ds \leq c F(\| u^0 \|),
$$

(3.85)

where the function $F$ is defined by (2.18).

On the other hand, multiplying (1.12) by $u_{xx}$ and integrating from 0 to 1 by parts, we have

$$
\dot{V}(t) = -2\varepsilon \int_0^1 u_{xx}^2 dx + \delta u_{xx}^2 \bigg|_0^1 + 2 \int_0^1 uu_{xx} dx
= -2\varepsilon \int_0^1 u_{xx}^2 dx + \delta [k_1 u(1, t)^3 + k_2 u(1, t)]^2 - \delta u_{xx}(0, t)^2
+ 2 \int_0^1 uu_{xx} dx.
$$

(3.86)

Since

$$u(x, t)^2 = \left( \int_0^x u_x(x, t) dx \right)^2 \leq x \int_0^1 u_x(x, t)^2 dx \leq V(t),
$$

(3.87)

and

$$2 \int_0^1 uu_{xx} dx \leq \frac{1}{\varepsilon} \int_0^1 u^2 u_x^2 dx + \varepsilon \int_0^1 u_{xx}^2 dx
\leq \frac{V^2(t)}{\varepsilon} + \varepsilon \int_0^1 u_{xx}^2 dx,
$$

(3.88)

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we deduce from (3.86) that

\[
\dot{V}(t) \leq -\varepsilon \int_0^1 u_{xx}^2 \, dx + 2\delta k_1^2 V(t)u(1, t)^4 + \frac{1}{\varepsilon} V^2(t) + 2\delta k_2^2 V(t) \tag{3.89}
\]

\[
\leq 2\delta k_1^2 V(t)u(1, t)^4 + \frac{1}{\varepsilon} V^2(t) + 2\delta k_2^2 V(t).
\]

Multiplying (3.89) by \(e^{ot}\), we obtain

\[
\frac{d}{dt}(e^{ot}V(t)) \leq \omega e^{ot}V(t) + \left[ 2\delta k_1^2 V(t)u(1, t)^4 + \frac{1}{\varepsilon} V^2(t) + 2\delta k_2^2 V(t) \right]e^{ot}. \tag{3.90}
\]

Integrating from 0 to \(t\) and using (3.85), we obtain

\[
e^{ot}V(t) \leq V(0) + (\omega + 2\delta k_2^2) \int_0^t e^{os} V(s) \, ds
\]

\[
+ \int_0^t \left[ \frac{V(s)}{\varepsilon} + 2\delta k_1^2 u(1, s)^4 \right] V(s)e^{os} \, ds \leq V(0) + cF(\|u^0\|) \tag{3.91}
\]

\[
+ \int_0^t \left[ \frac{V(s)}{\varepsilon} + 2\delta k_1^2 u(1, s)^4 \right] V(s)e^{os} \, ds \leq cF(\|u^0\|_{H^1})
\]

\[
+ \int_0^t \left[ \frac{V(s)}{\varepsilon} + 2\delta k_1^2 u(1, s)^4 \right] V(s)e^{os} \, ds.
\]

It therefore follows from (3.85) and Gronwall-Bellman's inequality ([9, p.63]) that

\[
e^{ot}V(t) \leq cF(\|u^0\|_{H^1}) \exp \left[ \int_0^t \left( \frac{V(s)}{\varepsilon} + 2\delta k_1^2 u(1, s)^4 \right) \, ds \right] \tag{3.92}
\]

\[
\leq cF(\|u^0\|) \exp \left[ cF(\|u^0\|) \right].
\]

Consequently, (2.20) follows from the embedding theorem (see [1, p.97]).

We now prove (2.21). As in (3.74), we have

\[
\frac{d}{dt} \int_0^1 (x + 1)u(t)^2 \, dx = -2\varepsilon \int_0^1 (x + 1)u_{xt}(t)^2 \, dx
\]

\[
- 2\varepsilon \int_0^1 u_{xt}(t)u_x(t) \, dx - 3\delta \int_0^1 u_{xt}(t)^2 \, dx
\]

\[
- 2 \int_0^1 (x + 1)u_x(t)(u_x(t)u_x(t) + u(t)u_{xt}(t)) \, dx - \delta u_{xx}(0, t)^2
\]
\[ -4\varepsilon \int_0^1 u_x(t)^2 \, dx + \varepsilon \int_0^1 u_t(t)^2 \, dx - 3\delta \int_0^1 u_{xt}(t)^2 \, dx \]
\[ \leq -\varepsilon \int_0^1 u_t(t)^2 \, dx - 2\delta \int_0^1 u_{xt}(t)^2 \, dx + \frac{16}{\delta} \|u_t(t)\|^2 \|u_x(t)\|^2 \]
\[ \leq -(4\delta + \varepsilon) \int_0^1 u_t(t)^2 \, dx + \frac{16}{\delta} \|u_t(t)\|^2 \|u_x(t)\|^2 \]
\[ \leq -\frac{1}{2} (4\delta + \varepsilon) \int_0^1 (x + 1) u_t(t)^2 \, dx + \frac{16}{\delta} \|u_t(t)\|^2 \|u_x(t)\|^2. \]

Multiplying (3.93) by \(e^{\omega t}\), we obtain
\[
\frac{d}{dt} \left( e^{\omega t} \int_0^1 (x + 1) u_t(t)^2 \, dx \right) \leq \omega e^{\omega t} \int_0^1 (x + 1) u_t(t)^2 \, dx
\]
\[ - \frac{1}{2} (4\delta + \varepsilon) e^{\omega t} \int_0^1 (x + 1) u_t(t)^2 \, dx \]
\[ + \frac{16}{\delta} \|u_x(t)\|^2 e^{\omega t} \int_0^1 (x + 1) u_t(t)^2 \, dx \]
\[ \leq \frac{16}{\delta} \|u_x(t)\|^2 e^{\omega t} \int_0^1 (x + 1) u_t(t)^2 \, dx. \]

It therefore follows from (3.85) that
\[
\|u_t(t)\|^2 \leq 2\|u_t(0)\|^2 \exp \left( \int_0^t \frac{16\|u_x(s)\|^2}{\delta} \, ds \right) e^{-\omega t}
\]
\[ \leq c(\|u^0\|_{H^3}^2 + \|u^0\|_{H^1}^4) \exp \left( c F(\|u^0\|) \right) e^{-\omega t} \]
\[ \leq c F(\|u_0\|_{H^3}) \exp \left( c F(\|u^0\|) \right) e^{-\omega t}. \]

By (3.71), we have
\[
u_{xx} = \frac{1}{\delta} \int_0^{1-x} [u_t(1-\tau, t) + u(1-\tau, t) u_x(1-\tau, t)] e^{\frac{x}{\delta}(1+\tau)} \, d\tau
\]
\[ + [k_1(u(1, t) + k_2 u(1, t)] e^{\frac{x}{\delta}(1-1)} \]

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which, combining with (2.20) and (3.95), implies that

\[ \|u_{xx}\|^2 \leq c(\|u_t\|^2 + \|u_x\|^2 + \|u_x\|^4 + \|u_x\|^6) \]

\[ \leq c \sum_{i=1}^{3} F_i(\|u^0\|_{H^3}) \exp\left[ c F(\|u^0\|) \right] e^{-\omega t}. \]  

(3.97)

By equation (1.12), we obtain

\[ \|u_{xxx}\|^2 \leq c(\|u_{xx}\|^2 + \|u_t\|^2 + \|u_x\|^4) \]

\[ \leq c \sum_{i=1}^{3} F_i(\|u^0\|_{H^3}) \exp\left[ c F(\|u^0\|) \right] e^{-\omega t}. \]  

(3.98)

It therefore follows that

\[ \|u\|^2_{H^3} = \|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxx}\|^2 \leq 3\|u_{xx}\|^2 + \|u_{xxx}\|^2 \]

\[ \leq c \sum_{i=1}^{3} F_i(\|u^0\|_{H^3}) \exp\left[ c F(\|u^0\|) \right] e^{-\omega t}. \]  

(3.99)

This is (2.21).

The proof of (2.13) and (2.15) is the same and we give an outline here. Set

\[ B_1(t) = \left(2\delta k_1 - \frac{1}{3}\right)u(1, t)^4 + \left(2\delta k_2 - \frac{1}{3}\right)u(1, t)^2. \]  

(3.100)

By (1.9), we deduce that

\[ \dot{E}(t) + 2\varepsilon V(t) + B_1(t) \leq 0. \]  

(3.101)

Multiplying (3.101) by \( e^{\varepsilon t} \) and integrating from 0 to \( \infty \), we obtain

\[ 2\varepsilon \int_0^\infty e^{\varepsilon s} V(s) ds + \int_0^\infty e^{\varepsilon s} B_1(s) ds \leq 2E(0). \]  

(3.102)

Multiplying (3.89) by \( e^{\varepsilon t} \), we obtain

\[ \frac{d}{dt}\left( e^{\varepsilon t} V(t) \right) \leq \varepsilon e^{\varepsilon t} V(t) \]

\[ + \left[ 2\delta k_1^2 V(t)u(1, t)^4 + \frac{1}{\varepsilon} V^2(t) + 2\delta k_2^2 V(t) \right] e^{\varepsilon t}, \]  

(3.103)
from which, as in the proof of (2.20), we can deduce (2.13). Further, as in the proof of (3.93) (actually simpler than (3.93)), we have

\[
\frac{d}{dt}(\|u_t(t)\|^2) \leq -\varepsilon \|u_t(t)\|^2 + \frac{\|u_x\|^2}{\varepsilon} \|u_t(t)\|^2. \tag{3.104}
\]

Multiplying (3.104) by \(e^{\varepsilon t}\), we obtain

\[
\frac{d}{dt}(\|u_t(t)\|^2 e^{\varepsilon t}) \leq \frac{\|u_x\|^2}{\varepsilon} \|u_t(t)\|^2 e^{\varepsilon t}. \tag{3.105}
\]

It therefore follows from (3.102) that

\[
\|u_t(t)\|^2 \leq \|u_t(0)\|^2 \exp \left( \int_0^t \frac{\|u_x(s)\|^2}{\varepsilon} ds \right) e^{-\varepsilon t} \tag{3.106}
\]

\[
\leq c(\|u_0^0\|^2_{H^3} + \|u_0^0\|^4_{H^3}) \exp(\|u^0\|^2/\varepsilon^2) e^{-\varepsilon t}.
\]

Then, as in the proof of (2.21), we can easily obtain (2.15).

By a density argument, we now show that estimates (2.12), (2.13), (2.19) and (2.20) hold for \(u^0 \in \mathcal{H}^1_0(0, 1)\) and (2.15) and (2.21) hold for \(u^0(x) \in H^3_{bc}(0, 1)\). For this, we have to establish the continuous dependence of solutions with respect to initial data, that is

\[
\|u_1(t) - u_2(t)\|^2 \leq C(\|u_1^0\|, \|u_2^0\|) \|u_1^0 - u_2^0\|^2, \quad t \geq 0, \tag{3.107}
\]

\[
\|u_{1x}(t) - u_{2x}(t)\|^2 \leq C(\|u_1^0\|, \|u_2^0\|) \|u_{1x}^0 - u_{2x}^0\|^2, \quad t \geq 0, \tag{3.108}
\]

\[
\|u_1(t) - u_2(t)\|_{H^3}^2 \leq C(\|u_1^0\|_{H^3}, \|u_2^0\|_{H^3}) \|u_1^0 - u_2^0\|_{H^3}^2, \quad t \geq 0, \tag{3.109}
\]

We first establish (3.107). Replacing \(w_1, w_2\) and \(\eta\) in (3.48)–(3.51) by \(u_1, u_2\) and \(z\), respectively, we obtain

\[
z_t - \varepsilon z_{xx} + \delta z_{xxx} + u_1 z_x + u_2 z_{xx} = 0, \quad 0 < x < 1, \quad 0 < t < T, \tag{3.110}
\]

\[
z(0, t) = z_x(1, t) = 0, \quad 0 < t < T, \tag{3.111}
\]

\[
z_{xx}(1, t) = k_1(u_1(1, t)^3 - u_2(1, t)^3) + k_2 z(1, t), \quad 0 < t < T, \tag{3.112}
\]

\[
z(x, 0) = z^0, \quad 0 < x < 1. \tag{3.113}
\]

Noting that

\[
u_1^2 + u_1 u_2 + u_2^2 \geq \frac{1}{2} (u_1^2 + u_2^2) \geq 0, \tag{3.114}
\]
we obtain
\[
\frac{d}{dt} \int_0^1 z(t)^2 \, dx = 2 \int_0^1 z(ez_{xx} - \delta z_{xxx} - u_1z_x - zu_2) \, dx \\
\leq -2\delta z(1, t)z_{xx}(1, t) - \delta z_x(0, t)^2 - 2\varepsilon \|z_x\|^2 \\
\quad + 2\|z\| (\|u_1\| \|z_x\| + \|z_x\| \|u_2\|) \\
\leq -2\delta z(1, t)^2[k_1(u_1^2 + u_1u_2 + u_2^2) + k_2] \\
\quad - \delta z_x(0, t)^2 - \varepsilon \|z_x\|^2 + c\|z\|^2 (\|u_1\|^2 + \|u_2\|^2) \\
\leq c\|z\|^2 (\|u_1\|^2 + \|u_2\|^2).
\] (3.115)

Using Gronwall's inequality and (3.85), we obtain
\[
\|z(t)\|^2 = \|u_1(t) - u_2(t)\|^2 \\
\leq \|u_1^0 - u_2^0\|^2 \exp \left( c \int_0^t (\|u_1(s)\|^2 + \|u_2(s)\|^2) \, ds \right) \\
\leq C(\|u_1^0\|, \|u_2^0\|) \|u_1^0 - u_2^0\|^2, \quad t \geq 0.
\] (3.116)

We now establish (3.108). Integrating (3.115) from 0 to $t$ and using (3.116), we obtain
\[
\|z(t)\|^2 - \|z(0)\|^2 + \int_0^t (\delta z_x(0, s)^2 + \varepsilon \|z_x(s)\|^2) \, ds \\
\leq c \int_0^t \|z(s)\|^2 (\|u_1(s)\|^2 + \|u_2(s)\|^2) \, ds \\
\leq C(\|u_1^0\|, \|u_2^0\|) \|u_1^0 - u_2^0\|^2,
\] (3.117)

which implies
\[
\int_0^\infty (z_x(0, t)^2 + \|z_x(t)\|^2) \, dt \leq C(\|u_1^0\|, \|u_2^0\|) \|u_1^0 - u_2^0\|^2.
\] (3.118)

Further, we have
\[
\frac{d}{dt} \int_0^1 z_x(t)^2 \, dx = -2 \int_0^1 z_x(ez_{xx} - \delta z_{xxx} - u_1z_x - zu_2) \, dx \\
\leq 2\delta z_{xx}(1, t)^2 - 2\delta z_{xx}(0, t)^2 - 2\varepsilon \|z_{xx}\|^2 \\
\quad + 2\|z_{xx}\| (\|u_1\| \|z_x\| + \|z_x\| \|u_2\|) \\
\leq c\|z_x\|^2 + c\|z_x\|^2 (u_1(1, t)^4 + \|u_1\|^2 + u_2(1, t)^4 + \|u_2\|^2).
\] (3.119)
It therefore follows from (3.118) that
\[
\|z_x(t)\|^2 \leq \|z_x(0)\|^2 + c \int_0^\infty \|z_x(t)\|^2 dt \\
+ c \int_0^t (\|z_x(s)\|^2 (u_1(1, s))^4 + \|u_{1x}(s)\|^2 + u_2(1, s)^4 + \|u_{2x}(s)\|^2) ds \\
\leq C(\|u_1^0\|, \|u_2^0\|) \|u_1^0 - u_2^0\|^2 \\
+ c \int_0^t (\|z_x(s)\|^2 (u_1(1, s))^4 + \|u_{1x}(s)\|^2 + u_2(1, s)^4 + \|u_{2x}(s)\|^2) ds,
\]
which, combining with (3.85) and Gronwall's inequality, implies that
\[
\|z_x(t)\|^2 = \|u_{1x}(t) - u_{2x}(t)\|^2 \\
\leq C(\|u_1^0\|, \|u_2^0\|) \|u_1^0 - u_2^0\|^2, \quad t \geq 0.
\]
To establish (3.109), we need to estimate \(\|z_x(t)\|\). Using (3.110) and integrating by parts, we obtain
\[
\frac{d}{dt} \int_0^1 z_x(t)^2 dx \\
= 2 \int_0^1 z_x(\varepsilon z_{xxt} - \delta z_{xxt} - u_{1x} z_x - u_1 z_{xt} - z_t u_{2x} - z_t u_{2xt}) dx \\
\leq -2\delta z_t(1, t) z_{xxt}(1, t) - \delta z_{xt}(0, t)^2 - 2\varepsilon \|z_{xt}\|^2 \\
+ 2\|z_x\| (\|u_{1x}\| \|z_x\| + \|u_{1x}\| \|z_t\| + \|u_{2x}\| \|z_t\| + \|z\| \|u_{2xt}\|).
\]
Since for a generic function \(\phi\)
\[
\phi(1)^2 = \int_0^1 \phi(1)^2 dx \\
= \int_0^1 (\phi(x))^2 + 2 \int_x^1 \phi(\xi) \phi(\xi) d\xi dx \\
\leq \|\phi\|^2 + 2 \|\phi\| \|\phi\|,
\]
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we obtain
\[-z_t(1, t)z_{xxt}(1, t) = -z_t(1, t)^2[k_1(u_1(1, t)^2 + u_1(1, t)u_2(1, t)
+ u_2(1, t)^2) + k_2] - k_1z_t(1, t)z(1, t)[2u_1(1, t)u_{1t}(1, t)
+ u_{1t}(1, t)u_2(1, t) + u_1(1, t)u_{2t}(1, t) + 2u_2(1, t)u_{2t}(1, t)]\]
(drop out the first negative term)

\[\leq c(\|z_t\| + \|z_t\|^{1/2}\|z_{xt}\|^{1/2})(\|z\| + \|z\|^{1/2}\|z_x\|^{1/2})\]

\[\times \sum_{i,j=1}^2 \|u_{ix}\| (\|u_{jt}\| + \|u_{jt}\|^{1/2}\|u_{jxt}\|^{1/2}).\]

We now use Young’s inequality to estimate above every term as follows (the following \(\sigma\) is a generic positive constant depending on \(\sigma\)):

\[\|z_t\|\|z\|\|u_{ix}\|\|u_{jt}\| \leq \frac{1}{2}\|z_t\|^2\|u_{ix}\|^2 + \frac{1}{2}\|z\|^2\|u_{jt}\|^2, \quad (3.125)\]

\[\|z_t\|\|z\|\|u_{ix}\|\|u_{jt}\|^{1/2}\|u_{jxt}\|^{1/2}\]

\[\leq \frac{1}{2}\|z_t\|^2\|u_{ix}\|^2 + \frac{1}{4}\|z\|^2(\|u_{jt}\|^2 + \|u_{jxt}\|^2), \quad (3.126)\]

\[\|z_t\|\|z\|^{1/2}\|z_x\|^{1/2}\|u_{ix}\|\|u_{jt}\|\]

\[\leq \frac{1}{2}\|z_t\|^2\|u_{ix}\|^2 + \frac{1}{2}\|z\|\|z_x\|\|u_{jt}\|^2 \quad (3.127)\]

\[\leq \frac{1}{2}\|z_t\|^2\|u_{ix}\|^2 + \frac{1}{4}(\|z\|^2 + \|z_x\|^2)\|u_{jt}\|^2,\]

\[\|z_t\|\|z\|^{1/2}\|z_x\|^{1/2}\|u_{ix}\|\|u_{jt}\|^{1/2}\|u_{jxt}\|^{1/2}\]

\[\leq \frac{1}{2}\|z_t\|^2\|u_{ix}\|^2 + \frac{1}{2}\|z\|\|z_x\|\|u_{jt}\|\|u_{jxt}\| \quad (3.128)\]

\[\leq \frac{1}{2}\|z_t\|^2\|u_{ix}\|^2 + \frac{1}{4}(\|z\|^2\|u_{jxt}\|^2 + \|z_x\|^2\|u_{jt}\|^2),\]
\[ \|z_t\|^{1/2} \|z_{xt}\|^{1/2} \|z\| \|u_{ix}\| \|u_{jr}\| \]
\[ \leq \frac{1}{2} \|z\|^2 \|u_{jr}\|^2 + \frac{1}{2} \|z_t\| \|z_{xt}\| \|u_{ix}\|^2 \]
\[ \leq \frac{1}{2} \|z\|^2 \|u_{jr}\|^2 + \sigma \|z_{xt}\|^2 + c(\sigma) \|z_t\|^2 \|u_{ix}\|^4, \]

(3.129)

\[ \|z_t\|^{1/2} \|z_{xt}\|^{1/2} \|z\| \|u_{ix}\| \|u_{jr}\| \|u_{jx_t}\|^{1/2} \]
\[ \leq \frac{1}{2} \|z_t\| \|z_{xt}\| \|u_{ix}\|^2 + \frac{1}{2} \|z\| \|u_{jr}\| \|u_{jx_t}\| \]
\[ \leq \sigma \|z_{xt}\|^2 + c(\sigma) \|z_t\|^2 \|u_{ix}\|^4 + \frac{1}{4} \|z\|^2 (\|u_{jr}\|^2 + \|u_{jx_t}\|^2), \]

(3.130)

\[ \|z_t\|^{1/2} \|z_{xt}\|^{1/2} \|z\| \|z_{xx}\|^{1/2} \|z_{x}\|^{1/2} \|u_{ix}\| \|u_{jr}\| \]
\[ \leq \frac{1}{2} \|z_t\| \|z_{xt}\| \|u_{ix}\|^2 + \frac{1}{2} \|z\| \|z_{x}\| \|u_{jr}\|^2 \]
\[ \leq \sigma \|z_{xt}\|^2 + c(\sigma) \|z_t\|^2 \|u_{ix}\|^4 + \frac{1}{4} (\|z\|^2 + \|z_{x}\|^2) \|u_{jr}\|^2, \]

(3.131)

\[ \|z_t\|^{1/2} \|z_{xt}\|^{1/2} \|z\| \|z_{xx}\|^{1/2} \|z_{x}\|^{1/2} \|u_{ix}\| \|u_{jr}\| \|u_{jx_t}\|^{1/2} \]
\[ \leq \frac{1}{2} \|z_t\| \|z_{xt}\| \|u_{ix}\|^2 + \frac{1}{2} \|z\| \|z_{x}\| \|u_{jr}\| \|u_{jx_t}\| \]
\[ \leq \sigma \|z_{xt}\|^2 + c(\sigma) \|z_t\|^2 \|u_{ix}\|^4 + \frac{1}{4} (\|z\|^2 \|u_{jx_t}\|^2 + \|z_{x}\|^2 \|u_{jr}\|^2). \]

(3.132)

By taking \( \sigma \) small enough, we deduce
\[ -z_t(1, t)z_{xx}(1, t) \leq \varepsilon \|z_{xt}\|^2 \]
\[ + c(z) \sum_{i=1}^{2} (\|u_{ix}\|^2 + \|u_{ix}\|^2) + c(z) \sum_{i=1}^{2} \|u_{ix}\|^2 \]
\[ + c(z) \sum_{i=1}^{2} (\|u_{ix}\|^2 + \|u_{ix}\|^4). \]

(3.133)
Hence, by (3.122), we obtain

$$\frac{d}{dt} \int_0^1 z_i(t)^2 dx \leq c\|z\|^2 \sum_{i=1}^2 (\|u_{it}\|^2 + \|u_{ixt}\|^2) + c\|z_x\|^2 \sum_{i=1}^2 \|u_{it}\|^2$$

$$+ c\|z_t\|^2 \sum_{i=1}^2 (\|u_{ix}\|^2 + \|u_{ix}\|^4).$$

(3.134)

Moreover, by the first part of (3.93), we have

$$\frac{d}{dt} \int_0^1 (x+1)u_{it}(t)^2 dx$$

$$+ 2\delta \int_0^1 u_{ixt}(t)^2 dx \leq \frac{16}{\delta} \|u_{it}(t)\|^2 \|u_{ix}(t)\|^2.$$

(3.135)

Integrating from 0 to $\infty$, we deduce that

$$\int_0^\infty \sum_{i=1}^2 \|u_{ixt}\|^2 dt \leq C(\|u_1^0\|_{H^3}, \|u_2^0\|_{H^3}).$$

(3.136)

It therefore follows from (2.20), (3.95), (3.116), (3.118) and (3.134) that

$$\|z_t(t)\|^2 \leq \|z_t(0)\|^2$$

$$+ c \int_0^\infty \|z(s)\|^2 \sum_{i=1}^2 (\|u_{is}(s)\|^2 + \|u_{ixs}(s)\|^2) ds$$

$$+ c \int_0^\infty \|z_x(s)\|^2 \sum_{i=1}^2 \|u_{is}(s)\|^2 ds$$

$$+ c \int_0^t \|z(s)\|^2 \sum_{i=1}^2 (\|u_{ix}(s)\|^2 + \|u_{ix}(s)\|^4) ds \leq \|z_t(0)\|^2$$

$$+ C(\|u_1^0\|, \|u_2^0\|) \|u_1^0 - u_2^0\|^2 \int_0^\infty \sum_{i=1}^2 (\|u_{is}(s)\|^2 + \|u_{ixs}(s)\|^2) ds$$

$$+ C(\|u_1^0\|, \|u_2^0\|) \|u_1^0 - u_2^0\|^2 \int_0^\infty \sum_{i=1}^2 \|u_{is}(s)\|^2 ds$$

$$+ c \int_0^t \|z(s)\|^2 \sum_{i=1}^2 (\|u_{ix}(s)\|^2 + \|u_{ix}(s)\|^4) ds$$

(3.137)

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\[ \leq C(\|u_1^0\|_{H^3}, \|u_2^0\|_{H^3})\|u_1^0 - u_2^0\|_{H^3}^2 \]
\[ + c \int_0^t \|z_2(s)\|^2 \sum_{i=1}^2 (\|u_{ix}(s)\|^2 + \|u_{ix}(s)\|^4) ds, \]
which, by (2.20) and Gronwall’s inequality, implies that
\[ \|u_{1t}(t) - u_{2t}(t)\|^2 \leq C(\|u_1^0\|_{H^3}, \|u_2^0\|_{H^3})\|u_1^0 - u_2^0\|_{H^3}^2, \quad t \geq 0. \] (3.138)

Hence, as in the proof of (3.99), we can obtain
\[ \|u_1(t) - u_2(t)\|_{H^3}^2 \leq C(\|u_1^0\|_{H^3}, \|u_2^0\|_{H^3})\|u_1^0 - u_2^0\|_{H^3}^2, \quad t \geq 0. \] (3.139)
Inequalities (3.116), (3.121) and (3.139) and Lemma 3.2 are sufficient for us to use the density argument.

**Step 2: Existence and uniqueness of global classical solutions.** Using inequalities (3.138) and (3.139), by a density argument, we can prove that problem (1.12)–(1.15) has a unique global classical solution for \( u^0 \in H^3_{bc}(0, 1) \), respectively. Indeed, for \( u^0 \in H^3_{bc}(0, 1) \), it follows from Lemma 3.2 that there exists a sequence \( \{u_n^0\} \subset H_{bc}^7(0, 1) \) such that \( u_n^0 \) converges to \( u^0 \) in \( H^3(0, 1) \). By (3.138) and (3.139), the solution \( u_n \) corresponding to the initial condition \( u_n^0 \) converges to a function \( u \) in \( C^1([0, T]; L^2(0, 1)) \cap C([0, T]; H^3_{bc}(0, 1)) \). Thus we can pass to the limit in (1.12)–(1.15) for \( u_n \) and then \( u \) satisfies (1.12)–(1.15). Furthermore, since estimate (2.21) shows that the solution never blows up in finite time, the local solution can be continued to \( (0, +\infty) \). The uniqueness is the direct consequence of (3.139).

**Step 3: Existence and uniqueness of global weak solutions.** For \( u^0 \in \mathcal{H}_0^1(0, 1) \), it follows from Lemma 3.2 that there exists a sequence \( \{u_n^0\} \subset H^3_{bc}(0, 1) \) such that \( u_n^0 \) converges to \( u^0 \) in \( H^1(0, 1) \). By (3.121), the solution \( u_n \) corresponding to the initial condition \( u_n^0 \) converges to a function \( u \) in \( C([0, T]; \mathcal{H}_0^1(0, 1)) \). By Definition 2.1, \( u \) is a weak solution of (1.12)–(1.15). Furthermore, since estimate (2.20) shows that the solution never blows up in finite time, the local solution can be continued to \( (0, +\infty) \). The uniqueness is the direct consequence of (3.121).

This completes the proof. \( \square \)

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Remark 3.1. From the definition (3.77) of \( T_0 = T_0(\epsilon, \delta, \|u^0\|) \) we see that \( T_0 = 0 \) if \( \|u^0\| \leq \frac{1}{2}\delta(4\delta + \epsilon) \). Thus Theorem 2.2 still holds for \( \epsilon = 0 \) and \( \|u^0\| \leq 2\delta^2 \) (we need to make a slight change in the above proof by considering \( \frac{d}{dt} \int_0^1 (1+x)u_x^2 \, dx \) instead of \( \dot{V} \) in (3.86)). This means that the KdV equation (i.e., \( \epsilon = 0 \)) is locally stabilizable. However, if \( \|u^0\| \) is large, then \( T_0(\epsilon, \delta, \|u^0\|) \to \infty \) as \( \epsilon \to 0 \). Hence the problem of global boundary stabilization of the KdV equation is open. On the other hand, if we introduce one more control so that the problem (1.12)–(1.15) becomes

\begin{align}
  u_t + \delta u_{xxx} + uu_x &= 0, & 0 < x < 1, & t > 0, \\
  u(0, t) &= 0, & t > 0, \\
  u_x(0, t) &= \sqrt{3u_x(1, t)^2 + \int_0^1 u^2 \, dx}, & t > 0, \\
  u_{xx}(1, t) &= k_1u^3(1, t) + k_2u(1, t)], & t > 0, \\
  u(x, 0) &= u^0(x), & 0 < x < 1,
\end{align}

the controlled system is \( H^1 \) exponentially stable. But we are not able to prove that it is well posed. The \( H^1 \) stability can be seen from the following inequalities

\begin{align}
  \frac{d}{dt} \int_0^1 u^2 \, dx &\leq \left( \frac{1}{3} - 2\delta k_1 \right) u(1, t)^4 + \left( \frac{1}{3} - 2\delta k_2 \right) u(1, t)^2 \\
  \quad - \delta \int_0^1 u^2 \, dx - 2u_x(1, t)^2 &\leq -\delta \int_0^1 u^2 \, dx,
\end{align}

\begin{align}
  \frac{d}{dt} \int_0^1 (1+x)u^2 \, dx &\leq 2\left( \frac{1}{3} - 2\delta k_1 \right) u(1, t)^4 \\
  \quad + 2\left( \frac{1}{3} - 2\delta k_2 \right) u(1, t)^2 + 2\delta u(1, t)u_x(1, t) \\
  \quad + 2\delta u_x(1, t)^2 - \delta u_x(0, t)^2 - 3\delta \int_0^1 u_x^2 \, dx + \frac{2}{3} \int_0^1 u^3 \, dx,
\end{align}
and

\[
\frac{d}{dt} \int_0^1 (1 + x)u_x^2 \, dx \leq 3\delta \left[ k_1 u(1, t)^3 + k_2 u(1, t) \right]^2 \\
+ \delta \left[ u_x(1, t)^2 + u_x(0, t)^2 \right] - 3\delta \int_0^1 u_{xx}^2 \, dx \\
+ 4\|u_x\|^2(\|u_x\| + \|u_{xx}\|).
\]

(3.147)

**Remark 3.2.** If we define the \( C([0, T]; L^2(0, 1)) \)-limit \( u \) of the sequence \( \{u_n\} \) of smooth solutions as an ultraweak solution of problem (1.12)–(1.15), then inequality (3.116) shows that problem (1.12)–(1.15) has a unique ultraweak solution \( u \in C([0, T]; L^2(0, 1)) \) for the initial data \( u^0 \in L^2(0, 1) \).

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