# Stabilization of Stochastic Nonlinear Systems Driven by Noise of Unknown Covariance

Hua Deng, Miroslav Krstić, and Ruth J. Williams

***Abstract*—This paper poses and solves a new problem of stochastic (nonlinear) disturbance attenuation where the task is to make the system solution bounded (in expectation, with appropriate nonlinear weighting) by a monotone function of the supremum of the covariance of the noise. This is a natural stochastic counterpart of the problem of input-to-state stabilization in the sense of Sontag. Our development starts with a set of new global stochastic Lyapunov theorems. For an exemplary class of stochastic strict-feedback systems with vanishing nonlinearities, where the equilibrium is preserved in the presence of noise, we develop an adaptive stabilization scheme (based on tuning functions) that requires no *a priori* knowledge of a bound on the covariance. Next, we introduce a control Lyapunov function formula for stochastic disturbance attenuation. Finally, we address optimality and solve a differential game problem with the control and the noise covariance as opposing players; for strict-feedback systems the resulting Isaacs equation has a closed-form solution.**

***Index Terms*—Adaptive backstepping, control Lyapunov functions, input-to-state stability (ISS), inverse optimality, Sontag formula, stability in probability, stochastic disturbance attenuation.**

## I. Introduction

### A. Prior Work

EVER since the emergence of stochastic stabilization theory in the 1960s [31], progress has been plagued by a fundamental technical obstacle in the Lyapunov analysis—the Itô differentiation introduces not only the gradient but also the Hessian of the Lyapunov function. This diverted the attention from stabilization to optimization, including the risk-sensitive control problem [3], [12], [13], [25], [34], [38] and other problems [22], [23], effectively replacing the Lyapunov problem by an even more difficult problem of solving a Hamilton–Jacobi–Bellman PDE.

Progress on stabilization of *deterministic* systems was equally discouraging until the advances in differential geometric theory of the 1980s [24] and the discovery of a simple constructive formula for Lyapunov stabilization [41], which have created a flurry of activity in robust, adaptive, and optimal nonlinear control [18], [29], [39]. These advances have naturally led to re-examining the stochastic stabilization problem. It would be fair to say that it was Florchinger [14]–[17], who revamped the area of stochastic stabilization. However, Pan, and Başar [36] were the first to solve the stabilization problem for the class of strict-feedback systems representative of (robust and adaptive) stabilization results for deterministic systems [29]. Even though their starting point was a risk-sensitive cost criterion, their result guarantees global asymptotic stability in probability. Deng and Krstić [6] developed a simpler (algorithmic) design for strict feedback systems and then extended the results on inverse optimal stabilization for general systems to the stochastic case [6]. They also designed the first scheme for stochastic output-feedback nonlinear systems [7]. Based on his new concept of "stochastic expISS," Tsinias [45] developed both state-feedback and output-feedback backstepping schemes for systems with unity intensity noise.

### B. Motivation

We consider systems of the form

$$dx = f(x)\,dt + g_1(x)\Sigma(t)\,dw + g_2(x)u\,dt \qquad (1.1)$$

where

| | |
|---|---|
| $w$ | standard Wiener process; |
| $\Sigma(\cdot)$ | time-dependent, nonnegative–definite matrix valued function; |
| $\Sigma(\cdot)\Sigma(\cdot)^{\mathrm{T}}$ | infinitesimal covariance function of the driving noise $\Sigma(\cdot)dw$. |

In all of the results that guarantee global asymptotic stability in probability [5]–[7], [15]–[17], [36] it is assumed that $g_1(0)=0$ and $\Sigma(\cdot)\equiv I$. The assumption $g_1(0)=0$ excludes linear systems $dx = Ax\,dt + B_1\Sigma(t)\,dw + B_2u\,dt$ where the noise is additive and nonvanishing. Also, in linear quadratic control, the assumption $\Sigma(\cdot)\equiv I$ is avoided by absorbing the noise covariance into the value function, which allows $\Sigma(\cdot)$ to be unknown and the control design to be independent of $\Sigma(\cdot)$ and $B_1$. This is not possible in the nonlinear case and $g_1(x)$ must be accounted for in the control design to allow arbitrary unknown $\Sigma(\cdot)$.

The above discussion leads to the following objective: design a feedback control law for system (1.1) that makes some positive–definite, radially unbounded function of the solution $x(t)$ bounded (in expectation) by some monotone function of $\sup_t |\Sigma(t)\Sigma(t)^{\mathrm{T}}|$. This is a natural objective when no bound on $\Sigma(\cdot)$ is known to the designer and/or $g_1(0)\neq 0$. This objective is a stochastic counterpart of the deterministic input-to-state stability (ISS) [42] where $|x(t)|$ is bounded by a monotone function of the supremum of the disturbance. Since in the stochastic case it would make no sense to bound the solutions by the supremum

Manuscript received October 6, 2000; revised January 9, 2001. Recommended by Associate Editor Q. Zhang. This work was supported in part by the National Science Foundation under Grants ECS-9624386, DMS-0071408, and in part by the Air Force Office of Scientific Research under Grant F496209610223. Finalist for the Student Best Paper Award, 1998 American Control Conference.

H. Deng and M. Krstić are with the Department of MAE, University of California at San Diego, La Jolla, CA 92093 USA (e-mail: huadeng@mae.ucsd.edu; krstic@ucsd.edu).

R. J. Williams is with the Department of Mathematics, University of California at San Diego, La Jolla, CA 92093 USA (e-mail: williams@math.ucsd.edu).

Publisher Item Identifier S 0018-9286(01)07681-4.

of the noise which may be unbounded, we view the bounding by the supremum of the norm of the covariance as the most natural *disturbance attenuation* problem in the stochastic setting.

### C. Results of the Paper

Our presentation starts with stochastic Lyapunov theorems in Section II with proofs that refine those in Kushner [31] and Khasminskii [28], with an emphasis on the *global* aspects, with a stochastic version of the convergence result of LaSalle [32] and Yoshizawa [48], and with an elegant class $\mathcal{K}$ [21], [27] formalism.

In Section III we let $g_1(0) = 0$ in which case the equilibrium at the origin can be preserved in the presence of noise. We use an adaptive control technique which estimates $\sup_t |\Sigma(t)\Sigma(t)^{\mathrm{T}}|$ and tunes one control parameter to achieve regulation of $x(t)$ (in probability) in stochastic strict-feedback systems. This class of systems was dealt with in [5], [36] under the assumption that $\Sigma(\cdot) \equiv I$. Our design is based on adaptive backstepping with tuning functions [29].

In Section IV, we develop a control algorithm for stochastic disturbance attenuation in strict-feedback systems. The resulting system solutions are bounded (in expectation) by a monotone function of the supremum of the norm of the noise covariance (plus a decaying effect of initial conditions). This concept is related to various ergodic concepts in the literature [9], [28] and is different from Tsinias' stochastic ISS [45] where the solution of a stochastic system with two disturbances, one stochastic and one deterministic, is bounded by a bound on the deterministic disturbance. Our approach employs *quartic* Lyapunov functions introduced in [5]. Nevertheless, when applied to the linear case, the control law remains linear.

In Section V, we introduce the concept of a noise-to-state Lyapunov function (ns-lf) which is a stochastic extension of Sontag's ISS Lyapunov functions [42]. In Section V we also define an ns-control Lyapunov function and show that a continuous feedback always exists that makes it an ns-lf. This result is the stochastic version of Sontag's "universal formula" [41] and its several extensions to systems with uncertainties [18], [29], [43]; it also strengthens the formula of Florchinger [16] which applies only when $g_1(0) = 0$ and $\Sigma(\cdot) \equiv I$.

In Section VI, we prove that the formula given in Section V to guarantee the existence of ns-lf for the system (1.1) is optimal with respect to a differential game of the form

$$\inf_u \sup_\Sigma \limsup_{r\to\infty} E \cdot \left[ S(x(\tau_r)) + \int_0^{\tau_r} \cdot \left( l(x) + \gamma_2\left(\left|R_2(x)^{1/2}u\right|\right) - \gamma_1\left(\left|\Sigma\Sigma^{\mathrm{T}}\right|\right)\right) dt \right] \quad (1.2)$$

where

| | |
|---|---|
| $\tau_r$ | $= \inf\{t \geq 0: \|x(t)\| \geq r\}$; |
| $S(x)$ | positive–definite and radially unbounded; |
| $l(x)$ | positive–definite; |
| $R_2(x)$ | strictly positive; |
| $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ | class $\mathcal{K}_\infty$ functions. |

This result is a stochastic version of [30], motivated by the inverse optimality results in [18], [39]. It is important to compare the differential game problem (1.2) with the risk-sensitive problems and "stochastic differential games" [3]. The difference from the risk sensitive problem, in which $\Sigma$ is fixed and known, is obvious. The difference from stochastic differential games is that, rather than keeping the covariance known/fixed and letting another *deterministic* disturbance be the opposing player,[1] we leave the role of the opposing player to the covariance. This results in a stochastic form of disturbance attenuation where we achieve an energy-like bound[2]

$$\int_0^\infty E\left[l(x) + \gamma_2\left(\left|R_2(x)^{1/2}u\right|\right)\right] dt \leq \int_0^\infty \gamma_1\left(\left|\Sigma\Sigma^{\mathrm{T}}\right|\right) dt.$$

A comparison with the $LQG/\mathcal{H}_2$ problems is also in order. By proclaiming $\Sigma$ as a player in a differential game, we avoid the anomaly seen in $LQG/\mathcal{H}_2$ where the controller does not depend on the noise input matrix $B_1$.

*Example 1.1:* This example gives an idea about what type of stabilization problems are pursued in this paper. Consider the scalar system

$$dx = u\,dt + x\sigma(t)\,dw \quad (1.3)$$

where $w$ is a standard Wiener process and $\sigma$: $\mathbb{R}_+ \to \mathbb{R}_+$ is bounded. Consider the following two control laws:

$$u = -x - x^3 \quad (1.4)$$

$$u = -x - \xi x, \qquad \dot{\xi} = x^2. \quad (1.5)$$

It can be shown that they guarantee, respectively, that

$$E\left\{x(t)^2\right\} \leq e^{-2t} E\left\{x(0)^2\right\} + \tfrac{1}{16} \sup_{0\leq s\leq t} \sigma(s)^4 \quad (1.6)$$

$$\frac{d}{dt} E\left\{x^2 + \left(\frac{\|\sigma^2\|_\infty}{2} - \xi\right)^2\right\} \leq -2E\left\{x^2\right\}. \quad (1.7)$$

The controller (1.4) is a disturbance attenuation controller. The controller (1.5) is an adaptive controller. The stability types guaranteed by these controllers will become clear in the subsequent sections of the paper. We return to this example in Section VI. □

### D. Notation

The following will be used throughout this paper. For $n \geq 1$, $\mathbb{R}^n$ will denote the $n$-dimensional Euclidean space and $\mathbb{R}_+ = [0, \infty)$. For a vector $x \in \mathbb{R}^n$, $|x|$ will denote the Euclidean norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. All vectors will be column vectors unless indicated otherwise. The transpose of a vector or matrix will be denoted with a superscript of $\mathrm{T}$. The space of $n \times m$ matrices with real entries will be denoted by $\mathbb{R}^{n\times m}$, $|X|$ will denote the *Frobenius* norm of $X \in \mathbb{R}^{n\times m}$:

$$|X| = \left(\sum_{i=1}^n \sum_{j=1}^m X_{ij}^2\right)^{1/2} = \left(\mathrm{Tr}\{X^{\mathrm{T}}X\}\right)^{1/2} = \left(\mathrm{Tr}\{XX^{\mathrm{T}}\}\right)^{1/2} \quad (1.8)$$

[1] This disturbance becomes stochastic only in its state-dependent worst-case form because the state is stochastic, but it is not itself the source of stochasticity.

[2] For zero initial condition.

where Tr denotes the trace operator, and $col(X)$ will denote the $n \cdot m$-dimensional column vector obtained by stacking the columns of $X$ vertically, end-to-end. For a bounded function $X$: $\mathbb{R}_+ \to \mathbb{R}^{n\times m}$

$$\|X\|_\infty \equiv \sup_{t\in\mathbb{R}_+} |X(t)|. \tag{1.9}$$

A function $V$: $\mathbb{R}^n \to \mathbb{R}$ is said to be $C^k$ if it is $k$-times continuously differentiable. For a $C^1$ function $V$, $\partial V/\partial x$ will denote the gradient of $V$ (written as a *row* vector) and for a $C^2$ function $V$, $\partial^2 V/\partial x^2$ will denote the Hessian of $V$, the $n \times n$ matrix of second-order partial derivatives of $V$. A function $V$: $\mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if $V(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $V(0) = 0$.

## II. Global Lyapunov Theorems for Stochastic Systems

This section reviews some basic notation and stability theory for stochastic nonlinear systems. Even though an extensive coverage of stochastic Lyapunov theorems already exists in Khasminskii [28], Kushner [31], and Mao [33], in this section the reader will find many refinements and improvements.

1) A rigorous treatment of the *global* case (for example, compare the estimates in (2.20), (2.25) with [28], [31], [33]).
2) A presentation based on class $\mathcal{K}$ functions rather than on the $\epsilon$–$\delta$ format in [28], [31], [33] shows a clearer connection between modern deterministic stability results in the style of Hahn [21] or Khalil [27] and stochastic stability results.
3) A stochastic version of the convergence theorem due to LaSalle [32] and Yoshizawa [48]. This theorem (Theorem 2.1) is the cornerstone of our approach. It is used in the analysis of the adaptive systems in Section III and also to obtain Theorem 2.2 for global asymptotic stability in probability.

Consider the nonlinear stochastic system

$$dx = f(x,t)\,dt + g(x,t)\Sigma(t)\,dw, \qquad x(0) = x_0 \in \mathbb{R}^n \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $w$ is an $m$-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$, the Borel measurable functions $f$: $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g$: $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ are locally bounded and locally Lipschitz continuous in $x \in \mathbb{R}^n$ (uniformly in $t \in \mathbb{R}_+$) with $f(0,t) = 0$, $g(0,t) = 0$ for all $t \geq 0$, and $\Sigma$: $\mathbb{R}_+ \to \mathbb{R}^{m\times m}$ is Borel measurable and bounded, and the matrix $\Sigma(t)$ is nonnegative–definite for each $t \geq 0$. The above conditions ensure uniqueness and local existence (up to an explosion time) of strong solutions to (2.1) [26, Ch. 5]. Since all the issues we discuss in this paper are uniform in $t$, we do not stress the initial time, instead, we use 0 and $x_0$ to denote the initial time and initial state of the system. We also use $V_0$ to denote the initial value of a Lyapunov function.

*Definition 2.1:* A function $\gamma$: $\mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$.

*Definition 2.2:* The equilibrium $x = 0$ of the system (2.1) is

- globally stable in probability if $\forall \epsilon > 0$ there exists a class $\mathcal{K}$ function $\gamma(\cdot)$ such that
$$P\{|x(t)| < \gamma(|x_0|)\} \geq 1 - \epsilon \qquad \forall t \geq 0,\ \forall x_0 \in \mathbb{R}^n \setminus \{0\} \tag{2.2}$$
- globally asymptotically stable in probability *if it is globally stable in probability and*
$$P\left\{\lim_{t\to\infty} |x(t)| = 0\right\} = 1 \qquad \forall x_0 \in \mathbb{R}^n. \tag{2.3}$$

*Theorem 2.1:* Consider system (2.1) and suppose there exists a $C^2$ function $V$: $\mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\alpha_1$ and $\alpha_2$, such that for all $x \in \mathbb{R}^n$, $t \geq 0$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \tag{2.4}$$

$$\mathcal{L}V(x,t) = \frac{\partial V}{\partial x} f(x,t) + \frac{1}{2}\operatorname{Tr}\left\{\Sigma(t)^{\mathrm{T}} g(x,t)^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g(x,t)\Sigma(t)\right\} \leq -W(x) \tag{2.5}$$

where $W$: $\mathbb{R}^n \to \mathbb{R}$ is continuous and nonnegative. Then there is a unique strong solution of (2.6) for each $x_0 \in \mathbb{R}^n$, the equilibrium $x \equiv 0$ is globally stable in probability and

$$P\left\{\lim_{t\to\infty} W(x(t)) = 0\right\} = 1, \qquad \forall x_0 \in \mathbb{R}^n. \tag{2.6}$$

*Proof:* Since $\mathcal{L}V \leq 0$ and $V$ is radially unbounded, for each $x_0 \in \mathbb{R}^n$, there exists globally a unique strong solution to (2.1) [28, p. 84, Th. 4.1] with probability one (that is, the probability of escape in finite time is zero and the probability that two solutions starting from the same initial condition are different is zero).

In the following, (super)martingales will be defined relative to the usual augmented filtration $\{\mathcal{F}_t\}$ generated by $w(\cdot)$. Since $\mathcal{L}V(x,t) \leq 0$ and $V(x) \geq 0$, $V_t = V(x(t))$ is a supermartingale. By a supermartingale inequality [37, p. 154, (54.5)], for any class $\mathcal{K}_\infty$ function $\delta(\cdot)$, we have

$$P\left\{\sup_{0\leq s\leq t} V_s \geq \delta(V_0)\right\} \leq \frac{21V_0}{\delta(V_0)}, \qquad \forall t \geq 0,\ \forall V_0 \neq 0 \tag{2.7}$$

thus,

$$P\left\{\sup_{0\leq s\leq t} V_s < \delta(V_0)\right\} \geq 1 - \frac{21V_0}{\delta(V_0)}, \qquad \forall t \geq 0, \forall V_0 \neq 0. \tag{2.8}$$

Denote $\rho = \alpha_1^{-1} \circ \delta \circ \alpha_2$. Then $\sup_{0\leq s\leq t} V_s < \delta(V_0)$ implies $\sup_{0\leq s\leq t} |x(s)| < \rho(|x_0|)$, and thus

$$P\left\{\sup_{0\leq s\leq t} |x(s)| < \rho(|x_0|)\right\} \geq 1 - \frac{21V_0}{\delta(V_0)} \qquad \forall t \geq 0, \forall V_0 \neq 0. \tag{2.9}$$

For a given $\epsilon > 0$, choose $\delta(\cdot)$ such that

$$\delta(V_0) \geq \frac{21V_0}{\epsilon}, \qquad \forall V_0 \geq 0. \tag{2.10}$$

Then we have

$$P\left\{\sup_{0\leq s\leq t} |x(s)| < \rho(|x_0|)\right\} \geq 1 - \epsilon \qquad \forall t \geq 0, \forall x_0 \in \mathbb{R}^n \setminus \{0\} \tag{2.11}$$

which implies

$$P\{|x(t)| < \rho(|x_0|)\} \geq 1-\epsilon, \qquad \forall t \geq 0, \forall x_0 \in \mathbb{R}^n \backslash \{0\} \tag{2.12}$$

and the global stability in probability is proved.

By (2.5) and the vanishing of $f$, $g$ at $x = 0$, $W(0) = 0$, and since $x(\cdot) \equiv 0$ a.s. for $x_0 = 0$, we see that (2.6) holds for $x_0 = 0$. For $x_0 \in \mathbb{R}^n \setminus \{0\}$, to prove the a.s. convergence of $W(x(t))$ to zero as $t \to \infty$, we decompose the sample space into three mutually exclusive events

1. $A_1 = \left\{\omega\colon \limsup_{t\to\infty} W(x(t, \omega)) = 0\right\}$
2. $A_2 = \left\{\omega\colon \liminf_{t\to\infty} W(x(t, \omega)) > 0\right\}$
3. $A_3 = \left\{\omega\colon \liminf_{t\to\infty} W(x(t, \omega)) = 0 \text{ and } \limsup_{t\to\infty} W(x(t, \omega)) > 0\right\}.$

We will show that given $x_0 \in \mathbb{R}^n \setminus \{0\}$, $P\{A_2\} = P\{A_3\} = 0$ and hence $P\{A_1\} = 1$ which implies the desired result since $W(x(t)) \geq 0$ for all $t$.

For $r > 0$, let $\tau_r = \inf\{t \geq 0\colon x(t) \notin B\}$ where $B = \{x \in \mathbb{R}^n\colon |x| \leq r\}$. For $t \geq 0$, let $t_r = \min\{t, \tau_r\}$. Since $\mathcal{L}V(x, s)$ is bounded on $B\times[0, \infty)$, and because the local martingale term in Itô's formula when evaluated at $t_r$ is a martingale in $t$ [since $\Sigma(\cdot)$ is bounded and $\partial V/\partial x$ and $g(x, \cdot)$ are bounded whenever $x$ is restricted to a compact set], we have

$$\begin{aligned} E\{V_{t_r}\} &= V_0 + E\left\{\int_0^{t_r} \mathcal{L}V(x, s)\, ds\right\} \\ &\leq V_0 - E\left\{\int_0^{t_r} W(x(s))\, ds\right\} \end{aligned} \tag{2.13}$$

where the last inequality is by (2.5). Thus, since $V(\cdot) \geq 0$

$$E\left\{\int_0^{t_r} W(x(s))\, ds\right\} \leq V_0. \tag{2.14}$$

Since $W \geq 0$, letting $r \to \infty$, $t \to \infty$ and applying Fatou's lemma yields

$$E\left\{\int_0^{\infty} W(x(s))\, ds\right\} \leq V_0. \tag{2.15}$$

Hence

$$\int_0^{\infty} W(x(s))\, ds < \infty \qquad \text{a.s.} \tag{2.16}$$

and it follows immediately that $P\{A_2\} = 0$.

Now we turn to proving that $P\{A_3\} = 0$. We proceed by contradiction. Suppose $P\{A_3\} > 0$, then there exist $\varepsilon_1 > 0$ and $\epsilon_0 > 0$ such that

$$P\{W(x(\cdot)) \text{ crosses from below } \varepsilon_1 \text{ to above } 2\varepsilon_1 \text{ and back infinitely many times}\} \geq \epsilon_0. \tag{2.17}$$

For $r > 0$ and $s \geq 0$, let $s_r = \min\{s, \tau_r\}$, and define two functions $\rho_1$ and $\rho_2$ by

$$\rho_1(r) \triangleq \max_{|x|\leq r} \sup_{t\geq 0} |f(x, t)|, \tag{2.18}$$

$$\rho_2(r) \triangleq \max_{|x|\leq r} \sup_{t\geq 0} |g(x, t)\Sigma(t)| \tag{2.19}$$

where we recall that $f(x, t)$, $g(x, t)$ are locally bounded in $x$ (uniformly in $t$), $\Sigma(\cdot)$ is bounded and $|g(x, t)\Sigma(t)|$ is the Frobenius norm of $g(x, t)\Sigma(t)$. From (2.1) we compute

$$\begin{aligned} &E\left\{\sup_{0\leq s\leq h} |x(s_r) - x(0)|^2\right\} \\ &\quad = E\left\{\sup_{0\leq s\leq h} \left|\int_0^{s_r} f(x, t)\, dt + \int_0^{s_r} g(x, t)\Sigma(t)\, dw\right|^2\right\} \\ &\quad \leq 2E\left\{\sup_{0\leq s\leq h} \left|\int_0^{s_r} f(x, t)\, dt\right|^2\right\} \\ &\qquad + 2E\left\{\sup_{0\leq s\leq h} \left|\int_0^{s_r} g(x, t)\Sigma(t)\, dw\right|^2\right\} \\ &\quad \leq 2\rho_1(r)^2h^2 + 2E\left\{\sup_{0\leq s\leq h} \left|\int_0^{s_r} g(x, t)\Sigma(t)\, dw\right|^2\right\}. \end{aligned} \tag{2.20}$$

Applying Doob's maximal inequality and the Itô isometry, with simple manipulations applied to the right member of the inequality, we have

$$\begin{aligned} &2E\left\{\sup_{0\leq s\leq h} \left|\int_0^{s_r} g(x, t)\Sigma(t)\, dw\right|^2\right\} \\ &\quad \leq 8E\left\{\left|\int_0^{h_r} g(x, t)\Sigma(t)\, dw\right|^2\right\} \\ &\quad = 8E\left\{\int_0^{h_r} |g(x, t)\Sigma(t)|^2\, dt\right\} \leq 8\rho_2(r)^2h \end{aligned} \tag{2.21}$$

where $h_r = \inf\{h, \tau_r\}$. Combining the above two inequalities, we get

$$E\left\{\sup_{0\leq s\leq h} |x(s_r) - x(0)|^2\right\} \leq 2\rho_1(r)^2h^2 + 8\rho_2(r)^2h \tag{2.22}$$

and by Chebyshev's inequality, we have for any $\eta > 0$,

$$\begin{aligned} &P\left\{\sup_{0\leq s\leq h} |x(s_r) - x(0)| > \eta\right\} \\ &\quad \leq \frac{E\left\{\sup_{0\leq s\leq h} |x(s_r) - x(0)|^2\right\}}{\eta^2} \\ &\quad \leq \frac{2\rho_1(r)^2h^2 + 8\rho_2(r)^2h}{\eta^2}. \end{aligned} \tag{2.23}$$

Given $\epsilon > 0$, let $\rho$ be as in (2.11). By the uniform continuity of $W$ on the closed ball $B$ of radius $\rho(r)$ centered at the origin [18, Corollary A.5], there exists a class $\mathcal{K}$ function $\gamma$ such that for all $y$, $z$ in $B$, $|y - z| \le \gamma(u)$ implies $|W(y) - W(z)| \le u$ for all $u > 0$. Thus, for $|x_0| \le r$ and $\varepsilon_2 > 0$,

$$\begin{aligned} P&\left\{\sup_{0\le s\le h} |W(x(s)) - W(x(0))| > \varepsilon_2\right\} \\ &\le P\left\{\sup_{0\le s\le h} |x(s) - x(0)| > \gamma(\varepsilon_2) \text{ and } \sup_{0\le s\le h} |x(s)| < \rho(r)\right\} + P\left\{\sup_{0\le s\le h} |x(s)| \ge \rho(r)\right\} \\ &\le P\left\{\sup_{0\le s\le h} |x(s_{\rho(r)}) - x(0)| > \gamma(\varepsilon_2)\right\} + \epsilon \\ &\le \frac{2\rho_1(\rho(r))^2h^2 + 8\rho_2(\rho(r))^2 h}{\gamma(\varepsilon_2)^2} + \epsilon \end{aligned} \tag{2.24}$$

where in the last inequality we have used (2.23) with $\eta = \gamma(\varepsilon_2)$ and $\rho(r)$ in place of $r$. Now, setting $\epsilon = 1/2$, for every $r > 0$ and $\varepsilon_2 > 0$, we can find an $h^* = h^*(r, \varepsilon_2) > 0$, such that, for all $|x_0| \le r$

$$P\left\{\sup_{0\le s\le h} |W(x(s)) - W(x(0))| \le \varepsilon_2\right\} \ge \tfrac{1}{4} \quad \forall\, h \in (0, h^*]. \tag{2.25}$$

Now, let $T^1_{\varepsilon_1} = \inf\{t \ge 0\colon W(x(t)) \in B_{\varepsilon_1}\}$ where $B_{\varepsilon_1} = \{x \in \mathbb{R}^n\colon W(x) \le \varepsilon_1\}$, $T^1_{2\varepsilon_1} = \inf\{t \ge T^1_{\varepsilon_1}\colon W(x(t)) \notin B_{2\varepsilon_1}\}$ where $B_{2\varepsilon_1} = \{x \in \mathbb{R}^n\colon W(x) \le 2\varepsilon_1\}$, and, similarly, $T^i_{\varepsilon_1} = \inf\{t \ge T^{i-1}_{2\varepsilon_1}\colon W(x(t)) \in B_{\varepsilon_1}\}$, $T^i_{2\varepsilon_1} = \inf\{t \ge T^i_{\varepsilon_1}\colon W(x(t)) \notin B_{2\varepsilon_1}\}$ for all $i \ge 2$. By the continuity of $W(x(\cdot))$, we have that $T^i_{\varepsilon_1}, T^i_{2\varepsilon_1} \to \infty$ a.s. as $i \to \infty$. From (2.15), we have

$$\begin{aligned} V_0 &\ge E\left\{\int_0^\infty W(x(s))\,ds\right\} \\ &\ge \sum_{i=1}^\infty E\left\{1_{\{T^i_{2\varepsilon_1} < \tau_r\}} \int_{T^i_{2\varepsilon_1}}^{T^{i+1}_{\varepsilon_1}} W(x(s))\,ds\right\} \\ &\ge \sum_{i=1}^\infty E\left\{1_{\{T^i_{2\varepsilon_1} < \tau_r\}} \varepsilon_1 \left(T^{i+1}_{\varepsilon_1} - T^i_{2\varepsilon_1}\right)\right\} \\ &= \sum_{i=1}^\infty \varepsilon_1 E\left\{1_{\{T^i_{2\varepsilon_1} < \tau_r\}} E\left\{T^{i+1}_{\varepsilon_1} - T^i_{2\varepsilon_1} \,\middle|\, \mathcal{F}_{T^i_{2\varepsilon_1}}\right\}\right\}. \end{aligned} \tag{2.26}$$

Now, by the strong Markov property of solutions of (2.1), on $\{T^i_{2\varepsilon_1} < \tau_r\}$, the law of $\tilde{x}(\cdot) \equiv x(\cdot + T^i_{2\varepsilon_1})$ under the conditional distribution $P(\cdot|\mathcal{F}_{T^i_{2\varepsilon_1}})$ is the same as that of a solution of (2.27) with $t + T^i_{2\varepsilon_1}$ in place of $t$ and initial position satisfying $|\tilde{x}(0)| < r$. Since $\rho_1$, $\rho_2$ are defined by supremums over all $t$, and (2.5) holds for all $t$, the estimate (2.25) applies with $\tilde{x}(\cdot)$ in place of $x(\cdot)$ and $P(\cdot|\mathcal{F}_{T^i_{2\varepsilon_1}})$ in place of $P(\cdot)$, on $\{T^i_{2\varepsilon_1} < \tau_r\}$. Setting $\varepsilon_2 = \varepsilon_1/2$ there, we obtain the following on $\{T^i_{2\varepsilon_1} < \tau_r\}$:

$$\begin{aligned} E&\left\{T^{i+1}_{\varepsilon_1} - T^i_{2\varepsilon_1} \,\middle|\, \mathcal{F}_{T^i_{2\varepsilon_1}}\right\} \\ &\ge h^* P\left\{\sup_{0\le s\le h^*} |W(\tilde{x}(s)) - W(\tilde{x}(0))| \le \frac{\varepsilon_1}{2} \,\middle|\, \mathcal{F}_{T^i_{2\varepsilon_1}}\right\} \\ &\ge \frac{h^*}{4} \end{aligned} \tag{2.27}$$

where $h^* = h^*(r, \varepsilon_1/2)$.

Substituting this into (2.26) yields

$$\infty > V_0 \ge \sum_{i=1}^\infty \frac{h^*}{4} \varepsilon_1 P\left\{T^i_{2\varepsilon_1} < \tau_r\right\}. \tag{2.28}$$

It then follows from the Borel–Cantelli lemma that

$$P\left\{T^i_{2\varepsilon_1} < \tau_r \text{ for infinitely many } i\right\} = 0. \tag{2.29}$$

Thus

$$P\left\{T^i_{2\varepsilon_1} < \infty \text{ for infinitely many } i \text{ and } \tau_r = \infty\right\} = 0. \tag{2.30}$$

Since the sets $\{\tau_r = \infty\}$ are increasing with $r$, if we show that $P\{\tau_r = \infty\} \to 1$ as $r \to \infty$, it will follow that:

$$P\left\{T^i_{2\varepsilon_1} < \infty \text{ for infinitely many } i\right\} = 0 \tag{2.31}$$

and this contradicts (2.17). This yields the desired convergence of $W(x(\cdot))$.

By letting $t \to \infty$ in the supermartingale inequality (2.17), we obtain

$$P\left\{\sup_{s\ge 0} |x(s)| \ge r\right\} \le P\left\{\sup_{s\ge 0} V_s \ge \alpha_1(r)\right\} \le \frac{21 V_0}{\alpha_1(r)} \quad \forall\, r > 0. \tag{2.32}$$

Hence

$$P\{\tau_r = \infty\} \ge P\left\{\sup_{s\ge 0} |x(s)| < r\right\} \ge 1 - \frac{21 V_0}{\alpha_1(r)} \quad \forall\, r > 0 \tag{2.33}$$

which implies that $P\{\tau_r = \infty\} \to 1$ as $r \to \infty$. This completes the proof. □

*Theorem 2.2:* Consider system (2.1) and suppose there exists a $C^2$ function $V\colon \mathbb{R}^n \to \mathbb{R}_+$, class $\mathcal{K}_\infty$ functions $\alpha_1$ and $\alpha_2$, and a class $\mathcal{K}$ function $\alpha_3$, such that for all $x \in \mathbb{R}^n$, $t \ge 0$

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|), \tag{2.34}$$

$$\begin{aligned} \mathcal{L}V(x, t) &= \frac{\partial V}{\partial x} f(x, t) + \frac{1}{2} \operatorname{Tr}\left\{\Sigma(t)^{\mathrm{T}} g(x, t)^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g(x, t)\Sigma(t)\right\} \\ &\le -\alpha_3(|x|). \end{aligned} \tag{2.35}$$

Then the equilibrium $x = 0$ is globally asymptotically stable in probability.

*Proof:* This theorem is a direct corollary of Theorem 2.1. The fact that $W(x) = \alpha_3(|x|)$ is strictly positive if $x \neq 0$ implies $P\{\lim_{t\to\infty} |x(t)| = 0\} = 1$ for all $x_0 \in \mathbb{R}^n$. Combining this with global stability in probability established in Theorem 2.1, implies the equilibrium is globally asymptotically stable in probability. □

## III. Adaptive Stabilization of Strict-Feedback Systems

In this section, we address the stabilization problem for the system

$$dx = f(x)\,dt + g_1(x)\Sigma(t)\,dw + g_2(x)u\,dt \tag{3.1}$$

where $w$ is a standard Wiener process and $f(0) = 0$, $g_1(0) = 0$. For the sake of discussion, let us assume that $\Sigma$ is constant. For deterministic systems with constant parameters, the usual approach is adaptive control [29], which allows the treatment of unknown parameters multiplying known nonlinearities. In the stochastic case here, we have the noise $\Sigma(\cdot)\,dw$ with unknown covariance multiplying the known nonlinearity $g_1(x)$. As we shall see in this section, the presence of noise does not prevent stabilization as long as $g_1(0) = 0$, i.e., as long as the equilibrium is preserved in the presence of noise. Note that this is a strong condition which is usually not imposed in the so-called "stochastic (linear) adaptive control," where the noise is additive and nonvanishing (see, e.g., [10] and the reference therein). However, in the problem pursued here, the additional generality is that the noise can be of unknown (and, in fact, time-varying) covariance and it can multiply a nonlinearity.

In this section, we deal with strict-feedback systems given by nonlinear stochastic differential equations

$$dx_i = x_{i+1}\,dt + \varphi_i(\overline{x}_i)^{\mathrm{T}}\Sigma(t)\,dw, \qquad i = 1, \ldots, n-1 \tag{3.2}$$

$$dx_n = u\,dt + \varphi_n(\overline{x}_n)^{\mathrm{T}}\Sigma(t)\,dw \tag{3.3}$$

where

| | |
|---|---|
| $\overline{x}_i$ | $= [x_1, \ldots, x_i]^{\mathrm{T}}$; |
| $\varphi_i(\overline{x}_i)$ | $m$-vector valued smooth ($C^\infty$) functions with $\varphi_i(0) = 0$; |
| $w$ | $m$-dimensional standard Wiener process; |
| $\Sigma: \mathbb{R}_+ \to \mathbb{R}^{m\times m}$ | bounded Borel measurable function where $\Sigma(t)$ is nonnegative definite for each $t$. |

As we shall see in the sequel, to achieve adaptive stabilization in the presence of unknown $\Sigma$, for this class of systems, it is not necessary to estimate the entire matrix $\Sigma$ and, in fact, it is possible to allow $\Sigma$ to be time-varying. Instead we will estimate only one unknown parameter $\theta = \|\Sigma\Sigma^{\mathrm{T}}\|_\infty$ using the estimate $\hat{\theta}(t)$ at time $t$. We employ the adaptive backstepping technique with tuning functions [29]. Our presentation is very concise: instead of introducing the stabilizing functions $\alpha_i(\overline{x}_i, \hat{\theta})$ and tuning functions $\tau_i(\overline{x}_i, \hat{\theta})$ for $i = 0, 1, \ldots, n$, in a step-by-step fashion, we derive these smooth functions simultaneously. The control $u$ and parameter estimate $\hat{\theta}$ will be given by $u = \alpha_n(\overline{x}_n, \hat{\theta})$, $\dot{\hat{\theta}} = \gamma\tau_n(\overline{x}_n, \hat{\theta})$ for some $\gamma > 0$.

We start with several important preparatory comments. Since $\varphi_i(0) = 0$, we will require the $\alpha_i$'s to vanish at $\overline{x}_i = 0$. Define the error variables

$$z_i = x_i - \alpha_{i-1}\left(\overline{x}_{i-1}, \hat{\theta}\right), \qquad i = 1, \ldots, n. \tag{3.4}$$

Let $\alpha_0 = 0$. Then, by the mean value theorem for integrals, $\alpha_i(\overline{x}_i, \hat{\theta})$ can be expressed as

$$\begin{aligned}\alpha_i(\overline{x}_i, \hat{\theta}) &= \sum_{l=1}^{i} x_l h_{il}\left(\overline{x}_i, \hat{\theta}\right)\\ &= \sum_{l=1}^{i}\left(z_l + \alpha_{l-1}\left(\overline{x}_{l-1}, \hat{\theta}\right)\right)h_{il}\left(\overline{x}_i, \hat{\theta}\right)\\ &= \sum_{l=1}^{i} z_l\alpha_{il}\left(\overline{x}_i, \hat{\theta}\right)\end{aligned} \tag{3.5}$$

where $\alpha_{il}(\overline{x}_i, \hat{\theta})$ are smooth functions. Similarly, we can now write $\varphi_i(\overline{x}_i)$ as

$$\varphi_i(\overline{x}_i) = \sum_{k=1}^{i} z_k\psi_{ik}(\overline{x}_i, \hat{\theta}) \tag{3.6}$$

where $\psi_{ik}(\overline{x}_i, \hat{\theta})$ are smooth functions. Then, according to Itô's differentiation rule, the system (3.2), (3.3) can be written as

$$\begin{aligned}dz_i &= d(x_i - \alpha_{i-1})\\ &= \Bigg(x_{i+1} - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1}\\ &\qquad - \frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}(\Sigma\Sigma^{\mathrm{T}})(t)\varphi_q - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\Bigg)dt\\ &\quad + \left(\varphi_i^{\mathrm{T}} - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l^{\mathrm{T}}\right)\Sigma(t)\,dw, \qquad i = 1, \ldots, n\end{aligned} \tag{3.7}$$

where $x_{n+1} = u = \alpha_n(\overline{x}_n, \hat{\theta})$. We employ a Lyapunov function of the form

$$V(z, \tilde{\theta}) = \sum_{i=1}^{n}\frac{1}{4}z_i^4 + \frac{1}{2\gamma}\tilde{\theta}^2 \tag{3.8}$$

where $\tilde{\theta} = \|\Sigma\Sigma^{\mathrm{T}}\|_\infty - \hat{\theta}$ is the parameter estimation error, and we set out to select the functions $\alpha_i(\overline{x}_i, \hat{\theta})$ and $\tau_i(\overline{x}_i, \hat{\theta})$ to make $\mathcal{L}V(z, \tilde{\theta}, t)$ nonpositive. Along the solutions of (3.7), we have

$$\begin{aligned}\mathcal{L}V = \sum_{i=1}^{n} z_i^3\Bigg(&x_{i+1} - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1}\\ &- \frac{1}{2}\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm{T}}\Sigma\Sigma^{\mathrm{T}}\varphi_q - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{2}\sum_{i=1}^{n} z_i^2 \left(\varphi_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l\right)^{\mathrm T}\Sigma\Sigma^{\mathrm T}\\
&\cdot\left(\varphi_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l\right) - \frac{\tilde\theta\dot{\hat\theta}}{\gamma}\\
\le &\sum_{i=1}^{n} z_i^3\left(\alpha_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} - \frac{\partial\alpha_{i-1}}{\partial\hat\theta}\dot{\hat\theta}\right)\\
&+\sum_{i=1}^{n} z_i^3 z_{i+1} - \frac{1}{2}\sum_{i=1}^{n} z_i^3\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm T}\Sigma\Sigma^{\mathrm T}\varphi_q\\
&+\frac{3}{2}\sum_{i=1}^{n} z_i^2\left(\varphi_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l\right)^{\mathrm T}\\
&\cdot\left(\varphi_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\varphi_l\right)\|\Sigma\Sigma^{\mathrm T}\|_\infty - \frac{\tilde\theta\dot{\hat\theta}}{\gamma}
\end{aligned}
\tag{3.9}
$$

where $z_{n+1}=0$. Consider the third term

$$
\begin{aligned}
-&\frac{1}{2}\sum_{i=1}^{n} z_i^3\sum_{p,q=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\varphi_p^{\mathrm T}\Sigma\Sigma^{\mathrm T}\varphi_q\\
&\le \frac{1}{2}\sum_{i=1}^{n}\sum_{p,q=1}^{i-1}|z_i|^3\left|\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\right|\left|\sum_{k=1}^{p} z_k\psi_{pk}\right|\\
&\quad\cdot\left|\sum_{l=1}^{q} z_l\psi_{ql}\right|\left|\Sigma\Sigma^{\mathrm T}\right|\\
&\le \frac{1}{2}\sum_{i=1}^{n}\sum_{p,q=1}^{i-1}\sum_{k=1}^{p}\sum_{l=1}^{q}\left|\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\right|\\
&\quad\cdot|\psi_{pk}|\,|\psi_{ql}||z_i|^3|z_k||z_l|\,\left|\Sigma\Sigma^{\mathrm T}\right|\\
&\le \tfrac{1}{4}\sum_{i=1}^{n}\sum_{p,q=1}^{i-1}\sum_{k=1}^{p}\sum_{l=1}^{q}\\
&\quad\cdot\left\{\left(\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\right)^2|\psi_{pk}|^2|\psi_{ql}|^2 z_i^6 + \frac{1}{2}z_k^4 + \frac{1}{2}z_l^4\right\}\left|\Sigma\Sigma^{\mathrm T}\right|\\
&= \frac{1}{4}\sum_{i=1}^{n} z_i^6\sum_{p,q=1}^{i-1}\sum_{k=1}^{p}\sum_{l=1}^{q}\left(\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\right)^2\\
&\quad\cdot\psi_{pk}^{\mathrm T}\psi_{pk}\psi_{ql}^{\mathrm T}\psi_{ql}\left|\Sigma\Sigma^{\mathrm T}\right|\\
&\quad+\tfrac{1}{8}\sum_{i=1}^{n-1} z_i^4\sum_{k=i+1}^{n} k(k-1)(k-i)\left|\Sigma\Sigma^{\mathrm T}\right|
\end{aligned}
\tag{3.10}
$$

and employing the inequalities in [5, eqs. (3.13) and (3.15–3.20)], we have for

$$
\beta_{ik}\equiv\psi_{ik} - \sum_{l=k}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}\psi_{lk},\quad \epsilon_i>0\colon i=1,\ldots,n-1
$$

$$
\epsilon_n=0,\ \epsilon_0=\infty,\ \epsilon_{ikl}>0,\ \epsilon_{ikl}=\epsilon_{ilk}
$$

$$
\begin{aligned}
\mathcal{L}V \le &\sum_{i=1}^{n} z_i^3\left(\alpha_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} - \frac{\partial\alpha_{i-1}}{\partial\hat\theta}\dot{\hat\theta}\right)\\
&+\frac{3}{4}\sum_{i=1}^{n}\epsilon_i^{4/3}z_i^4 + \sum_{i=1}^{n}\frac{1}{4\epsilon_{i-1}^4}z_i^4\\
&+\frac{1}{4}\sum_{i=1}^{n} z_i^6\sum_{p,q=1}^{i-1}\sum_{k=1}^{p}\sum_{l=1}^{q}\left(\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\right)^2\\
&\cdot\psi_{pk}^{\mathrm T}\psi_{pk}\psi_{ql}^{\mathrm T}\psi_{ql}\|\Sigma\Sigma^{\mathrm T}\|_\infty\\
&+\tfrac{1}{8}\sum_{i=1}^{n-1} z_i^4\sum_{k=i+1}^{n} k(k-1)(k-i)\|\Sigma\Sigma^{\mathrm T}\|_\infty\\
&+\Bigg[\tfrac{3}{2}\sum_{i=1}^{n}\left(z_i^4\beta_{ii}^{\mathrm T}\beta_{ii} + 2z_i^3\beta_{ii}^{\mathrm T}\sum_{k=1}^{i-1} z_k\beta_{ik}\right)\\
&\quad+\frac{3}{4}\sum_{i=1}^{n} z_i^4\left(\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}\frac{1}{\epsilon_{ikl}^2}\beta_{ikj}^2\beta_{ilj}^2\right)\\
&\quad+\frac{3m}{4}\sum_{i=1}^{n-1} z_i^4\left(\sum_{k=i+1}^{n}\sum_{l=1}^{k-1}\epsilon_{kil}^2\right)\Bigg]\left(\hat\theta+\tilde\theta\right) - \frac{\tilde\theta\dot{\hat\theta}}{\gamma}\\
= &\sum_{i=1}^{n} z_i^3\Bigg[\alpha_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} + \frac{1}{4}z_i^3\sum_{p,q=1}^{i-1}\sum_{k=1}^{p}\sum_{l=1}^{q}\\
&\quad\cdot\left(\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\right)^2\psi_{pk}^{\mathrm T}\psi_{pk}\psi_{ql}^{\mathrm T}\psi_{ql}\hat\theta + \frac{1}{8}z_i\sum_{k=i+1}^{n}\\
&\quad\cdot k(k-1)(k-i)\hat\theta + \frac{3}{4}\epsilon_i^{4/3}z_i + \frac{1}{4\epsilon_{i-1}^4}z_i\\
&\quad+\left(\frac{3}{2}z_i\beta_{ii}^{\mathrm T}\beta_{ii} + 3\beta_{ii}^{\mathrm T}\sum_{k=1}^{i-1} z_k\beta_{ik}\right.\\
&\quad\quad+\frac{3}{4}z_i\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}\frac{1}{\epsilon_{ikl}^2}\beta_{ikj}^2\beta_{ilj}^2\\
&\quad\quad\left.+\frac{3m}{4}z_i\sum_{k=i+1}^{n}\sum_{l=1}^{k-1}\epsilon_{kil}^2\right)\hat\theta - \frac{\partial\alpha_{i-1}}{\partial\hat\theta}\dot{\hat\theta}\Bigg]\\
&-\tilde\theta\left[\frac{\dot{\hat\theta}}{\gamma} - \frac{1}{4}\sum_{i=1}^{n} z_i^6\sum_{p,q=1}^{i-1}\sum_{k=1}^{p}\sum_{l=1}^{q}\left(\frac{\partial^2\alpha_{i-1}}{\partial x_p\partial x_q}\right)^2\right.\\
&\quad\cdot\psi_{pk}^{\mathrm T}\psi_{pk}\psi_{ql}^{\mathrm T}\psi_{ql}\\
&\quad-\tfrac{1}{8}\sum_{i=1}^{n} z_i^4\sum_{k=i+1}^{n} k(k-1)(k-i)\\
&\quad-\tfrac{3}{2}\sum_{i=1}^{n}\left(z_i^4\beta_{ii}^{\mathrm T}\beta_{ii} + 2z_i^3\beta_{ii}^{\mathrm T}\sum_{k=1}^{i-1} z_k\beta_{ik}\right)
\end{aligned}
$$

$$-\frac{3}{4}\sum_{i=1}^{n} z_i^4\left(\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}\frac{1}{\epsilon_{ikl}^2}\beta_{ikj}^2\beta_{ilj}^2\right) -\frac{3m}{4}\sum_{i=1}^{n-1} z_i^4\left(\sum_{k=i+1}^{n}\sum_{l=1}^{k-1}\epsilon_{kil}^2\right)\Bigg]. \tag{3.11}$$

Here, $\beta_{ikj}$ is the $j$th component of $\beta_{ik}$. Let

$$\dot{\hat{\theta}} = \gamma\tau_n \tag{3.12}$$

$$\tau_i = \tau_{i-1} + \omega_i z_i^3, \qquad i = 1, \ldots, n \tag{3.13}$$

where $\tau_0 = 0$ and

$$\begin{aligned}\omega_i =& \frac{1}{4} z_i^3 \sum_{p,q=1}^{i-1}\sum_{k=1}^{p}\sum_{l=1}^{q}\left(\frac{\partial^2\alpha_{i-1}}{\partial x_p \partial x_q}\right)^2 \psi_{pk}^{\mathrm{T}}\psi_{pk}\psi_{ql}^{\mathrm{T}}\psi_{ql} \\ &+ \tfrac{1}{8} z_i \sum_{k=i+1}^{n} k(k-1)(k-i) \\ &+ \tfrac{3}{2} z_i\beta_{ii}^{\mathrm{T}}\beta_{ii} + 3\beta_{ii}^{\mathrm{T}}\sum_{k=1}^{i-1} z_k\beta_{ik} \\ &+ \frac{3}{4} z_i\left(\sum_{j=1}^{m}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}\frac{1}{\epsilon_{ikl}^2}\beta_{ikj}^2\beta_{ilj}^2\right) \\ &+ \frac{3m}{4} z_i \sum_{k=i+1}^{n}\sum_{l=1}^{k-1}\epsilon_{kil}^2. \end{aligned}\tag{3.14}$$

Then

$$\begin{aligned}\mathcal{L}V \le& \sum_{i=1}^{n} z_i^3\Bigg(\alpha_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} + \frac{3}{4}\epsilon_i^{4/3} z_i \\ &+ \frac{1}{4\epsilon_{i-1}^4} z_i + \omega_i\hat{\theta} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\sum_{j=1}^{n}\gamma z_j^3\omega_j\Bigg) \\ =& \sum_{i=1}^{n} z_i^3\Bigg(\alpha_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} + \frac{3}{4}\epsilon_i^{4/3} z_i \\ &+ \frac{1}{4\epsilon_{i-1}^4} z_i + \omega_i\hat{\theta} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\sum_{j=1}^{i}\gamma z_j^3\omega_j\Bigg) \\ &- \sum_{i=1}^{n-1} z_i^3\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\sum_{j=i+1}^{n}\gamma z_j^3\omega_j \\ =& \sum_{i=1}^{n} z_i^3\Bigg(\alpha_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} + \frac{3}{4}\epsilon_i^{4/3} z_i \\ &+ \frac{1}{4\epsilon_{i-1}^4} z_i + \omega_i\hat{\theta} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\sum_{j=1}^{i}\gamma z_j^3\omega_j\Bigg) \\ &- \sum_{i=2}^{n} z_i^3\sum_{j=1}^{i-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\gamma z_j^3\omega_i \\ =& \sum_{i=1}^{n} z_i^3\Bigg(\alpha_i - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} + \frac{3}{4}\epsilon_i^{4/3} z_i \\ &+ \frac{1}{4\epsilon_{i-1}^4} z_i + \omega_i\hat{\theta} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\sum_{j=1}^{i}\gamma z_j^3\omega_j \\ &- \sum_{j=1}^{i-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\gamma z_j^3\omega_i\Bigg). \end{aligned}\tag{3.15}$$

Letting

$$\alpha_0 = 0, \tag{3.16}$$

$$\begin{aligned}\alpha_i =& -c_i z_i + \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1} - \frac{3}{4}\epsilon_i^{4/3} z_i \\ &- \frac{1}{4\epsilon_{i-1}^4} z_i - \omega_i\hat{\theta} + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\sum_{j=1}^{i}\gamma z_j^3\omega_j \\ &+ \sum_{j=1}^{i-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\gamma z_j^3\omega_i, \qquad i = 1, \ldots, n, \end{aligned}\tag{3.17}$$

$$u = \alpha_n \tag{3.18}$$

where $c_i > 0$, the infinitesimal generator of the system (3.7) becomes negative–definite

$$\mathcal{L}V(z, \tilde{\theta}) \le -\sum_{i=1}^{n} c_i z_i^4. \tag{3.19}$$

*Theorem 3.1:* The equilibrium $x = 0$, $\hat{\theta} = \|\Sigma\Sigma^{\mathrm{T}}\|_\infty$, of the closed-loop system defined by (3.2)–(3.4), (3.12)–(3.14), and (3.16)–(3.18), is globally stable in probability and for each $(x_0, \hat{\theta}_0) \in \mathbb{R}^{n+1}$,

$$P\left\{\lim_{t\to\infty} |x(t)| = 0\right\} = 1, \tag{3.20}$$

$$P\left\{\lim_{t\to\infty} \hat{\theta}(t) \text{ exists and is finite}\right\} = 1. \tag{3.21}$$

*Proof:* By applying Theorem 2.1 to the pair $(z, \tilde{\theta})$, we conclude that in these coordinates the equilibrium point $(0, 0)$ is globally stable in probability and

$$P\left\{\lim_{t\to\infty} |z(t)| = 0\right\} = 1, \qquad \forall (z_0, \tilde{\theta}_0) \in \mathbb{R}^{n+1}. \tag{3.22}$$

Furthermore, for $(z_0, \tilde{\theta}_0) \in \mathbb{R}^{n+1}$, since $\mathcal{L}V(z, \tilde{\theta}, t) \le 0$ and $V(z, \tilde{\theta}) \ge 0$, $V_t = V(z(t), \tilde{\theta}(t))$ is a nonnegative supermartingale and so it converges a.s. as $t \to \infty$. In view of (3.22) and the definition of $V$, it follows that a.s., $\tilde{\theta}(t)$ converges to a finite (possibly random) limit $\tilde{\theta}_\infty$ as $t \to \infty$. Now

$$x_1 = z_1, \tag{3.23}$$

$$x_2 = z_2 + \alpha_1(x_1, \hat{\theta}) \tag{3.24}$$
$$\vdots \tag{3.25}$$
$$x_n = z_n + \alpha_{n-1}(\overline{x}_{n-1}, \hat{\theta}) \tag{3.26}$$
$$\hat{\theta} = \|\Sigma\Sigma^{\mathrm{T}}\|_\infty - \tilde{\theta} \tag{3.27}$$

defines a continuous map $\Gamma$: $(x, \hat{\theta}) = \Gamma(z, \tilde{\theta})$, where $\Gamma(0, \tilde{\theta}) = (0, \|\Sigma\Sigma^{\mathrm{T}}\|_\infty - \tilde{\theta})$ for all $\tilde{\theta}$, by the definition of the $\alpha_i$s. Similarly, $\Gamma^{-1}$ is well defined and continuous. By Corollary A.15 of Freeman and Kokotović [18], $\Gamma$ and $\Gamma^{-1}$ are $C\mathcal{K}$-continuous and it follows that the global stability in probability of $(z, \tilde{\theta}) = (0, 0)$ implies the global stability in probability of $(x, \hat{\theta}) = (0, \|\Sigma\Sigma^{\mathrm{T}}\|_\infty)$ in $(x, \hat{\theta})$-coordinates. In addition, for each $(x_0, \hat{\theta}_0) \in \mathbb{R}^{n+1}$, a.s., as $t \to \infty$

$$\begin{aligned}\left(x(t), \hat{\theta}(t)\right) &= \Gamma\left(z(t), \tilde{\theta}(t)\right) \to \Gamma\left(0, \tilde{\theta}_\infty\right)\\ &= \left(0, \|\Sigma\Sigma^{\mathrm{T}}\|_\infty - \tilde{\theta}_\infty\right)\end{aligned} \tag{3.28}$$

which yields (3.20)–(3.21). □

## IV. Stochastic Disturbance Attenuation for Strict-Feedback Systems

In this section, we relax the assumption from Section III that the noise vector field is vanishing at the origin. This prevents equilibrium stabilization but still allows disturbance attenuation which we pursue using robust nonlinear control tools.

We first prove a general technical result to be used in analyzing these systems. For this, $f$ and $g$ are as in (2.1).

*Theorem 4.1:* Suppose there exists a $C^2$ function $V$: $\mathbb{R}^n \to \mathbb{R}_+$, a constant $c > 0$, class $\mathcal{K}_\infty$ functions $\alpha_1$, $\alpha_2$, and a Borel measurable, increasing function $\gamma$: $\mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\begin{aligned}&\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)\\ &\mathcal{L}V(x, t, \Sigma) \stackrel{\Delta}{=} \frac{\partial V}{\partial x} f(x, t)\\ &\qquad + \frac{1}{2}\mathrm{Tr}\left\{\Sigma^{\mathrm{T}} g(x, t)^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g(x, t)\Sigma\right\}\\ &\qquad \le -cV(x) + \gamma\left(|\Sigma\Sigma^{\mathrm{T}}|\right)\end{aligned} \tag{4.1}$$

for all $x \in \mathbb{R}^n$, $t \ge 0$ and all nonnegative definite matrices $\Sigma \in \mathbb{R}^{m\times m}$. Then, there is a unique strong solution of (2.1) for each $x_0 \in \mathbb{R}^n$ and it satisfies

$$E[V(x(t))] \le e^{-ct}V(x_0) + c^{-1}\gamma\left(\sup_{0\le s\le t} |\Sigma(s)\Sigma(s)^{\mathrm{T}}|\right) \quad \forall t \ge 0. \tag{4.2}$$

*Proof:* Since $\Sigma$: $\mathbb{R}_+ \to \mathbb{R}^{m\times m}$ is bounded, $\mathcal{L}V(x, t, \Sigma(t)) \le C$ for all $x \in \mathbb{R}^n$, $t \ge 0$, for some $C > 0$, and it follows from [28, proof of Th. 4.1] that there exists globally a unique strong solution to (2.1) for each $x_0 \in \mathbb{R}^n$. Applying Itô's formula to such a solution yields, for all $t \ge 0$

$$\begin{aligned}&V(x(t))e^{ct}\\ &= V(x_0) + \int_0^t e^{cs}\frac{\partial V}{\partial x}(x(s))g(x(s), s)\Sigma(s)\,dw(s)\\ &\quad + \int_0^t e^{cs}\left(\mathcal{L}V(x(s), s, \Sigma(s)) + cV(x(s))\right) ds.\end{aligned} \tag{4.3}$$

If $t$ is replaced by $t_r = \min\{t, \tau_r\}$ in the above, where $\tau_r = \inf\{s \ge 0\text{: } |x(s)| \ge r\}$, then the stochastic integral (first integral) in (4.3) defines a martingale (with $r$ fixed and $t$ varying), not just a local martingale. Thus, on taking expectations in (4.3) with $t_r$ in place of $t$ and then using (4.1) on the right, we obtain

$$\begin{aligned}&E\left[V(x(t_r))e^{ct_r}\right]\\ &\le V(x_0) + E\left[\int_0^{t_r} e^{cs}\gamma\left(|\Sigma(s)\Sigma(s)^{\mathrm{T}}|\right) ds\right].\end{aligned} \tag{4.4}$$

On letting $r \to \infty$ and using Fatou's lemma on the left and monotone convergence on the right, we obtain

$$\begin{aligned}&E\left[V(x(t))e^{ct}\right]\\ &\le V(x_0) + E\left[\int_0^{t} e^{cs}\gamma\left(|\Sigma(s)\Sigma(s)^{\mathrm{T}}|\right) ds\right].\end{aligned} \tag{4.5}$$

The result (4.2) follows immediately from this using the fact that $\gamma$ is an increasing function and simple integration. □

We apply the above to strict-feedback systems driven by a stochastic process with time varying but bounded incremental covariance with an *unknown bound*. This class of systems is given by nonlinear stochastic differential equations

$$dx_i = x_{i+1}\,dt + \varphi_i(\overline{x}_i)^{\mathrm{T}}\Sigma(t)\,dw, \qquad i = 1, \ldots, n-1 \tag{4.6}$$
$$dx_n = u\,dt + \varphi_n(\overline{x}_n)^{\mathrm{T}}\Sigma(t)\,dw \tag{4.7}$$

where

| | |
|---|---|
| $\overline{x}_i$ | $= [x_1, \ldots, x_i]^{\mathrm{T}}$; |
| $\varphi_i(\overline{x}_i)$ | $m$-vector-valued smooth ($C^\infty$) functions; |
| $w$ | $m$-dimensional standard Wiener process; |
| $\Sigma$: $\mathbb{R}_+ \to \mathbb{R}^{m\times m}$ | bounded Borel measurable function where $\Sigma(t)$ is nonnegative–definite for each $t$. |

To obtain a Lyapunov function, we employ the backstepping technique [29]. Our presentation here is concise, we derive the virtual controls $\alpha_i(\overline{x}_i)$, $i = 0, 1, \ldots, n$, simultaneously. We start with the transformation $z_i = x_i - \alpha_{i-1}$, and according to Itô's differentiation rule, we rewrite the system (4.6), (4.7) as

$$\begin{aligned} dz_i &= d\left(x_i - \alpha_{i-1}\right) \\ &= \left(x_{i+1} - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, x_{l+1} \right. \\ &\qquad \left. - \frac{1}{2} \sum_{p,\,q=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q}\, \varphi_p^{\mathrm{T}} (\Sigma\Sigma^{\mathrm{T}})(t) \varphi_q \right) dt \\ &\quad + \left(\varphi_i^{\mathrm{T}} - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, \varphi_l^{\mathrm{T}} \right) \Sigma(t)\, dw, \qquad i = 1, \ldots, n \end{aligned} \tag{4.8}$$

where $x_{n+1} = u = \alpha_n(\overline{x}_n)$. We employ a Lyapunov function of the form

$$V(z) = \sum_{i=1}^{n} \tfrac{1}{4}\, z_i^4. \tag{4.9}$$

We stress the difference between the deterministic case [29], where the Lyapunov function is quadratic, and the stochastic case here where the Lyapunov function is chosen as quartic to accommodate the Hessian term $\partial^2 V / \partial x^2$. Now we set out to select the functions $\alpha_i(\overline{x}_i)$ to make $\mathcal{L}V \leq -cV + k|\Sigma\Sigma^{\mathrm{T}}|^2$, where $c$ and $k$ are positive constants. Along the solutions of (4.8), we have

$$\begin{aligned} \mathcal{L}V &= \sum_{i=1}^{n} z_i^3 \left( x_{i+1} - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, x_{l+1} \right. \\ &\qquad \left. - \frac{1}{2} \sum_{p,\,q=1}^{i-1} \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q}\, \varphi_p^{\mathrm{T}} \Sigma\Sigma^{\mathrm{T}} \varphi_q \right) \\ &\quad + \frac{3}{2} \sum_{i=1}^{n} z_i^2 \left( \varphi_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, \varphi_l \right)^{\mathrm{T}} \\ &\quad \cdot \Sigma\Sigma^{\mathrm{T}} \left( \varphi_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, \varphi_l \right) \\ &\leq \sum_{i=1}^{n} z_i^3 \left( \alpha_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, x_{l+1} \right) + \sum_{i=1}^{n} z_i^3 z_{i+1} \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} |z_i|^3 \sum_{p,\,q=1}^{i-1} \left| \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q} \right| |\varphi_p|\, |\varphi_q|\, \left|\Sigma\Sigma^{\mathrm{T}}\right| \\ &\quad + \frac{3}{2} \sum_{i=1}^{n} z_i^2 \left( \varphi_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, \varphi_l \right)^{\mathrm{T}} \\ &\quad \cdot \left( \varphi_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, \varphi_l \right) \left|\Sigma\Sigma^{\mathrm{T}}\right| \end{aligned} \tag{4.10}$$

where $z_{n+1} = 0$. Employing the inequality (3.13) in [5], we have

$$\begin{aligned} \mathcal{L}V &\leq \sum_{i=1}^{n} z_i^3 \left( \alpha_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, x_{l+1} \right) \\ &\quad + \frac{3}{4} \sum_{i=1}^{n} \epsilon_i^{4/3} z_i^4 + \sum_{i=1}^{n} \frac{1}{4\epsilon_{i-1}^4}\, z_i^4 \\ &\quad + \frac{1}{4} \sum_{i=1}^{n} z_i^6 \sum_{p,\,q=1}^{i-1} \left( \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q} \right)^2 \varphi_p^{\mathrm{T}} \varphi_p \varphi_q^{\mathrm{T}} \varphi_q \\ &\quad + \tfrac{1}{4} \sum_{i=1}^{n} \sum_{p,\,q=1}^{i-1} \left|\Sigma\Sigma^{\mathrm{T}}\right|^2 + \tfrac{3}{4} \sum_{i=1}^{n} z_i^4 \\ &\quad \cdot \left( \left( \varphi_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, \varphi_l \right)^{\mathrm{T}} \left( \varphi_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, \varphi_l \right) \right)^2 \\ &\quad + \tfrac{3}{4} \sum_{i=1}^{n} \left|\Sigma\Sigma^{\mathrm{T}}\right|^2 \\ &= \sum_{i=1}^{n} z_i^3 \left\{ \alpha_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, x_{l+1} + \frac{3}{4}\, \epsilon_i^{4/3} z_i \right. \\ &\qquad + \frac{1}{4\epsilon_{i-1}^4}\, z_i + \frac{1}{4}\, z_i^3 \sum_{p,\,q=1}^{i-1} \left( \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q} \right)^2 \\ &\qquad \cdot \varphi_p^{\mathrm{T}} \varphi_p \varphi_q^{\mathrm{T}} \varphi_q + \tfrac{3}{4}\, z_i \\ &\qquad \cdot \left( \left( \varphi_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l} \varphi_l \right)^{\mathrm{T}} \right. \\ &\qquad \left. \left. \cdot \left( \varphi_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, \varphi_l \right) \right)^2 \right\} \\ &\quad + \left( \frac{(n-1)n(2n-1)}{24} + \frac{3}{4}\, n \right) \left|\Sigma\Sigma^{\mathrm{T}}\right|^2. \end{aligned} \tag{4.11}$$

where $\epsilon_i > 0$, $i = 1, \ldots, n-1$, $\epsilon_n = 0$ and $\epsilon_0 = \infty$. Letting $c > 0$

$$\alpha_0 = 0 \tag{4.12}$$

$$\begin{aligned} \alpha_i &= -\frac{1}{4}\, c z_i + \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, x_{l+1} - \frac{3}{4}\, \epsilon_i^{\frac{4}{3}} z_i \\ &\quad - \frac{1}{4\epsilon_{i-1}^4}\, z_i - \frac{1}{4}\, z_i^3 \sum_{p,\,q=1}^{i-1} \left( \frac{\partial^2 \alpha_{i-1}}{\partial x_p \partial x_q} \right)^2 \varphi_p^{\mathrm{T}} \varphi_p \varphi_q^{\mathrm{T}} \varphi_q \\ &\quad - \frac{3}{4}\, z_i \left( \left( \varphi_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, \varphi_l \right)^{\mathrm{T}} \right. \\ &\qquad \left. \cdot \left( \varphi_i - \sum_{l=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_l}\, \varphi_l \right) \right)^2 \\ &\qquad i = 1, \ldots, n-1 \end{aligned} \tag{4.13}$$

$$u = \alpha_n \tag{4.14}$$

$$k = \frac{(n-1)n(2n-1)}{24} + \frac{3}{4}n \tag{4.15}$$

we have

$$\mathcal{L}V \le -cV + k\left|\Sigma\Sigma^{\mathrm{T}}\right|^2. \tag{4.16}$$

Thus, $V$ satisfies the assumptions of Theorem 4.1 with $z$ in place of $x$ and $\gamma(v) = kv^2$, and so we have the following theorem.

*Theorem 4.2:* The system (4.6), (4.7) with feedback (4.14) satisfies

$$E\left\{|z(t)|_4^4\right\} \le e^{-ct}|z(0)|_4^4 + \frac{4k}{c}\sup_{s\in[0,t]}|\Sigma(s)\Sigma(s)^{\mathrm{T}}|^2. \tag{4.17}$$

where the transformation $x \to z(x)$ is smoothly invertible and origin-preserving, and $|z|_4 = (\sum_i z_i^4)^{1/4}$.

When we set the nonlinearities $\varphi_i(\overline{x}_i)$ in (4.6), (4.7) to constant values, we get a linear system in the controllable canonical (chain of integrators) form. In this case, the above procedure actually results in a linear control law. This is easy to see by noting that $\alpha_1(x_1)$ is linear, which inductively implies that the first partial derivatives of $\alpha_i$ are constant and that the second partial derivatives are zero. The linearity of the control law comes as somewhat of a surprise because of the quartic form of the Lyapunov function.

## V. Noise-to-State Lyapunov Functions for General Systems

This section extends the disturbance attenuation ideas from Section IV to general stochastic nonlinear systems. Consider first the uncontrolled system

$$dx = f(x)\,dt + g(x)\Sigma\,dw \tag{5.1}$$

where

| | |
|---|---|
| $x \in \mathbb{R}^n$ | state; |
| $w$ | $m$-dimensional standard Wiener process; |
| $\Sigma$ | takes values in the $m \times m$ nonnegative–definite matrices; |
| $f$: $\mathbb{R}^n \to \mathbb{R}^n$ and $g$: $\mathbb{R}^n \to \mathbb{R}^{n\times m}$ | Borel measurable and locally bounded. |

*Definition 5.1:* The system (5.1) is said to have an ns-lf if there exists a $C^2$ function $V$: $\mathbb{R}^n \to \mathbb{R}_+$, class $\mathcal{K}_\infty$ functions $\alpha_1$, $\alpha_2$ and $\rho$, and a positive definite function $W$, such that for each $x \in \mathbb{R}^n$ and nonnegative–definite $m \times m$ matrix $\Sigma$

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{5.2}$$

$$\begin{gathered}|x| \ge \rho\left(\left|\Sigma\Sigma^{\mathrm{T}}\right|\right)\\ \Downarrow\\ \mathcal{L}V(x,\Sigma) = \frac{\partial V}{\partial x}f(x)\\ +\frac{1}{2}\operatorname{Tr}\left\{\Sigma^{\mathrm{T}}g(x)^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g(x)\Sigma\right\} \le -W(x).\end{gathered} \tag{5.3}$$

*Remark 5.1:* A function $V(x)$ satisfying the conditions of Theorem 4.1 is an ns-lf. The converse is not true. □

Now we turn our attention to the system

$$dx = f(x)\,dt + g_1(x)\Sigma\,dw + g_2(x)u\,dt \tag{5.4}$$

where

| | |
|---|---|
| $w$ and $\Sigma$ | as in (5.1); |
| $u$ | $p$-dimensional control input; |
| $f$: $\mathbb{R}^n \to \mathbb{R}^n$ | continuous; |
| $g_1$: $\mathbb{R}^n \to \mathbb{R}^{n\times m}$ | continuous; |
| $g_2$: $\mathbb{R}^n \to \mathbb{R}^{n\times p}$ | continuous. |

We study the problem of finding continuous feedback that guarantees that the system has an ns-lf. The case without any disturbances was solved by Sontag [41] who derived the "universal formula" used in most of the subsequent work. The formulas for systems with deterministic affine disturbances were derived by Freeman and Kokotović [18], Krstić *et al.* [29], and Sontag and Wang [43]. A formula for the stochastic case with unity intensity noise and vanishing $g_1(x)$ was given by Florchinger [16]. Our result here (Theorem 5.1) is for the case where the incremental covariance $\Sigma\Sigma^{\mathrm{T}}$ is time-varying, unknown, and bounded with an unknown bound, and where $g_1(x)$ may be nonvanishing at the origin.

*Definition 5.2:* A $C^2$ function $V$: $\mathbb{R}^n \to \mathbb{R}_+$ is called an *ns-control Lyapunov function* (ns-clf) for system (5.4), if there exist class $\mathcal{K}_\infty$ functions $\alpha_1$, $\alpha_2$ and $\rho$ such that (5.2) holds for all $x \in \mathbb{R}^n$ and the following implication holds for all $x \in \mathbb{R}^n \setminus \{0\}$ and nonnegative–definite $\Sigma \in \mathbb{R}^{m\times m}$:

$$\begin{gathered}|x| \ge \rho\left(\left|\Sigma\Sigma^{\mathrm{T}}\right|\right)\\ \Downarrow\\ \inf_{u\in\mathbb{R}^p}\left\{L_fV(x) + \frac{1}{2}\operatorname{Tr}\left\{\Sigma^{\mathrm{T}}g_1(x)^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1(x)\Sigma\right\}\right.\\ \left.+L_{g_2}V(x)u\right\} < 0\end{gathered} \tag{5.5}$$

where

$$L_fV = \frac{\partial V}{\partial x}f, \quad L_{g_2}V = \frac{\partial V}{\partial x}g_2.$$

*Lemma 5.1:* A 4-tuple $(V, \alpha_1, \alpha_2, \rho)$ satisfies Definition 5.2 if and only if (5.2) holds for all $x \in \mathbb{R}^n$ and whenever $x \ne 0$

$$\begin{aligned}(L_{g_2}V)(x) = 0 \Rightarrow L_fV(x)&\\ +\frac{1}{2}\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right|(x)\rho^{-1}(|x|) &< 0.\end{aligned} \tag{5.6}$$

*Proof:* (Necessity) By Definition 5.2, if $x \ne 0$ and $L_{g_2}V(x) = 0$, then for any nonnegative–definite $m \times m$ matrix $\Sigma$

$$\begin{gathered}|x| \ge \rho\left(\left|\Sigma\Sigma^{\mathrm{T}}\right|\right)\\ \Downarrow\\ L_fV + \frac{1}{2}\operatorname{Tr}\left\{\Sigma^{\mathrm{T}}g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\Sigma\right\} < 0.\end{gathered} \tag{5.7}$$

Consider the incremental covariance given by the feedback law

$$\Sigma\Sigma^{\mathrm{T}} = \rho^{-1}(|x|)\frac{g_1^{\mathrm{T}}\dfrac{\partial^2 V}{\partial x^2}g_1}{\left|g_1^{\mathrm{T}}\dfrac{\partial^2 V}{\partial x^2}g_1\right|} \tag{5.8}$$

where the quotient is defined to be $1/\sqrt{m}$ times the $m \times m$ identity matrix if the denominator is zero, and $\Sigma$ is a nonnegative definite square root of $\Sigma\Sigma^{\mathrm{T}}$. This then satisfies the condition in (5.6)

$$\rho\left(\left|\Sigma\Sigma^{\mathrm{T}}\right|\right) = |x|. \tag{5.9}$$

So, using (5.8)

$$\begin{aligned} L_f V &+ \frac{1}{2}\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right|\rho^{-1}(|x|) \\ &= L_f V + \frac{1}{2}\operatorname{Tr}\left\{g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\Sigma\Sigma^{\mathrm{T}}\right\} \\ &= L_f V + \frac{1}{2}\operatorname{Tr}\left\{\Sigma^{\mathrm{T}}g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\Sigma\right\} < 0. \end{aligned} \tag{5.10}$$

(Sufficiency) For $x \neq 0$, $|x| \geq \rho(|\Sigma\Sigma^{\mathrm{T}}|)$

$$\begin{aligned} &\inf_{u\in\mathbb{R}^p}\left\{L_f V + \frac{1}{2}\operatorname{Tr}\left\{\Sigma^{\mathrm{T}}g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\Sigma\right\} + L_{g_2}Vu\right\} \\ &\quad\leq \inf_{u\in\mathbb{R}^p}\left\{L_f V + \frac{1}{2}\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right|\left|\Sigma\Sigma^{\mathrm{T}}\right| + L_{g_2}Vu\right\} \\ &\quad\leq \inf_{u\in\mathbb{R}^p}\left\{L_f V + \frac{1}{2}\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right|\rho^{-1}(|x|) + L_{g_2}Vu\right\} < 0 \end{aligned} \tag{5.11}$$

where to show the last inequality, one chooses $u$ according to the expression given in (5.12) below. □

*Theorem 5.1:* If there exists an ns-clf for system (5.4), then there exists a feedback law continuous away from the origin that guarantees that the ns-clf is an ns-lf.

*Proof:* Consider the Sontag-type control law [41]

$$\alpha_s(x) = \begin{cases} -\dfrac{\omega + \sqrt{\omega^2 + \left(L_{g_2}V(L_{g_2}V)^{\mathrm{T}}\right)^2}}{L_{g_2}V(L_{g_2}V)^{\mathrm{T}}} & \\ (L_{g_2}V)^{\mathrm{T}} & L_{g_2}V \neq 0 \\ 0, & L_{g_2}V = 0 \end{cases} \tag{5.12}$$

where

$$\omega = L_f V + \frac{1}{2}\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right|\rho^{-1}(|x|). \tag{5.13}$$

From the results in [41], it follows that $\alpha_s(x)$ is continuous away from $x=0$, so it remains to prove that it makes $V(x)$ an ns-lf. Substituting (5.12) into $\mathcal{L}V$, we have

$$\begin{aligned} \mathcal{L}V &= L_f V + \frac{1}{2}\operatorname{Tr}\left\{\Sigma^{\mathrm{T}}g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\Sigma\right\} + L_{g_2}V\alpha_s(x) \\ &\leq -\sqrt{\omega^2 + \left(L_{g_2}V(L_{g_2}V)^{\mathrm{T}}\right)^2} \\ &\quad - \frac{1}{2}\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g_1\right|\left(\rho^{-1}(|x|) - \left|\Sigma\Sigma^{\mathrm{T}}\right|\right). \end{aligned} \tag{5.14}$$

If $|x| \geq \rho(|\Sigma\Sigma^{\mathrm{T}}|)$, we have

$$\mathcal{L}V \leq -\sqrt{\omega^2 + \left(L_{g_2}V(L_{g_2}V)^{\mathrm{T}}\right)^2} \triangleq -W(x) \tag{5.15}$$

where, according to Lemma 5.1, $W(x)$ is positive definite. [We have $W(0)=0$ because $V$ has a minimum at $x=0$ and hence $\partial V/\partial x = 0$ there.] By Definition 5.1, $V(x)$ is an ns-lf. □

In addition to the continuity away from the origin, the formula (5.12) will be continuous at the origin provided the ns-clf $V(x)$ satisfies a *small control property*: there exists a continuous control law $u = \alpha_c(x)$ which guarantees that $V(x)$ is an ns-lf. The proof of this fact directly follows from [41].

## VI. ns-clfs as Inverse Optimal Value Functions for General Systems

In contrast to most of the work in stochastic nonlinear control where the starting point is an optimal (risk-sensitive) control problem [3], [12], [13], [25], [34], [36], [38], our approach in the previous sections was directed toward stability. In this section, we establish connections with optimality. For general stochastic nonlinear systems (affine in control and noise) that have an ns-clf, we design controllers that solve a meaningful optimal control problem. This "inverse optimal" approach where the cost functional is not given *a priori*, and thus the task of solving Hamilton–Jacobi PDE's is avoided, has recently soared in popularity in the *robust nonlinear control* literature [18], [30], [39].

Consider the general nonlinear stochastic system affine in the noise $\Sigma\,dw$ and control $u$:

$$dx = f(x)\,dt + g_1(x)\Sigma\,dw + g_2(x)u\,dt \tag{6.1}$$

where

| | |
|---|---|
| $w$ | $m$-dimensional standard Wiener process; |
| $\Sigma$ | takes values in the $m\times m$ nonnegative–definite matrices; |
| $u$ | $p$-dimensional control; |
| $f\colon \mathbb{R}^n \to \mathbb{R}^n$ | continuous; |
| $g_1\colon \mathbb{R}^n \to \mathbb{R}^{n\times m}$ | continuous; |
| $g_2\colon \mathbb{R}^n \to \mathbb{R}^{n\times p}$ | continuous. |

*Definition 6.1:* The *inverse optimal stochastic gain assignment* problem for system (6.1) is solvable if there exist class $\mathcal{K}_\infty$ functions $\gamma_1$ and $\gamma_2$ whose derivatives $\gamma_1'$ and $\gamma_2'$ are also class $\mathcal{K}_\infty$ functions, a matrix-valued function $R_2(x)$ such that $R_2(x) = R_2(x)^{\mathrm{T}} > 0$ for all $x$, a positive definite function $l(x)$, a positive–definite radially unbounded function $S(x)$, and a feedback control law $u = \alpha(x)$ continuous away from the origin with $\alpha(0)=0$, which minimizes the cost functional

$$\begin{aligned} J(u) = \sup_{\Sigma\in\mathcal{D}}\Bigg\{\limsup_{r\to\infty} E\Bigg[S(x(\tau_r)) + \int_0^{\tau_r} &\\ \cdot\left(l(x) + \gamma_2\left(\left|R_2(x)^{1/2}u\right|\right)\right.& \\ \left. -\gamma_1\left(\left|\Sigma\Sigma^{\mathrm{T}}\right|\right)\right)dt\Bigg]\Bigg\}& \end{aligned} \tag{6.2}$$

where $\mathcal{D}$ is the set of locally bounded functions of $(x,\,t)$ taking values in the nonnegative–definite $m\times m$ matrices, $x(\cdot)$ is a solution of (6.1) with $\Sigma = \Sigma(x(\cdot),\,\cdot)$, $u = \alpha(x(\cdot))$ there, and $\tau_r = \inf\{t\geq 0\colon |x(t)| \geq r\}$.

*Remark 6.1:* The class $\mathcal{D}$ includes functions of $t$ to cover the case where $\Sigma = \Sigma(t)$, $t \geq 0$. □

This optimal control problem looks different than other problems considered in the literature. First, in the jargon of the risk-sensitive theory, (6.2) is a risk-neutral problem. Second, to see the main difference, consider the problem

$$I(u) = \lim_{t\to\infty} E\left[S(x(t)) + \int_0^t \cdot \left(l(x) + \gamma_2\left(\left|R_2(x)^{1/2}u\right|\right)\right) ds\right] \quad (6.3)$$

which appears as a direct nonlinear extension of the standard linear stochastic control problem [2] (a division by time $t$ would lead to the optimal $\mathcal{H}_2$ problem [19]). This problem would be appropriate if $\Sigma$ were constant and known. In that case, the term $\int_0^t \gamma_1(|\Sigma\Sigma^{\mathrm{T}}|)\, ds$ would be included in the value function. However, when $\Sigma$ is unknown and/or time varying, it is more reasonable to pose the problem as a differential game (6.2). (Further clarification is given in Remark 6.2). Note that this differential game is very different from stochastic differential games [3, Sec. 4.7.2] where the player opposed to control is another *deterministic* disturbance (see footnote 1 in Section I). In our case the opposing player is the stochastic disturbance $\Sigma dw$ through its incremental covariance $\Sigma\Sigma^{\mathrm{T}}$.

The next theorem allows a solution to the inverse optimal stochastic gain assignment problem provided a solution to a certain Hamilton–Jacobi–Isaacs equation is available. Before we state the theorem, we introduce the so-called Legendre–Fenchel transform which is the key tool for the results in this section. Let $\gamma$ be a class $\mathcal{K}_\infty$ function whose derivative $\gamma'$ is also a class $\mathcal{K}_\infty$ function, then $\ell\gamma$ denotes the Legendre–Fenchel transform

$$\ell\gamma(r) = \int_0^r (\gamma')^{-1}(s)\, ds \quad (6.4)$$

where $(\gamma')^{-1}(r)$ stands for the inverse function of $d\gamma(r)/dr$. The reader is referred to the Appendix for some useful facts on the Legendre–Fenchel transform.

*Theorem 6.1:* Consider the control law

$$u = \alpha(x) = -R_2^{-1}(L_{g_2}V)^{\mathrm{T}} \frac{\ell\gamma_2\left(|L_{g_2}VR_2^{-1/2}|\right)}{\left|L_{g_2}VR_2^{-1/2}\right|^2} \quad (6.5)$$

where $V\colon \mathbb{R}^n \to \mathbb{R}_+$ is a $C^2$ function such that (5.2) holds for two class $\mathcal{K}_\infty$ functions $\alpha_1$, $\alpha_2$, $\gamma_1$ and $\gamma_2$ are class $\mathcal{K}_\infty$ functions whose derivatives are also class $\mathcal{K}_\infty$ functions, and $R_2(x)$ is a matrix-valued function such that $R_2(x) = R_2(x)^{\mathrm{T}} > 0$. If the control law (6.5), when used for the system

$$dx = f(x)\, dt + g_1(x)\overline{\Sigma}\, dw + g_2(x)u\, dt \quad (6.6)$$

where $w$ is a standard $m$-dimensional Wiener process and $\overline{\Sigma} \in \mathbb{R}^{m\times m}$ is nonnegative definite satisfying

$$\overline{\Sigma\Sigma}^{\mathrm{T}} = 2g_1^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g_1 \frac{\ell\gamma_1\left(\left|g_1^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g_1\right|\right)}{\left|g_1^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g_1\right|^2} \quad (6.7)$$

is such that

$$\overline{\mathcal{L}}V(x)|_{(6.5)} = L_fV(x) + \frac{1}{2}\mathrm{Tr}\left\{\overline{\Sigma}^{\mathrm{T}} g_1(x) \frac{\partial^2 V}{\partial x^2} g_1(x)\overline{\Sigma}\right\} + L_{g_2}V\alpha(x) \leq -W(x) \quad (6.8)$$

for all $x \in \mathbb{R}^n$ and a positive–definite function $W\colon \mathbb{R}^n \to \mathbb{R}_+$, then the control law

$$u^* = \alpha^*(x) = -\frac{\beta}{2} R_2^{-1}(L_{g_2}V)^{\mathrm{T}} \frac{(\gamma_2')^{-1}\left(\left|L_{g_2}VR_2^{-1/2}\right|\right)}{\left|L_{g_2}VR_2^{-1/2}\right|}, \quad \beta \geq 2 \quad (6.9)$$

solves the problem of inverse optimal stochastic gain assignment for the system (6.1) by minimizing the cost functional

$$J(u) = \sup_{\Sigma\in\mathcal{D}}\left\{\limsup_{r\to\infty} E\left[2\beta V(x(\tau_r)) + \int_0^{\tau_r} \cdot\left(l(x) + \beta^2\gamma_2\left(\frac{2}{\beta}\left|R_2(x)^{1/2}u\right|\right) - \beta\lambda\gamma_1\left(\frac{|\Sigma\Sigma^{\mathrm{T}}|}{\lambda}\right)\right) dt\right]\right\} \quad (6.10)$$

where $\lambda \in (0, 2]$ and

$$\begin{aligned} l(x) = {} & 2\beta\left[\ell\gamma_2\left(\left|L_{g_2}VR_2^{-1/2}\right|\right) - L_fV - \ell\gamma_1\left(\left|g_1^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g_1\right|\right)\right] \\ & + \beta(\beta-2)\ell\gamma_2\left(\left|L_{g_2}VR_2^{-1/2}\right|\right) \\ & + \beta(2-\lambda)\ell\gamma_1\left(\left|g_1^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g_1\right|\right). \end{aligned} \quad (6.11)$$

*Remark 6.2:* Even though not explicit in the statement of Theorem 6.1, $V(x)$ solves the following family of Hamilton–Jacobi–Isaacs equations parameterized by $\beta \in [2, \infty)$ and $\lambda \in (0, 2]$

$$L_fV + \frac{\lambda}{2}\ell\gamma_1\left(\left|g_1^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g_1\right|\right) - \frac{\beta}{2}\ell\gamma_2\left(\left|L_{g_2}VR_2^{-1/2}\right|\right) + \frac{l(x)}{2\beta} = 0. \quad (6.12)$$

This equation, which depends only on known quantities, helps explain why we are pursuing a differential game problem with $\Sigma$ as a player. If we set (6.3) as the cost, the resulting HJB equation is

$$L_fV + \frac{1}{2}\mathrm{Tr}\left\{\Sigma^{\mathrm{T}} g_1^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2} g_1\Sigma\right\} - \frac{\beta}{2}\ell\gamma_2\left(\left|L_{g_2}VR_2^{-1/2}\right|\right) + \frac{l(x)}{2\beta} = 0. \quad (6.13)$$

If $\Sigma$ is unknown (and allowed to take any value), it is clear that this equation cannot be solved. There is only one exception—linear systems. In the linear case $g_1(x)$ would be constant and $V(x)$ would be quadratic, which would make $g_1^{\mathrm{T}}(\partial^2 V/\partial x^2)g_1$ constant. For a constant $\Sigma$, even if it is unknown, one would absorb the term

$$\frac{1}{2}\,\mathrm{Tr}\left\{\Sigma^{\mathrm{T}} g_1^{\mathrm{T}} \frac{\partial^2 V}{\partial x^2}\, g_1 \Sigma\right\}$$

into the value function. It is obvious that this can not be done when $g_1$ depends on $x$ and/or $V(x)$ is nonquadratic. Thus, we pursue a differential game problem in which $\Sigma$ is a player and its actions are penalized. □

*Proof of Theorem 6.1:* From (6.8), under the control (6.5) for (6.6) and (6.7)

$$\begin{aligned}\overline{\mathcal{L}}V =& L_f V + \ell\gamma_1\left(\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}\, g_1\right|\right)\\ &- \ell\gamma_2\left(\left|L_{g_2} V R_2^{-1/2}\right|\right) \le -W. \end{aligned}\tag{6.14}$$

Then we have

$$\begin{aligned} l(x) \ge& 2\beta W(x) + \beta(\beta-2)\ell\gamma_2\left(\left|L_{g_2} V R_2^{-1/2}\right|\right)\\ &+ \beta(2-\lambda)\ell\gamma_1\left(\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}\, g_1\right|\right). \end{aligned}\tag{6.15}$$

Since $W(x)$ is positive–definite, $\beta \ge 2$, $\lambda \in (0, 2]$ and $\ell\gamma_2$ and $\ell\gamma_1$ are class $\mathcal{K}_\infty$ functions (Lemma A.1), $l(x)$ is bounded below by a positive definite function. Therefore, $J(u)$ is a meaningful cost functional.

Now we prove optimality. According to Dynkin's formula and by substituting $l(x)$ into $J(u)$, we have

$$\begin{aligned} J(u) =& \sup_{\Sigma\in\mathcal{D}}\left\{\limsup_{r\to\infty} E\left[2\beta V(x(\tau_r)) + \int_0^{\tau_r}\right.\right.\\ &\cdot\left(l(x) + \beta^2\gamma_2\left(\frac{2}{\beta}\left|R_2^{1/2}u\right|\right)\right.\\ &\left.\left.\left.-\beta\lambda\gamma_1\left(\frac{\left|\Sigma\Sigma^{\mathrm{T}}\right|}{\lambda}\right)\right) dt\right]\right\}\\ =& \sup_{\Sigma\in\mathcal{D}}\left\{\limsup_{r\to\infty} E\left[2\beta V(x(0)) + \int_0^{\tau_r}\right.\right.\\ &\cdot\left(2\beta\mathcal{L}V|_{(6.1)} + l(x)\right.\\ &+ \beta^2\gamma_2\left(\frac{2}{\beta}\left|R_2^{1/2}u\right|\right)\\ &\left.\left.\left.-\beta\lambda\gamma_1\left(\frac{\left|\Sigma\Sigma^{\mathrm{T}}\right|}{\lambda}\right)\right) dt\right]\right\}\\ =& \sup_{\Sigma\in\mathcal{D}}\left\{2\beta E\{V(x(0))\} + \limsup_{r\to\infty} E\int_0^{\tau_r}\right.\\ &\cdot\left[\beta^2\gamma_2\left(\frac{2}{\beta}\left|R_2^{1/2}u\right|\right)\right.\\ &+ \beta^2\ell\gamma_2\left(\left|L_{g_2} V R_2^{-1/2}\right|\right)\\ &+ 2\beta L_{g_2} V u - \beta\lambda\gamma_1\left(\frac{\left|\Sigma\Sigma^{\mathrm{T}}\right|}{\lambda}\right)\\ &- \beta\lambda\ell\gamma_1\left(\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}\, g_1\right|\right)\\ &\left.\left.+\beta\mathrm{Tr}\left\{\Sigma^{\mathrm{T}} g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}\, g_1\Sigma\right\}\right] dt\right\}. \end{aligned}\tag{6.16}$$

Using Lemma A.2 we have

$$\begin{aligned} &-2\beta L_{g_2} V u\\ &\quad= \beta^2\left(\frac{2}{\beta}\, R_2^{1/2}u\right)^{\mathrm{T}}\left(-R_2^{-1/2}(L_{g_2}V)^{\mathrm{T}}\right)\\ &\quad\le \beta^2\gamma_2\left(\frac{2}{\beta}\left|R_2^{1/2}u\right|\right) + \beta^2\ell\gamma_2\left(\left|L_{g_2} V R_2^{-1/2}\right|\right) \end{aligned}\tag{6.17}$$

$$\begin{aligned} &\beta\mathrm{Tr}\left\{\Sigma^{\mathrm{T}} g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}\, g_1\Sigma\right\}\\ &\quad= \beta\left(\mathrm{col}(\Sigma\Sigma^{\mathrm{T}})\right)^{\mathrm{T}}\left(\mathrm{col}\left(g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}\, g_1\right)\right)\\ &\quad\le \beta\lambda\gamma_1\left(\frac{\left|\Sigma\Sigma^{\mathrm{T}}\right|}{\lambda}\right) + \beta\lambda\ell\gamma_1\left(\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}\, g_1\right|\right) \end{aligned}\tag{6.18}$$

and the equalities hold when

$$u^* = -\frac{\beta}{2}\, R_2^{-1/2}(\gamma_2')^{-1}\left(\left|L_{g_2} V R_2^{-1/2}\right|\right)\frac{R_2^{-1/2}(L_{g_2}V)^{\mathrm{T}}}{\left|L_{g_2} V R_2^{-1/2}\right|}\tag{6.19}$$

and

$$\left(\Sigma\Sigma^{\mathrm{T}}\right)^* = \lambda(\gamma_1')^{-1}\left(\left|g_1^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}\, g_1\right|\right)\frac{g_1^{\mathrm{T}}\dfrac{\partial^2 V}{\partial x^2}\, g_1}{\left|g_1^{\mathrm{T}}\dfrac{\partial^2 V}{\partial x^2}\, g_1\right|}.\tag{6.20}$$

So the "worst case" unknown covariance is given by (6.20), the minimum of (6.16) is reached with $u = u^*$, and

$$\min_u J(u) = 2\beta E\{V(x(0))\}.\tag{6.21}$$

To satisfy the requirements of Definition 6.1, it only remains to prove that $\alpha^*(x)$ is continuous away from the origin and $\alpha^*(0) = 0$. This is proved in [6,proof of Th. 3.1]. □

The next theorem is the main result of this section. It constructs a controller that solves the problem posed in Definition 6.1.

*Theorem 6.2:* If the system (6.1) has an ns-clf $V(x)$ such that $g_1^{\mathrm{T}}(\partial^2 V/\partial x^2)g_1$ vanishes at the origin, then the problem of inverse optimal stochastic gain assignment is solvable.

*Proof:* To solve the problem of inverse optimal stochastic gain assignment, we should find the functions $V(x)$, $R_2(x)$, $l(x)$, $\gamma_1(\cdot)$, $\gamma_2(\cdot)$ that solve the Hamilton–Jacobi–Isaacs equation (6.12) for some $\beta \in [2, \infty)$ and $\lambda \in (0, 2]$. Then the inverse optimal controller would be given by (6.9). Since the system has an ns-clf, in particular, there exist $(V, \rho)$ that satisfy (5.6), consider the choice

$$R_2(x) = I \begin{cases} \dfrac{2L_{g_2}V(L_{g_2}V)^{\mathrm{T}}}{\omega + \sqrt{\omega^2 + \left(L_{g_2}V(L_{g_2}V)^{\mathrm{T}}\right)^2}}, & L_{g_2}V \neq 0 \\ \text{any positive number}, & L_{g_2}V = 0. \end{cases} \tag{6.22}$$

where $\omega$ is given by (5.13) and $\gamma_2(r) = (1/4)r^2$. In addition, let $\beta = \lambda = 2$, then $\ell\gamma_2(r) = r^2$, and after some computation we get

$$\begin{aligned} L_f V &+ \frac{\lambda}{2}\,\ell\gamma_1\left(\left|g_1^{\mathrm{T}}\,\frac{\partial^2 V}{\partial x^2}\,g_1\right|\right) - \frac{\beta}{2}\,\ell\gamma_2\left(\left|L_{g_2}VR_2^{-1/2}\right|\right) \\ &= -\tfrac{1}{2}\left[-\omega + \sqrt{\omega^2 + \left(L_{g_2}V(L_{g_2}V)^{\mathrm{T}}\right)^2}\right] \\ &\quad - \frac{1}{2}\left|g_1^{\mathrm{T}}\,\frac{\partial^2 V}{\partial x^2}\,g_1\right|\rho^{-1}(|x|) + \ell\gamma_1\left(\left|g_1^{\mathrm{T}}\,\frac{\partial^2 V}{\partial x^2}\,g_1\right|\right). \end{aligned} \tag{6.23}$$

Since $g_1^{\mathrm{T}}(\partial^2 V/\partial x^2)g_1$ vanishes at the origin, there exists a class $\mathcal{K}_\infty$ function $\pi(|x|)$ such that

$$\left|g_1^{\mathrm{T}}\,\frac{\partial^2 V}{\partial x^2}\,g_1\right| \leq \pi(|x|).$$

Let $\zeta(r)$ be a class $\mathcal{K}_\infty$ function, whose derivative $\zeta'$ is also in $\mathcal{K}_\infty$, and such that $\zeta(r) \leq (1/2)r\rho^{-1}(\pi^{-1}(r))$. Denoting $\gamma_1 = \ell\zeta$, since $\ell\ell\zeta = \zeta$, we have

$$\ell\gamma_1(r) = \zeta(r) \leq \tfrac{1}{2}\,r\rho^{-1}\left(\pi^{-1}(r)\right) \tag{6.24}$$

so

$$\ell\gamma_1\left(\left|g_1^{\mathrm{T}}\,\frac{\partial^2 V}{\partial x^2}\,g_1\right|\right) \leq \frac{1}{2}\left|g_1^{\mathrm{T}}\,\frac{\partial^2 V}{\partial x^2}\,g_1\right|\rho^{-1}(|x|). \tag{6.25}$$

Choose

$$\begin{aligned} l(x) &= 4\left\{\tfrac{1}{2}\left[-\omega + \sqrt{\omega^2 + \left(L_{g_2}V(L_{g_2}V)^{\mathrm{T}}\right)^2}\right]\right. \\ &\qquad + \frac{1}{2}\left|g_1^{\mathrm{T}}\,\frac{\partial^2 V}{\partial x^2}\,g_1\right|\rho^{-1}(|x|) \\ &\qquad \left. -\ell\gamma_1\left(\left|g_1^{\mathrm{T}}\,\frac{\partial^2 V}{\partial x^2}\,g_1\right|\right)\right\} \\ &\geq 2\left[-\omega + \sqrt{\omega^2 + \left(L_{g_2}V(L_{g_2}V)^{\mathrm{T}}\right)^2}\right], \end{aligned} \tag{6.26}$$

which is positive definite by Lemma 5.1 and (6.23). This completes the selection of $V(x)$, $R_2(x)$, $l(x)$, $\gamma_1(\cdot)$, $\gamma_2(\cdot)$ that solve the HJI equation (6.12). □

*Remark 6.3:* The condition in Theorem 6.2 that $g_1^{\mathrm{T}}(\partial^2 V/\partial x^2)g_1$ be vanishing at the origin excludes the possibility of a linear system ($g_1 = \text{const}$) with a quadratic ns-clf $V(x)$. This condition can be eliminated by modifying the cost functional (6.2) but then other issues arise, like radial unboundedness of $\sqrt{\omega^2 + (L_{g_2}V(L_{g_2}V)^{\mathrm{T}})^2}$. It is our opinion, supported by the results in Section IV, that, for stochastic systems, Lyapunov functions that are higher order at the origin are superior to quadratic Lyapunov functions. The peculiarity of the linear case [the fact that $(1/2)\mathrm{Tr}\,\{\Sigma^{\mathrm{T}} g_1^{\mathrm{T}}\,(\partial^2 V/\partial x^2)g_1\Sigma\}$ can be absorbed into the value function, making the controller independent of the noise vector field $g_1$!] has prevented the inadequacy of quadratic Lyapunov functions from being exposed for several decades now. □

## VII. EXAMPLE

In this brief section, we return to Example 1.1. From the results of the paper it is clear that (1.4) guarantees that system (1.3) has an ns-clf and (1.5) achieves stability of $x = \xi = 0$ and regulation of $x$ (in probability). The $x(t)$ time responses in Fig. 1 reveal the difference between the achieved stability properties. The simulations are performed for $\sigma(t) \equiv 2\sqrt{2}$. While the adaptive controller on the right achieves regulation of $x$, the nonadaptive controller on the left only forces $x$ to converge to an interval around zero proportional to $\sigma$. As is evident from the figure, the nonadaptive controller results in a residual error, whereas the adaptive controller does not. The variable $\xi$ is the estimate of $\|\sigma^2\|_\infty/2 = 4$. We see that $\xi(t)$ converges to about 2.5 and does not reach the true value 4. This is not unexpected as in adaptive regulation problems we seldom see convergence to the true parameter.

## VIII. CONCLUSION

We solved the problem of state-feedback attenuation of stochastic disturbances with *unknown covariance*. Our results are given for exemplary, rather than for the most general possible, classes of stochastic nonlinear systems. For example, it is straightforward to add known nonlinearities and deterministic disturbances, as well as zero dynamics with appropriate input-to-state properties. The output-feedback problem for the class of systems in [7] should also be straightforward.

A major difficulty specific to the stochastic case is that $l(x)$ in Section VI cannot be guaranteed to be radially unbounded as in the deterministic case [30]. The reason for this obstacle is the term $\partial^2 V/\partial x^2$ which prevents easy modifications of the Lyapunov function (in many cases this term acts to make $\mathcal{L}V$ less negative).

As we stated in Sections I and VI, this design cures the anomaly in the $LQG/\mathcal{H}_2$ design where the controller does not depend on the noise input matrix $B_1$. A linear design that does take $B_1$ into account is *covariance control* [40], however, in covariance control, a bound on $\Sigma$ needs to be known.

When applied to linear systems, the design in Section III solves the stabilization problem with multiplicative noise. A

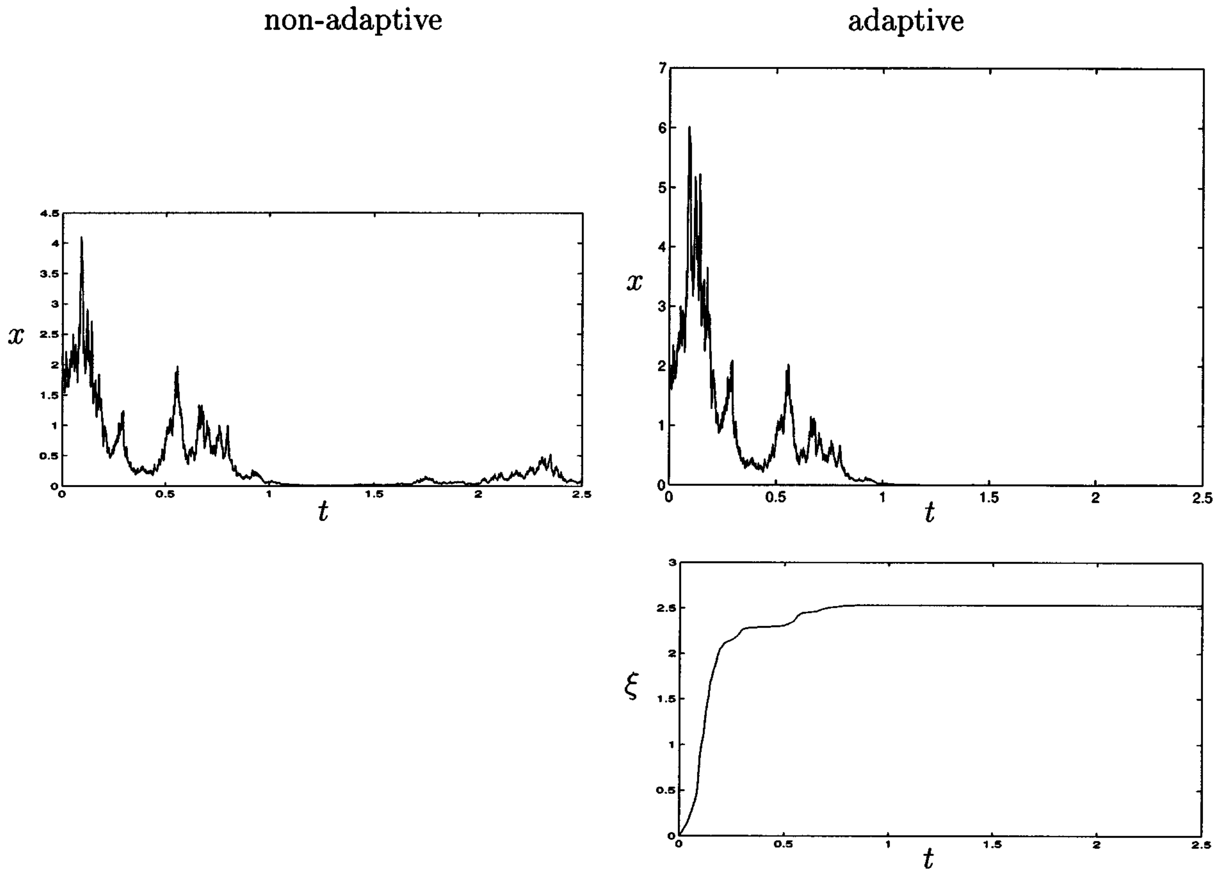


Fig. 1. The time responses with the nonadaptive and the adaptive controller.

sizeable body of literature on this problem was reviewed in [4]. All of the previous results assume either restrictive geometric conditions as, e.g., in [46] (their conditions are not satisfied by linear strict-feedback systems) or require the knowledge of a bound on the noise covariance [11], [47]. Our adaptive design requires no *a priori* knowledge of a bound on the covariance.

## APPENDIX

*Lemma A.1 (Krstić and Li [30]):* If $\gamma$ and its derivative $\gamma'$ are class $\mathcal{K}_\infty$ functions, then the Legendre–Fenchel transform satisfies the following properties:

$$(a) \qquad \ell\gamma(r) = r(\gamma')^{-1}(r) - \gamma\left((\gamma')^{-1}(r)\right) = \int_0^r (\gamma')^{-1}(s)\,ds \tag{A.1}$$

$$(b) \qquad \ell\ell\gamma = \gamma \tag{A.2}$$

$$(c) \qquad \ell\gamma \text{ is a class } \mathcal{K}_\infty \text{ function} \tag{A.3}$$

$$(d) \qquad \ell\gamma\left(\gamma'(r)\right) = r\gamma'(r) - \gamma(r). \tag{A.4}$$

*Lemma A.2 (Young's Inequality [20, Theorem 156]):* For any two vectors $x$ and $y$, the following holds:

$$x^{\mathrm{T}}y \le \gamma(|x|) + \ell\gamma(|y|) \tag{A.5}$$

and the equality is achieved if and only if

$$y = \gamma'(|x|)\,\frac{x}{|x|}, \quad \text{that is, for } x = (\gamma')^{-1}(|y|)\,\frac{y}{|y|}. \tag{A.6}$$

## REFERENCES

[1] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*. Upper Saddle River, NJ: Prentice-Hall, 1990.
[2] K. J. Astrom, *Introduction to Stochastic Control Theory*. New York: Academic, 1970.
[3] T. Başar and P. Bernhard, *$H^\infty$ Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, 2nd ed. Boston, MA: Birkhäuser, 1995.
[4] D. S. Bernstein, "Robust static and dynamic output-feedback stabilization: Deterministic and stochastic perspectives," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 1076–1084, 1987.
[5] H. Deng and M. Krstić, "Stochastic nonlinear stabilization—Part I: A backstepping design," *Syst. Control Lett.*, vol. 32, pp. 143–150, 1997.
[6] ——, "Stochastic nonlinear stabilization—Part II: Inverse optimality," *Syst. Control Lett.*, vol. 32, pp. 151–159, 1997.
[7] ——, "Output-feedback stochastic nonlinear stabilization," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 328–333, Feb. 1999.
[8] J. L. Doob, *Stochastic Processes*. New York: Wiley, 1953.
[9] D. Down, S. P. Meyn, and R. L. Tweedie, "Exponential and uniform ergodicity of Markov processes," *Ann Probab*, vol. 23, pp. 1671–1691, 1995.
[10] T. E. Duncan and B. Pasik-Duncan, "Stochastic adaptive control," in *The Control Handbook*, W. S. Levine, Ed. Boca Raton, FL: CRC Press, 1996, pp. 1127–1136.
[11] L. El Ghaoui, "State-feedback control of systems with multiplicative noise via linear matrix inequalities," *Syst. Control Lett.*, vol. 24, pp. 223–228, 1995.
[12] W. H. Fleming and W. M. McEneaney, "Risk-sensitive control on an infinite time horizon," *SIAM J. Control Optim.*, vol. 33, no. 6, pp. 1881–1915, 1995.
[13] W. H. Fleming and H. M. Soner, *Control Markov Processes and Viscosity Solutions*. New York: Springer-Verlag, 1993.
[14] P. Florchinger, "Lyapunov-like techniques for stochastic stability," *SIAM J. Control Optim.*, vol. 33, pp. 1151–1169, 1995.
[15] ——, "Global stabilization of cascade stochastic systems," in *Proc. 34th Conf. Decision Control*, New Orleans, LA, 1995, pp. 2185–2186.

[16] ——, "A universal formula for the stabilization of control stochastic differential equations," *Stoch. Analy. Appl.*, vol. 11, pp. 155–162, 1993.

[17] ——, "Feedback stabilization of affine in the control stochastic differential systems by the control Lyapunov function method," *SIAM J. Control Optim.*, vol. 35, pp. 500–511, 1997.

[18] R. A. Freeman and P. V. Kokotović, *Robust Nonlinear Control Design—State-Space and Lyapunov Techniques*. Boston, MA: Birkhäuser, 1996.

[19] M. Green and D. J. N. Limebeer, *Linear Robust Control*. Upper Saddle River: Prentice-Hall, 1995.

[20] G. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, 2nd ed. Cambridge, U.K.: Cambridge University Press, 1989.

[21] W. Hahn, *Stability of Motion*. New York: Springer-Verlag, 1967.

[22] U. G. Haussmann and W. Suo, "Singular optimal stochastic controls—Part I: Existence," *SIAM J. Control Optimiz.*, vol. 33, pp. 916–936, 1995.

[23] U. G. Haussmann and W. Suo, "Singular optimal stochastic controls—Dynamic programming," *SIAM J. Control Optimiz.*, vol. 33, pp. 937–959, 1995.

[24] A. Isidori, *Nonlinear Control Systems*, 3rd ed. New York: Springer-Verlag, 1995.

[25] M. R. James, J. Baras, and R. J. Elliott, "Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems," *IEEE Trans. Automatic. Contr.*, vol. 39, pp. 780–792, Apr. 1994.

[26] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed. New York: Springer, 1991.

[27] H. K. Khalil, *Nonlinear Systems*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1996.

[28] R. Z. Khas'minskii, *Stochastic Stability of Differential Equations*. Rockville, MD: S & N International, 1980.

[29] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.

[30] M. Krstić and Z. H. Li, "Inverse optimal design of input-to-state stabilizing nonlinear controllers," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 336–350, Mar. 1998.

[31] H. J. Kushner, *Stochastic Stability and Control*. New York: Academic, 1967.

[32] J. P. LaSalle, "Stability theory for ordinary differential equations," *J. Differential Equations*, vol. 4, pp. 57–65, 1968.

[33] X. Mao, *Stability of Stochastic Differential Equations with Respect to Semimartingales*. White Plains, NY: Longman, 1991.

[34] H. Nagai, "Bellman equations of risk-sensitive control," *SIAM J. Control Optim.*, vol. 34, pp. 74–101, 1996.

[35] B. Øksendal, *Stochastic Differential Equations—An Introduction with Applications*. New York: Springer-Verlag, 1995.

[36] Z. Pan and T Başar, "Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criterion," *SIAM J. Control Optim.*, vol. 37, pp. 957–995, 1999.

[37] L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales*, 2nd ed. New York: Wiley, 1994, vol. 1.

[38] T. Runolfsson, "The equivalence between infinite horizon control of stochastic systems with exponential-of-integral performance index and stochastic differential games," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1551–1563, Aug. 1994.

[39] R. Sepulchre, M. Janković, and P. V. Kokotović, *Constructive Nonlinear Control*. New York: Springer-Verlag, 1997.

[40] R. Skelton, T. Iwasaki, and K. Grigoriadis, *A Unified Algebraic Approach to Control Design*. London, U.K.: Taylor and Francis, 1997.

[41] E. D. Sontag, "A 'universal' construction of Artstein's theorem on nonlinear stabilization," *Syst. Control Lett.*, vol. 13, pp. 117–123, 1989.

[42] ——, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 435–443, Apr. 1989.

[43] E. D. Sontag and Y. Wang, "On characterizations of input-to-state stability property," *Syst. Control Lett.*, vol. 24, pp. 351–359, 1994.

[44] A. R. Teel, "A nonlinear small gain theorem for the analysis of control systems with saturation," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1256–1270, Sept. 1996.

[45] J. Tsinias, "The concept of 'exponential ISS' for stochastic systems and applications to feedback stabilization," *Syst. Control Lett.*, vol. 36, pp. 221–229, 1999.

[46] J. L. Willems and J. C. Willems, "Feedback stabilizability for stochastic systems with state and control dependent noise," *Automatica*, vol. 12, pp. 277–283, 1976.

[47] W. M. Wonham, "Optimal stationary control of a linear system with state-dependent noise," *SIAM J. Contr.*, vol. 5, pp. 486–500, 1967.

[48] T. Yoshizawa, *Stability Theory by Lyapunov's Second Method*, Japan: The Mathematical Society of Japan, 1966.


**Hua Deng** received the B.E. and M.E. degrees from Xi'an Jiaotong University, Xi'an, China, and the Ph.D. degree from the University of California at San Diego, with M. Krstić as her advisor, in 1990, 1993, and 2001, respectively.

She is Senior Engineer at Western Digital Corporation, Orange County, CA, and is coauthor of the book *Stabilization of Nonlinear Uncertain Systems* (New York: Springer-Verlag, 1998). She was a finalist for the Student Best Paper Award (1998 ACC). Her research interests include nonlinear, adaptive, and stochastic control.

**Miroslav Krstić** received the Ph.D. degree from the University of California at Santa Barbara (UCSB), with P. Kokotovic as his advisor, in 1994, and received the UCSB Best Dissertation Award.

He is Professor of Mechanical and Aerospace Engineering at the University of California, San Diego. He is a coauthor of the books *Nonlinear and Adaptive Control Design* (New York: Wiley, 1995) and *Stabilization of Nonlinear Uncertain Systems* (New York: Springer-Verlag, 1998). He holds one patent on control of aeroengine compressors, and has served as Associate Editor of *International Journal of Adaptive Control and Signal Processing, Systems and Control Letters*, and *Journal for Dynamics of Continuous, Discrete, and Impulsive Systems*. His research interests include nonlinear, adaptive, robust, and stochastic control theory for finite dimensional and distributed parameter systems, and applications to propulsion systems and flows.

Dr. Krstić is a recipient of several paper prize awards, including the George S. Axelby Outstanding Paper Award of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and the O. Hugo Schuck Award for the best paper at the American Control Conference. He has also received the National Science Foundation Career Award, the Office of Naval Research Young Investigator Award, and is the only recipient of the Presidential Early Career Award for Scientists and Engineers (PECASE) in the area of control theory. He has served as Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL.

**Ruth J. Williams** received the B.S. (Honors) and M.S. degrees from the University of Melbourne, Australia, and the Ph.D. degree in mathematics from Stanford University, Standford, CA, in 1977, 1979, and 1983, respectively.

She is currently a Professor of Mathematics at the University of California, San Diego. Her research interests are in probability, stochastic processes and their applications.

Dr. Williams is a Fellow of the American Association for the Advancement of Science and the Institute of Mathematical Statistics. She is a past recipient of an NSF Presidential Young Investigator award (1987) and an Alfred P. Sloan Fellowship (1988).