Extremum Seeking for a Class of Wave Partial Differential Equations With Kelvin-Voigt Damping

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Abstract—This letter presents the design and analysis of gradient extremum seeking applied to scalar static maps within the context of infinite-dimensional dynamics governed by Partial Differential Equations (PDEs) of wave type featuring a small amount of Kelvin-Voigt damping. Notably, this particular class of PDEs for extremum seeking still needs to be explored in the existing literature. First, to compensate for the influence of PDE actuation dynamics, we employ a boundary control law via backstepping transformation and averaging-based estimates of the gradient and Hessian. Finally, we prove the local exponential convergence to a small neighborhood surrounding the unknown optimal point by means of an Input-to-State Stability analysis, as well as by employing Lyapunov functionals and averaging theory in infinite dimensions.

Index Terms—Adaptive control, backstepping, distributed parameter systems, extremum seeking, optimization.

I. INTRODUCTION

EXTREMUM seeking (ES) is a non-model-based approach in the field of adaptive control for real-time optimization which searches extremum points (maximum or minimum) of a performance function of nonlinear systems. The ES method has attracted significant attention in the control community, mainly because it addresses control system problems considering modeling inaccuracies or uncertain dynamics [15].

ES was initially introduced in [17] with the aim of maximizing power transfer to a tram car. Over time, the volume of publications related to ES remained relatively limited [24]. However, a significant turning point occurred with the presentation of the first general stability proof for ES in systems with unknown output maps, particularly for stable dynamic systems, as shown in [15]. Subsequently, numerous relevant studies have emerged, contributing significantly to both theoretical advances and practical applications [1], [5], [18].

Reference [21] marked a significant milestone as the first to apply the ES scheme to Partial Differential Equations (PDEs), addressing the design and analysis of multiparameter static maps in the presence of arbitrarily long time delays. The delays, as highlighted by the authors, can be represented as first-order hyperbolic transport PDEs. This approach has paved the way for further extensions to other classes of PDEs [20].

In this letter, we explore a wave PDE that incorporates a minor amount of Kelvin-Voigt damping, expanding the range of PDEs amenable to control through the ES approach. While previous research has addressed boundary control for PDEs with Kelvin-Voigt damping, it has primarily focused on stabilizing specific beam types, such as slender Timoshenko beams and shear beams, rather than optimizing them in real-time [13]. Notably, the wave equation with Kelvin-Voigt damping cannot be neatly categorized as a hyperbolic PDE. Instead, it exhibits characteristics of a parabolic/hyperbolic hybrid type, characterized by, at most, a finite number of conjugate-complex eigenvalues within its spectrum [14].

A. Notation and Terminology

We denote the partial derivatives of a function $u(x, t)$ as
\[
\partial_x u(x, t) = \frac{\partial u(x, t)}{\partial x}, \quad \partial_t u(x, t) = \frac{\partial u(x, t)}{\partial t}.
\]
We conveniently use the compact form $u_x(x, t)$ and $u_t(x, t)$, or simply $u_x$ and $u_t$, for the former and the latter, respectively. The subscript or superscript “av” is employed to denote the average value of a periodic variable with period $\Pi$. The Euclidean or 2-norm of a finite-dimensional ordinary differential equation (ODE) with state vector $X(t)$ is denoted by single bars, $|X(t)|$. In contrast, norms of functions (of $x$) are denoted by double bars. We denote the spatial $L_2^2[0, D]$ norm of the PDE state $u(x, t)$ as $\|u(t)\|^2_{L_2^2[0, D]} = \int_0^D u^2(x, t) \, dx$, where we drop the index $L_2^2[0, D]$ in the following, hence $\|\cdot\| = \|\cdot\|_{L_2^2[0, D]}$, if not otherwise specified. As defined in [10], a vector function $f(t, \epsilon) \in \mathbb{R}^n$ is said to be of order $O(\epsilon)$ over an interval $[t_1, t_2]$, if $\exists k, \delta : |f(t, \epsilon)| \leq k \epsilon, \forall t \in [t_1, t_2]$ and $\forall \epsilon \in [0, \delta]$. In most cases, we do not provide precise estimates for the constants $k$ and $\delta$, and we use $O(\epsilon)$ to be interpreted as an
order-of-magnitude relation for sufficiently small $\epsilon$. Depending on the context, “$s$” represents either the Laplace variable or the differential operator “$d/dt$.” For a transfer function $H_0(s)$ with a generic input $u$, the pure convolution $h_0(t) \ast u(t)$, where $h_0(t)$ is the impulse response of $H_0(s)$, is denoted by $H_0(s)u$. This notation aligns with conventions outlined in [7]. In this letter, the inequalities of Young, Poincaré, Agmon and Cauchy-Schwarz are used very frequently. Please see [12, Appendix A] or [20, Appendix E], to refer to them.

B. Motivating Example for Underwater Search

While the main focus of this letter is on the design and stability analysis of ES feedback subject to actuation dynamics of the wave PDE type with Kelvin-Voigt damping, the primary goal of this subsection is to establish a connection between the proposed ES strategy and a real-world application.

The application mentioned in the previous paragraph is illustrated in Fig. 1 and involves a deep-sea cable-actuated source seeking. In this scenario, a sensor is suspended on a cable and moved through it from the sea surface using a surface vessel. The sensor operates without position awareness, primarily due to the challenging undersea environment. The task at hand is to locate the source signal as closely as possible. No external fluid flow (e.g., water current) is considered, and the dynamics of the boat is ignored for simplicity [4].

Fig. 1. Motivating example - underwater search: $x_0$ represents the relative linear position of the surface vessel with respect to the sensor. The control task aims to drive the sensor to the source signal, meaning that $x_0(t) \rightarrow 0$ (or to a small neighborhood of zero) as $t \rightarrow +\infty$.

The control task is to find the source of a signal of an unknown concentration field, which can be chemical, acoustic, electromagnetic, etc. The sensor captures this field, and its source seeking is to find the source of a signal of an unknown concentration field, which can be chemical, acoustic, etc. The sensor captures this field, and its source of energy dissipation; instead, we use it as means of enhancing the controllability of the model (1)–(4).

The cable of this application is represented by a string described by the following PDE model over an interval $x \in [0, D]$:

\begin{align}
\epsilon \alpha_x &= (1 + d\partial_t)\alpha_{xx}, & (1) \\
\alpha_x(0, t) &= 0, & (2) \\
\alpha(0, t) &= \text{measured}, & (3) \\
\alpha(D, t) &= \text{controlled}. & (4)
\end{align}

Equations (1)–(4) represent the dynamics of a string controlled at the end $x = D$, pinned to the surface vessel, and with a free end at $x = 0$, where the sensor is located. The term $\alpha(x, t)$ in (1) represents the state variable of the PDE dynamics governing the motion of the cable. Equations (2)–(4) serve as boundary conditions. The constants $\epsilon, d$ and $D$ are positive. The constant $D$ physically corresponding to length of the cable. The value $1/\epsilon$ represents the “stiffness” of the string, which can be expressed as $E/\rho$, where $E$ denotes the Young’s modulus and $\rho$ the density of the material. The term $d\partial_t$ models the “Kelvin-Voigt” damping, representing the internal material damping, not the damping that arises due to the viscous interaction of the string with the surrounding medium. We assume that this model takes into account a small amount of damping ($d$), which is a realistic consideration in any material. We do not rely on the Kelvin-Voigt term as source of energy dissipation; instead, we use it as means of enhancing the controllability of the model (1)–(4).

II. PROBLEM STATEMENT

A. Basic Gradient-Based Extremum Seeking Without PDEs

The ES goal is to optimize an unknown static map $y = Q(\Theta)$ using a real-time optimization with optimal unknown output $y^*$ and optimizer $\Theta^*$, measurable output $y(t)$ and input $\Theta(t)$.

The method of ES [15] uses sinusoidal perturbation/probing and demolutation signals given by

\begin{equation}
S(t) = a \sin(\omega t) \quad \text{and} \quad M(t) = \frac{2}{a} \sin(\omega t),
\end{equation}

with amplitude $a$ and frequency $\omega$. Both signals are chosen in order to obtain estimates of the unknown gradient $\partial Q(\Theta)/\partial \Theta$ and negative Hessian $H := \partial^2 Q(\Theta)/\partial \Theta^2 < 0$ of the nonlinear map $Q(\Theta)$ to be maximized, respectively. The actual input $\Theta(t) := \hat{\Theta}(t) + S(t)$ is derived on the real-time estimate $\hat{\Theta}(t)$ of $\Theta^*$, but is perturbed by $S(t)$. The estimate $\hat{\Theta}$ is generated with the integrator $\hat{\Theta}(t) = K M(t) y(t)$ which locally approximates the gradient update law $\hat{\Theta}(t) = KH(\hat{\Theta}(t) - \Theta^*)$, tuning $\hat{\Theta}(t)$ to $\Theta^*$. Hence, by defining the estimation error $\hat{\Theta} = \Theta(t) - \Theta^*$, the average error-dynamics becomes $\overline{\partial \Theta^*} = KH \partial \Theta^*$, with an adaptation gain $K > 0$. As a result, the average system exhibits exponential stability, as indicated by the average theorem [10, Th. 10.4].
B. Scalar Maps With Actuation PDE Dynamics

Now, we consider actuation dynamics described by a wave equation containing Kelvin-Voigt damping with $\varepsilon = 1$, $\theta(t) \in \mathbb{R}$ and the sensor $\Theta(t) \in \mathbb{R}$ given by

$$\Theta(t) = \alpha(0, t),$$

$$\partial_t \alpha(x, t) = \partial_{xx} \alpha(x, t) + d \partial_{xxt} \alpha(x, t),$$

$$\partial_t \alpha(0, t) = 0,$$

$$\alpha(D, t) = \theta(t),$$

where $\alpha : [0, D] \times \mathbb{R}_+ \rightarrow \mathbb{R}$, and $D$ is the known domain length, as mentioned before. The output signal measured with the sensor is represented by the unknown static map

$$y(t) = Q(\Theta(t)).$$

with input $\Theta(t)$ in (6).

We are assuming the unknown nonlinear map is locally quadratic, such that

$$Q(\Theta) = y^* + \frac{H}{2}(\Theta - \Theta^*)^2,$$

where $\Theta^*, y^* \in \mathbb{R}$ and $H < 0$ represents the Hessian of the map. Hence, from (10) and (11), the output of the static map is given by

$$y(t) = y^* + \frac{H}{2}(\Theta(t) - \Theta^*)^2.$$

Adapting the proposed scheme in [21] and combining (6)–(9) with the ES approach in Section II-A, the closed-loop ES with actuation dynamics governed by the Kelvin-Voigt PDE is illustrated in Fig. 2.

C. Trajectory Generation for the Probing Signal

The perturbation $S(t)$ is adapted from the basic ES scheme described in Section II-A to accommodate actuation dynamics. The trajectory generation problem, as described in [14, Ch. 12], is outlined as follows:

$$S(t) := \beta(D, t),$$

$$\partial_t \beta(x, t) = \partial_{xx} \beta(x, t) + d \partial_{xxt} \beta(x, t),$$

$$\partial_t \beta(0, t) = 0,$$

$$\beta(0, t) = a \sin(\omega t),$$

where $\beta : [0, D] \times \mathbb{R}_+ \rightarrow \mathbb{R}$. The explicit solution of (13) is derived for the reference trajectory

$$\beta(D, t) := \beta^*(D, t) = S(t), \quad \beta(0, t) := \beta^*(0, t) = a \sin(\omega t).$$

This solution is found by postulating the reference solution $\beta^*(x, t)$ as a power series of the spatial variable with time dependent coefficients: $\beta^*(x, t) = \sum_{i=0}^{\infty} a_i(t) x^i$, as in [16]. The string reference solution is given by [23]

$$w(x, t) = u_{av}(x, t) - \int_0^x k(x, \sigma) u_{av}(\sigma, t) d\sigma - K \theta_{av}(t)$$

III. Boundary Extremum Seeking Control Law

We consider the PDE-ODE cascade shown in Fig. 3, and use the backstepping transformation

$$w(x, t) = u_{av}(x, t) - \int_0^x k(x, \sigma) u_{av}(\sigma, t) d\sigma - K \theta_{av}(t)$$
to transform the original (30)–(33) into the target system
\[
\dot{\vartheta}_{av}(t) = \overline{K}\vartheta_{av}(t) + w(0, t), \quad \overline{K} < 0, \quad (35)
\]
\[
w_g = (1 + d\vartheta)(w_{xx} - cw), \quad c > 0, \quad (36)
\]
\[
w_{1}(0, t) = 0, \quad \omega_{1}(D, t) = 0. \quad (38)
\]

The gain kernel PDE \(k(x, \sigma)\) come from the solution of (see [14, Sec. 4.2])
\[
k_{xx} = k_{x\sigma} + ck, \quad k_{x}(0, x) = 0, \quad k(x, x) = \frac{c}{2}x. \quad (39)
\]
The solution to the PDE in (39) is obtained through a summation of successive approximation series [14, Sec. 4.4]:
\[
k(x, \sigma) = -cx\left(\frac{I_1(\sqrt{c(x^2 - \sigma^2)})}{\sqrt{c(x^2 - \sigma^2)}}\right) \quad (40)
\]
and, from (34) and (38), the average control law is given by:
\[
u_{av}(D, t) = \overline{K}\vartheta_{av}(t) - \overline{K}\int_{0}^{\sigma} cD\frac{I_1(\sqrt{c(D^2 - \sigma^2)})}{\sqrt{c(D^2 - \sigma^2)}}u_{av}(\sigma, t)d\sigma. \quad (41)
\]
where \(I_1\) is the modified Bessel function [14, Appendix A.2].

Thus, introducing a result of [5], the averaged version of the gradient and Hessian estimate are calculated as
\[
G_{av}(t) = H\vartheta_{av}(t), \quad \dot{H}_{av}(t) = H. \quad (42)
\]

From (33) and (41), choosing \(\overline{K} = KH\) with \(K > 0\) and plugging the average gradient and Hessian estimates (42), we obtain
\[
U_{av}(t) = KG_{av}(t) - KH\int_{0}^{\sigma} cD\frac{I_1(\sqrt{c(D^2 - \sigma^2)})}{\sqrt{c(D^2 - \sigma^2)}}u_{av}(\sigma, t)d\sigma. \quad (43)
\]

We introduce a low-pass filter to obtain the non-average controller
\[
U(t) = \frac{\pi}{s + \pi} \left[\vartheta(0, t) - \vartheta_{av}(t)\int_{0}^{\sigma} cD\frac{I_1(\sqrt{c(D^2 - \sigma^2)})}{\sqrt{c(D^2 - \sigma^2)}}u_{av}(\sigma, t)d\sigma\right]. \quad (44)
\]
with \(\bar{c} \to +\infty\) sufficiently large.

### IV. Stability Analysis

The \(n^{th}\) pair of eigenvalues \(\sigma_n\) of the subsystem (36)–(38) satisfies the quadratic equation:
\[
\sigma_n^2 + d\left(c + \frac{\pi}{2} + \pi n\right)^2 \sigma_n + \left(c + \frac{\pi}{2} + \pi n\right)^2 = 0. \quad (45)
\]
where \(n = 0, 1, 2, \ldots\). There are two sets of eigenvalues: for lower values of \(n\), the eigenvalues are located on the circle
\[
\left(Re(\sigma_n) + \frac{1}{d}\right)^2 + (Im(\sigma_n))^2 = \frac{1}{d^2}, \quad (46)
\]
and for higher \(n\) the eigenvalues are real. One branch of these eigenvalues approaches \(-1/d\) as \(n \to \infty\), while the other branch converges to \(-\infty\).

The open-loop eigenvalues \((c = 0)\) are depicted in Fig. 4. As \(c\) increases, these eigenvalues move along the circle in the negative real direction and reduce in number on the circle until they become real. By setting a very high value for \(c\), it is possible to make all of the eigenvalues real. However, this may not be advisable, both in terms of transient response and control effort. Thus, it is essential to be cautious when enhancing damping through the backstepping transformation and controller. Specifically, lower values of \(c\) should be favored when \(d\) is already relatively high. Since (36)–(38) is exponentially stable according to the stability analysis based on the corresponding eigenvalues, and that (35) is Input-to-State Stable (ISS) [8] with respect to \(w(0, t)\), we can infer that \(|\vartheta_{av}(t)| \to 0\). To investigate stability using Lyapunov functionals and derive results in terms of \(J_2\) estimates, taking into account the initial conditions of the closed-loop system, we propose the next theorem.

**Theorem 1.** Consider the control system in Fig. 2, with control law \(U(t)\) given in (44). There exists \(\bar{c} > 0\) such that, \(\forall \sigma \geq \bar{c}^*, 3\omega^*(\bar{c}) > 0\) such that, \(\forall \omega \geq \omega^*, \) and \(K > 0\) sufficiently large, the closed-loop system (30)–(33) has a unique locally exponentially stable periodic solution in \(t\) with a period \(\Pi := 2\pi/\omega\), denoted as \(\hat{\sigma}_{\Pi}(t), w_{\Pi}(s, t)\). This solution satisfies the condition:
\[
\left(|\sigma_{\Pi}(t)|^2 + \|w_{\Pi}(t)\|^2 + \|u_{\Pi}(t)\|^2\right)^{1/2} \leq \sigma(1/\omega), \quad \forall t \geq 0. \quad (47)
\]
Moreover,
\[
\lim_{t \to \infty} |\sigma(t) - \Theta^*| = \sigma(a + 1/\omega), \quad (48)
\]
\[
\lim_{t \to \infty} |\sigma(t) - \Theta^*| = \sigma(a + 1/\omega), \quad (49)
\]
\[
\lim_{t \to \infty} |\sigma(t) - \Theta^*| = \sigma(a^2 + 1/\omega^2). \quad (50)
\]

**Proof.** We begin by introducing the Lyapunov functional
\[
V(t) = \frac{1}{2}\left[\vartheta_{av}(t) + (1 + \delta d)\left(\|w_k\|^2 + \|w_k\|^2\right)
\quad + \|w_k\|^2 + 2\delta(w, w_k)\right]. \quad (51)
\]
where \(\langle , \rangle\) denotes the inner product. Using Poincare’s inequality, it is easy to see that for sufficiently small positive \(\delta\), there exist positive constants \(m_1\) and \(m_2\) such that
\[
m_1\Psi(t) \leq V(t) \leq m_2\Psi(t) \quad (52)
\]
with \(\Psi(t) = \vartheta_{av}(t) + \|w_k\|^2 + \|w_k\|^2\).
Furthermore, after assuming $\dot{c} \to +\infty$ in (44) for simplicity, the time derivative of $V$ along the solution of (35)-(38), with $K = KH$ and $K > 0$, is given by

$$\dot{V}(t) = KH\partial_{\alpha}(t) + \partial_{\alpha}(t)w(0, t) + (1 + \delta d) \int_0^D w_x(x, t)w_t(x, t)dx + \int_0^D w_t(x, t)dw_{tx}(x, t) - dw_{tx}(x, t)dx + \delta \int_0^D w_x(x, t)(cw_x(x, t) - dw_{tx}(x, t))dx + \delta \int_0^D w(x, t)\left(w_{xx}(x, t) - cw(x, t) + dw_{tx}(x, t)\right)dx. \quad (53)$$

Integrating by parts the integral terms of (53) with second-order partial derivative in $x$, using the boundary conditions (37)-(38), and its time derivatives, we get that

$$\dot{V}(t) = KH\partial_{\alpha}(t) + \partial_{\alpha}(t)w(0, t) - \delta \left(\|w\|_2^2 + c\|w\|_2^2\right) - (cd - \delta)\|w\|_2^2 - d\|w\|_2^2. \quad (54)$$

By applying Young's inequality $(ab \leq \frac{a^2}{2} + \frac{b^2}{2}, \gamma > 0)$ to the second term in the right-hand side of (54) and thereafter applying Agmon's inequality to $w(0, t)$, one gets

$$\dot{V} \leq -\left(KH - \frac{\gamma}{2}\right)\partial_{\alpha}^2 - \delta\left[1 - \frac{1}{2\gamma}\right]\|w\|_2^2 + \left(c - \frac{1}{\gamma}\right)\|w\|_2^2 - (cd - \delta)\|w\|_2^2 - d\|w\|_2^2. \quad (55)$$

Now, for an appropriate $\gamma > 0$ and a sufficiently large $K$ (bearing in mind that $KH < 0$), and using (51) and (54), one can show that there exists a sufficiently small $\lambda > 0$ such that

$$\dot{V} \leq -\lambda V.$$

From this result, along with (52), it follows that $\hat{\Psi}(t) \leq Me^{-t/M}\hat{\Psi}(0)$, for a sufficiently large $M > 0$. From the invertibility of the backstepping transformation (34) and from the smoothness of its kernel $k(x, \sigma)$ [14], it follows that, $\forall t \geq 0$:

$$\hat{\partial}_{\alpha}(t)^2 + \|\partial_{\alpha}(t)^2\|^2 + \|\partial_{\alpha}(t)^2\|^2 \leq \hat{M}e^{-t/M}\|\partial_{\alpha}(0)^2\|^2 + \|\partial_{\alpha}(0)^2\|^2 + \|\partial_{\alpha}(0)^2\|^2, \quad \hat{M} > 0. \quad (56)$$

From (56), the origin of the average closed-loop system with wave PDE and Kelvin-Voigt damping is exponentially stable. Then, according to the averaging theory in infinite dimensions [6], for $\omega$ sufficiently large, the closed-loop system (30)-(33), with $U(t)$ in (44), has a unique exponentially stable periodic solution around its equilibrium (origin) satisfying (47).

On the other hand, the asymptotic convergence to a neighborhood of the extremum point is proved taking the absolute value of the second expression in (24) after replacing $\Theta(t) = \dot{\Theta}(t) + \Theta^*$ from (25), resulting in:

$$|\Theta(t) - \Theta^*| = |\dot{\Theta}(t) + a \sin(\omega t)|. \quad (57)$$

Considering (57) and writing it by adding and subtracting the periodic solution $\hat{\Theta}^*(t)$, it follows

$$|\Theta(t) - \Theta^*| = |\dot{\Theta}(t) - \hat{\Theta}^*(t) + \hat{\Theta}^*(t) + a \sin(\omega t)|. \quad (58)$$

By applying the average theorem [6], one can conclude that $\dot{\Theta}(t) - \hat{\Theta}^*(t) \to 0$ exponentially. Consequently, $\lim_{t \to \infty}|\Theta(t) - \Theta^*| = \lim_{t \to \infty}|\dot{\Theta}^*(t) + a \sin(\omega t)|. \quad (59)$

Finally, utilizing the relationship (47), we ultimately arrive at the result presented in (49).

Since $\dot{\Theta}(t) - \Theta^* = \dot{\Theta}(t) + S(t)$ from (24) and (25), and recalling that $S(t)$ is of order $O(\alpha e^{-t/\alpha})$, as shown in (13) and (19), we finally get, with $\lim_{t \to \infty}|\hat{\Theta}(t)| = \hat{O}(1/\omega)$, the ultimate bound in (48).

In order to show the convergence of the output $y(t)$, we can follow the same steps employed for $\Theta(t)$ by plugging (49) into (12), such that

$$\lim_{t \to \infty}|y(t) - y^*| = \lim_{t \to \infty}|H\dot{\Theta}^*(t) + Ha^2 \sin(\omega t)|. \quad (60)$$

Hence, by rewriting (60) in terms of $\dot{\Theta}^*(t)$ and again with the help of (47), we finally get (50).

Robustness to neglected (parasitic) Kelvin-Voigt damping has been studied in [9] and to neglected standard (viscous damping) in [22]. Such robustness properties should hold for the averaged ES system, in the presence of the ODE for the ES algorithm, just as they hold when neglected viscosity is present in a delay-compensating design for ODEs in [2]. The stability of the average system translates to the original system. Conversely, if a design is developed which relies on the presence of Kelvin-Voigt damping, robustness to slightly reduced Kelvin-Voigt damping holds, in analogy with [11, Sec. 3], but there is no basis to expect robustness to a complete loss of Kelvin-Voigt damping [19] in a design that leverages Kelvin-Voigt damping.

### V. Simulations

The numerical simulation considers the quadratic map described in (11) as a reference, with parameter values selected according to Table I.

Fig. 5 corresponds to the numerical plot of the closed-loop system evolution in a three-dimensional space, taking into account the domain $x \in [0, D]$ and the time $t$. The curves in blue and in red show the convergence of $\Theta(t)$ and $\dot{\Theta}(t)$ to a small neighborhood around the optimizer $\Theta^* = 8$, respectively.

### VI. Conclusion

In this letter, we introduced a methodology for maximizing static maps by seeking their optimal points. Our approach distinguishes itself by its ability to maximize maps in real-time without relying on prior knowledge of their parameters. The key innovation of our method lies in the utilization of a boundary control law via backstepping transformation and averaging-based estimates of the gradient and Hessian.
This allows us to effectively compensate for the infinite-dimensional dynamics introduced by the PDE actuation of wave type featuring a small amount of Kelvin-Voigt damping, with guaranteed exponential stability and convergence to a small neighborhood of the extremum point. While our work has made significant strides, it is essential to acknowledge its limitations. An exciting opportunity lies in extending the proposed approach to address real-time optimization problems, particularly in underwater search scenarios. Specifically, the model of the motivating system example accounts for its limitations. An exciting opportunity lies in extending the small neighborhood of the extremum point. While our work has made significant strides, it is essential to acknowledge this investigation holds the potential to enhance the applicability of our approach, highlighting its robustness in handling complex real-world dynamics.

REFERENCES