INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2000; **10**:821–848

Recursive scaling design for robust global nonlinear stabilization via output feedback

Hiroshi Ito[1,*,†] and Miroslav Krstić[2]

[1] *Department of Control Engineering and Science, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan*
[2] *Department of Mechanical and Aerospace Engineering, University of California, San Diego 9500 Gilman Dr., La Jolla, CA 92093-0411, U.S.A.*

SUMMARY

This paper proposes a novel approach to robust backstepping for global stabilization of uncertain nonlinear systems via output feedback. The design procedure developed in this paper is based on the concept of state-dependent scaling, which handles output-feedback stabilization problems of strict-feedback systems with various structures of uncertainties in a unified way. The proposed method is suitable for numerical computation. The theory of the method employs the Schur complements formula instead of Young's inequality and completing the squares. This paper shows a condition of allowable uncertainty size under which an uncertain system is globally stabilized by output feedback. A class of systems is shown to be always globally stabilizable for arbitrarily large nonlinear size of uncertainties. A recursive procedure of robust observer design for such a class of uncertain systems is presented. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: robust backstepping; state-dependent scaling; robust global stabilization; output feedback.

1. INTRODUCTION

Backstepping has become a popular paradigm for global stabilization of a wide class of uncertain nonlinear systems. Robust backstepping involves domination of uncertain nonlinearities at each step of its recursive procedure [1, 2]. Such domination is achieved through the choice of appropriate functions which satisfy certain inequalities in the Lyapunov derivative corresponding to the locations and characteristics of uncertain components. Recently, it was shown in Reference [3] that state-dependent scaling can provide a systematic and unified method for constructing suitable dominating functions for state-feedback control. The idea of state-dependent (SD) scaling design is motivated by the study [4] which demonstrated that scaling factors of small-gain conditions are allowed to be functions of state. The methodology of SD scaling is applicable not only to geometrically structured systems, but also to other general classes of nonlinear systems

* Correspondence to: Hiroshi Ito, Department of Control Engineering and Science, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan.
† E-mail: hiroshi@ces.kyutech.ac.jp

Received 6 August 1999
Accepted 21 April 2000

[5]. The concept of SD scaling design has flexibilities to formulate a wide variety of robust nonlinear control problems and it is suitable for computational optimization [6].

As for output feedback control where state variables are not available for feedback, one may simply give up seeking global stabilization and settle for semi-global stabilization. The idea of input saturation and high-gain observer has been successful for such semi-global stabilization [7–9]. The studies [10, 11] proposed a useful semi-global backstepping lemma and high-gain observers with saturating control for dynamic output feedback. By using these semi-global techniques, a robust stabilization problem was also considered intensively for a certain type and location of unstructured uncertainty, namely, robustness against unknown stable zero dynamics. It is possible to incorporate unknown parameters in such semi-global stabilization (e.g. Reference [12]). However, from another viewpoint, given an uncertain system, semi-global stabilization using high-gain and saturation may be meaningful only if the system cannot be globally stabilized. There are also global results for output-feedback stabilization of nonlinear systems in the strict-feedback form. Typical results (e.g. Reference [2]) are applicable only to nonlinear systems whose nonlinearities in the system equation do not depend on unmeasured states. It is not clear what is the essential ingredient of this assumption, apart from their technique of constructing observers and controllers. Aside from inverse optimality, discussion about robust global stabilization via this type of output feedback is absent in spite of their practical importance. Note that the feedback configuration meant by 'output feedback' in this paper and Reference [2] is different from partial-state measurement feedback considered in Reference [13].

The first objective of this paper is to propose a unified design procedure to tackle robust and global stabilization problems of output feedback for a class of uncertainties which are as broad as uncertainty considered in the state-feedback literature, namely, uncertain systems in the strict-feedback form with nonlinearly bounded uncertainties appearing in systems with various structures. This paper successfully extends the authors' state-dependent scaling design for state-feedback backstepping to the output-feedback case. Then, the robust backstepping is described as recursive selection of appropriate scaling factors. The proposed design procedures are amenable to *numerical* calculation based on computational optimization. Since the backstepping is performed by domination, it is unnecessary to use precise parameters of systems, which prevents the controller from having long and complicated terms. Another important feature of this paper is that output backstepping is shown to be feasible without using Young's inequality. The paper employs the Schur complements formula which gives a necessary and sufficient condition for negativity of a quadratic form on the transformed co-ordinate. It is demonstrated that the Schur complements formula is less conservative than other popular techniques such as Young's inequality and completing the squares. This paper also presents a novel recursive procedure of robust observer design which guarantees global solutions for a class of uncertain nonlinear systems.

The authors' position is seeking global stabilization instead of settling for semi-global stabilization. Thus, the second objective is to characterize the essential difference between nominal and robust global stabilization in output feedback control. The robustness in this paper is desirable in that the size and location of uncertainty is pre-specified *a priori*, which is completely different from the inverse optimal type of robustness. The backstepping procedure is developed without an assumption restricting nonlinearities to depend only on measured states, i.e., $\phi(y)$ where y is output. The feedback gain design does not need to exclude nonlinearities such as $\phi(y)x$, where x is unnecessarily measured. This paper describes what kind of task is required for observer design in such a case. Then, the paper shows a condition on uncertain nonlinear systems for which global

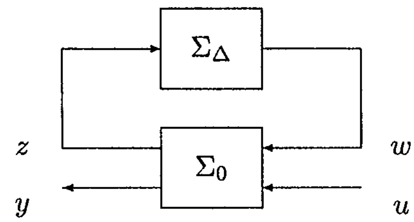

Figure 1. Uncertain nonlinear plant Σ_P.

robust stabilizability is always guaranteed. The condition vanishes if the whole state is available for control. It will be shown that, exclusively for nonlinear systems, 'nonlinear size' of uncertainties, appeared as coupling, is crucial for global robust stabilization, which cannot compensated globally by either feedback-gain or observer-gain independently. This situation contrasts with nominal stabilization in which the whole system is globally stabilizable by designing controller feedback-gain strong enough whenever the observer dynamics by itself design to be only stable (or, *vice versa*).

The standpoint of this paper is quite different from those of nonlinear adaptive control and many of backstepping papers. From a viewpoint of this paper, roles of SD scaling design are (1) provide a method of (trying to) solving the problem; (2) characterize a condition under which the robust stabilization is solvable; (3) provide information about how large size of uncertainty is allowable. The latter two roles are necessary since the problem by itself does not always have the solution. Note that the nominal system and structure and size of uncertainties are specified *a priori* regardless of solvability. The SD scaling design provides us with a way to obtain a control law even if achievable performance is not as good as originally desired. It also theoretically persuades us to give up seeking unreasonably large uncertainty. In addition, this paper demonstrates a class of uncertain systems which is always robustly stabilizable for arbitrarily large size and arbitrarily fast growth-order uncertainties. This standpoint is more common in the nonlinear literature.

In this paper, $F > 0$ stands for $F = F^{\mathrm{T}} > 0$. The maximum eigenvalue is denoted by $\lambda_{\max}(\cdot)$.

2. SD SCALING ANALYSIS FOR OBSERVER-FEEDBACK CONTROL

Consider the uncertain nonlinear system Σ_P shown in Figure 1. The system Σ_0 denotes a nominal plant and Σ_Δ represents the uncertainty. The nominal part Σ_0 is described by

$$\Sigma_0\colon \begin{cases} \dot{x} = A(y)x + B(y)w + G(y)u, & x(t) \in \mathscr{R}^n, u(t) \in \mathscr{R}^1 \\ z = C(y)x, & w(t) \in \mathscr{R}^p, z(t) \in \mathscr{R}^p \\ y = C_y x, & y(t) \in \mathscr{R}^r \end{cases} \tag{1}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad \begin{array}{c} w_i(t) \in \mathscr{R}^{p_i} \\ z_i(t) \in \mathscr{R}^{p_i} \\ p_i \geqslant 0, \quad p = \sum_{i=1}^{n} p_i \\ i = 1, 2, \ldots, n \end{array}$$

The matrix-valued functions A, B, C and G are assumed to be $\mathbf{C}^0$ functions of the output y. Suppose that the uncertain system Σ_Δ has the following structure of nonlinear mappings $\Delta: z \mapsto w$:

$$\Sigma_\Delta\colon\ \Delta = \text{block-diag}[\Delta_1, \Delta_2, \dots, \Delta_n] \tag{2}$$

Some of the mappings $\Delta_i: z_i \mapsto w_i$, $i = 1, 2, \dots, n$, can be zero in vector size. Each uncertainty Δ_i is defined as

$$\Delta_i: z_i = \begin{bmatrix} z_{i\mathrm{d}} \\ z_{i\mathrm{s}} \end{bmatrix} \mapsto w_i = \begin{bmatrix} w_{i\mathrm{d}} \\ w_{i\mathrm{s}} \end{bmatrix}, \quad w_i = \begin{bmatrix} \Delta_{i\mathrm{d}} & 0 \\ 0 & \Delta_{i\mathrm{s}} \end{bmatrix} z_i \tag{3}$$

Here, $\Delta_{i\mathrm{d}}$ and $\Delta_{i\mathrm{s}}$ represent a dynamic and a static system, respectively. It is unnecessary for each Δ_i to have the two types of uncertainty. These $\Delta_{i\mathrm{d}}$ and $\Delta_{i\mathrm{s}}$ are defined by

$$\Delta_{i\mathrm{d}}\colon\quad \begin{cases} \dot{x}_{\Delta_i} = f_{\Delta_{i\mathrm{d}}}(x_{\Delta_i}, z_{i\mathrm{d}}, t) \\ w_{i\mathrm{d}} = h_{\Delta_{i\mathrm{d}}}(x_{\Delta_i}, z_{i\mathrm{d}}, t) \end{cases} \tag{4}$$

$$\Delta_{i\mathrm{s}}\colon\ w_{i\mathrm{s}} = h_{\Delta_{i\mathrm{s}}}(z_{i\mathrm{s}}, t) \tag{5}$$

where vector-valued $\mathbf{C}^0$ functions $f_{\Delta_{i\mathrm{d}}}$, $h_{\Delta_{i\mathrm{d}}}$ and $h_{\Delta_{i\mathrm{s}}}$ satisfy $f_{\Delta_{i\mathrm{d}}}(0, 0, t) = 0$, $h_{\Delta_{i\mathrm{d}}}(0, 0, t) = 0$ and $h_{\Delta_{i\mathrm{s}}}(0, t) = 0$ for all $t \geqslant 0$. The state variable of Σ_Δ and $x_\Delta = [x_{\Delta_1}^{\mathrm{T}}, x_{\Delta_2}^{\mathrm{T}}, \dots, x_{\Delta_n}^{\mathrm{T}}]^{\mathrm{T}}$. For notational simplicity, we assume that $\Delta_{i\mathrm{d}}$ and $\Delta_{i\mathrm{s}}$ are square in size of input and output vectors.

Definition 1

The uncertainty Σ_Δ is said to be admissible if (i)–(ii) are satisfied for $i = 1, 2, \dots, n$: (i) The equilibrium $x_{\Delta_i} = 0$ of $\Delta_{i\mathrm{d}}$ is globally uniformly asymptotically stable and $\Delta_{i\mathrm{d}}$ *has* $\mathscr{L}_2$-gain less than or equal to one with a positive-definite radially unbounded $\mathbf{C}^1$ storage function $V_{\Delta i}(x_{\Delta_i})$. (ii) $\Delta_{i\mathrm{s}}$ satisfies $\|z_{i\mathrm{s}}\|^2 \geqslant \|w_{i\mathrm{s}}\|^2$ for all $t \in [0, \infty)$.

The uncertain system Σ_{P} has an equilibrium point at the origin when $u \equiv 0$. Uncertainty having super-linear growth (and thus unbounded gain) can be included by a judicious choice of $B(y)$ and $C(y)$. Indeed Σ_0 not only describes a nominal plant, but also contains information about input-output nonlinearities of uncertainty. The mainpulation to choose an appropriate pair of $(\Sigma_0, \Sigma_\Delta)$ taking nonlinearity into account is akin to the idea of nonlinear gain [14–17]. Remember that $B(y)$ and $C(y)$ specify the 'nonlinear size' (including size, nonlinearity, location and structure) of uncertainties.

To robustly stabilize nonlinear system Σ_{P}, we employ the full-order observer and feedback of the estimated state:

$$\begin{aligned} \dot{\hat{x}} &= A(y)\hat{x} + Y(y, \hat{x})(y - \hat{y}) + G(y)u \\ \hat{y} &= C_y \hat{x} \end{aligned} \tag{6}$$

$$u = K(y, \hat{x})\hat{x} \tag{7}$$

The closed-loop system is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & GK \\ YC_y & A - YC_y + GK \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w \tag{8}$$

Consider a diffeomorphism between $[\hat{x}^{\mathrm{T}}, \hat{x}^{\mathrm{T}} - x^{\mathrm{T}}]^{\mathrm{T}} \in \mathscr{R}^{2n}$ and $[\hat{\chi}^{\mathrm{T}}, \eta]^{\mathrm{T}} \in \mathscr{R}^{2n}$ as follows:

$$\begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} = \begin{bmatrix} S(y, \hat{x}) & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x} - x \end{bmatrix} \tag{9}$$

The time-derivative of $\hat{\chi}$ is obtained as

$$\dot{\hat{\chi}} = \left[\frac{\partial S}{\partial y_1}\hat{x}, \frac{\partial S}{\partial y_2}\hat{x}, \dots, \frac{\partial S}{\partial y_n}\hat{x} \right] C_y \dot{x} + \left[\frac{\partial S}{\partial \hat{x}_1}\hat{x}, \frac{\partial S}{\partial \hat{x}_2}\hat{x}, \dots, \frac{\partial S}{\partial \hat{x}_n}\hat{x} \right] \dot{\hat{x}} + S(y, \hat{x})\dot{\hat{x}} = V(y, \hat{x})\dot{x} + T(y, \hat{x})\dot{\hat{x}} \tag{10}$$

The square matrix W is constant and non-singular. The closed-loop system on the new co-ordinate $(\hat{\chi}, \eta)$ is

$$\begin{bmatrix} \dot{\hat{\chi}} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} (V+T)(A+GK)S^{-1} & -(VA + TYC_y)W^{-1} \\ 0 & W(A - YC_y)W^{-1} \end{bmatrix} \begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} + \begin{bmatrix} VB \\ -WB \end{bmatrix} w \tag{11}$$

$$z = C[S^{-1} \quad -W^{-1}] \begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} \tag{12}$$

For the dynamic uncertainty Δ_{id}, we define

$$\mathbf{L}_{id} = \{L_{id} = \lambda_{id} I_{id} \colon \lambda_{id} > 0\}. \tag{13}$$

Here, I_{id} denotes an identity matrix which is compatible in size with z_{id}. For the static uncertainty Δ_{is},

$$\mathbf{L}_{is} = \{L_{is} = \lambda_{is}(y, \hat{x}) I_{is} \colon \lambda_{is}(y, \hat{x}) > 0 \forall (y, \hat{x}) \in \mathscr{R}^r \times \mathscr{R}^n\}. \tag{14}$$

is defined. These matrices are SD scaling which estimates the worst case value of the time-derivative of Lyapunov functions [5]. The set of scaling matrices for the whole Σ_Δ is denoted by $\mathbf{L}$ as follows:

$$\mathbf{L} = \left\{ L = \operatorname*{block-diag}_{i=1}^{n} L_i(y, \hat{x}), L_i \in \mathbf{L}_i \right\} \tag{15}$$

$$\mathbf{L}_i = \left\{ L_i(y, \hat{x}) = \begin{bmatrix} L_{id} & 0 \\ 0 & L_{is}(y, \hat{x}) \end{bmatrix} : \begin{matrix} L_{id} \in \mathbf{L}_{id} \\ L_{is} \in \mathbf{L}_{is} \end{matrix} \right\} \tag{16}$$

Scaling matrices for static uncertainties are chosen as functions of output and state estimate, while constant scaling is necessary and sufficient for robust stabilization against time-varying uncertainty in linear system theory. Repeated uncertainties and corresponding SD scaling matrices can be incorporated in all materials of this paper as in Reference [3]. They are deleted for brevity. The following describes the main idea of the SD scaling approach to the output feedback stabilization.

Theorem 1

(i) Suppose that there exist constant matrices $P > 0$ and $\tilde{P} > 0$ such that

$$N(y, \hat{x}) = \begin{bmatrix} S^{-\mathrm{T}}(A+GK)^{\mathrm{T}}(V+T)^{\mathrm{T}}P + P(V+T)(A+GK)S^{-1} & -P(VA+TYC_y)W^{-1} \\ -W^{-\mathrm{T}}(VA+TYC_y)^{\mathrm{T}}P & W^{-\mathrm{T}}(A-YC_y)^{\mathrm{T}}W^{\mathrm{T}}\tilde{P} + \tilde{P}W(A-YC_y)W^{-1} \end{bmatrix} < 0 \tag{17}$$

is satisfied for all $(y, \hat{x}) \in \mathscr{R}^r \times \mathscr{R}^n$, then the nominal nonlinear system Σ_0 is globally uniformly asymptotically stabilized by the dynamic output feedback (6) and (7). Furthermore, a Lyapunov function is given by $V(x, \hat{x}) = \hat{\chi}^{\mathrm{T}} P \hat{\chi} + \eta^{\mathrm{T}} \tilde{P} \eta$.

(ii) Suppose that there exist constant matrices $P > 0$, $\tilde{P} > 0$ and a scaling function $L \in \mathbf{L}$ such that

$$M(y, \hat{x}) = \begin{bmatrix} S^{-\mathrm{T}}(A+GK)^{\mathrm{T}}(V+T)^{\mathrm{T}}P + P(V+T)(A+GK)S^{-1} & PVB & S^{-\mathrm{T}}C^{\mathrm{T}}L & -P(VA+TYC_y)W^{-1} \\ B^{\mathrm{T}}V^{\mathrm{T}}P & -L & 0 & -B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P} \\ LCS^{-1} & 0 & -L & -LCW^{-1} \\ -W^{-\mathrm{T}}(VA+TYC_y)^{\mathrm{T}}P & -\tilde{P}WB & -W^{-\mathrm{T}}C^{\mathrm{T}}L & W^{-\mathrm{T}}(A-YC_y)^{\mathrm{T}}W^{\mathrm{T}}\tilde{P} + \tilde{P}W(A-YC_y)W^{-1} \end{bmatrix} < 0 \quad (18)$$

is satisfied for all $(y, \hat{x}) \in \mathscr{R}^r \times \mathscr{R}^n$, then the uncertain nonlinear system Σ_{P} is globally uniformly asymptotically stabilized by the dynamic output feedback (6) and (7) for any admissible uncertainty Σ_Δ. Furthermore, a Lyapunov function is given by $V(x, \hat{x}, x_\Delta) = \hat{\chi}^{\mathrm{T}} P \hat{\chi} + \eta^{\mathrm{T}} \tilde{P} \eta + \sum_{i=1}^{n} \lambda_{id} V_{\Delta i}(x_{\Delta i})$.

Proof. (i) Note that co-ordinate transformation (9) is globally diffeomorphic. The function $V(x, \hat{x})$ defined in (i) is positive definite and radially unbounded. The time derivative of V satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x, \hat{x}) \leqslant \begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix}^{\mathrm{T}} N \begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix}$$

Since $N(y, \hat{x}) < 0$ is satisfied for all $(y, \hat{x}) \in \mathscr{R}^r \times \mathscr{R}^n$, $\mathrm{d}V/\mathrm{d}t < 0$ holds for all $(x, \hat{x}) \in \mathscr{R}^n \times \mathscr{R}^n \setminus \{0\}$. Thus, the equilibrium $[x^{\mathrm{T}}, \hat{x}^{\mathrm{T}}]^{\mathrm{T}} = 0$ of the closed-loop system is globally asymptotically stable. (ii) Due to assumptions of admissible uncertainties, the function $V(x, \hat{x})$ defined in (ii) is positive definite and radially unbounded. The time derivative of V satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} V \leqslant \begin{bmatrix} \hat{\chi} \\ \eta \\ w \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{N}} & PVB \\ & & -\tilde{P}WB \\ B^{\mathrm{T}}V^{\mathrm{T}}P & -B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P} & 0 \end{bmatrix} \begin{bmatrix} \hat{\chi} \\ \eta \\ w \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} w_{id} \\ z_{id} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -L_{id} & 0 \\ 0 & L_{id} \end{bmatrix} \begin{bmatrix} w_{id} \\ z_{id} \end{bmatrix}$$

Using properties of state-dependent scaling [18], it can be verified that $\mathrm{d}V/\mathrm{d}t < 0$ holds for all $(x, \hat{x}, w) \neq 0$ if

$$\begin{bmatrix} \hat{\chi} \\ \eta \\ w \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{N}} & PVB \\ & & -\tilde{P}WB \\ B^{\mathrm{T}}V^{\mathrm{T}}P & -B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P} & 0 \end{bmatrix} + \begin{bmatrix} 0 & S^{-\mathrm{T}}C^{\mathrm{T}} \\ 0 & -W^{-\mathrm{T}}C^{\mathrm{T}} \\ I & 0 \end{bmatrix} \times \begin{bmatrix} -L & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ CS^{-1} & -CW^{-1} & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{\chi} \\ \eta \\ w \end{bmatrix} < 0$$

is satisfied for all $(\hat{\chi}, \eta, w) \neq 0$. Rearranging this inequality, we obtain

$$\begin{bmatrix} & N & \begin{matrix} PVB \\ -\tilde{P}WB \end{matrix} \\ B^{\mathrm{T}}V^{\mathrm{T}}P & -B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P} & -L \end{bmatrix} + \begin{bmatrix} S^{-\mathrm{T}}C^{\mathrm{T}}L \\ -W^{-\mathrm{T}}C^{\mathrm{T}}L \\ 0 \end{bmatrix} L^{-1}[LCS^{-1} \quad -LCW^{-1} \; 0] < 0$$

Since $L > 0$, the above inequality is equivalent to $M < 0$. Hence, the inequality $M < 0$ is sufficient for global uniform asymptotic stability of the equilibrium $[x^{\mathrm{T}}, \hat{x}^{\mathrm{T}}, x_{\Delta}^{\mathrm{T}}]^{\mathrm{T}} = 0$. □

The analysis of robust stability is reduced into the existence of a scaling matrix which makes M negative. This is considered as the definition of the state-dependent scaling approach to output feedback control with the full-order observer. Although representation (11) seems to allow us to use a sort of separation between state-feedback stabilization and observer design, it is not true for nonlinear systems stabilization. To explain this point, we need the following.

Lemma 1

Consider a symmetric matrix

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^{\mathrm{T}} & F_{22} \end{bmatrix}, \quad F_{22} \in \mathscr{R}^{q \times q}$$

(i) Schur complements formula: $F < 0$ is equivalent to

$$F_{22} < 0, \quad F_{11} - F_{12}F_{22}^{-1}F_{12}^{\mathrm{T}} < 0 \tag{19}$$

(ii) Young's inequality: $F < 0$ is satisfied if

$$F_{22} + \Gamma^{-1} < 0, \quad F_{11} + F_{12}\Gamma F_{12}^{\mathrm{T}} < 0, \quad \Gamma = \operatorname*{diag}_{i=1}^{q} \gamma_i > 0 \tag{20}$$

Proof. The inequalities (ii) can be derived by using elementary linear algebra as follows:

$$0 > F_{11} + F_{12}\Gamma F_{12}^{\mathrm{T}} > F_{11} + F_{12}\Gamma F_{12}^{\mathrm{T}} - F_{12}(\Gamma + F_{22}^{-1})F_{12}^{\mathrm{T}} = F_{11} - F_{12}F_{22}^{-1}F_{12}^{\mathrm{T}} \tag{21}$$ □

Condition (20) is an alternative expression of Young's inequality for vectors or completing the squares:

$$2y^{\mathrm{T}}z \leqslant y^{\mathrm{T}}\Gamma y + z^{\mathrm{T}}\Gamma^{-1}z \tag{22}$$

where y and z are vectors. It is verified that (22) yields (20) as follows:

$$\begin{aligned} \begin{bmatrix} x \\ z \end{bmatrix}^{\mathrm{T}} F \begin{bmatrix} x \\ z \end{bmatrix} &= x^{\mathrm{T}}F_{11}x + 2x^{\mathrm{T}}F_{12}z + z^{\mathrm{T}}F_{22}z \\ &\leqslant x^{\mathrm{T}}F_{11}x + x^{\mathrm{T}}F_{12}\Gamma F_{12}^{\mathrm{T}}x + z^{\mathrm{T}}\Gamma^{-1}z + z^{\mathrm{T}}F_{22}z \\ &= \begin{bmatrix} x \\ z \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} F_{11} + F_{12}\Gamma F_{12}^{\mathrm{T}} & 0 \\ 0 & F_{22} + \Gamma^{-1} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \end{aligned}$$

This paper refers to (20) as Young's inequality in order to distinguish that from the Schur complements formula since (20) has been commonly used in terms of the Young's inequality (22) in nonlinear systems control [1, 2, 19]. The common role of Young's inequality is to get rid of products of two vectors in the Lyapunov derivative and to get a decoupled quadratic expression. The Schur complements formula looks at the negativity in terms of matrices in stead of the scalar value of quadratic forms. The Schur complements formula gives a necessary and sufficient condition while Young's inequality is only sufficient. From this viewpoint, this paper shows how to replace the task of Young's inequality with the superior Schur complements. In other words, this paper proposes backstepping procedures without introducing any conservatism *in solving problems recursively* except that Theorem 1 is a sufficient condition (note that recursive construction of a solution by itself may have unnecessary conservatism). This idea may not only allow the design to tolerate large size of uncertainties, but also prevent controllers from having unnecessary high gain and harmfully fast or slow growth order.

Corollary 1

Assume that there exists a constant matrix $\tilde{P} > 0$ such that (23) holds for all $(y, \hat{x}) \in \mathscr{R}^r \times \mathscr{R}^n$.

$$H(y, \hat{x}) = W^{-\mathrm{T}}(A - YC_y)^{\mathrm{T}} W^{\mathrm{T}} \tilde{P} + \tilde{P} W (A - YC_y) W^{-1} < 0 \tag{23}$$

(i) Suppose that there exists a constant $P > 0$ such that

$$\begin{aligned} \bar{N}(y, \hat{x}) &= N_{11}(y, \hat{x}) - N_{12}(y, \hat{x}) H^{-1}(y, \hat{x}) N_{12}^{\mathrm{T}}(y, \hat{x}) < 0 \\ N_{11} &= S^{-\mathrm{T}}(A + GK)^{\mathrm{T}}(V + T)^{\mathrm{T}} P + P(T + V)(A + GK) S^{-1} \\ N_{12} &= P(VA + TYC_y) W^{-1} \end{aligned} \tag{24}$$

is satisfied for all $(y, \hat{x}) \in \mathscr{R}^r \times \mathscr{R}^n$, then the nominal nonlinear system Σ_0 is globally uniformly asymptotically stabilized by the dynamic output feedback (6) and (7). Moreover, if Σ_0 is a linear system and if S is constant, the pair of (24) and (23) is equivalent to the existence of $P > 0$ and $\tilde{P} > 0$ satisfying

$$N_{11} < 0, \quad H < 0 \tag{25}$$

(ii) Suppose that there exist a constant matrix $P > 0$ and a scaling function $L \in \mathbf{L}$ such that

$$\bar{M}(y, \hat{x}) = M_{11}(y, \hat{x}) - M_{12}(y, \hat{x}) H^{-1}(y, \hat{x}) M_{12}^{\mathrm{T}}(y, \hat{x}) < 0$$

$$M_{11} = \begin{bmatrix} N_{11} & PVB & S^{-\mathrm{T}} C^{\mathrm{T}} L \\ B^{\mathrm{T}} V^{\mathrm{T}} P & -L & 0 \\ LCS^{-1} & 0 & -L \end{bmatrix}, \quad M_{12} = \begin{bmatrix} N_{12} \\ B^{\mathrm{T}} W^{\mathrm{T}} \tilde{P} \\ LCW^{-1} \end{bmatrix} \tag{26}$$

is satisfied for all $(y, \hat{x}) \in \mathscr{R}^r \times \mathscr{R}^n$, then the uncertain nonlinear system Σ_{P} is globally uniformly asymptotically stabilized by the dynamic output feedback (6) and (7) for any admissible uncertainty Σ_Δ.

Proof. (i) Conditions (24) and (23) are obtained by applying the Schur complements formula to (17). It is obvious that (24) and (23) imply (25). Conversely, suppose that $P > 0$ and $\tilde{P} > 0$ solve (25) with a constant S for a linear system Σ_0. If $\tilde{P}$ in (24) is replaced by $\beta \tilde{P}$, inequality (24) is satisfied for a sufficiently large constant $\beta > 0$.

(ii) This part is straightforward from the Schur complements formula. □

Inequalities (25) represent the separation principle for linear systems. Conditions (25), however do not guarantee global stability nonlinear Σ_0. If Σ_0 is nonlinear, β in the proof would be required to be unbounded as y or $\hat{x}$ goes to $\pm\infty$. If β is a function of $(y, \hat{x})$, there is no guarantee that there exists a Lyapunov function V which is consistent with

$$\frac{\partial V}{\partial[\hat{\chi}^{\mathrm{T}}, \eta^{\mathrm{T}}]^{\mathrm{T}}} = 2[\hat{\chi}^{\mathrm{T}}P, \eta^{\mathrm{T}}\beta(y, \hat{x})\tilde{P}]$$

for P and $\tilde{P}$ of (25). It is true that (24) can be satisfied semi-globally by a sufficiently large constant β. We may achieve semi-global stabilization by using (25) and taking into account the level set of $V(x, \hat{x}) = \chi^{\mathrm{T}}P\chi + \eta^{\mathrm{T}}\beta\tilde{P}\eta$. This paper does not pursue this direction of semi-global stabilization since it does not capture essential points required for global and nonlinear stabilization. As for robust stabilization, we cannot separate observer design completely from robust stabilization in a global sense. The separation argument in (i) of Corollary 1 is not applicable to (ii) even for linear Σ_0 because of the coupling term M_{12} (especially $B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}$) between feedback and observer. Linear robust control theory tells us that observer design must be coupled with robustification of stabilization. In other words, the observer should be designed strong enough, taking into account the effect of uncertainty.

The parameters P, $\tilde{P}$ and L solving the inequalities in Theorem 1 and Corollary 1 globally are not always guaranteed to be existent. The existence of the solutions depends on the geometric structure of Σ_0 and Σ_{P} as well as S and W. In the rest of this paper, classes of systems which admit the global solutions are presented.

3. A ROBUST STRICT-FEEDBACK FORM AND OBSERVERS

This section defines a class of uncertain nonlinear systems for which output backstepping design via SD scaling will be proposed. The output equation of Σ_0 is supposed to be

$$y = x_1$$

or equivalently $C_y = [1\ 0 \cdots 0]$. This case is sometimes called output feedback in the nonlinear control literature [2]. We assume that A and G can be written in the form

$$A(x_1) = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & & 0 \\ a_{21} & a_{22} & a_{23} & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & & & 0 \\ a_{n-1,1} & a_{n-1,2} & \cdots & & & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & & & a_{n,n} \end{bmatrix}, \quad G(x_1) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n,n+1} \end{bmatrix}$$

with $\mathbf{C}^0$ scalar functions $a_{ij}(x_1)$ required to satisfy

$$a_{i,i+1}(x_1) \neq 0, \quad 1 \leqslant i \leqslant n, \quad \forall x_1 \in \mathscr{R}$$

As for functions B and C, we assume

$$B(x_1)=\begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ B_{n1} & \cdots & B_{n,n-1} & B_{nn} \end{bmatrix},\quad C(x_1)=\begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ C_{21} & C_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ C_{n1} & \cdots & C_{n,n-1} & C_{nn} \end{bmatrix}$$

where $B_{ij}(x_1)\in\mathscr{R}^{1\times p_i}$ and $C_{ij}(x_1)\in\mathscr{R}^{p_i\times 1}$. Then, the uncertainty affects the system as

$$B(x)w=\begin{bmatrix} B_{11}\Delta_1 C_{11} & 0 & 0 & \cdots \\ B_{21}\Delta_1 C_{11}+B_{22}\Delta_2 C_{21} & B_{22}\Delta_2 C_{22} & 0 & \ddots \\ \vdots & \vdots & & \ddots \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix}$$

The nonlinear operator Δ_i can have dynamics with initial conditions. For simplicity, this paper assumes that the system does not have any uncertainties in the virtual control coefficients in the backstepping. It is possible to extend the idea of SD scaling easily to the uncertain system which has uncertain components Δ in a more general manner as in Reference [3].

Two types of properties of observers will be used in this paper.

Ordinary observer: The observer-gain $Y(x_1)$ is chosen as a $\mathbf{C}^0$ function matrix such that there exist a constant matrix $\tilde{P}>0$ and a $\mathbf{C}^0$ function matrix $Q_y(x_1)>0$ satisfying

$$(A-YC_y)^{\mathrm{T}}\tilde{P}+\tilde{P}(A-YC_y)<-Q_y \tag{27}$$

for all $x_1\in\mathscr{R}$.

Robust observer. Given a matrix-valued function $\Gamma(x_1)>0$, the $\mathbf{C}^0$ observer-gain function $Y(x_1)$ and the constant matrix W are chosen such that there exists a constant diagonal matrix $\tilde{P}>0$ satisfying

$$H(x_1)=W^{-\mathrm{T}}(A-YC_y)^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}+\tilde{P}W(A-YC_y)W^{-1}<-\Gamma^{-1} \tag{28}$$

for all $x_1\in\mathscr{R}$.

Note that $H<-\Gamma^{-1}<0$ is equivalent to $0<-H^{-1}<\Gamma$. A robust observer is an ordinary observer. The converse is not true. Suppose that $\tilde{P}>0$ is a solution to (27). We can always decompose the matrix into $\tilde{P}=W^{\mathrm{T}}\tilde{P}_{\mathrm{new}}W$ with a lower triangular matrix W and a diagonal matrix $\tilde{P}_{\mathrm{new}}$. This implies that (28) is satisfied by replacing $\tilde{P}$ and Γ^{-1} with $\tilde{P}_{\mathrm{new}}$ and $W^{-\mathrm{T}}Q_yW^{-1}$, respectively, whenever (27) holds with a symmetric matrix $\tilde{P}>0$. However, $\Gamma^{-1}\leqslant W^{-\mathrm{T}}Q_yW^{-1}$ is not guaranteed at all. The left-hand side of (28) corresponds to the Lyapunov derivative of the observer error system. The robust observer requires that the observer error system is stable to a degree prescribed by Γ. The smaller $\Gamma>0$ is, the more robust the observer is.

4. BACKSTEPPING DESIGN FOR OUTPUT FEEDBACK

This section extends the robust backstepping procedure in [3, 18] to output feedback design. The backstepping is carried out successfully by selecting SD scaling matrices recursively.

Let $\hat{x}_{[k]}$ denote the state of the observer $\hat{x}_1$ through $\hat{x}_k$:

$$\hat{x}_{[k]} = [\hat{x}_1, \hat{x}_2, \dots, \hat{x}_k]^{\mathrm{T}}$$

Consider smooth scalar-valued functions $s_1(x_1)$, $s_2(x_1, \hat{x}_1), \dots, s_{n-1}(x_1, \hat{x}_{[n-2]})$ which are to be determined in a recursive manner from s_1 through s_{n-1}. We define a matrix $S(x_1, \hat{x})$ as follows:

$$S^{-1}(x_1, \hat{x}_{[n-2]}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ s_1 & 1 & 0 & \cdots & 0 \\ 0 & s_2 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 1 \end{bmatrix} \tag{29}$$

The smooth functions V and T in (10) are obtained as

$$V(x_1, \hat{x}_{[n-1]}) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \star_{1,1} & 0 & \cdots & 0 \\ \star_{2,2} & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star_{n-1,n-1} & 0 & \cdots & 0 \end{bmatrix}$$

$$T(x_1, \hat{x}_{[n-1]}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \star_{0,1} & 1 & 0 & 0 & \cdots & 0 \\ \star_{2,2} & \star_{1,2} & 1 & 0 & \ddots & 0 \\ \star_{3,3} & \star_{3,3} & \star_{2,3} & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \star_{n-1,n-1} & \cdots & \cdots & \star_{n-1,n-1} & \star_{n-2,n-1} & 1 \end{bmatrix}$$

where $\star_{i,j}$ denotes any function depending only on $(x_1, \hat{x}_{[i]})$, and the functions s_1 through s_j and their partial derivatives. Let us choose the feedback gain (7) as

$$K = [(-1)^{n-1} s_1 \cdots s_n, \ \dots, \ -s_{n-1}s_n, \ s_n] \tag{30}$$

where $s_n(x_1, \hat{x}_{[n-1]})$ is another smooth function yet to be determined. We also consider P and W in the form of

$$P = \operatorname*{diag}_{i=1}^{n} P_i, \quad P_i > 0, \quad W = \begin{bmatrix} W_{11} & 0 & \cdots & 0 \\ W_{21} & W_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ W_{n1} & \cdots & W_{n,n-1} & W_{n,n} \end{bmatrix} \tag{31}$$

Scaling matrices are $\mathbf{C}^0$ functions chosen from the set

$$\mathbf{L} = \left\{ L = \operatorname*{block\text{-}diag}_{i=1}^{n} L_i(x_1, \hat{x}_{[i-2]}), L_i \in \mathbf{L}_i \right\} \tag{32}$$

The matrices in (17) and (18) which characterize the nominal and robust stability become

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^{\mathrm{T}} & H \end{bmatrix} \tag{33}$$

$$N_{11} = \hat{S}^{\mathrm{T}} \hat{A}^{\mathrm{T}} (V+T)^{\mathrm{T}} P + P(V+T) \hat{A}^{\mathrm{T}} \hat{\mathrm{S}}, \quad \hat{A}^{\mathrm{T}} = [A\ G], \ \hat{S} = \left[\frac{S^{-1}}{0 \ \cdots \ 0\, s_n} \right]$$

$$N_{12} = -P(VA + TYC_y) W^{-1}$$

$$H = W^{-\mathrm{T}} (A - YC_y)^{\mathrm{T}} W^{\mathrm{T}} \tilde{P} + \tilde{P} W (A - YC_y) W^{-1}$$

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\mathrm{T}} & H \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} N_{11} & PVB & S^{-\mathrm{T}} C^{\mathrm{T}} L \\ B^{\mathrm{T}} V^{\mathrm{T}} P & -L & 0 \\ LCS^{-1} & 0 & -L \end{bmatrix}, \quad M_{12} = \begin{bmatrix} N_{12} \\ -B^{\mathrm{T}} W^{\mathrm{T}} \tilde{P} \\ -LCW^{-1} \end{bmatrix} \tag{34}$$

where $\tilde{P}$ is a positive-definite symmetric matrix. Using the characteristic matrices N and M, we introduce two key matrices $\tilde{N}_{[k]}$ and $\tilde{M}_{[k]}$ as follows:

$$\tilde{N}_{[k]} = \begin{bmatrix} \bar{R}_k^{\mathrm{T}} & 0 \\ 0 & I_n \end{bmatrix} N \begin{bmatrix} \bar{R}_k & 0 \\ 0 & I_n \end{bmatrix} \tag{35}$$

$$\bar{R}_k = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in \mathscr{R}^{n \times k}, \quad \bar{R}_n = I_n$$

$$\tilde{M}_{[k]} = \begin{bmatrix} \bar{Q}_k^{\mathrm{T}} & 0 \\ 0 & I_n \end{bmatrix} M \begin{bmatrix} \bar{Q}_k & 0 \\ 0 & I_n \end{bmatrix}$$

$$\bar{Q}_k = \begin{bmatrix} \bar{R}_k & 0 & 0 \\ 0 & I_{\bar{q}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{\bar{q}} \\ 0 & 0 & 0 \end{bmatrix} \in \mathscr{R}^{(n+2p) \times (k+2\bar{q})}, \quad \bar{Q}_n = I_{n+2p} \tag{36}$$

where I_k denotes a $k \times k$ identity matrix and $\bar{q} = \sum_{i=1}^{k} p_i$. These $\tilde{N}_{[k]}$ and $\tilde{M}_{[k]}$ directly let us be ready for the backstepping design.

Before describing the backstepping procedure, we show several important properties of $\tilde{N}_{[k]}$ and $\tilde{M}_{[k]}$. It is seen that the two matrices satisfy

$$\tilde{N}_{[k]}(x_1, \hat{x}_{[k-1]}) = \left[\begin{array}{c|c} N_{[k]11}(x_1, \hat{x}_{[k-1]}) & N_{[k]12}(x_1, \hat{x}_{[k-1]})\, 0 \\ \hline N_{[k]12}^{\mathrm{T}}(x_1, \hat{x}_{[k-1]}) & \\ 0 & H(x_1) \end{array} \right]$$

$$\tilde{M}_{[k]}(x_1, \hat{x}_{[k-1]}) = \left[\begin{array}{c|c} M_{[k]11}(x_1, \hat{x}_{[k-1]}) & \bar{Q}_k^T M_{12}(x_1, \hat{x}_{[n-1]}) \\ \hline M_{12}^{\mathrm{T}}(x_1, \hat{x}_{[n-1]}) \bar{Q}_k & H(x_1) \end{array}\right]$$

Here, $N_{[k]11}$, $N_{[k]12}$ and $M_{[k]11}$ are given by

$$N_{[k]11} = \hat{S}_{[k]}^{\mathrm{T}} \hat{A}^{\mathrm{T}}{}_{[k]}^{\mathrm{T}} (V_{[k]} + T_{[k]})^{\mathrm{T}} P_{[k]} + P_{[k]} (V_{[k]} + T_{[k]}) \hat{A}^{\mathrm{T}}{}_{[k]} \hat{S}_{[k]}$$

$$N_{[k]12} = -P_{[k]} (V_{[k]} A_{[k]} + T_{[k]} Y_{[k]} C_{y[k]}) W_{[k]}^{-1}$$

$$M_{[k]11} = \begin{bmatrix} N_{[k]11} & P_{[k]} V_{[k]} B_{[k]} & S_{[k]}^{-\mathrm{T}} C_{[k]}^{\mathrm{T}} L_{[k]} \\ B_{[k]}^{\mathrm{T}} V_{[k]}^{\mathrm{T}} P_{[k]} & -L_{[k]} & 0 \\ L_{[k]} C_{[k]} S_{[k]}^{-1} & 0 & -L_{[k]} \end{bmatrix}$$

which consist of system matrices for the first k states and input–output components.

$$A_{[k]} = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & & 0 \\ a_{k-1,1} & a_{k-1,2} & \cdots & & \cdots & a_{k-1,k} \\ a_{k1} & a_{k2} & \cdots & & \cdots & a_{k,k} \end{bmatrix}, \quad P_{[k]} = \begin{bmatrix} P_1 & 0 & \\ 0 & \ddots & 0 \\ & 0 & P_k \end{bmatrix}$$

$$B_{[k]} = \begin{bmatrix} B_{11} & 0 & \ddots \\ \vdots & \ddots & 0 \\ B_{k1} & \cdots & B_{kk} \end{bmatrix}, \quad C_{[k]} = \begin{bmatrix} C_{11} & 0 & \ddots \\ \vdots & \ddots & 0 \\ C_{k1} & \cdots & C_{kk} \end{bmatrix}, \quad L_{[k]} = \begin{bmatrix} L_1 & 0 & \ddots \\ 0 & \ddots & 0 \\ & 0 & L_k \end{bmatrix}$$

Similarly, $S_{[k]}^{-1}(x_1, \hat{x}_{[k-2]})$, $V_{[k]}(x_1, \hat{x}_{[k-1]})$, $T_{[k]}(x_1, \hat{x}_{[k-1]})$ and $W_{[k]}$ are defined as $k \times k$ upper left parts of S^{-1}, V, T and W, respectively. In addition, the following matrices are used:

$$\hat{A}_{[k]} = \left[A_{[k]} \left| \begin{array}{c} 0 \\ a_{k,k+1} \end{array} \right. \right] = \left[\begin{array}{c|c} \hat{A}_{[k-1]}(x_1) & 0 \\ \hline \star_{0,0} & a_{k,k+1} \end{array}\right], \quad \hat{S}_{[k]}(x_1, \hat{x}_{[k-1]}) = \left[\begin{array}{c} S_{[k]}^{-1} \\ \hline 0 \cdots 0 \; s_k \end{array}\right]$$

$$Y_{[k]} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix}, \quad Y_{[n]} = Y, \quad C_{y[k]} = [1 \; \underbrace{0 \cdots 0}_{k-1 \text{ times}}]$$

We can verify the following properties of $\tilde{N}_{[k]}$ and $\tilde{M}_{[k]}$ easily.

Proposition 1

Suppose $1 \leqslant k \leqslant n$.

(i-a) $\tilde{N}_{[k]}$ does not include $\{s_{k+1}, s_{k+2}, \ldots, s_n\}$.
(i-b) Entries of $\tilde{N}_{[k]}$ are affine in s_k.
(i-c) Entries of $\tilde{N}_{[k]}$ are jointly affine in all the entries of $P_{[k]}$.
(i-d) $\tilde{N}_{[k]} < 0$ implies $\tilde{N}_{[k-1]} < 0$ unless $k = 1$.
(i-e) $\tilde{N}_{[n]} = N$.

(ii-a) $\tilde{M}_{[k]}$ does not include either $\{s_{k+1}, s_{k+2}, \ldots, s_n\}$ or $\{L_{k+1}, L_{k+2}, \ldots, L_n\}$.
(ii-b) Entries of $\tilde{M}_{[k]}$ are jointly affine in L_k and s_k.
(ii-c) Entries of $\tilde{M}_{[k]}$ are jointly affine in all the entries of $L_{[k]}$ and $P_{[k]}$.
(ii-d) $\tilde{M}_{[k]} < 0$ implies $\tilde{M}_{[k-1]} < 0$ unless $k = 1$.
(ii-e) $\tilde{M}_{[n]} = M$.

Although the system is nonlinear in state variables, the problem of SD scaling is recursively linear in design parameters. On the basis of Proposition 1, this paper proposes the following procedures of backstepping for feedback gain design.

Nominal backstepping: Solve

$$\tilde{N}_{[k]}(x_1, \hat{x}_{[k-1]}) < 0, \quad \forall (x_1, \hat{x}_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1} \tag{37}$$

for s_k from $k = 1$ through $k = n$.

Robust backstepping: Solve

$$\tilde{M}_{[k]}(x_1, \hat{x}_{[k-1]}) < 0, \quad \forall (x_1, \hat{x}_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1} \tag{38}$$

for $\{s_k, L_k\}$ from $k = 1$ through $k = n$.

Both the procedures suppose that P, $\tilde{P}$, Y and W are given. The procedures can be carried out recursively since the process of finding decision parameters at Step k does not require any decision parameters to be found at Step $k+1, k+2, \ldots, n$. The procedures is also justified in that Step k is a necessary step for accomplishing Step $k+1$, $k+2, \ldots, n$. The problem of finding $\{L_k, s_k\}$ satisfying $\tilde{N}_{[k]} < 0$ and $\tilde{M}_{[k]} < 0$ is convex in decision parameters. The nominal backstepping and the robust backstepping for output feedback design via SD scaling are amenable to numerical computation based on optimization as it has been shown for state-feedback design in Reference [3]. The recursive design this section proposes does not require precise knowledge of each system parameter since the design is based on domination instead of cancellation. An exactly canceling formula is considered as one special solution to the domination. Moreover, the domination approach can be exploited to get rid of the propagation of complicated and long terms in the control law K.

The subsequent sections investigate whether the solutions of the inductive problems exist or not. A condition of allowable size and nonlinearity of uncertainty will be derived. Furthermore, a class of systems which can be always robustly stabilizable against arbitrarily large uncertainties by output feedback will be shown.

5. EXISTENCE AND ANALYTICAL SOLUTION

This section transforms the nominal and robust backstepping into problems which are suitable for finding analytical solutions. No conservatism is introduced in this process. The section, thereby, proposes alternative procedures of nominal and robust backstepping and solves them analytically. These alternative backstepping procedures can be also done by numerical calculation. The transformed problem of robust stabilization is no longer affine in decision parameters. Nevertheless, the backstepping can be easily done by curve fitting of a real-valued function which is only required to lie in a certain interval.

Define the following two functions:

$$\bar{N}_{[k]}(x_1, \hat{x}_{[k-1]}) = \bar{R}_k^{\mathrm{T}}(N_{11} - N_{12}H^{-1}N_{12}^{\mathrm{T}})\bar{R}_k \tag{39}$$

$$\bar{M}_{[k]}(x_1, \hat{x}_{[k-1]}) = \bar{Q}_k^{\mathrm{T}}(M_{11} - M_{12}H^{-1}M_{12}^{\mathrm{T}})\bar{Q}_k \tag{40}$$

Application of the Schur complements to $\tilde{N}_{[k]}$ and $\tilde{M}_{[k]}$ yields the equivalence

$$\bar{N}_{[k]} < 0 \Leftrightarrow \tilde{N}_{[k]} < 0 \tag{41}$$

$$\bar{M}_{[k]} < 0 \Leftrightarrow \tilde{M}_{[k]} < 0 \tag{42}$$

on the assumption that $H < 0$. Note that $\bar{N}_{[n]} = \bar{N}$ and $\bar{M}_{[n]} = \bar{M}$ where $\bar{N}$ and $\bar{M}$ are defined in Theorem 1 and Corollary 1. The matrix $\bar{N}_{[k]}$ involves $(s_1, \dots, s_k)$ and their partial derivatives. The matrix $\bar{M}_{[k]}$ involves $(s_1, \dots, s_k)$, their derivatives and $L_{[k]}$. We partition $\bar{N}_{[k]}$ into four blocks as follows:

$$\bar{N}_{[k]}(x_1, \hat{x}_{[k-1]}) = \begin{bmatrix} \bar{N}_{[k-1]}(x_1, \hat{x}_{[k-2]}) & \tilde{\Phi}_k(x_1, \hat{x}_{[k-1]}) \\ \Phi_k^{\mathrm{T}}(x_1, \hat{x}_{[k-1]}) & \tilde{\Psi}_k(x_1, \hat{x}_{[k-1]}) \end{bmatrix} \quad \text{for } k = 2, \dots, n \tag{43}$$

$$\tilde{\Psi}_k = 2P_k(a_{kk} + a_{k,k+1}s_k + \star_{k-1,k-1}) \in \mathscr{R}, \quad \tilde{\Phi}_k = \star_{k-1,k-1} \tag{44}$$

$$\bar{N}_{[1]}(x_1) = \tilde{\Psi}_1(x_1) \tag{45}$$

Let Q_k be a non-singular matrix of the form

$$Q_k = \begin{bmatrix} I_{k-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_1 & 0 & 0 \\ 0 & I_q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{p_k} & 0 \\ 0 & 0 & I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{p_k} \end{bmatrix} \in \mathscr{R}^{(k+2\bar{q})\times(k+2\bar{q})}$$

where $q := \sum_{i=1}^{k-1} p_i$. In order to obtain an inductive expression for $\bar{M}_{[k]}$, which is similar to (43), we let $\tilde{P}$ in $\tilde{M}_{[k]}$ be a diagonal matrix

$$\tilde{P} = \operatorname*{diag}_{i=1}^{n} \tilde{P}_i, \quad \tilde{P}_i > 0 \tag{46}$$

This assumption is not a restriction at all since a matrix $\tilde{P} > 0$ can be always decomposed into $\tilde{P} = W_{\text{new}}^{\mathrm{T}}\tilde{P}_{\text{new}}W_{\text{new}}$ with lower triangular W_{new} and diagonal $\tilde{P}_{\text{new}}$, so that the design parameter W absorbs W_{new}. The definition of (46) and Q_k allows $\bar{M}_{[k]}$ to be partitioned in the form of

$$Q_k^{\mathrm{T}}\bar{M}_{[k]}(x_1, \hat{x}_{[k-1]})Q_k = \begin{bmatrix} \bar{M}_{[k-1]}(x_1, \hat{x}_{[k-2]}) & \Phi_k(x_1, \hat{x}_{[k-1]}) \\ \Phi_k^{\mathrm{T}}(x_1, \hat{x}_{[k-1]}) & \psi_k(x_1, \hat{x}_{[k-1]}) \end{bmatrix} \quad \text{for } k = 2, \dots, n \tag{47}$$

$$\Psi_k = \begin{bmatrix} 2P_k(a_{kk} + a_{k,k+1}s_k + \star_{k-1,k-1}) & \star_{k-1,k-1} & (\star_{k-1,k-1} + C_{kk}^{\mathrm{T}})L_k \\ * & -L_k + U_{kk} & \star_{0,0}C_{k,-}^{\mathrm{T}}L_k \\ * & * & -L_k - L_kC_{k,-}W_{[k]}^{-1}[H^{-1}]_{[k]}W_{[k]}^{-\mathrm{T}}C_{k,-}^{\mathrm{T}}L_k \end{bmatrix}$$

$$\in \mathscr{R}^{(1+2p_k)\times(1+2p_k)} \tag{48}$$

$$\Phi_k = \begin{bmatrix} \star_{k-1,k-1} & \star_{k-2,k-2} & \star_{k-2,k-1}C_{k,-}^{\mathrm{T}}L_k \\ \star_{k-1,k-1} & U_{*,k} & \star_{0,0}C_{k,-}^{T}L_k \\ \diamondsuit_{k-1,k-1} & \diamondsuit_{k-3,k-1} & \diamondsuit_{k-3,k-1}C_{k,-}^{T}L_k \end{bmatrix} \tag{49}$$

$$\bar{M}_{[1]}(x_1) = \Psi_1(x_1) \tag{50}$$

where $\diamondsuit_{i,j}$ denotes any function depending only on $(x_1, \hat{x}_{[i]}, L_{[j]})$, $(s_1, \ldots, s_j)$ and their partial derivatives. The following matrices are used in (48) and (49):

$$U_{kk} = -[B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}]_k H^{-1}[B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}]_k^{\mathrm{T}}, \quad U_{*,k} = -[B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}]_{[k-1]}H^{-1}[B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}]_k^{\mathrm{T}}$$

$$[B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}]_{[k]} = \begin{bmatrix} [B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}]_1 \\ \vdots \\ [B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}]_k \end{bmatrix}, \quad B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P} = [B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}]_{[n]}$$

$$[H^{-1}]_{[k]} = \left[\begin{array}{c|c} [H^{-1}]_{[k-1]} & \star_{0,0} \\ \hline \star_{0,0} & [H^{-1}]_{kk} \end{array}\right], \quad [H^{-1}]_{[n]} = H^{-1}$$

$$C = \left[\begin{array}{l} C_{1,-}|0\cdots 0 \\ \hline \quad C_{2,-}|0 \;\vdots \\ \hline \quad\vdots \quad \ddots\; 0 \\ \hline \quad C_{n,-} \end{array}\right], \quad C_{1,-} = C_{11}$$

The calculation to obtain the explicit expressions of (44), (48) and (49) is straightforward, and it is omitted. Define $\tilde{J}_k(x_1, \hat{x}_{[k-1]}) \in \mathscr{R}$ conformably with

$$\begin{aligned} &\tilde{\Psi}_k - \tilde{\Phi}_k^{\mathrm{T}}\bar{N}_{[k-1]}^{-1}\tilde{\Phi}_k = \tilde{J}_k \quad \text{for } k \geqslant 2 \\ &\tilde{\Psi}_1 = \tilde{J}_1 \qquad\qquad\qquad\quad\; \text{for } k = 1 \end{aligned} \tag{51}$$

We also define $J_k(x_1, \hat{x}_{[k-1]}) \in \mathscr{R}$, $E_k(x_1, \hat{x}_{[k-1]}) \in \mathscr{R}^{1\times 2p_k}$ and $F_k(x_1, \hat{x}_{[k-1]}) \in \mathscr{R}^{2p_k\times 2p_k}$ as

$$\Psi_k - \Phi_k^{\mathrm{T}}\bar{M}_{[k-1]}^{-1}\Phi_k = \begin{bmatrix} J_k & E_k \\ E_k^{\mathrm{T}} & F_k \end{bmatrix} \quad \text{for } k \geqslant 2$$

$$\Psi_1 = \begin{bmatrix} J_1 & E_1 \\ E_1^{\mathrm{T}} & F_1 \end{bmatrix} \quad \text{for } k = 1 \tag{52}$$

Here, J_k is identical with $\tilde{J}_k$ if $p = 0$. Applying the Schur complements formula to (51) and (52), we obtain the following lemma.

Lemma 2

Suppose that $H(x_1) < 0$ holds for all $x_1 \in \mathscr{R}$. Let k be any integer belonging to $[1, n]$.

(i) Assume that $\tilde{N}_{[k-1]} < 0$ is satisfied for all $(x_1, \hat{x}_{[k-2]}) \in \mathscr{R} \times \mathscr{R}^{k-2}$ unless $k = 1$. Then, $\tilde{N}_{[k]} < 0$ holds for all $(x_1, \hat{x}_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1}$ if and only if

$$\tilde{J}_k < 0 \tag{53}$$

is satisfied for all $(x_1, \hat{x}_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1}$.

(ii) Assume that $\tilde{M}_{[k-1]} < 0$ is satisfied for all $(x_1, \hat{x}_{[k-2]}) \in \mathscr{R} \times \mathscr{R}^{k-2}$ unless $k = 1$. Then, $\tilde{M}_{[k]} < 0$ holds for all $(x_1, \hat{x}_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1}$ if and only if

$$F_k < 0,\ J_k - E_k F_k^{-1} E_k^{\mathrm{T}} < 0 \quad \text{when } p_k \neq 0 \tag{54}$$

$$J_k < 0 \quad \text{when } p_k = 0 \tag{55}$$

are satisfied for all $(x_1, \hat{x}_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1}$.

The backstepping in Section 4 is reduced to either (53), (54) or (55).

We are now in the position to state an existence result of nominal stabilization via output feedback. The function $\tilde{J}_k$ for nominal stabilization is calculated as

$$\tilde{J}_k = 2P_k(a_{kk} + a_{k,k+1}s_k + \bigstar_{k-1,k-1}) \tag{56}$$

This equation leads us to the following.

Theorem 2

Given an ordinary observer, the nominal system Σ_0 can be globally uniformly asymptotically stabilized by the output-feedback law (6) and (7) with a smooth function K.

Proof. Remember that $a_{k,k+1}(x_1)$ is non-zero for all $x_1 \in \mathscr{R}$. Since $a_{k,k+1}$ and other functions in (56) are $\mathbf{C}^0$ functions, for each $k = 1, 2, \ldots, n$ there exist a smooth function $s_k(x_1, \hat{x}_{[k-1]})$ such that $\tilde{J}_k < 0$ is satisfied for all $(x_1, \hat{x}_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1}$. Let $W = I$. Suppose that s_1 satisfies $\tilde{J}_1 < 0$. Then, $\tilde{N}_{[1]}(x_1) < 0$ holds for all $x_1 \in \mathscr{R}$ by Lemma 2. Suppose that $\tilde{N}_{[k-1]} < 0$ is satisfied for all $(x_1, \hat{x}_{[k-2]}) \in \mathscr{R} \times \mathscr{R}^{k-2}$. Then, we can find s_k so as to achieve $\tilde{J}_k < 0$ again. Lemma 2 implies $\tilde{N}_{[k]} < 0$ for all $(x_1, \hat{x}_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1}$. By induction, Theorem 1 and Proposition 1(i-e) prove the claim. □

For the robust stabilization problem, J_k is given as follows:

$$J_k = 2P_k(a_{kk} + a_{k,k+1}s_k + \bigstar_{k-1,k-1}) + \diamondsuit_{k-1,k-1} \tag{57}$$

The matrices E_k and F_k in (52) are calculated as

$$E_k = [\diamondsuit_{k-1,k-1} \ \ (C_{kk}^{\mathrm{T}} + \diamondsuit_{k-1,k-1})L_k] \tag{58}$$

$$F_k = \begin{bmatrix} -L_k + Z_a & Z_b L_k \\ L_k Z_b^{\mathrm{T}} & -L_k + L_k Z_c L_k \end{bmatrix} \tag{59}$$

where the following expressions are used:

$$Z_a = -B_{-,k}^{\mathrm{T}} W_{\langle k \rangle}^{\mathrm{T}} \tilde{P}_{\langle k \rangle}([H^{-1}]_{\langle k \rangle} + \bar{F}_{k11}) \tilde{P}_{\langle k \rangle} W_{\langle k \rangle} B_{-,k} \quad Z_b = (\bigstar_{0,0} - \bar{F}_{k12}^{\mathrm{T}} \tilde{P}_{\langle k \rangle} W_{\langle k \rangle} B_{-,k})^{\mathrm{T}} C_{k,-}^{\mathrm{T}}$$

$$Z_c = -C_{k,-}(W_{[k]}^{-1}[H^{-1}]_{[k]}W_{[k]}^{-\mathrm{T}} + \bar{F}_{k22})C_{k,-}^{\mathrm{T}}$$

$$\bar{F}_k(x_1, \hat{x}_{[k-2]}) = \begin{bmatrix} \bar{F}_{k11} & \bar{F}_{k12} \\ \bar{F}_{k12}^{\mathrm{T}} & \bar{F}_{k22} \end{bmatrix} = \begin{bmatrix} \star_{k-2,k-2} & \star_{k-2,k-1} \\ \bar{U}_k & \star_{0,0} \\ \diamond_{k-3,k-1} & \diamond_{k-3,k-1} \end{bmatrix}^{\mathrm{T}} \bar{M}_{[k-1]}^{-1} \begin{bmatrix} \star_{k-2,k-2} & \star_{k-2,k-1} \\ \bar{U}_k & \star_{0,0} \\ \diamond_{k-3,k-1} & \diamond_{k-3,k-1} \end{bmatrix}$$

$$\bar{F}_{1,11} = 0, \quad \bar{F}_{1,12} = 0, \; \bar{F}_{1,22} = 0$$

$$\bar{U}_k = - \; [B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}]_{[k-1]}H^{-1}\begin{bmatrix} 0 \\ I_{n-k+1} \end{bmatrix}$$

$$B = \left[\begin{array}{c|c|c|c} B_{-,1} & \begin{array}{c} 0 \\ \hline B_{-,2} \end{array} & \begin{array}{c} \cdots \\ \ddots \\ \hline \cdots \end{array} & \begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline B_{-,n} \end{array} \end{array}\right], \quad B_{-,n} = B_{nn}$$

$$[H^{-1}]_{\langle k \rangle} = \left[\begin{array}{c|c} [H^{-1}]_{kk} & \star_{0,0} \\ \hline \star_{0,0} & [H^{-1}]_{\langle k-1 \rangle} \end{array}\right], \quad [H^{-1}]_{\langle 1 \rangle} = H^{-1}$$

$$\tilde{P}_{\langle k \rangle} = \left[\begin{array}{c|c} \tilde{P}_k & 0 \\ \hline 0 & \tilde{P}_{\langle k+1 \rangle} \end{array}\right], \quad W_{\langle k \rangle} = \left[\begin{array}{c|c} W_{kk} & 0 \\ \hline \star & W_{\langle k+1 \rangle} \end{array}\right]$$

where the component denoted by $\star$ is constant. Note that $\bar{F}_{k11} \leqslant 0$ and $\bar{F}_{k22} \leqslant 0$ hold if $\tilde{M}_{[k-1]} < 0$ is satisfied.

Lemma 3

Let k be any integer belonging to $[1, n]$.

(i) F_k is supposed to be invertible for all $(x_1, \hat{x}_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1}$. Then, there always exists a smooth function $s_k(x_1, \hat{x}_{[k-1]})$ such that $J_k - E_k F_k^{-1} E_k^{\mathrm{T}} < 0$ is satisfied for all $(x_1, \hat{x}_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1}$.

(ii) Suppose that $p_k \neq 0$. Assume that $\bar{M}_{[k-1]}(x_1, \hat{x}_{[k-2]}) < 0$ holds for all $(x_1, \hat{x}_{[k-2]}) \in \mathscr{R} \times \mathscr{R}^{k-2}$ unless $k = 1$. Assume that $H(x_1) < 0$ holds for all $x_1 \in \mathscr{R}$ *if* $k = 1$. Then, there always exists a $\mathbf{C}^0$ function $\lambda_k(x_1, \hat{x}_{[k-2]})$ such that

$$\lambda_k(x_1, \hat{x}_{[k-2]}) > 0, \quad F_k(x_1, \hat{x}_{[k-2]}) < 0 \tag{60}$$

are satisfied for all $(x_1, \hat{x}_{[k-2]}) \in \mathscr{R} \times \mathscr{R}^{k-2}$ with $L_k(x_1, \hat{x}_{[k-2]}) = \lambda_k(x_1, \hat{x}_{[k-2]})I_{p_k}$ if

$$\begin{aligned} &\lambda_{\max}(-B_{-,k}^{T}W_{\langle k \rangle}^{\mathrm{T}}\tilde{P}_{\langle k \rangle}([H^{-1}]_{\langle k \rangle} + \bar{F}_{k11})\tilde{P}_{\langle k \rangle}W_{\langle k \rangle}B_{-,k}) \\ &\qquad \times \lambda_{\max}(-C_{k,-}(W_{[k]}^{-1}[H^{-1}]_{[k]}W_{[k]}^{-\mathrm{T}} + \bar{F}_{k22})C_{k,-}^{\mathrm{T}}) \leqslant \tfrac{1}{4} \end{aligned} \tag{61}$$

holds for all $(x_1, \hat{x}_{[k-2]}) \in \mathscr{R} \times \mathscr{R}^{k-2}$.

Proof. (i) This part is almost the same as Theorem 2.

(ii) Define $Z_{[k]} \in \mathscr{R}^{(k+2q)\times(k\times 2q)}$ and $[Z_{[k]}]_{22} \in \mathscr{R}^{2p_k \times 2p_k}$ with

$$Z_{[k]} = -Q_k^{\mathrm{T}}\bar{Q}_k^{\mathrm{T}}M_{12}H^{-1}M_{12}^{\mathrm{T}}\bar{Q}_kQ_k = \begin{bmatrix} \diamond_{k-1,k-1} & \diamond_{k-1,k} \\ \diamond_{k-1,k} & [Z_{[k]}]_{22} \end{bmatrix}$$

The assumption $\tilde{M}_{[k-1]} < 0$ implies $H < 0$ and $[Z_{[k]}]_{22} \geqslant 0$. According to (52), we write F_k as

$$F_k = [\Psi_k]_{22} - [\Phi_k \bar{M}_{[k-1]}^{-1} \Phi_k]_{22}$$

Then, $[\Phi_k \bar{M}_{[k-1]}^{-1} \Phi_k]_{22} \leqslant 0$ follows from $\bar{M}_{[k-1]} < 0$ implied by $\tilde{M}_{[k-1]} < 0$. Using (59), (47) and $Z_{[k]} = Q_k^{\mathrm{T}}(\bar{M}_{[k]} - \bar{Q}_k^{\mathrm{T}} M_{11} \bar{Q}_k) Q_k$, it is verified that

$$Z_0 = \begin{bmatrix} Z_a & Z_b \\ Z_b^{\mathrm{T}} & Z_c \end{bmatrix}$$

satisfies the following.

$$\begin{bmatrix} I & 0 \\ 0 & L_k \end{bmatrix} Z_0 \begin{bmatrix} I & 0 \\ 0 & L_k \end{bmatrix} = [Z_{[k]}]_{22} - [\Phi_k \bar{M}_{[k-1]}^{-1} \Phi_k]_{22}$$

Thus, we obtain $Z_0 \geqslant 0$ for all $(x_1, \hat{x}_{[k-2]}) \in \mathscr{R} \times \mathscr{R}^{k-2}$. Defining $\bar{a} = \lambda_{\max}(Z_a)$, $\bar{b} = \lambda_{\max}(Z_b^T Z_b)$ and $\bar{c} = \lambda_{\max}(Z_c)$, the inequality $Z_0 \geqslant 0$ directly proves that non-negative numbers $\bar{a}$, $\bar{b}$ and $\bar{c}$ satisfy $\bar{b} \leqslant \bar{a}\bar{c}$ for all $(x_1, \hat{x}_{[k-2]}) \in \mathscr{R} \times \mathscr{R}^{k-2}$. Now, we take $L_k = \lambda_k I$ and consider F_k at an arbitrary point $(x_1, \hat{x}_{[k-2]}) \in \mathscr{R} \times \mathscr{R}^{k-2}$. From Young's inequality, the inequality $F_k < 0$ is implied by the existence of $q > 0$ satisfying

$$-\lambda_k I + Z_a + q^{-1} I < 0, \quad \lambda_k^2 Z_c - \lambda_k I + q \lambda_k^2 Z_b^{\mathrm{T}} Z_b < 0 \tag{62}$$

Obviously, (62) is met if $q^{-1} < \lambda_k - \bar{a}$ and $\bar{b} < q^{-1}(\lambda_k^{-1} - \bar{c})$ are satisfied. Thus, $F_k < 0$ holds if

$$\lambda_k > \bar{a} \tag{63}$$

$$\lambda_k^2 \bar{c} + (\bar{b} - \bar{a}\bar{c} - 1)\lambda_k + \bar{a} < 0 \tag{64}$$

are satisfied. By manipulating the determinant of (64) together with (63), it is verified that there exists a real number λ_k such that (63) and (64) are satisfied if and only if

$$\bar{b} \leqslant \bar{a}\bar{c} + 1 - 2\sqrt{\bar{a}\bar{c}}, \quad \bar{a}\bar{c} < 1 \tag{65}$$

hold. Now we show that under the conditions (65), any solution $\lambda_k \in \mathscr{R}$ to (64) satisfies (63). To this end, first note that $\bar{b} \leqslant 1 - \bar{a}\bar{c}$ is obtained from $\bar{a} > 0$, $\bar{c} > 0$ and $\bar{a}\bar{c} < 1$. Let the set of all solutions

$$\left(\frac{(1 + \bar{a}\bar{c} - \bar{b}) - \sqrt{(1 + \bar{a}\bar{c} - \bar{b})^2 - 4\bar{a}\bar{c}}}{2\bar{c}}, \frac{(1 + \bar{a}\bar{c} - \bar{b}) + \sqrt{(1 + \bar{a}\bar{c} - \bar{b})^2 - 4\bar{a}\bar{c}}}{2\bar{c}} \right) \tag{66}$$

to (64) be denoted by the interval (l_λ, r_λ). Suppose that $l_\lambda < \bar{a}$, or, equivalently

$$-\sqrt{(1 + \bar{a}\bar{c} - \bar{b})^2 - 4\bar{a}\bar{c}} < \bar{b} - 1 + \bar{a}\bar{c}$$

This yields that

$$\begin{aligned} (1 + \bar{a}\bar{c} - \bar{b})^2 - 4\bar{a}\bar{c} &> (\bar{b} - 1 + \bar{a}\bar{c})^2 \\ &= (1 + \bar{a}\bar{c} - \bar{b})^2 - 4\bar{a}\bar{c}(1 - \bar{b}) \end{aligned}$$

The above inequality contradicts $1 \geqslant 1 - \bar{b}$. Thus, we conclude $\bar{a} \leqslant l_\lambda$. Next, recall that $\bar{b} \leqslant \bar{a}\bar{c}$. It is seen that two conditions of (65) are met if $1 - 4\bar{a}\bar{c} > 0$ is satisfied. Hence, if (61) holds and λ_k belongs to (l_λ, r_λ), then $L_k = \lambda_k I$ solves $F_k < 0$. Finally, note that (66) becomes $(\bar{a}, +\infty)$ as $\bar{c}$ goes to 0. The solution λ_k satisfying $F_k < 0$ always exists in such a case. Since all functions $\bar{a}$, $\bar{b}$ and $\bar{c}$ are $\mathbf{C}^0$ functions defined on $\mathscr{R}^k \times \mathscr{R}^{k-2}$, there exits $\mathbf{C}^0$ function $\lambda_k(x_1, \hat{x}_{[k-2]})$ such that two inequalities in (60) hold for all $(x_1, \hat{x}_{[k-2]}) \in \mathscr{R}^k \times \mathscr{R}^{k-2}$ on the assumption that (61) or (65) holds. □

Condition (61) is only sufficient for the existence of L_k in the backstepping procedure. The sufficiency is for only the purpose of obtaining a simple and explanatory condition. To check whether $\{L_k, s_k\}$ are chosen properly or not, one only has to evaluate $\widetilde{M}_{[k]} < 0$. If $\sum_{\mathrm{P}}$ does not have uncertainties at its kth state-space equation, requirement (61) vanishes at the kth steps of backstepping. Since B and C represent the nonlinear bounds of uncertainties, condition (61) is considered as the nonlinear size of tolerable uncertainties. For instance, if C and B are block diagonal, uncertainties appear in Σ_{P} as $B_{ii}\Delta_i C_{ii}$, and (61) with $W = I$ becomes

$$\lambda_{\max}(B_{kk}^{\mathrm{T}} B_{kk})\lambda_{\max}(C_{kk}C_{kk}^{\mathrm{T}}) \leqslant \frac{1}{4\gamma_k(\gamma_k - \bar{f}_k)\tilde{P}_k^2}$$

$$\bar{f}_1 = 0, \quad \bar{f}_k(x_1, \hat{x}_{[k-2]}) = \Diamond_{k-2,k-1} \leqslant 0 \quad \text{for } k \geqslant 2$$

where $\gamma_k(x_1)$ is any function satisfying $\mathrm{diag}_{i=1}^n \gamma_i > -H^{-1} > 0$.

A solution λ_k to (60) is any $\mathbf{C}^0$ function whose value lies between

$$\frac{1 + \bar{a}\bar{c} - \bar{b} \pm \sqrt{(1 + \bar{a}\bar{c} - \bar{b})^2 - 4\bar{a}\bar{c}}}{2\bar{c}} \tag{67}$$

where

$$\bar{a} = \lambda_{\max}(Z_a), \quad \bar{b} = \lambda_{\max}(Z_b^{\mathrm{T}} Z_b), \quad \bar{c} = \lambda_{\max}(Z_c)$$

A solution s_k is calculated from the affine inequality $J_k - E_k F_k^{-1} E_k^{\mathrm{T}} < 0$. We thereby arrive at the following result for robust stabilizability of Σ_{P}.

Theorem 3

Suppose that a robust observer is chose such that (61) is satisfied for all $k = 1, 2, \ldots, n$.

(i) Assume that the uncertainty Σ_Δ has only static components Δ_{is}. The system Σ_{P} can be globally uniformly asymptotically stabilized for any admissible uncertainty by the output-feedback law (6) and (7) with a smooth function K.

(ii) Assume that the uncertainty Σ_Δ has dynamic components Δ_{id}. If there exists a constant λ_k belonging to interval (67) for each $k = 1, 2, \ldots, n$. then, the system Σ_{P} can be globally uniformly asymptotically stabilized for any admissible uncertainty by the output-feedback law (6) and (7) with a smooth function K.

Proof. Suppose that a robust observer is constructed with H satisfying (61). Lemma 3(ii) guarantees the existence of $L_k \in \mathbf{L}_k$ satisfying $F_k < 0$. Due to Lemma 3(i), we can always find s_k achieving (54) or (55). Hence, we have $\widetilde{M}_{[k]} < 0$ from Lemma 2(ii). Repeating this procedure from $k = 1$ through $k = n$, we obtain $\widetilde{M}_{[n]} < 0$. Thus, Theorem 1 and Proposition 1(ii-e) prove the asymptotic stability. Finally, scaling matrices for dynamic uncertainties are required to be constant. □

Condition (61) may be satisfied for $k \leqslant 2$ by sufficiently small $\mathbf{C}^0$ functions $\gamma_i(x_1) > 0$ and $\mathrm{diag}_{i=1}^n \gamma_i > -H^{-1}$. However, the argument is valid only if an observer fulfilling $H < -\Gamma^{-1}$ is constructed for such a large Γ^{-1}. The smaller γ_k puts a heavier burden on the observer. The required *strong* observers may not exist unless the full information of x is available. The components of Y and W might become very large when γ_i is too small. Since $\bar{F}_{k11}$ and $\bar{F}_{k22}$ depend on Y and W, the condition (61) indicates a strong coupling between observer-gain design and feedback-gain design. The output-feedback robust stabilization problem is not always solvable globally in a backstepping manner for arbitrarily large uncertainties. This situation contrasts with state-feedback control by which global stabilization can be always achieved for arbitrarily large uncertainties [3].

6. RECURSIVE DESIGN OF GLOBALLY ROBUST OBSERVER

The ordinary observer defined in Section 3 can be constructed easily whenever the $\mathbf{C}^0$ function $A(x_1)x$ satisfies

$$A(x_1)x = A_0 x + \psi(x_1)$$

with a constant matrix A_0 [2]. However, the observer provided in Reference [2] is not guaranteed to be a robust observer. We need to develop a method of constructing the robust observer gain. This section demonstrates the existence of the robust observer gain for a class of diagonal matrices $\Gamma(x_1)$.

Given $\Gamma(x_1)$, it is required to find the coordinate transformation W and the observer gain $Y(x_1)$ such that (28) is satisfied for all $x_1 \in \mathscr{R}$ with a diagonal matrix $\tilde{P} > 0$. First, we choose W as

$$W = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ w_2 & 1 & 0 & \cdots & 0 \\ 0 & w_3 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & w_n & 1 \end{bmatrix} \tag{68}$$

The entries w_i for $2 \leqslant i \leqslant n$ are constant. Now define

$$\hat{W} = \left[\frac{w_1(x_1)\, 0 \cdots 0}{W^{\mathrm{T}}} \right] \tag{69}$$

where $w_1(x_1)$ is a $\mathbf{C}^0$ function defined on $x_1 \in \mathscr{R}$ yet to be determined. Let the observer gain be

$$Y(x_1) = -W^{-1} \begin{bmatrix} w_1(x_1) \\ 0 \end{bmatrix} = - \begin{bmatrix} w_1 \\ -w_1 w_2 \\ \vdots \\ (-1)^{n-1} w_1 w_2 \cdots w_n \end{bmatrix} \tag{70}$$

Then, we obtain

$$[C_y^{\mathrm{T}} \quad A^{\mathrm{T}}]\, \hat{W} = -C_y^{\mathrm{T}} Y^{\mathrm{T}} W^{\mathrm{T}} + A^{\mathrm{T}} W^{\mathrm{T}} = (A^{\mathrm{T}} - C_y^{\mathrm{T}} Y^{\mathrm{T}}) W^{\mathrm{T}}$$

Inequality (28) is equivalent to

$$\begin{aligned} \bar{H}(x_1) &= \hat{W}^{\mathrm{T}}\bar{A}^{\mathrm{T}}W^{-1}\tilde{P}^{-1} + \tilde{P}^{-1}W^{-\mathrm{T}}\bar{A}\hat{W} + \tilde{P}^{-1}\Gamma^{-1}\tilde{P}^{-1} < 0 \\ \bar{A} &= [C_y^{\mathrm{T}}\ A^{\mathrm{T}}] \end{aligned} \tag{71}$$

The structure of (71) is the same as that of feedback-gain backstepping design except that the lower triangular structure is replaced by the upper triangular one. In order to demonstrate that W can be determined recursively from w_n to w_1, the following notation:

$$\bar{H}_{\langle k\rangle}(x_1) = \hat{W}_{\langle k\rangle}^{\mathrm{T}}\bar{A}_{\langle k\rangle}^{\mathrm{T}}W_{\langle k\rangle}^{-1}\tilde{P}_{\langle k\rangle}^{-1} + \tilde{P}_{\langle k\rangle}^{-1}W_{\langle k\rangle}^{-\mathrm{T}}\bar{A}_{\langle k\rangle}\hat{W}_{\langle k\rangle} + \tilde{P}_{\langle k\rangle}^{-1}\Gamma_{\langle k\rangle}^{-1}\tilde{P}_{\langle k\rangle}^{-1} \tag{72}$$

is useful for $k = 1, 2, \ldots, n$, where

$$\bar{A}_{\langle k\rangle} = \left[\begin{array}{c|c} a_{k-1,k} & \star \\ \hline 0 & \bar{A}_{\langle k+1\rangle} \end{array}\right], \quad \bar{A}_{\langle n\rangle} = [a_{n-1,n}\ \ a_{n,n}]$$

$$\Gamma_{\langle k\rangle} = \left[\begin{array}{c|c} \gamma_k & 0 \\ \hline 0 & \Gamma_{\langle k+1\rangle} \end{array}\right], \quad \Gamma_{\langle n\rangle} = \gamma_n$$

$$\hat{W}_{\langle k\rangle} = \left[\begin{array}{c} w_k\, 0 \cdots 0 \\ \hline W_{\langle k\rangle}^{\mathrm{T}} \end{array}\right], \quad \hat{W}_{\langle n\rangle} = \begin{bmatrix} w_1 \\ 1 \end{bmatrix}$$

The matrix $\bar{H}_{\langle k\rangle}$ has properties which are almost the same as those in Proposition 1. Obviously, $\bar{H} = \bar{H}_{\langle 1\rangle}$ holds. The matrix $\bar{H}_{\langle k\rangle}$ does not include $\{w_{k-1}, w_{k-2}, \ldots, w_1\}$ and it is affine in w_k. In addition, we have

$$\bar{H}_{\langle k\rangle} = \left[\begin{array}{c|c} \bar{H}_{kk} & \star_{0,0} \\ \hline \star_{0,0} & \bar{H}_{\langle k+1\rangle} \end{array}\right]$$

Due to these properties, the following design procedure makes sense.

Recursive observer design: Solve

$$\bar{H}_{\langle k\rangle}(x_1) < 0, \quad \forall x_1 \in \mathscr{R} \tag{73}$$

for w_k from $k = n$ through $k = 1$.

Recall that $a_{k-1,k} \neq 0$ holds for all $x_1 \in \mathscr{R}$ by assumption. Applying the Schur complements formula to $\bar{H}_{\langle k\rangle}$, we can obtain the following answer to the existence of the solution.

Theorem 4

Suppose that $\bar{A}_{\langle 3\rangle}$ and $\Gamma_{\langle 3\rangle}$ are constant matrices. Given an integer $k \in [1, n]$, assume that $\bar{H}_{\langle k+1\rangle}(x_1) < 0$ holds for all $x_1 \in \mathscr{R}$ unless $k = n$.

(i) For $k = n, n-1, \ldots, 3$: There always exists a constant w_k such that $\bar{H}_{\langle k\rangle} < 0$ is satisfied.

(ii) For $k = 2$: There always exists a constant w_2 such that $\bar{H}_{\langle 2\rangle}(x_1) < 0$ is satisfied for all $x_1 \in \mathscr{R}$ if there exist positive constants c_i such that

$$\left|\frac{a_{i2}^2(x_1)}{a_{12}(x_1)}\right| \leqslant c_i, \quad i = 2, 3, \ldots, n, \quad \left|\frac{1}{\gamma_2(x_1)a_{12}(x_1)}\right| \leqslant c_1 \tag{74}$$

hold for all $x_1 \in \mathscr{R}$.

(iii) For $k = 1$: There always exists a smooth function $w_1(x_1)$ such that $\bar{H}_{\langle 1\rangle}(x_1) < 0$ is satisfied for all $x_1 \in \mathscr{R}$.

The idea of the proof is almost the same as Theorem 2. An expression for constructing w_k can be obtained easily as a scalar inequality $\bar{H}_{kk} - \star_{0,0}\bar{H}^{-1}_{\langle k+1\rangle}\star_{0,0} < 0$ which is affine in w_k. The constant and growth constraints on A and Γ guarantee w_k to be constant for $2 \leqslant k \leqslant n$. It may be worth noting that $\bar{H} < 0$ is always semi-globally achievable by constants w_k for all $1 \leqslant k \leqslant n$ without condition (74). The second inequality in (74) is unnecessary for the existence of ordinary observers since $\Gamma(x_1)$ is not a fixed parameter. The first inequality in (74) is met if $\bar{A}_{\langle 2\rangle}$ is constant. Indeed, the recursive observer design includes the ordinary observer design in Reference [2] as a special case.

The recursive design of observers in this section resembles backstepping. The observer design starts with a parameter away from observer-gain and back to the actual observer-gain. This is the unique feature of the observer design procedure in this paper. This design procedure of robust nonlinear observer is suitable for *numerical* calculation as well. Since the design is based on domination instead of cancellation, it is amenable to robustification. The following is established using this fact.

Proposition 2

Suppose that the uncertain system Σ_P is linear. Then, Σ_P is robustly stabilizable for arbitrarily large static uncertainties by the output-feedback law (6) and (7) with constant K and Y.

Proof. Due to the block lower triangular structure of B and C, the closed-loop system Σ_P with static uncertainties can be described as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix} = \begin{bmatrix} S(A_\delta + GK)S^{-1} & -SYC_yW^{-1}\\ 0 & W(A_\delta - YC_y)W^{-1}\end{bmatrix}\begin{bmatrix}\hat{\chi}\\ \eta\end{bmatrix}$$

with an uncertain matrix

$$A_\delta = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & & 0\\ a_{21} & a_{22} & a_{23} & 0 & & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ & & & & & 0\\ a_{n-1,1} & a_{n-1,2} & \cdots & & & a_{n-1,n}\\ a_{n,1} & a_{n,2} & \cdots & & & a_{n,n}\end{bmatrix} + \begin{bmatrix} \delta_{11} & 0 & 0 & \cdots & & 0\\ \delta_{21} & \delta_{22} & 0 & 0 & & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ & & & & & 0\\ \delta_{n-1,1} & \delta_{n-1,2} & \cdots & & & 0\\ \delta_{n,1} & \delta_{n,2} & \cdots & & & \delta_{n,n}\end{bmatrix}$$

where each δ_{ij} is a uniformly bounded function of t. Using the observer design method in this section,

$$W^{-\mathrm{T}}(A_\delta - YC_y)^{\mathrm{T}}W^{\mathrm{T}}\tilde{P} + \tilde{P}W(A_\delta - YC_y)W^{-1} < 0$$

can be achieved uniformly in t for all admissible uncertainties δ_{ij}. It is also seen that

$$S^{-\mathrm{T}}(A_\delta + GK)^{\mathrm{T}}S^{\mathrm{T}}P + PS(A_\delta + GK)S^{-1} < 0$$

can be satisfied by constant S and K [1, 3]. According to the argument in (i) of Corollary 1, $N < 0$ is satisfied for A_δ. □

7. UNCERTAIN SYSTEMS GUARANTEEING SOLUTIONS

This section focuses on a special class of uncertain systems, which will be shown to be always robustly stabilized by output feedback.

Assumption 1

The function $A(x_1)x$ satisfies

$$A(x_1)x = A_0x + \psi(x_1) + \phi(x_1)x_2 \tag{75}$$

with a constant matrix A_0 and $\mathbf{C}^0$ functions ψ and ϕ. There exist positive constants c_i such that

$$|a_{i2}^2(x_1)/a_{12}(x_1)| \leqslant c_i, \quad i = 2, 3, \ldots, n \tag{76}$$

hold for all $x_1 \in \mathscr{R}$. The matrices B and C satisfy

$$B(x_1) = \begin{bmatrix} B_{11}(x_1) \\ B_{21}(x_1) \\ \vdots \\ B_{n1}(x_1) \end{bmatrix}, \quad C(x_1) = [C_{11}(x_1)\, 0 \cdots 0] \tag{77}$$

where $B_{i1}(x_1) \in \mathscr{R}^{1 \times p_1}$, $C_{11}(x_1) \in \mathscr{R}^{p_1 \times 1}$ and $p_1 = p$.

Identity (75) equating its left- and right-hand side implies not only $\psi(0) = 0$, but also that assumption (75) is equivalent to

$$A(x_1)x = A_0x + A_1(x_1)x_1 + A_2(x_1)x_2$$

with $\mathbf{C}^0$ functions $A_1(x_1)$ and $A_2(x_1)$. This assumption is weaker than a common assumption

$$A(x_1)x = A_0x + \psi(x_1), \quad \psi(0) = 0$$

in which the nonlinearities are allowed to depend only on the measured state. Note that if $\phi_1(x_1)$(the first entry of the vector ϕ) is a constant, we do not need constraint (76) since such a system can be transformed to a system with $\phi = 0$ by using co-ordinate transformation [20]. Lemma 3(ii) reduces to the following on Assumption 1.

Lemma 4

Assume that $H(x_1) < 0$ holds for all $x_1 \in \mathscr{R}$. There always exists a $\mathbf{C}^0$ function $\lambda_1(x_1)$ such that

$$\lambda_1(x_1) > 0, \quad F_1(x_1) < 0 \tag{78}$$

are satisfied for all $x_1 \in \mathscr{R}$ with $L_1(x_1) = \lambda_1(x_1)I_{p_1}$ if

$$-[H^{-1}]_{11}\lambda_{\max}(-B^{\mathrm{T}}W^{\mathrm{T}}\tilde{P}H^{-1}\tilde{P}WB)\lambda_{\max}(C_{11}C_{11}^{\mathrm{T}}) \leqslant \tfrac{1}{4} \tag{79}$$

hold for all $x_1 \in \mathscr{R}$.

For only the purpose of deriving a compact condition (79), the simple form $L_1 = \lambda_1 I$ is used in Lemma 4. In actual design, one can exploit the freedom of $\mathbf{L}_i$ to avoid unnecessary high-gain and fast growth order of control laws. For example, if uncertain parameters in Σ_{P} are scalar-valued functions, L_1 can be a diagonal matrix consisting of independent entries such as $L_1 = \mathrm{diag}_{i=1}^{p_1}\lambda_{1,i}$. Condition (79) can be always satisfied for any B and C by a sufficiently small positive function $-[H^{-1}(x_1)]_{11}$. If $H < -\Gamma^{-1}$ is satisfied by an observer, $[-H^{-1}]_{11} < \gamma_1$

holds. Section 6 has shown that such a strong observer can be always constructed for arbitrary $\gamma_1(x_1) > 0$ and arbitrary constants γ_k, $k = 3, 4, \dots, n$ under Assumption 1. Recall that W is independent of γ_1 in the observer design.

Theorem 5

Suppose that Σ_P satisfies Assumption 1 and that the uncertainty Σ_Δ has only static components Δ_{is}. Then, the system Σ_P can be globally uniformly asymptotically stabilized for any admissible uncertainty by the output-feedback law (6) and (7) with a smooth function K.

Proof. Choose γ_i, $i = 3, 4, \dots, n$ as any positive numbers. We can define a $\mathbf{C}^0$ function $\gamma_2(x_1)$ satisfying

$$\gamma_2(x_1) = \frac{1}{\sqrt{a_{12}^2(x_1)}}, \quad \left|\frac{1}{\gamma_2(x_1)a_{12}(x_1)}\right| = 1, \quad \gamma_2(x_1) > 0, \ \forall x_1 \in \mathscr{R} \tag{80}$$

since $a_{12}(x_1) \neq 0$ is true for all $x_1 \in \mathscr{R}$ by assumption. In addition, let $\gamma_1(x_1)$ be a function satisfying

$$\gamma_1 \lambda_{\max}(B^{\mathrm{T}} W^{\mathrm{T}} \tilde{P} \Gamma \tilde{P} W B)\, \lambda_{\max}(C_{11} C_{11}^{\mathrm{T}}) \leqslant \tfrac{1}{4} \tag{81}$$

for all $x_1 \in \mathscr{R}$. Due to Theorem 4 and Assumption 1, there always exists $Y(x_1)$ such that $\bar{H}(x_1) < 0$ holds for all $x_1 \in \mathscr{R}$. Finally, combining Lemmas 3(i) and 4, it is possible to find s_k and L_1 achieving $\tilde{M}(x_1, \hat{x}_{[k-1]}) < 0$ for all $(x_1, x_{[k-1]}) \in \mathscr{R} \times \mathscr{R}^{k-1}$ from $k = 1$ through $k = n$. $\square$

Combining Theorem 2 with the recursive observer design in the previous section, we obtain the following which is a special case of the above theorem.

Corollary 2

Suppose that Σ_0 satisfies (75) and (76). Then, the nominal system Σ_0 can be globally uniformly asymptotically stabilized by the output-feedback law (6) and (7) with a smooth function K.

If the system Σ_0 fulfills $\phi = 0$, requirements (75) and (76) vanish. The combination of the nominal backstepping formulated by $\tilde{N}_{[k]} < 0$ and the recursive observer design formulated by $\bar{H}_{\langle k \rangle} < 0$ is an extension of the observer backstepping design presented in Reference [2]. This paper replaces Young's inequality in the observer backstepping design with the Schur complements formula. In addition, $\phi \neq 0$ is allowed by the result of this paper.

Assumptions (75) and (76) in Theorem 5 and Corollary 2 comes from observer design to ensure *globalness* of the observer. Global robust stabilization against dynamic uncertainties via output feedback is not always achievable if the uncertainty structure and size of uncertainty is arbitrarily prescribed *a priori*. Stability robustness in terms of input-to-state stability (ISS) can be obtained as follows.

Corollary 3

In addition to Assumption 1, assume that there exist constants $d_i > 0$ such that

$$\frac{B_{11}(x_1)B_{11}^{\mathrm{T}}(x_1)}{\sqrt{a_{12}^2(x_1)}} \leqslant d_0, \quad \frac{B_{21}(x_1)B_{21}^{\mathrm{T}}(x_1)}{\sqrt{a_{12}^2(x_1)}} \leqslant d_1, \quad B_{i1}(x_1)B_{i1}^{\mathrm{T}}(x_1) \leqslant d_i, \ i = 2, 3, \dots, n \tag{82}$$

hold for all $x_1 \in \mathcal{R}$. Then, the system Σ_0 can be made ISS by the output-feedback law (6) and (7) with a smooth function K.

Proof. The matrix F_1 is calculated as

$$F_1 = \begin{bmatrix} -\lambda_1 I + Z_a & \lambda_1 Z_b \\ \lambda_1 Z_b^{\mathrm{T}} & -\lambda_1 I + \lambda_1^2 Z_c \end{bmatrix}$$

$$Z_a = -B^{\mathrm{T}} W^{\mathrm{T}} \tilde{P} H^{-1} \tilde{P} W B, \quad Z_b = \star_{0,0} C_{11}^{\mathrm{T}}, \quad Z_c = -C_{11}[H^{-1}]_{11} C_{11}^{\mathrm{T}}$$

Letting $B_l = [B_{21}^{\mathrm{T}}, \dots, B_{n1}]^{\mathrm{T}}$, we have the following equation.

$$\begin{aligned} B^{\mathrm{T}} W^{\mathrm{T}} \tilde{P} \Gamma \tilde{P} W B = {} & \gamma_1 B_{11}^{\mathrm{T}} \tilde{P}_1^2 B_{11} + \gamma_2 (w_2 B_{11} + B_{21})^{\mathrm{T}} \tilde{P}_2^2 (w_2 B_{11} + B_{21}) \\ & + B_l^{\mathrm{T}} \hat{W}_{\langle 3 \rangle} \tilde{P}_{\langle 3 \rangle} \Gamma_{\langle 3 \rangle} \tilde{P}_{\langle 3 \rangle} \hat{W}_{\langle 3 \rangle}^{\mathrm{T}} B_l \end{aligned} \tag{83}$$

Pick arbitrary positive numbers γ_i, $i = 3, 4, \dots, n$. Choose $\gamma_2(x_1)$ as (80) so that

$$0 \leqslant \lambda_{\max}(\gamma_2 (w_2 B_{11} + B_{21})^{\mathrm{T}} \tilde{P}_2^2 (w_2 B_{11} + B_{21})) < \alpha_0$$

holds with a constant α_0 for all $x_1 \in \mathcal{R}$. Since B_l is uniformly bounded, there exist a constant $\alpha_1 > 0$ and a $\mathbf{C}^0$ function $\gamma_1(x_1)$ satisfying

$$\gamma_1 \lambda_{\max}(B_{11}^{\mathrm{T}} \tilde{P}_1^2 B_{11}) + \alpha_0 + \lambda_{\max}(B_l^{\mathrm{T}} \hat{W}_{\langle 3 \rangle} \tilde{P}_{\langle 3 \rangle} \Gamma_{\langle 3 \rangle} \tilde{P}_{\langle 3 \rangle} \hat{W}_{\langle 3 \rangle}^{\mathrm{T}} B_l) \leqslant \alpha_1$$

and $\gamma_1 > 0$ for all $x_1 \in \mathcal{R}$. The existence of a robust observer solving $H < -\Gamma^{-1}$ for these γ_i, $i = 1, \dots, n$ is guaranteed by Theorem 4. Here, it is important that W is constructed independently of γ_1 in the observer design. From (83) and $0 < -H^{-1} < \Gamma$ it follows that

$$\alpha_1 \geqslant \lambda_{\max}(B^{\mathrm{T}} W^{\mathrm{T}} \tilde{P} \Gamma \tilde{P} W B) \geqslant \lambda_{\max}(-B^{\mathrm{T}} W^{\mathrm{T}} \tilde{P} H^{-1} \tilde{P} W B) = \bar{a}$$

Let $C_{11}(x_1) = \beta_1(x_1) \tilde{C}_{11}$ where $\tilde{C}_{11}$ is a constant satisfying $\lambda_{\max}(\tilde{C}_{11} \tilde{C}_{11}^{\mathrm{T}}) = 1$. Pick arbitrary real numbers $v > 0$ and $\varepsilon > 0$ and choose a $\mathbf{C}^0$ function $\beta_1(x_1)$ such that

$$(-\varepsilon [H^{-1}]_{11} \beta_1^2(x_1) + \bar{b}(x_1)) < \frac{\varepsilon(\alpha_1 + \varepsilon)}{(\alpha_1 + v + \varepsilon)^2} \tag{84}$$

holds for all $x_1 \in \mathcal{R}$. There exists such a function β_1 since $\bar{b}(x_1) = \lambda_{\max}(C_{11} \star_{0,0}^2 C_{11}^{\mathrm{T}}) \geqslant 0$ and $\varepsilon [H^{-1}]_{11} < 0$. Now, choose λ_1 as a positive constant $\lambda_1 = \alpha_1 + v + \varepsilon$. Noting $\bar{c} = -[H^{-1}]_{11} \beta_1^2$, inequality (84) is rewritten as

$$\bar{c} < (\lambda_1^{-1} - v \lambda_1^{-2}) - \bar{b}(\lambda_1 - \alpha_1 - v)^{-1}, \quad \lambda_1 - \alpha_1 - v > 0$$

The above condition is equivalent to

$$\lambda_1 - \alpha_1 - v > 0, \quad \bar{b} < (\lambda_1^{-1} - \bar{c} - \lambda_1^{-2} v)(\lambda_1 - \alpha_1 - v)$$

Therefore, there exists a function $q(x_1) > 0$ such that

$$q^{-1} < \lambda_1 - \alpha_1 - v, \quad \bar{b} < q^{-1}(\lambda_1^{-1} - \bar{c} - \lambda_1^{-2} v)$$

Hence, using Young's inequality and $\alpha_1 \geqslant \bar{a} \geqslant 0$, we have

$$\begin{bmatrix} -\lambda_1 I + Z_a & \lambda_1 Z_b \\ \lambda_1 Z_b^{\mathrm{T}} & -\lambda_1 I + \lambda_1^2 Z_c \end{bmatrix} + vI < 0$$

Thus, the constant $\lambda_1 = \alpha_1 + v + \varepsilon > 0$ solves $F_1(x_1) + vI < 0$ for all $x_1 \in \mathscr{R}$. Since J_k, $k = 1, 2, \ldots, n$ is affine in s_k, it is possible to achieve $M(x_1, \hat{x}_{[n-1]}) + vI < 0$ for all $(x_1, \hat{x}_{[n-1]}) \in \mathscr{R} \times \mathscr{R}^{n-1}$ by selecting s_k. Using the Schur complements formula of M, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x, \hat{x}) \leqslant -v \begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \hat{\chi} \\ \eta \end{bmatrix} + w_1^{\mathrm{T}}(L_1 - vI)w_1$$

for all $(x, \hat{x}) \in \mathscr{R}^n \times \mathscr{R}^n$. Here, the constant matrix $L_1 - vI = (\lambda_1 - v)I$ is positive definite. Since S and W define a diffeomorphism globally, the closed system is ISS. □

An example of the class of systems Σ_0 that satisfy Assumption 1 is

$$\begin{aligned} \dot{x}_1 &= x_2 + \psi_1(x_1) + \phi_1(x_1)x_2 + b_1(x_1)w \\ \dot{x}_2 &= x_3 + \psi_2(x_1) + \phi_2(x_1)x_2 + b_2(x_1)w \\ &\vdots \quad \vdots \\ \dot{x}_n &= g(x_1)u + \psi_n(x_1) + \phi_n(x_1)x_2 + b_n(x_1)w \\ y &= x_1 \end{aligned} \tag{85}$$

where ϕ has positive constants c_i, $i = 2, 3, \ldots, n$ such that

$$|\phi_i(x_1)/(1 + \phi_1(x_1))| \leqslant c_i, \quad \forall x_1 \in \mathscr{R} \tag{86}$$

The class of uncertain systems Σ_{P} includes

$$\begin{aligned} \dot{x}_1 &= x_2 + \psi_1(x_1) + \phi_1(x_1)x_2 + \delta_1(x_1, t) \\ \dot{x}_2 &= x_3 + \psi_2(x_1) + \phi_2(x_1)x_2 + \delta_2(x_1, \mathrm{t}) \\ &\vdots \quad \vdots \\ \dot{x}_n &= g(x_1)u + \psi_n(x_1) + \phi_n(x_1)x_2 + \delta_n(x_1, t) \\ y &= x_1 \end{aligned} \tag{87}$$

where ϕ_i satisfies (86). For each $i \in [1, n]$, it is assumed that there exists a $\mathbf{C}^0$ function f_i such that

$$|\delta_i(x_1, t)| \leqslant |f_i(x_1)|, f_i(0) = 0, \quad \forall x_1 \in \mathscr{R}, \forall t \geqslant 0 \tag{88}$$

The function $\delta_i(x_1, t)$ can be replaced by $\delta_i(x_1, \theta)$ in (87) and (88), where $\theta(t)$ is an uncertain parameter vector.

8. CONCLUSIONS

We have proposed a new method of robust backstepping for output-feedback global stabilization of uncertain nonlinear systems. The procedure and constructive proof of robust backstepping are very simple, which only uses Schur complements and domination by scaling functions recursively. This strategy is quite different from other backstepping techniques available in the literature. This paper also has presented a recursive procedure of robust observer design for a class of uncertain strict-feedback nonlinear systems. Since this paper assumes that observer design is completed before feedback design, two recursive designs of feedback and observer in this paper cannot interlace with each other. Integration of the backstepping for feedback and the observer design is an interesting direction of further research.

REFERENCES

1. Freeman RA, Kokotović PV. *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Birkhäuser: Boston, 1996.
2. Krstić M, Kanellakopoulos I, Kokotović PV. *Nonlinear and Adaptive Control Design*. Wiley: New York, 1995.
3. Ito H, Freeman R. A new look at robust backstepping through state-dependent scaling design. *Proceedings of the American Control Conference* 1999; 11–16.
4. Ito H. Robust control for nonlinear systems with structured $\mathscr{L}_2$-gain bounded uncertainty. *Systems & Control Letters* 1996; **28**:167–172.
5. Ito H. State-dependent scaling for robust nonlinear control: techniques and effectiveness. *Proceedings of the 37th IEEE Conference on Decision and Control* 1998; 4111–4114.
6. Ito H. Region maximization technique in robust nonlinear control via state-dependent scaling design. *Proceedings of the 1999 American Control Conference* 1999; 1390–1394.
7. Esfandiari F, Khalil HK. Output feedback stabilization of fully linearizable systems. *International Journal of Control* 1992; **56**:1007–1037.
8. Khalil HK, Esfandiari F. Semi-global stabilization of a class of nonlinear systems using output feedback. *IEEE Transactions on Automatic Control* 1993; **38**:1412–1415.
9. Lin Z, Saberi A. Robust semi-global stabilization of minimum-phase input-output linearizable systems via partial state and output feedback. *IEEE Transactions on Automatic Control* 1995; **40**:1029–1041.
10. Teel AR, Praly L. Tools for semi-global stabilization by partial state feedback and output feedback. *SIAM Journal of Control and Optimization* 1995; **33**:1443–1488.
11. Teel AR, Praly L. Global stabilizability and observability imply semi-global stabilizability by output feedback. *Systems of Control Letters* 1994; **22**:313–325.
12. Lin W, Qian C. Semi-global robust stabilization of nonlinear systems by partial state and output feedback. *Proceedings of the 37st IEEE Conference on Decision Control* 1998; 3105–3110.
13. Jiang Z-P, Mareels I, Hill D. Robust control of uncertain nonlinear systems via measurement feedback. *IEEE Transactions on Automatic Control* 1999; **44**:807–812.
14. Sontag E, Wang Y. New characterizations of input to state stability. *IEEE Transactions on Automatic Control* 1996; **41**:1283–1294.
15. Sontag E. Notions of integral input-to-state stability. *Proceedings of the American Control Conference* 1998; 3210–3214.
16. Mareels IM, Hill DJ. Monotone stability of nonlinear feedback systems. *Journal of Mathematical System Estimation and Control* 1992; **2**:275–291.
17. Jiang ZP, Mareels MY. A small-gain control method for nonlinear cascaded systems with dynamic uncertainties. *IEEE Transactions on Automatic Control* 1997; **42**:292–308.
18. Ito H, Freeman R. State-dependent scaling design for a unified approach to robust backstepping. *Automatica* 2000, to appear.
19. Sepulchre R, Janković M, Kokotović PV. *Constructive Nonlinear Control*. Springer: New York, 1997.
20. Tan Y, Kanellakopoulos I, Jiang Z-P. Nonlinear observer/controller design for a class of nonlinear systems. *Proceedings of the 37th IEEE Conference on Decision and Control* 1998; 2503–2508.

