Inverse optimal extremum seeking under delays

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Abstract
We establish the inverse optimality in the average sense of our earlier Gradient- and Newton-based extremum seeking algorithms for maximizing unknown locally quadratic maps in the presence of constant delays. To compensate the delay, these algorithms employ a predictor feedback with a perturbation-based estimate for the unknown Hessian (or its inverse). The algorithm’s inverse optimality is the result of running the predictor through a simple first-order filter with a fast enough pole.

1. Introduction
Extremum Seeking (ES) is a real-time optimization method with the ability of determining the extremum of a nonlinear convex map which may be related to a performance index of the plant. ES does not assume the explicit knowledge of the plant or the nonlinear map to be optimized, but only the existence of an extremum, being a maximum or minimum point [1]. The sign of the Hessian (second derivative) of the map defines if we have a maximization or minimization optimization problem.

In the ES literature there are many publications applying high-pass and low-pass filters in order to improve the closed-loop system performance and to facilitate the tuning parameters [2–6]. However, they do not present any theoretical support that justifies the inclusion of such filters, on the contrary, only heuristic arguments are given.

In this paper, the proof of inverse optimality and its influence on the Gradient- and Newton-based extremum seeking feedback [5] is proved in the presence of delays. The results are valid, in particular, also in the absence of delays. We show that the basic predictor feedback controller originally proposed in [7,8], when applied through a low-pass filter, is inverse optimal and study its robustness to the low-pass filter time constant.

Inverse optimality was defined by Kalman in [9] as follows: “Given a dynamic system and a known control law, find performance criteria (if any) for which this control law is optimum.”

Inverse optimality can be related to a control Lyapunov function [10], with a particular control law. In this sense, the inverse optimality is guaranteed when a stabilizing controller is optimal for some criteria and, for a given Lyapunov function. Hence, it is possible to show that the feedback control law is optimal with respect to some cost function. In general, this functional includes a control input penalty (or the time-derivative of the control signal or its rate of variation) and has an infinite gain margin [11,12].

In this paper, we study the inverse optimality of the average system, which we do via Lyapunov method, guaranteeing the minimization of a functional which depends on the state variables as well as the rate of variation of the control input over the infinite-time horizon. Results and simulations illustrate the benefit of endowing ES with an inverse optimality property and its advantages for providing improved closed-loop responses. Exponential stability and convergence to a small neighborhood of the unknown extremum point are still preserved by using backstepping transformation and averaging theory in infinite dimensions.

This paper is organized as follows. In Section 2, the advantages of inverse optimal controllers are properly introduced. Section 3 revisits the Gradient-based extremum seeking scheme under delays, proposed in [7,8]. In Section 4, the optimality proof is developed along with the stability analysis, while Section 5 shows the analogous steps on the extension of the inverse optimal results for the Newton-based extremum seeking under delays. Section 6 presents the simulations of the ES control feedback verifying the inverse optimality for the resulting closed-loop system. Finally, Section 7 concludes the paper by highlighting the contributions and indicating potential future works.
2. Inverse optimality: What is it and why aim for it?

Extremum seeking is a method for optimization by control but it is not a method of optimal control. This statement, if not read carefully, may seem like a contradiction. But it is not. Let us recall the distinction between optimization, which ES pursues, and optimal control. ES pursues the optimization of a static map\(^1\)—whether in the presence or in the absence of a dynamic system. Hence, all that matters in ES is the optimality as the time approaches infinity. The trajectory through which ES approaches an optimal steady state may be quick and straight, or slow, roundabout, and complex, but the trajectory is not what is being penalized or rewarded. Only the asymptotic value of the output is what is being minimized or maximized.

In optimal control, the asymptotic value of the output is not what is being optimized—this value must be taken to zero. It is, instead, the transient of the output, state, and the input that is being minimized, in a suitable temporal norm.

Hence, while ES is performing optimization, it is not necessarily optimal in the sense of optimal control, i.e., in the sense of the transient of its estimate, or of the update rate, being minimized.

This realization brings up the question whether this transient—towards the optimum—can itself be made optimal.

The answer to this question is, in general, negative. The reason for the impossibility of optimizing the ES transient is that optimal control is model based and, under ES, at least the output map is unknown.

However, we are still unwilling to give up on optimality, even if it is not achieved by deliberate minimization of a cost functional of the transient but is, instead, of a serendipitous nature.

This insistence on obtaining optimality, by accident if not by design, has its root in Kalman’s question (paraphrased) “Is every stabilizing feedback law optimal?” Kalman’s answer to this question, in his 1964 paper [9], was negative but, in providing this answer, he provided something very useful: a characterization of “inverse optimal controllers”, i.e., all the controllers that happen to be optimal with respect to a cost that is meaningful (positive definite) though not chosen at will by the user.

Kalman’s notion of inverse optimality has spread from LTI systems to nonlinear control [13,14], ISS stabilization and differential games [15], stochastic nonlinear stabilization [16,17], adaptive control [18], PDE control [19], and control of delay systems [11].

The success of inverse optimality in the context of control of delay systems, specifically, in the context of predictor feedback for systems with input delays [11], is what inspires our interest in inverse optimality of ES in the presence of delays.

In this paper, we revisit the Gradient-based extremum seeking algorithm of [7] in the presence of a delay. We show that the inclusion of a filter with a single sufficiently negative pole guarantees inverse optimality of the average ES system. The infinite-horizon cost functional that ES happens to minimize includes a positive definite cost on the ES parameter estimation error, the parameter estimation rate over a moving time window of length equal to the delay \(D\), and the derivative of the parameter estimation error rate (i.e., parameter estimation “acceleration”).

This “fast enough single-pole low-pass filter” is a part of the common ES implementation anyway, for two reasons, one practical and one mathematical. First, the practical reason for including a low-pass filter is to attenuate higher-order perturbation terms of frequencies \(\omega\) and higher, which arise due to injected perturbation and the (locally) quadratic nature of the map being optimized. Second, the mathematical reason for including the low-pass filter is to make the predictor-compensated ES feedback amenable to the application of the existing averaging theorem, by Hale and Lunel [20], for functional differential equations, which includes systems with delays on the state.

The same inverse optimality considerations can be conducted for the Newton-based extremum seeking approach presented in [21], as discussed in the final part of the manuscript.

Norms and Notations: The 2-norm of the state vector \(x(t)\) for a finite-dimensional system described by an Ordinary Differential Equation (ODE) is denoted by single bars, \(|x(t)|\). In contrast, norms of functions (of \(x\)) are denoted by double bars. By default, \(\| \cdot \|\) denotes the spatial \(L_2[0, D]\) norm, i.e., \(\| \cdot \| = \| \cdot \|_{L_2[0, D]}\).

Since the state variable \(u(x, t)\) of the infinite-dimensional system governed by a Partial Differential Equation (PDE) is a function of two arguments, we should emphasize that taking a norm in one of the variables makes the norm a function of the other variable. For example, the \(L_2[0, D]\) norm of \(u(x, t)\) in \(x \in [0, D]\) is \(\|u(t)\| = \left(\int_0^D u^2(x, t) dx\right)^{1/2}\).

The partial derivatives of \(u(x, t)\) are denoted by \(u_t(x, t)\) and \(u_x(x, t)\) or, occasionally, by \(\partial_t u(x, t)\) and \(\partial_x u(x, t)\) to refer to the operator for its average signal \(u_{avg}(x, t)\). As defined in [23], big \(O(\epsilon)\) notation is used to quantify approximations or order of magnitude relation of vector functions, valid for “\(\epsilon\) sufficiently small”.

3. Gradient-based extremum seeking under delays

The goal of scalar ES is to maximize (or minimize) the output \(y \in \mathbb{R}\) of an unknown nonlinear static map \(Q(\theta)\) by changing its input \(\theta \in \mathbb{R}\). We further assume there is a constant and known delay \(D > 0\) in the measurement system such that the delayed output signal is

\[y(t) = Q(\theta(t - D))\]  

For the sake of clarity, we illustrate the complete output-delayed system in the block diagram of Fig. 1. Without loss of generality, we also consider the maximum seeking problem. For the sake of simplicity, we assume the nonlinear map to be optimized is quadratic as

\[Q(\theta) = y^* + \frac{H}{2}(\theta - \theta^*)^2\]  

with \(\theta^* \in \mathbb{R}, y^* \in \mathbb{R}\) and \(H \in \mathbb{R}\) being unknown scalar constants. The maximizer of the input parameter \(\theta\) is given by \(\theta^*\), while \(y^*\) is the extremum point and \(H < 0\) is the unknown Hessian of the map.

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\(^1\) or of a steady non-stationary motion, such as in the minimization of a limit cycle amplitude.
Substituting (2) into (1), lead us to the following quadratic static map under delay:

$$y(t) = y^* + \frac{H}{2} (\theta(t - D) - \theta^*)^2.$$ \hfill (3)

3.1. Probing and demodulation signals

Let us define the estimation error:

$$\hat{\theta}(t) = \hat{\theta}(t) - \theta^*$$ \hfill (4)

where \( \hat{\theta} \) is the estimate of \( \theta^* \). According to Fig. 1, one can write the error dynamics as

$$\dot{\hat{\theta}}(t - D) = U(t - D).$$ \hfill (5)

In addition, one gets

$$G(t) = M(t)y(t), \quad \theta(t) = \hat{\theta}(t) + S(t),$$ \hfill (6)

with additive and multiplicative dithers given by

$$S(t) = a \sin(\omega(t + D)) \quad M(t) = \frac{2}{a} \sin(\omega t).$$ \hfill (7)

In (7), the amplitude \( a \) and frequency \( \omega \) must be nonzero, assuming small and large values (as discussed later on), respectively.

The estimate of the unknown Hessian \( H \) is given by

$$\hat{H}(t) = N(t)y(t)$$ \hfill (8)

with the demodulating signal \( N(t) \) being

$$N(t) = -\frac{8}{a^2} \cos(2\omega t).$$ \hfill (9)

From [5], it is possible to show

$$\frac{1}{\Pi} \int_0^\Pi N(\sigma) \sigma d\sigma = H, \quad \Pi = 2\pi/\omega,$$ \hfill (10)

for a quadratic map, as assumed in (2). The expression (9) gives us an averaging-based estimate of \( H \) by means of (10), i.e., \( H_{av} = (N y)_{av} = H \).

3.2. Predictor feedback with averaging-based estimates for gradient and Hessian

From the averaging analysis, we can obtain the average version of \( G(t) \) in (6) as follows

$$G_{av}(t) = H\hat{\theta}_{av}(t - D).$$ \hfill (11)

Moreover, from (5), we can also derive the average models

$$\dot{\hat{\theta}}_{av}(t - D) = U_{av}(t - D),$$ \hfill (12)

$$\dot{G}_{av}(t) = H U_{av}(t - D),$$ \hfill (13)

with \( U_{av} \in \mathbb{R} \) being the average version of the control signal \( U \in \mathbb{R} \).

As shown in [7,8], by computing the variation of constants formula of (13), we can write the future state as

$$G_{av}(t + D) = G_{av}(t) + H \int_{t - D}^t U_{av}(\tau) d\tau,$$ \hfill (14)

which is given in terms of the average control signal \( U_{av}(\tau) \) from the past window \([t - D, t]\). It results in the next predictor-feedback law

$$U_{av}(t) = k \left[ G_{av}(t) + H \int_{t - D}^t U_{av}(\tau) d\tau \right], \quad k > 0,$$ \hfill (15)

which is able to make the equilibrium \( \hat{\theta}_{av} = 0 \) of the closed-loop system (12) and (15) exponentially stable.

In [7,8], it was shown the same stability objectives could be guaranteed if a simple modification of the above basic predictor-based controller (15) employing a low-pass filter [11] was applied.

In this case, the following infinite-dimensional filtered predictor feedback, computed from its non average version (15), is again considered:

$$U(t) = \frac{c}{s + c} \left\{ k \left[ G(t) + \hat{H}(t) \int_{t - D}^t U(\tau) d\tau \right] \right\}$$ \hfill (16)

where \( c > 0 \) is a design constant chosen sufficiently large.

This low-pass filtering was particularly required in the stability analysis of our earlier publications [7,8] when the averaging theorem in infinite dimensions was invoked [20].

Now, in the next section, we are going to demonstrate the advantages of such a filtering procedure go beyond to merely solve technical limitations in the analysis, but it can also improved the control performance of the closed-loop ES system, which is rigorously justified through the concept of inverse optimality [9].

4. Inverse optimal analysis

In this section, the stability analysis is carried out and the proof of inverse optimality is presented.

In the formulation of the inverse optimality problem we will consider \( U_{av}(t) \) as the input to the system, whereas \( U_{av}(t) \) is still the actuated variable. Hence, our inverse optimal design will be implementable after integration in time, i.e., as dynamic feedback. Treating \( U_{av}(t) \) as an input is the same as adding an integrator, which has been observed as being beneficial in the control design for delay systems in [24].

Theorem 1. There exists \( c^* \) such that the average feedback system of (5) and (16) is exponentially stable in the sense of the norm

$$\Psi(t) = \left( \| \hat{\theta}_{av}(t - D) \|^2 + \int_{t - D}^t U_{av}(\tau)^2 d\tau + U_{av}(t)^2 \right)^{1/2}$$ \hfill (17)

for all \( c > c^* \). Furthermore, there exists \( c^{**} > c^* \) such that for any \( c \geq c^{**} \), the feedback (16), with \( k \) sufficiently small, minimizes the cost functional

$$J = \int_0^\infty (L(t) + \hat{U}_{av}^2(t)) dt,$$ \hfill (18)

where \( L(t) \) is a functional of \( \hat{\theta}_{av}(t - D), U(\tau), \tau \in [t - D, t] \) and such that

$$L(t) \geq \mu \Psi(t)^2$$ \hfill (19)

for some \( \mu(c) > 0 \) with a property that \( \mu(c) \to \infty \) as \( c \to \infty \).

Proof. The proof is structured into Step 1 to Step 6, analogously to what has been done in [8]. Some overlaps with respect to Ref. [7] must be understood as a pedagogical tool. Some minimal technical details must be given otherwise the reader would miss the keypoints in the demonstration of the main novelty concerning the original inverse optimality result in Step 6.

4.1. Step 1: Transport PDE for delay representation

Considering [22], the delay in (5) is represented using a transport PDE such as

$$\hat{\theta}(t - D) = u(0, t),$$ \hfill (20)

$$u_i(x, t) = u_x(x, t), \quad x \in [0, D],$$ \hfill (21)

$$u(D, t) = U(t),$$ \hfill (22)

with the solution of (21)–(22) being

$$u(x, t) = U(t + x - D).$$ \hfill (23)
4.2. Step 2: Average model of the closed-loop system

By denoting
\[ \tilde{\theta}(t) := \tilde{\theta}(t-D), \quad \tilde{\vartheta}(t) := \tilde{\vartheta}(t-D), \quad (24) \]
the average version of system (20)–(22), with \( U(t) \) in (16) is:
\[ \tilde{\vartheta}_w(t) = u_w(0, t), \quad (25) \]
\[ \partial_t u_w(x, t) = \tilde{\vartheta}_u w(x, t), \quad x \in [0, D], \quad (26) \]
\[ \frac{d}{dt} u_w(D, t) = -c u_w(D, t) + D_k H \left[ \tilde{\vartheta}_w(t) + \int_0^D u_w(\sigma, t) d\sigma \right], \quad (27) \]
where the filter \( c/(\delta + c) \) in (16) was also represented in the state–space form. The solution of the transport PDE (26)–(27) is
\[ u_w(x, t) = U_w(t + x - D). \quad (28) \]

4.3. Step 3: Backstepping transformation, its inverse and the target system

Since we are not able to prove directly the stability for the average closed-loop system (25)–(27), we consider the infinite-dimensional backstepping transformation of the delay state
\[ \tilde{w}(x, t) = u_w(x, t) - k H \left[ \tilde{\vartheta}_w(t) + \int_0^x u_w(\sigma, t) d\sigma \right], \quad (29) \]
with inverse given by
\[ u_w(x, t) = \tilde{w}(x, t) + k H \left[ e^{kh} \tilde{\vartheta}_w(t) + \int_0^x e^{kh(x-\sigma)} w(\sigma, t) d\sigma \right]. \quad (30) \]
The transformation (29) maps the system (25)–(27) into the target system:
\[ \tilde{\vartheta}_w(t) = k H \tilde{\vartheta}_w(t) + w(0, t), \quad (31) \]
\[ w_1(x, t) = w(x, t), \quad x \in [0, D], \quad (32) \]
\[ w(D, t) = -\frac{1}{c} \partial_x u_w(D, t). \quad (33) \]

4.4. Step 4: Lyapunov–Krasovskii functional

Consider the following Lyapunov functional
\[ V(t) = \frac{\tilde{\vartheta}_w^2(t)}{2} + a \int_0^D (1 + x) w^2(x, t) dx + \frac{1}{2} w^2(D, t), \quad (34) \]
where the parameter \( a = -\frac{1}{k^2} \) and \( kH < 0 \). Computing the time-derivative of (34) along with (31)–(33), we have
\[ \dot{V}(t) = k H \tilde{\vartheta}_w^2(t) + \tilde{\vartheta}_w(t) w(0, t) \]
\[ + a \int_0^D (1 + x) w^2(x, t) dx + w(D, t) w_1(D, t) \]
\[ = k H \tilde{\vartheta}_w^2(t) + \tilde{\vartheta}_w(t) w(0, t) + \frac{a(1 + D)}{2} w^2(D, t) \]
\[ - \frac{a}{2} w^2(0, t) - \frac{a}{2} \int_0^D w^2(x, t) dx + w(D, t) w_1(D, t) \]
\[ \leq k H \tilde{\vartheta}_w^2(t) + \tilde{\vartheta}_w(t) w(0, t) \]
\[ - \frac{a}{2} \int_0^D w^2(x, t) dx \]
\[ + \left[ w_1(D, t) + \frac{a(1 + D)}{2} w(D, t) \right]. \quad (35) \]
Now, following the same procedure given in [7], we get
\[ \dot{V}(t) \leq -\frac{1}{4a} \tilde{\vartheta}_w^2(t) - \frac{a}{4(1 + D)} \int_0^D (1 + x) w^2(x, t) dx \]
\[ - (c - c^*) w^2(D, t), \quad (36) \]
where
\[ c^* = \frac{a(1 + D)}{2} - k H + a \left[ (k H)^2 e^{khD} \right] \]
\[ + \frac{1}{a} \left[ (k H)^2 e^{kh(D-a)} \right]^2. \quad (37) \]
According to (37), an upper bound for \( c^* \) can be obtained from the known delay \( D \) as well as some lower and upper bounds of the Hessian \( H \). In addition, from (36), it is clear the Hessian-dependence of the Gradient approach in terms of the convergence rate since the parameter \( a \) depends on \( H \).

Thus, from (36), if we chose \( c \) such that \( c > c^* \), we arrive at
\[ V(t) \leq -\mu^* V(t), \quad (38) \]
for some \( \mu^* > 0 \). Hence, the closed-loop system is exponentially stable in the sense of the full-state norm
\[ \left( |\tilde{\vartheta}_w(t)|^2 + \int_0^D w^2(x, t) dx + w^2(D, t) \right)^{1/2}, \quad (39) \]
i.e., in the transformed variable \( \tilde{\vartheta}_w(w) \).

4.5. Step 5: Average system exponential stability estimate (in L^2 norm)

In order to assure exponential stability for the average system (25)–(27) in the sense of the norm
\[ \left( |\tilde{\vartheta}_w(t)|^2 + \int_0^D u_w^2(x, t) dx + u_w^2(D, t) \right)^{1/2}, \]
we need to show there exist constants \( \alpha_1 > 0 \) and \( \alpha_2 > 0 \) such that
\[ \alpha_1 \Psi(t) \leq V(t) \leq \alpha_2 \Psi(t), \quad (40) \]
where
\[ \Psi(t) = |\tilde{\vartheta}_w(t)|^2 + \int_0^D u_w^2(x, t) dx + u_w^2(D, t), \]
or using (24) and (28),
\[ \Psi(t) = |\tilde{\vartheta}_w(t - D)|^2 + \int_{t-D}^t U_w^2(\tau) d\tau + U_w^2(t). \quad (41) \]

Inequality (40) can be directly established from (29), (30), (34), by using the Cauchy–Schwarz inequality and other calculations, such as in the proof of Theorem 2.1 in [22]. Thus, taking into account (38), we obtain
\[ \Psi(t) \leq \frac{\alpha_2}{\alpha_1} e^{-\mu^* t} \Psi(0), \quad (42) \]
which concludes the proof of exponential stability in the original variables \( \langle \tilde{\vartheta}_w(u_w) \rangle \).

4.6. Step 6: Inverse optimality

Based on the proof of Theorem 6 in [19] and Theorem 2.8 in [25], we chose \( c^* = 4c^* \), \( c = 2c^* \) and define \( \zeta(t) \) as:
\[ \zeta(t) = -2c \dot{V}(t) + c(c - 4c^*) w^2(D, t) \]
\[ \geq c \left( \frac{1}{2a} \tilde{\vartheta}_w^2(t) + \frac{a}{2} \int_0^D w^2(x, t) dx + (c - 2c^*) w^2(D, t) \right) \quad (43) \]
where \( \tilde{\vartheta}_w(t) := \tilde{\vartheta}_w(t - D) \), according to (24).

Using (29) for \( x = D \) and the fact that \( u_w(D, t) = U_w(t) \), from (33) we get (27). Let us now consider \( w(D, t) \). From (29) and (30), it is easy to see that
\[ w_1(D, t) = \tilde{u}_w(D, t) - k Hu_w(D, t), \quad (44) \]
where \( \partial_t w(t) = -c w(t) - kH w(t) \). Plugging (30) and (33) into (44), we get
\[
\begin{align*}
 w(t) + c w(t) - kH w(t) & = \frac{1}{2} (1 + D) w^2(D, t) \\
 - (kH)^2 w^2(t) & = \left( \int_0^D e^{kH(\eta - \sigma)} w(\sigma, t) d\sigma \right)^2.
\end{align*}
\] (45)

By plugging (45) into the derivative of the Lyapunov functional (35), one has
\[
\dot{V}(t) = kH \tilde{\varphi}_w(t) + \tilde{\varphi}_w(t) (w(0, t) + \frac{a(1 + D)}{2} w^2(D, t))
\]
\[
- \frac{a}{2} w^2(0, t) - \frac{a}{2} \int_0^D w^2(x, t) dx - 2c^2 w^2(D, t)
- kH w^2(D, t) - (kH)^2 w^2(D, t) e^{kH(\tilde{\varphi}_w(t))}
- (kH)^2 w(D, t) \int_0^D e^{kH(\eta - \sigma)} w(\sigma, t) d\sigma.
\] (46)

Then, by applying (46) to (43), \( L(t) \) can be written as:
\[
\begin{align*}
 L(t) & = -2c kH \tilde{\varphi}_w(t) - 2c \tilde{\varphi}_w(t) w(0, t) - 2c a (1 + D) w^2(D, t) \\
 & + \frac{a}{2} w^2(0, t) + c a \int_0^D w^2(x, t) dx + 2c kH w^2(D, t)
 & + 2c kH w(D, t) e^{kH(\tilde{\varphi}_w(t))}
 & + 2c kH w(D, t) \int_0^D e^{kH(\eta - \sigma)} w(\sigma, t) d\sigma + c^2 w^2(D, t).
\end{align*}
\] (47)

On the other hand, substituting the average version of the system (25) into the target system (31), we obtain
\[
u_w(0, t) = kH \tilde{\varphi}_w(t) + w(0, t).
\] (48)

Rearranging (48) in order to isolate \( w(0, t) \), we can write:
\[
w(0, t) = u_w(0, t) - kH \tilde{\varphi}_w(t).
\] (49)

Then, plugging (49) into (47), and adding–subtracting the term \( c \tilde{\varphi}_w(t) \) (in blue) in the right-hand side of the resulting equation, lead us to
\[
\begin{align*}
 L(t) & = c \left( a kH \tilde{\varphi}_w(t) - 2a kH w(0, t) \tilde{\varphi}_w(t)
 & - a(1 + D) w^2(D, t)
 & - c^2 w^2(D, t)
 & + 2c kH \tilde{\varphi}_w(t)
 \right)
 \times \left( e^{kH(\tilde{\varphi}_w(t))} + \int_0^D e^{kH(\eta - \sigma)} w(\sigma, t) d\sigma \right)

 + a w^2(0, t) + \frac{a}{2} \int_0^D w^2(x, t) dx + w^2(D, t) [2c^2 + 2kH]

 + c \left( c \tilde{\varphi}_w(t) \right) + \frac{a}{2} \int_0^D w^2(x, t) dx + c^2 w^2(D, t).
\end{align*}
\] (50)

Reminding that \( \alpha = -\frac{1}{D} \), and replacing \( kH \) by \( \frac{1}{a} \) in (50), one has
\[
\begin{align*}
 L(t) & = c \left( \left[ \frac{1}{a} - \gamma \right] \tilde{\varphi}_w(t) (2c^* - a(1 + D) - \frac{2}{a}) w^2(D, t)
 & + a w^2(0, t) + \frac{a}{2} \int_0^D w^2(x, t) dx + \frac{2}{a^2} w^2(D, t)
 \right)
 \times \left( e^{kH(\tilde{\varphi}_w(t))} + \int_0^D e^{kH(\eta - \sigma)} w(\sigma, t) d\sigma \right)

 + c \left( c \tilde{\varphi}_w(t) \right) + \frac{a}{2} \int_0^D w^2(x, t) dx + c^2 w^2(D, t).
\end{align*}
\] (51)

After some mathematical manipulations, the term \( L(t) \) in (51) can be rewritten as:
\[
L(t) = \mathcal{V}(t) + c \left( c \tilde{\varphi}_w(t) \right) + \frac{a}{2} \int_0^D w^2(x, t) dx
\]
\[
+ (c - c^*) w^2(D, t).
\] (52)

where \( \mathcal{V}(t) \) is given by:
\[
\mathcal{V}(t) = c \left( \left[ \frac{1}{a} - \gamma \right] \tilde{\varphi}_w(t) + (2c^* - a(1 + D) - \frac{2}{a}) w^2(D, t)
 + a w^2(0, t) + \frac{a}{2} \int_0^D w^2(\sigma, t) d\sigma + \frac{2}{a^2} w^2(D, t) e^{kH(\tilde{\varphi}_w(t))}
 + \frac{2}{a^2} w(D, t) \int_0^D e^{kH(\eta - \sigma)} w(\sigma, t) d\sigma \right).
\] (53)

In order to satisfy inequality (19), it is necessary to ensure \( \mathcal{V}(t) \geq 0 \). To assure the latter condition, we will analyze the terms in (53) with undefined signs so that we can guarantee they are non-negative. After adding and subtracting the terms \( \frac{1}{\alpha} [\tilde{\varphi}_w + w^2(\eta, t)] \) and \( \frac{2\sqrt{D}}{a^2} [w^2(D, t) + \int_0^D w^2(\sigma, t) d\sigma] \) (in blue and red) into (53), \( \mathcal{V}(t) \) can be rewritten as:
\[
\begin{align*}
\mathcal{V}(t) & = c \left( \left[ \frac{1}{a} - \gamma \right] \tilde{\varphi}_w(t)
 + (2c^* - a(1 + D) - \frac{2}{a}) w^2(D, t)
 + a w^2(0, t) + \frac{a}{2} \int_0^D w^2(\sigma, t) d\sigma + \frac{2}{a^2} w^2(D, t) \int_0^D e^{kH(\eta - \sigma)} w(\sigma, t) d\sigma
 + \frac{2}{a^2} w(D, t) \int_0^D e^{kH(\eta - \sigma)} w(\sigma, t) d\sigma \right)
\end{align*}
\] (54)

By employing the Young and Cauchy–Schwarz inequalities, it is possible verify valid lower bounds for the terms which were added and subtracted in (54), so that:
\[
\begin{align*}
\frac{1}{a^2} w^2(D, t) + \frac{1}{a^2} \tilde{\varphi}_w(t) & \geq \frac{2}{a^2} \left| w(D, t) e^{kH(\tilde{\varphi}_w(t))} \right| 
\end{align*}
\] (55)

and
\[
\begin{align*}
\frac{2\sqrt{D}}{a^{2}} \left( w^2(D, t) + \int_0^D w^2(\sigma, t) d\sigma \right)
\end{align*}
\] (56)

Analyzing \( \mathcal{V}(t) \) in terms of the lower bounds in (55) and (56), we get
\[
\begin{align*}
\mathcal{V}(t) & \geq c \left( \left[ \frac{1}{a} - \gamma \right] \tilde{\varphi}_w(t)
 + (2c^* - a(1 + D) - \frac{2}{a}) w^2(D, t)
 + a w^2(0, t) + \frac{a}{2} \int_0^D w^2(\sigma, t) d\sigma + \frac{2}{a^2} w^2(D, t) \int_0^D e^{kH(\eta - \sigma)} w(\sigma, t) d\sigma
 + \frac{2}{a^2} w(D, t) \int_0^D e^{kH(\eta - \sigma)} w(\sigma, t) d\sigma \right).
\end{align*}
\] (57)
Then, to ensure $\gamma(D, t) \geq 0$ it is necessary to satisfy the following conditions.

**1st Condition:**
\[
\frac{1}{a} - \frac{1}{a^2} - \gamma > 0, \\
\gamma < \frac{a - 1}{a^2}.
\]
\[
(58)
\]

**2nd Condition:**
\[
c = 2c^*, \\
2c^* - a(1 + D) - \frac{2}{a} - \frac{1}{a^2} - \frac{2\sqrt{D}}{a^2} > 0, \\
c > a(1 + D) + \frac{2}{a} + \frac{1}{a^2} + \frac{2\sqrt{D}}{a^2}.
\]
\[
(59)
\]

**3rd Condition:**
\[
a = \frac{2}{a} - \frac{2\sqrt{D}}{a^2} > 0, \\
a > \frac{\sqrt{4\sqrt{D}}}{a}.
\]
\[
(60)
\]

Therefore, considering $\mathcal{L}(t)$ given in (52) and $\gamma(D, t)$ in (53), under the conditions imposed for $\gamma$, $a$ and $c$, one can conclude $\gamma(D, t) \geq 0$ and
\[
\mathcal{L}(t) \geq c \left( \frac{1}{2a} \theta^2_{av}(t) + \frac{a}{2} \int_0^t w^2(x, t)dx + (\epsilon - 2c^*)w^2(D, t) \right),
\]
with $\gamma = \frac{\sqrt{4\sqrt{D}}}{a}$.

Hence, we have $\mathcal{L}(t) \geq \mu \psi(t)^2$, for the same reason that (40) holds, completing the proof of inverse optimality.

**Corollary 1.** For the delay-free case ($D = 0$), consider the control law
\[
\hat{U}(t) = -cU(t) + cG(t).
\]
\[
(61)
\]
There exists $c^*$ such that the average feedback system of (5), with $D = 0$, and (61) is exponentially stable in the sense of the norm
\[
\psi(t) = \left( \bar{\theta}_{av}(t)^2 + U_{av}(t)^2 \right)^{1/2},
\]
\[
(62)
\]
for all $c > c^*$. Hence, there will exist $c^{**} > c^* > 0$ such that for any $c \geq c^{**}$, the feedback (61) minimizes the cost functional
\[
J = \int_0^\infty \mathcal{L}(t) + \hat{U}_{av}(t)dt,
\]
\[
(63)
\]
where $\mathcal{L}(t)$ is a functional of $(\bar{\theta}_{av}(t), U(t))$, and such that
\[
\mathcal{L}(t) \geq \mu \psi(t)^2
\]
\[
(64)
\]
for some $\mu(c) > 0$ with a property that $\mu(c) \to \infty$ as $c \to \infty$.

The proof of Corollary 1 can be straightforwardly obtained from Theorem 1 with the Lyapunov function $V(t) = \bar{\theta}_{av}(t)^2 + U_{av}(t)^2$.

The feedback (15) is not inverse optimal, however the feedback (16) is, for any $c \in [c^{**}, \infty)$. Its optimality holds for a relevant cost functional, which is lower bounded by the temporal $L_2[0, \infty]$ norm of the ODE state $\bar{\theta}_{av}(t)$, the norm of the average control $U_{av}(t)$, as well as the norm of its derivative $\hat{U}_{av}(t)$—in addition to $\int_0^t U_{av}(t)^2dt$ which is fixed because feedback has no influence on it. The controller (16) is stabilizing for $c = \infty$, namely, in its nominal form (15), however, since $\mu(\infty) = \infty$, it is not optimal with respect to a cost functional that includes a penalty on $U_{av}(t)$.

In our inverse optimality results of Theorem 1 and Corollary 1, we also want to minimize the update rate $U_{av}(t)$ of the filtered-predictor feedbacks (16) or (61) over the infinite interval [11] in order to improve the closed-loop performance in terms of transient responses and smooth control signals. However, we could study the inverse optimality of the average system with a different cost functional and the “input” in (18) or (63) not being the average update rate. For instance, the controller in [12, Theorem 3], in the context of stabilization rather than ES, does not employ a filter and the cost functional employs (only) the control instead of its derivative. Unfortunately, this result cannot be used for ES since time-varying delays are considered there, and we would not be able to apply the averaging theorem by [20] to complete the proof, as discussed in [26].

Finally, analogously to the Steps 6 and 7 performed for the proof of Theorem 1 in [7], we can invoke the averaging theorem in infinite dimensions by [20] and still conclude the following results for constant delays:
\[
\limsup_{t \to +\infty} |\theta(t) - \theta^*| = O(a + 1/\omega),
\]
\[
(65)
\]
\[
\limsup_{t \to +\infty} |y(t) - y^*| = O(a^2 + 1/\omega^2).
\]
\[
(66)
\]

5. Inverse optimality for Newton-based extremum seeking under delays

Let us define the measurable signal
\[
z(t) = \Gamma(t)G(t),
\]
\[
(67)
\]
where $\Gamma(t)$ is updated by the following Riccati differential equation [5]:
\[
\dot{\Gamma} = \omega R \Gamma^2,
\]
\[
(68)
\]
with $\omega_R > 0$ being a design constant. Eq. (68) generates an estimate of the Hessian’s inverse, avoiding inversions of the Hessian estimates that may be zero during the transient phase.

The estimate error of the Hessian’s inverse can be defined as
\[
\bar{\Gamma}(t) = \Gamma(t) - H^{-1},
\]
\[
(69)
\]
and its dynamic equation can be written from (68) and (69) by
\[
\dot{\bar{\Gamma}} = \omega R \bar{\Gamma}^2 | \Gamma + H^{-1} | \times | 1 - \bar{H} | (\bar{\Gamma} + H^{-1})).
\]
\[
(70)
\]

By using the averaging analysis, we can verify from (6) and (67) that
\[
z_{av}(t) = \Gamma_{av}(t)H\bar{\theta}_{av}(t - D).
\]
\[
(71)
\]
From (69), Eq. (71) can be written in terms of $\bar{\Gamma}_{av}(t) = \Gamma_{av}(t) - H^{-1}$ as
\[
z_{av}(t) = \bar{\theta}_{av}(t - D) + \bar{\Gamma}_{av}(t)H\bar{\theta}_{av}(t - D).
\]
\[
(72)
\]
The second term in the right side of (72) is quadratic in $(\bar{\Gamma}_{av}, \bar{\theta}_{av})$, thus, the linearization of $\Gamma_{av}(t)$ at $H^{-1}$ results in the linearized version of (71) given by
\[
z_{av}(t) = \bar{\theta}_{av}(t - D).
\]
\[
(73)
\]

5.1. Predictor feedback with averaging-based estimates using the Hessian’s inverse

From (5) and (73), the following average models can be written:
\[
\hat{\theta}_{av}(t - D) = U_{av}(t - D),
\]
\[
(74)
\]
\[
\dot{z}_{av}(t) = U_{av}(t - D),
\]
\[
(75)
\]
where $U_{av} \in \mathbb{R}$ is the resulting average control for $U \in \mathbb{R}$.
In order to motivate the predictor feedback design, the idea here is to compensate for the delay by feeding back the future state $z(t + D)$, or $z_d(t + D)$ in the equivalent average system.

To obtain $z_d(t + D)$ with the variation of constants formula to (75), the future state is written as

$$z_d(t + D) = z_d(t) + \int_{t-D}^{t} U_d(\sigma)d\sigma, \quad (76)$$

in terms of the control signal $U_d(\sigma)$ from the past window $[t-D, t]$. Given any stabilizing gain $k > 0$, the average control would be given by

$$U_d(t) = -k\left[z_d(t) + \int_{t-D}^{t} U_d(\sigma)d\sigma\right], \quad (77)$$

resulting in the average control feedback

$$U_d(t) = -kz_d(t + D), \quad \forall t \geq 0, \quad (78)$$

as desired. Hence, the average system would be, $\forall t \geq D$:

$$\frac{d\tilde{z}_d(t)}{dt} = -k\tilde{z}_d(t) - k\Gamma \hat{H}_d(z_d(t)). \quad (79)$$

Since $k\Gamma \hat{H}_d$ is quadratic in $(\tilde{z}_d, \hat{H}_d)$, the linearization of the system (79) has all its eigenvalues determined by $-k$. The (local) exponential stability of the algorithm can be guaranteed with a convergence rate which is independent of the unknown Hessian $H$, being user-assignable.

In the next sub-section, we show that the inverse optimal control objectives can still be achieved if a simple modification of the above basic predictor-based controller, which employs a low-pass filter, is applied. In this case, we propose the following infinite-dimensional and averaging-based predictor feedback in order to compensate the delay [11]:

$$U(t) = \frac{c}{s + c} \left\{ -k\left[z(t) + \int_{t-D}^{t} U(\tau)d\tau\right]\right\}, \quad (80)$$

where $c > 0$ is sufficiently large, i.e., the predictor feedback is of the form of a low-pass filtered version of the non average version of (77). Note that we mix again the time and frequency domain notation in (80) by using the braces $\{}$ to denote that the transfer function acts as an operator on a time-domain function.

The predictor feedback (80) is infinite-dimensional because the integral involves the control history over the interval $[t-D, t]$ and averaging-based (perturbation-based) because $z$ in (67) is updated according to the estimate $\Gamma$ for the unknown Hessian’s inverse $H^{-1}$ given by (68), with $\hat{H}(t)$ in (81) satisfying the averaging property (10).

Fig. 2 shows the predictor feedback with a perturbation-based estimate of the Hessian’s inverse and the low-pass filter in order to guarantee the inverse optimality for the closed-loop system.

5.2. Hessian-independent inverse optimality

Differently from the Gradient-based ES approach, where the inverse optimality is guaranteed with $c$ satisfying some upper bounds $c > c^*$ with $c^* = 4c^*$ for $c^*$ in (37) depending explicitly on the Hessian $H$ (or some upper bound for its induced norm), the inverse optimal design for the Newton case can be obtained independent of that. We provide such an intuitive result in the next theorem.

**Theorem 2.** There exists $c^*$ such that the average feedback system of (5) with (68) and (80) is exponentially stable in the sense of the norm

$$\Psi(t) = \left( |\tilde{z}_d(t)|^2 + |\tilde{z}_d(t - D)|^2 + \int_{t-D}^{t} U_d(\tau)^2d\tau + U_d(t)^2 \right)^{1/2} \quad (81)$$

for all $c > c^*$. Furthermore, there exists $c^{**} > c^*$ such that for any $c \geq c^{**}$, the feedback (80), with $k$ sufficiently small and $\alpha, \beta$ in (68) sufficiently large, minimizes the cost functional

$$J = \int_{0}^{\infty} \{\mathcal{L}(t) + \tilde{U}_d(t)^2\}dt, \quad (82)$$

where $\mathcal{L}(t)$ is a functional of $(\tilde{z}_d(t), \tilde{z}_d(t - D), U(\tau)), \forall t \in [t-D, t]$ and such that

$$\mathcal{L}(t) \geq \mu \Psi(t)^2 \quad (83)$$

for some $\mu(c) > 0$ with a property that $\mu(c) \to \infty$ as $c \to \infty$.

**Proof.** The proof of Theorem 2 follows steps similar to those employed to prove Theorem 1. In this sense, we will simply point out the main differences, instead of giving the full independent proof.

First of all, we give the average closed-loop system (20)–(22) with the predictor-based control law (80) with the following PDE-representation:

$$\dot{\tilde{z}}_d(t - D) = u_d(0, t), \quad (84)$$

$$\partial_t u_d(x, t) = \partial_x u_d(x, t), \quad x \in [0, D], \quad (85)$$

$$u_d(D, t) = \frac{c}{s + c} \left\{ -k\Gamma \hat{H}_d(z_d(t - D) + \int_{0}^{D} u_d(\sigma, t)d\sigma \right\}. \quad (86)$$

From (71) and (72), we can conclude that the linearization of $\Gamma \hat{H}_d(t)$ at $H^{-1}$ leads to the linearized version of (71) given by (73), i.e., $z_d(t) = \tilde{z}_d(t - D)$. Thus, the term $\Gamma \hat{H}_d(t)$ in (86) can...
be replaced simply by \(\tilde{\theta}_w(t-D)\) in this linearized model. Now, by denoting
\[
\tilde{\theta}(t) = \tilde{\theta}(t-D),
\] (87)
one has \(\tilde{\theta}_w(t) = z_w(t) = \tilde{\theta}_w(t-D)\) and the following linearized average model of (84)–(86) can be obtained:
\[
\begin{align*}
\dot{\tilde{\theta}}_w(t) & = u_w(0, t), \\
\partial_t u_w(x, t) & = \partial_t u_w(x, t), \quad x \in [0, D], \\
\frac{d}{dt} u_w(D, t) & = -c_t u_w(D, t) - ck \left[ \tilde{\theta}_w(t) \\
& \quad + \int_0^D u_w(\sigma, t) d\sigma \right],
\end{align*}
\] (88)
where the low-pass filter \(c/(s+c)\) is also represented in the state-space form. The solution of the transport PDE (89)–(90) is given by \(u_w(x, t) = U_w(t-x-D)\). On the other hand, the average model for the estimation error associated to the inverse of the Hessian in (70) is \(\frac{d\bar{V}_w(t)}{dt} = -\omega_D \bar{F}_w(t) - \omega_D H \bar{F}_w(t)\) and its linearized version is described by
\[
\frac{d\bar{V}_w(t)}{dt} = -\omega_D \bar{F}_w(t). 
\] (91)
The next infinite-dimensional backstepping transformation
\[
w(x, t) = u_w(x, t) + k \left[ \tilde{\theta}_w(t) + \int_0^x u_w(\sigma, t) d\sigma \right] 
\] (92)
maps the system (88)–(90) into the target system:
\[
\begin{align*}
\dot{\tilde{\theta}}_w(t) & = -k \tilde{\theta}_w(t) + w(0, t), \\
\partial_t w(x, t) & = w(x, t), \quad x \in [0, D], \\
w(D, t) & = -c \partial_t u_w(D, t).
\end{align*}
\] (93)
(94)
(95)
The following Lyapunov–Krasovskii functional
\[
V(t) = \frac{\tilde{\theta}_w^2(t)}{2} + \frac{\bar{F}_w^2(t)}{2} + a \int_0^D (1+\chi)w^2(x, t)dx + \frac{1}{2} w^2(D, t), 
\] (96)
with the parameter\(^2\) \(a = 1/k\), lead us to
\[
\dot{V}(t) \leq -\frac{1}{4a} \tilde{\theta}_w^2(t) - \omega_D \bar{F}_w^2(t) \\
- \frac{a}{2} (1+D) \int_0^D (1+\chi)w^2(x, t)dx - (c-c^*)w^2(D, t), 
\] (97)
where \(c^* = \frac{c(1+D) + k}{k} + a \left[ k|e^{-kD}\bar{F}_w^2| + \frac{1}{2} \left[ k^2 e^{-kD} - \bar{F}_w^2 \right]\right]\), independent of the Hessian \(\bar{F}_w\). Hence, from (97), if \(c\) is chosen such that \(c > c^*\), we obtain
\[
\dot{V}(t) \leq -\mu V(t), 
\] (98)
for some appropriate \(\mu > 0\). Consequently, the closed-loop system is exponentially stable in the sense of the following norm for the state
\[
\left( \bar{F}_w(t)^2 + \tilde{\theta}_w(t)^2 \right)^{1/2} + \int_0^D w^2(x, t)dx + w^2(D, t), 
\] (99)
i.e., in the transformed state variable \((\tilde{\theta}_w, w)\), or equivalently in the norm of the state for the original system \((\tilde{\theta}_w(t-D), U_w(t))\):
\[
\Psi(t) := \bar{F}_w(t)^2 + \tilde{\theta}_w(t-D)^2 + \int_0^D U_w^2(\tau) d\tau + U_w^2(t), 
\] (100)
using (87).

Now, we can choose \(c^* = 4c^* \text{ and } c = 2c^*\) as well as define \(\zeta(t)\) such that:
\[
\zeta(t) = -2c \dot{V}(t) + c(c - 4c^*)w^2(D, t) \\
\geq c \left( \frac{1}{2}k \tilde{\theta}_w^2(t) + \frac{\bar{F}_w^2(t)}{2} + a \int_0^D w^2(x, t)dx + (c-2c^*)w^2(D, t) \right), 
\] (101)
where \(\tilde{\theta}_w(t) := \tilde{\theta}_w(t-D)\). Plugging the derivative of the backstepping transformation (92) at \(x = D\) into the derivative of the Lyapunov–Krasovskii in (96), one has
\[
\begin{align*}
\dot{V}(t) & = -k \tilde{\theta}_w^2(t) + \bar{F}_w^2(t)w(0, t) - \omega_D F_w^2(t) + \frac{a(1+D)}{2} w^2(D, t) \\
& \quad - \frac{a}{2} w^2(0, t) - a \int_0^D w^2(x, t)dx + 2c^*w^2(D, t) \\
& \quad + k \bar{F}_w^2(D, t) - k \bar{F}_w^2(D, t)e^{-kD} \tilde{\theta}_w(t) \\
& \quad - k^2 \bar{F}_w^2(t) + \frac{a}{2} \int_0^D w^2(x, t)dx + (c-2c^*)w^2(D, t). 
\end{align*}
\] (102)
Thus, \(\zeta(t)\) is simply write as:
\[
\begin{align*}
\zeta(t) & = 2c k \tilde{\theta}_w^2(t) - 2c \bar{F}_w^2(t)w(0, t) + 2ca \bar{F}_w^2(t) \\
& \quad - 2c \frac{a(1+D)}{2} w^2(D, t) + ca w^2(0, t) - ca \int_0^D w^2(x, t)dx \\
& \quad - 2ck \bar{F}_w^2(D, t) + 2ck \bar{F}_w^2(D, t)e^{-kD} \tilde{\theta}_w(t) \\
& \quad + 2ck^2 \bar{F}_w^2(D, t) + \frac{a}{2} \int_0^D w^2(x, t)dx + c^2 w^2(D, t). 
\end{align*}
\] (103)
However, by replacing the average version of (88) in the target system (93), we get
\[
u_w(0, t) = -\bar{F}_w(t) + w(0, t). 
\] (104)
Isolating \(w(0, t)\) in (104), one achieves:
\[
w(0, t) = u_w(0, t) + k \bar{F}_w(t). 
\] (105)
Then, substituting (105) into (103) as well as adding–subtracting the terms \(\gamma \tilde{\theta}_w^2(t)\) and \(\bar{F}_w^2(t)\) over there, yields to:
\[
\begin{align*}
\zeta(t) & = c \left( ak \bar{F}_w^2(t)^2 - 2(\gamma - 2c^*)w^2(D, t) \right) \\
& \quad - \alpha(1+D)w^2(D, t) - \gamma \tilde{\theta}_w^2(t) - (\gamma - 2c^*)w^2(D, t) + 2ck^2 \bar{F}_w^2(D, t) \\
& \quad \times \left[ e^{-kD} \tilde{\theta}_w(t) + \frac{a}{2} \int_0^D w^2(x, t)dx + (c-2c^*)w^2(D, t) \right] \\
& \quad + \frac{a}{2} \int_0^D w^2(x, t)dx + (c-2c^*)w^2(D, t). 
\end{align*}
\] (106)
Recalling that \(a = 1/k\) and substituting \(k = \frac{1}{a}\) in (106), results in
\[
\zeta(t) = c \left( \left( \frac{1}{a} - \gamma \right) \tilde{\theta}_w^2(t) + (2c^* - a(1+D) - \frac{2}{a}c) w^2(D, t) \right) \\
- (\gamma - 2c^*)\bar{F}_w^2(t) + a u_w(0, t) + \frac{a}{2} \int_0^D w^2(x, t)dx + \frac{a}{2} w^2(D, t) \\
\times \left[ e^{-kD} \tilde{\theta}_w(t) + \frac{a}{2} \int_0^D w^2(x, t)dx + (c-2c^*)w^2(D, t) \right] \\
+ \frac{a}{2} \int_0^D w^2(x, t)dx + (c-2c^*)w^2(D, t). 
\] (107)
\(^2\) With some abuse of notation, such a parameter must not be understood as the amplitude “\(a\)” of the probing and demodulation signals (7) and (9).
Hence, $\mathcal{L}(t)$ can be rewritten as:

$$
\mathcal{L}(t) = \mathcal{Y}(D, t) + c(\gamma \bar{\Delta}_{\psi}(t) + \bar{\gamma} \bar{F}_{\psi}(t))
+ \frac{a}{2} \int_0^D u^2(x, t) dx + (c - 2c^*)w^2(D, t)),
$$

(108)

with $\mathcal{Y}(D, t)$ given by

$$
\mathcal{Y}(D, t) = c \left( \left[ \left( \frac{1}{a} - \frac{1}{a^2} \right) \gamma \tilde{\Delta}_{\psi}(t) \right] - (\gamma - 2\omega_\gamma) \bar{F}_{\psi}(t) + \frac{a}{2} \int_0^D w^2(\sigma, t) d\sigma \right)
+ \left[ \left( \frac{1}{a} - \frac{1}{a^2} \right) \tilde{\Delta}_{\psi}(t) \right] + \left( \frac{1}{a} - \frac{1}{a^2} \right) \int_0^D w^2(\sigma, t) d\sigma
+ \frac{2}{a^2} w(D, t)e^{-k\Delta_{\psi}(t)} + \frac{2}{a^2} w(D, t)e^{-k(\Delta_{\psi} - \omega_{\Delta_{\psi}})} \right) d\sigma.

(109)

In order to satisfy inequality (101), it is necessary to guarantee that $\mathcal{Y}(D, t) \geq 0$. In this sense, there will be analyzed the non-quadric terms in (109) such that we can assure they are in fact equal or greater than zero. After adding–subtracting the terms $\frac{1}{a} [\tilde{\Delta}_{\psi} + u(D, t)]$ and $\frac{2}{a^2} w^2(D, t) + \int_0^D w^2(\sigma, t) d\sigma$ into (109), we can rewrite $\mathcal{Y}(D, t)$ as:

$$
\mathcal{Y}(D, t) = c \left( \left[ \left( \frac{1}{a} - \frac{1}{a^2} \right) \gamma \tilde{\Delta}_{\psi}(t) \right] - (\gamma - 2\omega_\gamma) \bar{F}_{\psi}(t) + \frac{a}{2} \int_0^D w^2(\sigma, t) d\sigma \right)
+ \left[ \left( \frac{1}{a} - \frac{1}{a^2} \right) \tilde{\Delta}_{\psi}(t) \right] + \left( \frac{1}{a} - \frac{1}{a^2} \right) \int_0^D w^2(\sigma, t) d\sigma
+ \frac{2}{a^2} w(D, t)e^{-k\Delta_{\psi}(t)} + \frac{2}{a^2} w(D, t)e^{-k(\Delta_{\psi} - \omega_{\Delta_{\psi}})} \right) d\sigma.

(110)

By means of the Young and Cauchy–Schwarz inequalities, it is possible to check that the upper bounds for the included terms in (110) satisfy:

$$
\frac{1}{a^2} w^2(D, t) + \frac{1}{a^2} \tilde{\Delta}_{\psi}(t) \geq \frac{1}{a^2} \left| w(D, t)e^{-k\Delta_{\psi}(t)} \right|,
$$

(111)

and

$$
\frac{2}{a^2} \left( w^2(D, t) + \int_0^D w^2(\sigma, t) d\sigma \right)
\geq \frac{2}{a^2} \left| w(D, t) \int_0^D e^{-k(\Delta_{\psi} - \omega_{\Delta_{\psi}})} w(\sigma, t) d\sigma \right|.
$$

(112)

From the lower bounds of $\mathcal{Y}(D, t)$, we conclude that

$$
\mathcal{Y}(D, t) \geq c \left( \left[ \left( \frac{1}{a} - \frac{1}{a^2} \right) \gamma \tilde{\Delta}_{\psi}(t) \right] + (\gamma - 2\omega_\gamma) \bar{F}_{\psi}(t) \right)
+ \left( \frac{1}{a} - \frac{1}{a^2} \right) \tilde{\Delta}_{\psi}(t) + \left( \frac{1}{a} - \frac{1}{a^2} \right) \int_0^D w^2(\sigma, t) d\sigma
+ \frac{2}{a^2} w(D, t)e^{-k\Delta_{\psi}(t)} + \frac{2}{a^2} w(D, t)e^{-k(\Delta_{\psi} - \omega_{\Delta_{\psi}})} \right) d\sigma.
$$

(113)

In order to obtain $\mathcal{Y}(D, t) \geq 0$, we need to verify the conditions.

1st Condition:

$$
\left\{ \begin{array}{l}
\frac{1}{a} - \frac{1}{a^2} - \gamma > 0, \\
\gamma < \frac{a - 1}{a^2}.
\end{array} \right.
$$

2nd Condition:

$$
2\omega_\gamma - \bar{\gamma} > 0, \\
\bar{\gamma} < 2\omega_\gamma.
$$

3rd Condition: Reminding that $c = 2c^*$

$$
2c^* - a(1 + D) - \frac{2}{a} \frac{1}{a^2} - \frac{2}{a^2} \sqrt{D} > 0,
\left\{ \begin{array}{l}
c > a(1 + D) + \frac{2}{a} + \frac{1}{a^2} + \frac{2}{a^2} \sqrt{D}.
\end{array} \right.
$$

4th Condition:

$$
\frac{a}{2} - \frac{2}{a^2} \sqrt{D} > 0,
\left\{ \begin{array}{l}
a > \frac{\sqrt{4\sqrt{D}}}{2}.
\end{array} \right.
$$

Finally, considering $\mathcal{L}(t)$ in (108) and $\mathcal{Y}(D, t)$ in (109) satisfying the four conditions above for $\gamma$, $\tilde{\gamma}$, $\tilde{\gamma}$, and $c$, one can conclude that $\mathcal{Y}(D, t) \geq 0$. Consequently,

$$
\mathcal{L}(t) \geq c \left( \int_0^D \frac{1}{2} k\Delta_{\psi}(t) + \frac{\bar{F}_{\psi}(t)}{2} + \frac{a}{2} \int_0^D w^2(x, t) dx + (c - 2c^*)w^2(D, t) \right),
$$

(114)

with $\gamma = k/2$ and $\tilde{\gamma} = 1/2$. Thus, $\mathcal{L}(t) \geq \mu\Psi(t)^2$ can be written in terms of the norm in (100), thus completing the proof of inverse optimality for the Newton algorithm.

In addition to (65) and (66), which can also be proved according to [21, Theorem 1], we can also write

$$
\limsup_{t \to +\infty} |\Gamma(t) - H^{-1}| = o(a + 1/\omega).
$$

(115)

Corollary 2. For the delay-free case ($D = 0$), consider the control law

$$
U(t) = -cU(t) + c\kappa(t).
$$

(116)

There exists $c^*$ such that the average feedback system of (5), with $D = 0$, (68) and (116) is exponentially stable in the sense of the norm

$$
\Psi(t) = \left( |\tilde{T}_{\psi}(t)|^2 + |\tilde{\theta}_{\psi}(t)|^2 + U_{\psi}(t)^2 \right)^{1/2},
$$

(117)

for all $c \geq c^*$. Thus, there will exist $c^{**} > c^* > 0$ such that for any $c \geq c^{**}$, the feedback (61) minimizes the cost functional

$$
\tilde{J} = \int_0^{\infty} \mathcal{L}(t) + \tilde{\mathcal{Y}}^2_{\psi}(t) dt,
$$

(118)

where $\mathcal{L}(t)$ is a functional of $(\tilde{T}_{\psi}(t), \tilde{\theta}_{\psi}(t), U(t))$, and such that $\mathcal{L}(t) \geq \mu\Psi(t)^2$

(119)

for some $\mu(c) > 0$ with a property that $\mu(c) \to \infty$ as $c \to \infty$.

The proof of the result of Corollary 2 can be straightforwardly obtained from Theorem 2 with the Lyapunov function $V(t) = \tilde{T}_{\psi}(t)/2 + \tilde{\theta}_{\psi}(t)/2 + U_{\psi}(t)/2$. 

6. Numerical simulations

In order to evaluate the effects of the inverse optimality for the ES feedback under delays, the next quadratic map (1)–(2) is considered: 

\[ Q(\theta) = 5 - 0.1(\theta - 3)^2, \]

with an output delay of \( D = 5 \text{s} \). The extremum point is \((\theta^*; y^*) = (3; 5)\) and the Hessian of the corresponding static map is \( H = -0.2 \). For the simulation tests, we have considered only the Gradient-based ES approach, where the following parameters were employed: \( \omega = 10 \text{ rad/s}, k = 0.8, \dot{\theta}(0) = -5 \) and \( a = 0.2 \). The time constant of the low-pass filter \( c = 40 \) was chosen to satisfy the Conditions 1–3 in the Step 6 of the proof of Theorem 1.

Fig. 3 presents a numerical comparison between the ES fundamental variables with and without using the filter \( \frac{1}{1+cs} \) in feedback law (16). As it can be observed, in the first case where the inverse optimality is guaranteed, the input–output signals \( \theta(t) \) and \( y(t) \) converge monotonically rather than swinging up-and-down, thus improving the transient responses. Moreover, the undesired oscillations in the Hessian estimate \( \dot{H}(t) \) are reduced as well as the amplitude of the control signal \( U(t) \).

Although the monotonic convergence to the extremum could be observed, this monotonicity is not to be expected in general because we do not know how to prove it for the average system, composed by a complicated cascade connection of ODE–PDE–ODE systems. In addition, Fig. 4 brings numerical results employing the Newton-based approach in order to compare the effect of various values of the filter parameter \( c \) on inverse optimality. There, we can noticed that the monotonicity is not always preserved.

7. Conclusions

In this paper, we derived inverse optimality results for extremum seeking feedback (both Gradient and Newton versions) with the low-pass filtered modification of the predictor-based feedback for delay compensation proposed in [7] and [21]. Extremum seeking is studied with Lyapunov tools and has a control input whose cost can be optimized over infinite time. We have established the stability robustness to the varying parameter \( c \) from some large value \( c^* \) to \( +\infty \), recovering in the limit, the basic, unfiltered predictor-based feedback (15). This robustness property might be intuitively expected from a singular perturbation idea, though an off-the-shelf theorem for establishing this property would be highly unlikely to be found in the literature, due to the infinite dimensionality and the special hybrid (ODE–PDE) structure of the system at hand.
Fig. 4. Time evolution of $\theta(t)$ under variation of the filter parameter $c \in [0.01, 0.1]$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Although the delay-case has been the main focus throughout the paper, the inverse optimality can also be assured for the delay-free case. The inverse optimality properties of the basic predictor feedback controller are illustrated by a numerical example.

For future research, we indicate the study of inverse optimality in the presence of delay mismatch [11,27] and for multiple-input maps [28], or even considering more general classes of PDE systems [29]. It will be also interesting to find whether
performance criteria exist under which other extremum seeking designs [30–33] are inverse optimal as well.

CRediT authorship contribution statement

**Denis Cesar Ferreira:** Conceptualization, Formal analysis, Investigation, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Tiago Roux Oliveira:** Conceptualization, Formal analysis, Investigation, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **Miroslav Krstic:** Formal analysis, Investigation, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

References